ROB-UY 2004

Robotic Manipulation & Locomotion

Lecture 07A - Velocities

NASA.gov

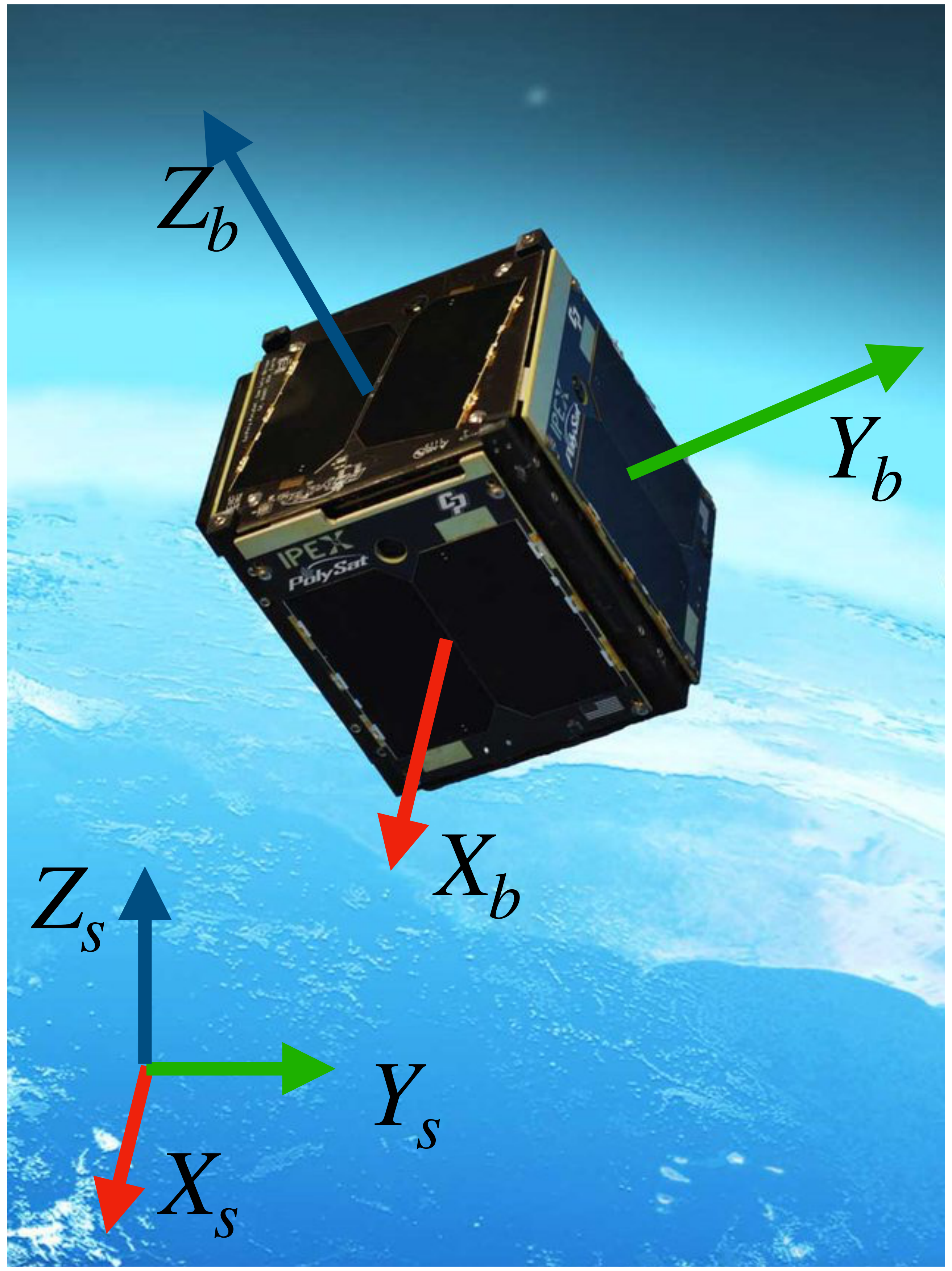
Agenda

1. Motivation
2. Helpful properties
