Final Exam Spring 2017: Question 4

Two countries have the same production functions $Y = K^{\alpha}(AL)^{1-\alpha}$. They also have the same steady state levels of income per capita. Finally, they have the same current levels of income per capita, which are below the steady state. However, they have different values of investment rates. Specifically, country 1 has a higher rate of investment (γ) than does country 2.

Which country has higher growth? Why? (hint: think in terms of $\triangle k$. In which country is it higher?).

Note: you can answer this question by drawing a simple graph. You don't have to go through a big algebraic proof or anything.

I) Algebraic solution: Simpler and suggested version

Assuming $\gamma_1 > \gamma_2$ we want to compare $\triangle k_t^1$ and $\triangle k_t^2$.

• Step 1: derive the law of motion of capital per worker in its approximate version: $\triangle k_t \approx \gamma y_t - (n+\delta)k_t > 0$, because $k_t < k^*$.

• Step 2: in steady state $\triangle k^* = \gamma y^* - (n+\delta)k^* = 0 \Leftrightarrow (n+\delta) = \gamma \frac{y^*}{k^*}$

• Step 3: Replacing $(n+\delta)$ in (1) by (2): $\triangle k_t = \gamma y_t - \gamma \frac{y^*}{k^*} k_t = \gamma \left(y_t - \frac{y^*}{k^*} k_t \right) > 0$

NOTE that $\left(y_t - \frac{y^*}{k^*} k_t \right)$ is the same for both countries, and is positive.

• Step 4: $\triangle k_t^1 - \triangle k_t^2 = \gamma_1 \left(y_t - \frac{y^*}{k^*} k_t \right) - \gamma_2 \left(y_t - \frac{y^*}{k^*} k_t \right) = (\gamma_1 - \gamma_2) \left(y_t - \frac{y^*}{k^*} k_t \right) > 0 \blacksquare$

II) Algebraic solution: Other version (more correct, but "not recommended" for an exam)

Assuming $\gamma_1 > \gamma_2$ we want to compare $\triangle k_t^1$ and $\triangle k_t^2$.

• Step 1: derive the law of motion of capital per worker (not approximated): $\triangle k_t = \frac{1}{1+n} [\gamma y_t - (n+\delta)k_t] > 0$, because $k_t < k^*$.

• Step 2: in steady state $\triangle k^* = 0 \Leftrightarrow \gamma y^* - (n+\delta)k^* = 0 \Leftrightarrow (n+\delta) = \gamma \frac{y^*}{k^*}$

• Step 3: Replacing $(n+\delta)$ in (1) by (2):
$\triangle k_t = \frac{1}{1+n} \left[\gamma y_t - \gamma \frac{y^*}{k^*} k_t \right] = \frac{\gamma}{1+n} \left(y_t - \frac{y^*}{k^*} k_t \right) > 0$

NOTE that $\left(y_t - \frac{y^*}{k^*}k_t\right)$ is the same for both countries and is positive; and that $\frac{\gamma}{n+\delta}$ is the same for both countries because the steady state is the same.

• Step 4: $\triangle k_t^1 > \triangle k_t^2 \Leftrightarrow \frac{\gamma_1}{1+n_1}\left(y_t - \frac{y^*}{k^*}k_t\right) > \frac{\gamma_2}{1+n_2}\left(y_t - \frac{y^*}{k^*}k_t\right) \Leftrightarrow \frac{\gamma_1}{1+n_1} > \frac{\gamma_2}{1+n_2}$
$\Leftrightarrow \frac{1+n_2}{\gamma_2} > \frac{1+n_1}{\gamma_1} \Leftrightarrow \frac{1-\delta+\delta+n_2}{\gamma_2} > \frac{1-\delta+\delta+n_1}{\gamma_1} \Leftrightarrow \frac{1-\delta}{\gamma_2} + \frac{\delta+n_2}{\gamma_2} > \frac{1-\delta}{\gamma_1} + \frac{\delta+n_1}{\gamma_1}$
$\Leftrightarrow \frac{1-\delta}{\gamma_2} > \frac{1-\delta}{\gamma_1} \Leftrightarrow \gamma_1 > \gamma_2, TRUE \blacksquare$