3. Rotation Rate Representations

Agenda

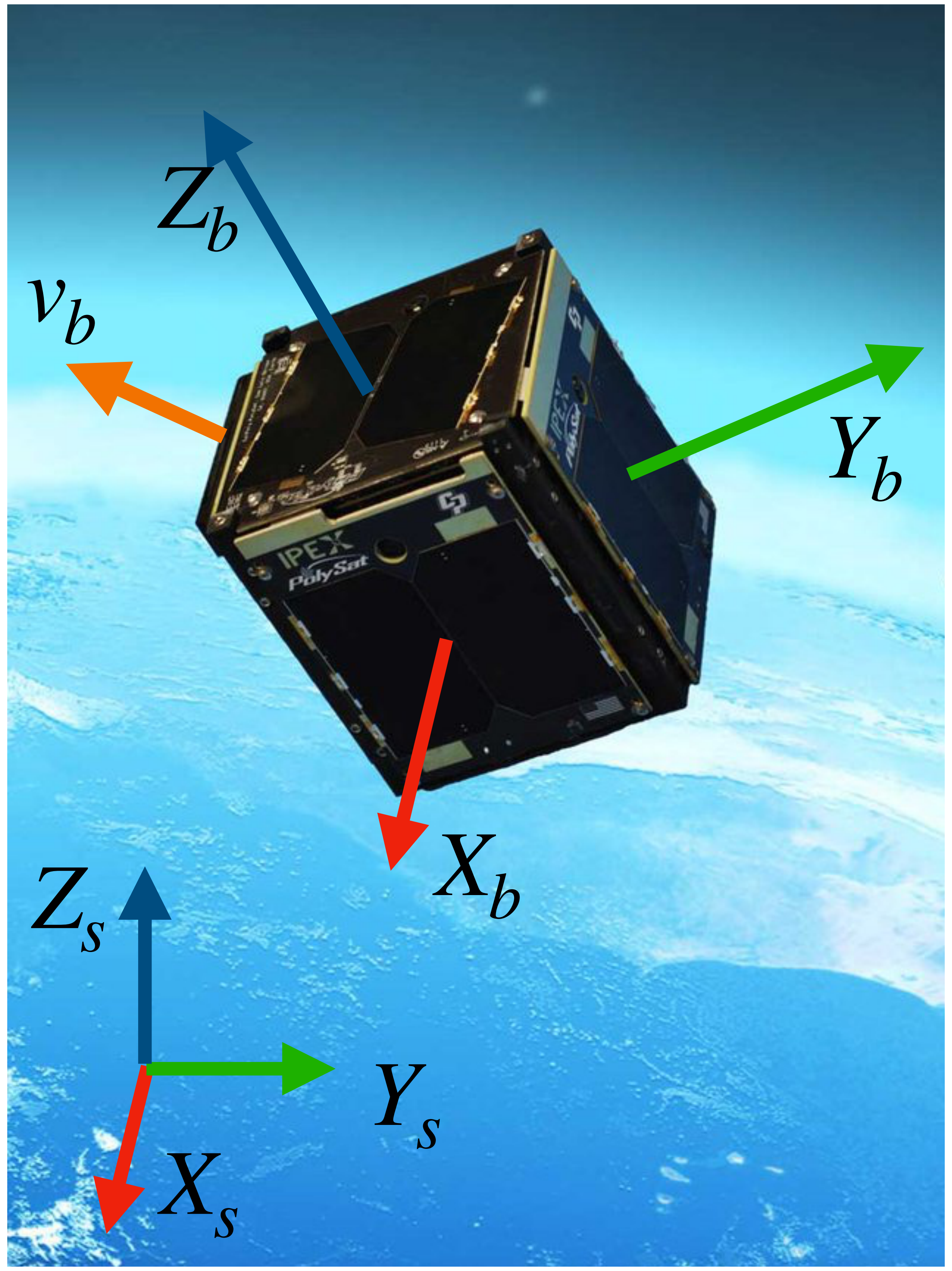
1. **Motivation**
2. Helpful properties
3. Rotation Rate Representations

Object Motion



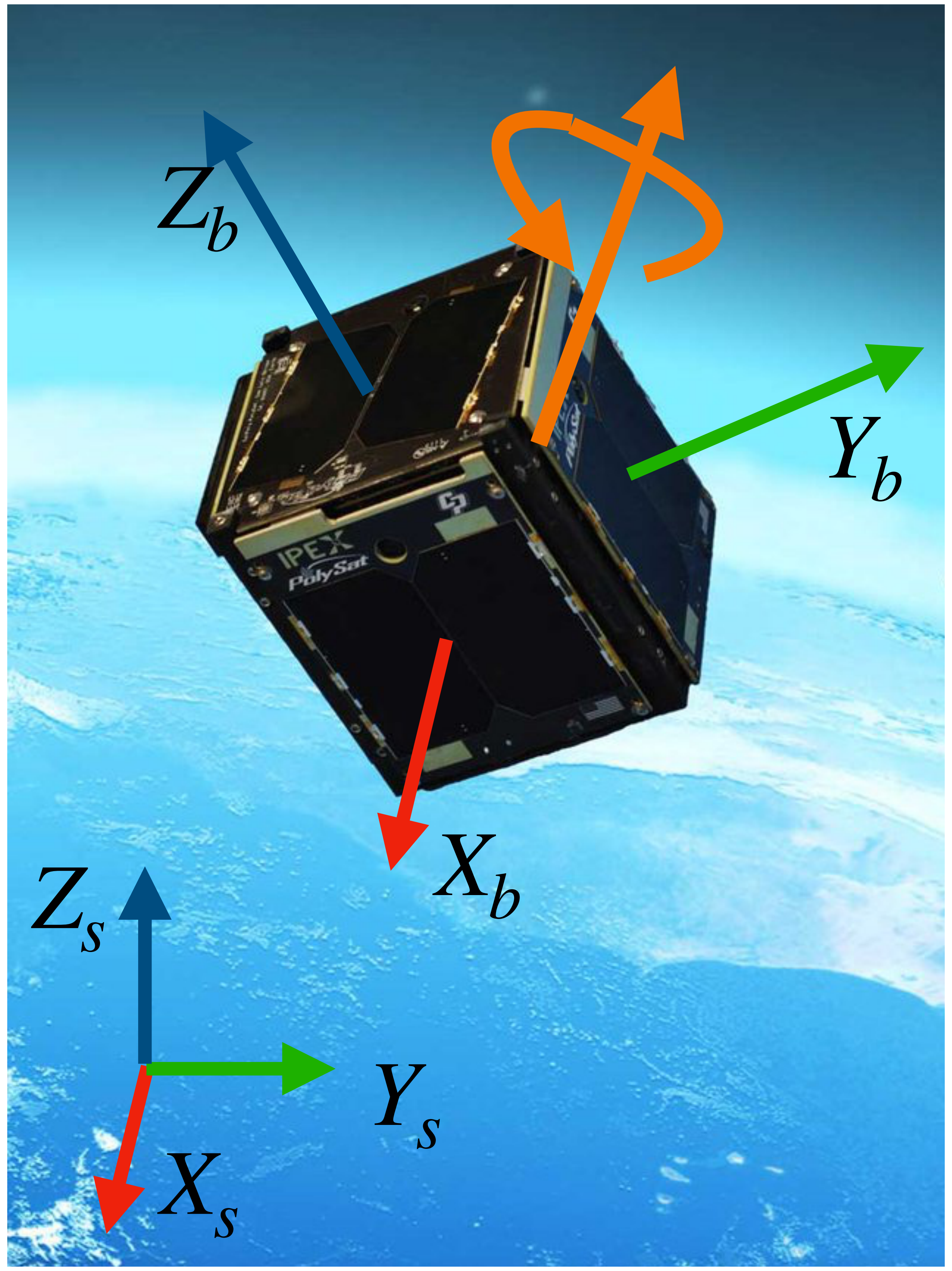
Object Motion

- We want to describe the **translation** rate of the object in a frame



Object Motion

- We want to describe the **rotation** rate of the object in a frame



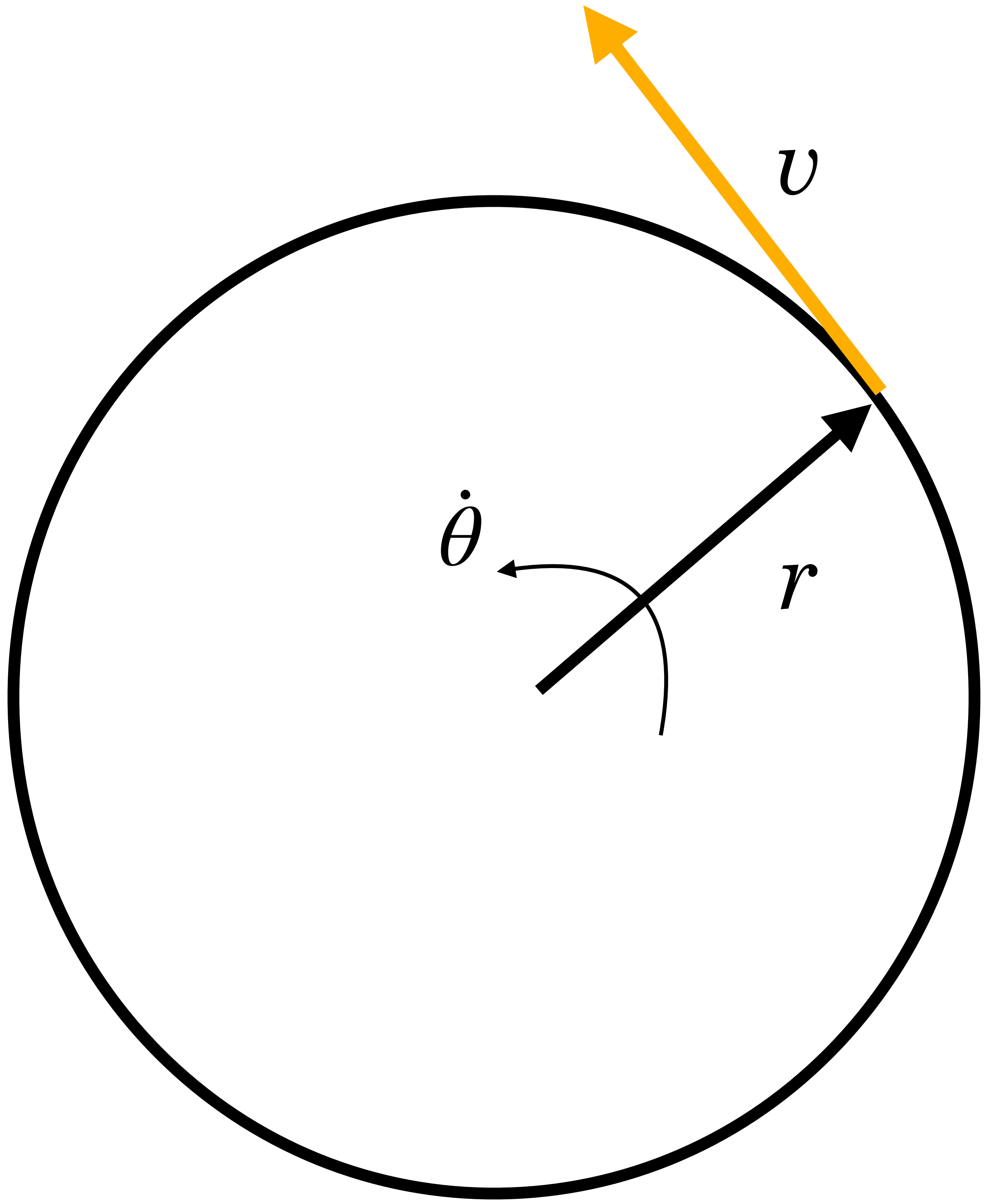
Agenda

1. Motivation
2. **Helpful properties**
3. Rotation Rate Representations

Property 1: Rotational velocity of a point

- The (linear) velocity v of a point rotating around a circle of radius r with rotational rate $\dot{\theta}$ is:

$$v = \dot{\theta} r$$

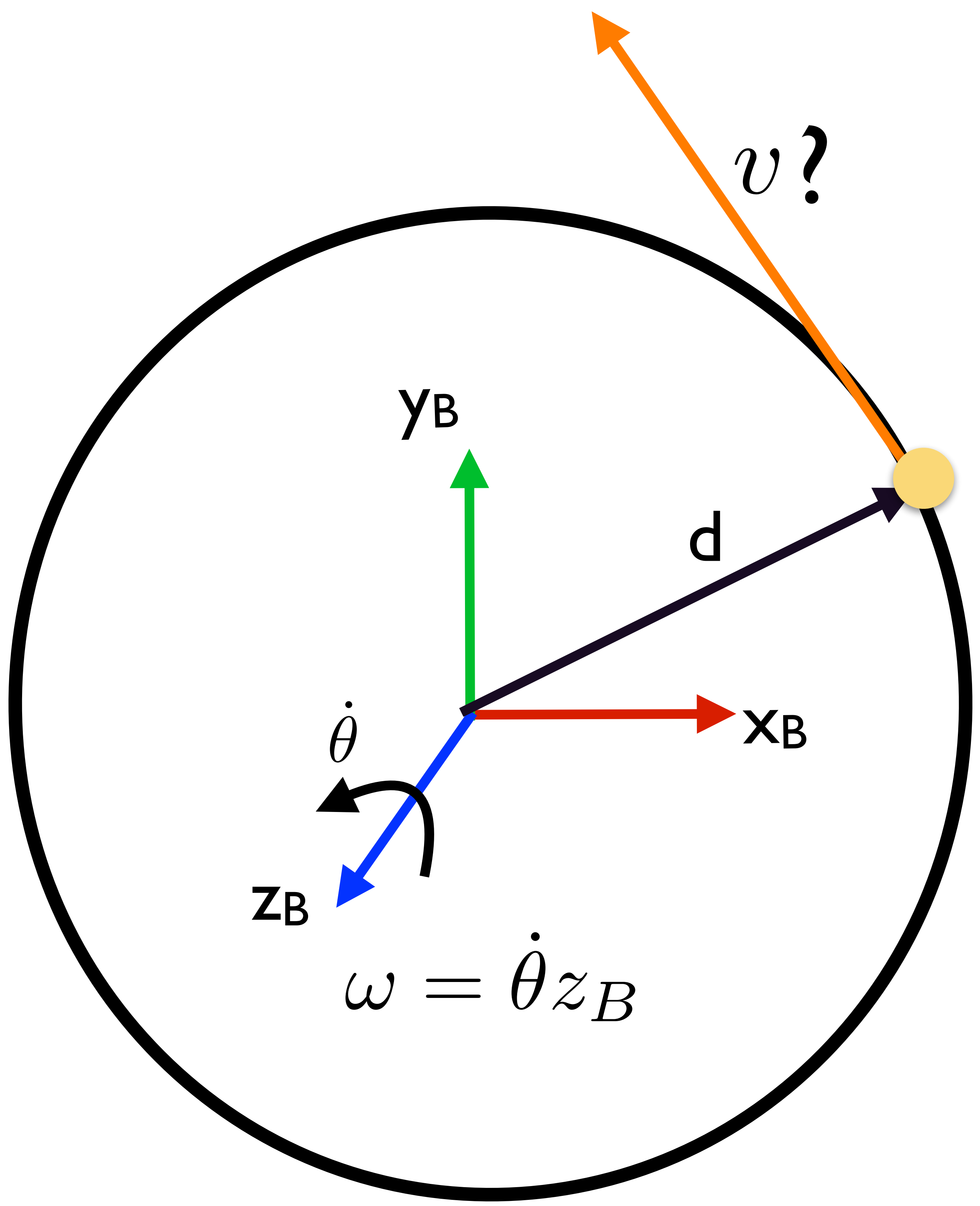


Property 1: Rotational velocity of a point

- The (linear) velocity of a point rotating at an angular velocity $\omega \in R^3$ is:

$$v = \omega \times d$$

- where ω encodes both the axis of rotation, its direction, and the speed of rotation $||\omega||$, and d is a vector that goes from the axis of rotation to the position of the point.



Property 2: Cross Product

$$a \times b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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$$a \times b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Property 2: Cross Product

$$a \times b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Property 2: Cross Product - Bracket Notation

$$a \times b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{[a]} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Property 2: Cross Product - Bracket Notation

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$$a \times b = [a]b$$

Property 3: Proposition 3.81

Proof. Letting r_i^{T} be the ith row of R, we have

$$\begin{aligned} R[\omega]R^{\mathrm{T}} &= \begin{bmatrix} r_1^{\mathrm{T}}(\omega \times r_1) & r_1^{\mathrm{T}}(\omega \times r_2) & r_1^{\mathrm{T}}(\omega \times r_3) \\ r_2^{\mathrm{T}}(\omega \times r_1) & r_2^{\mathrm{T}}(\omega \times r_2) & r_2^{\mathrm{T}}(\omega \times r_3) \\ r_3^{\mathrm{T}}(\omega \times r_1) & r_3^{\mathrm{T}}(\omega \times r_2) & r_3^{\mathrm{T}}(\omega \times r_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -r_3^{\mathrm{T}}\omega & r_2^{\mathrm{T}}\omega \\ r_3^{\mathrm{T}}\omega & 0 & -r_1^{\mathrm{T}}\omega \\ -r_2^{\mathrm{T}}\omega & r_1^{\mathrm{T}}\omega & 0 \end{bmatrix} \\ &= [R\omega], \end{aligned} \tag{3.32}$$

where the second line makes use of the determinant formula for 3×3 matrices, i.e., if M is a 3×3 matrix with columns $\{a, b, c\}$, then $\det M = a^{\mathrm{T}}(b \times c) = c^{\mathrm{T}}(a \times b) = b^{\mathrm{T}}(c \times a)$. $\square$

$$\downarrow$$

$$[R\omega] = R^T[\omega]R$$

Modern Robotics (K. Lynch

Property 3: Proposition 3.81

Proof. Letting r_i^{T} be the ith row of R, we have

$$\begin{aligned} R[\omega]R^{\mathrm{T}} &= \begin{bmatrix} r_1^{\mathrm{T}}(\omega \times r_1) & r_1^{\mathrm{T}}(\omega \times r_2) & r_1^{\mathrm{T}}(\omega \times r_3) \\ r_2^{\mathrm{T}}(\omega \times r_1) & r_2^{\mathrm{T}}(\omega \times r_2) & r_2^{\mathrm{T}}(\omega \times r_3) \\ r_3^{\mathrm{T}}(\omega \times r_1) & r_3^{\mathrm{T}}(\omega \times r_2) & r_3^{\mathrm{T}}(\omega \times r_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -r_3^{\mathrm{T}}\omega & r_2^{\mathrm{T}}\omega \\ r_3^{\mathrm{T}}\omega & 0 & -r_1^{\mathrm{T}}\omega \\ -r_2^{\mathrm{T}}\omega & r_1^{\mathrm{T}}\omega & 0 \end{bmatrix} \\ &= [R\omega], \end{aligned} \tag{3.32}$$

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↓

$$[R\omega] = R^T[\omega]R$$

$$[R_{SB}\omega_B] = R_{SB}^T[\omega_B]R_{SB}$$

Modern Robotics (K. Lynch

Agenda

1. Motivation
2. Helpful properties
3. Rotation Rate Representations

$$\omega_S = ?$$

$$\omega_B = ?$$

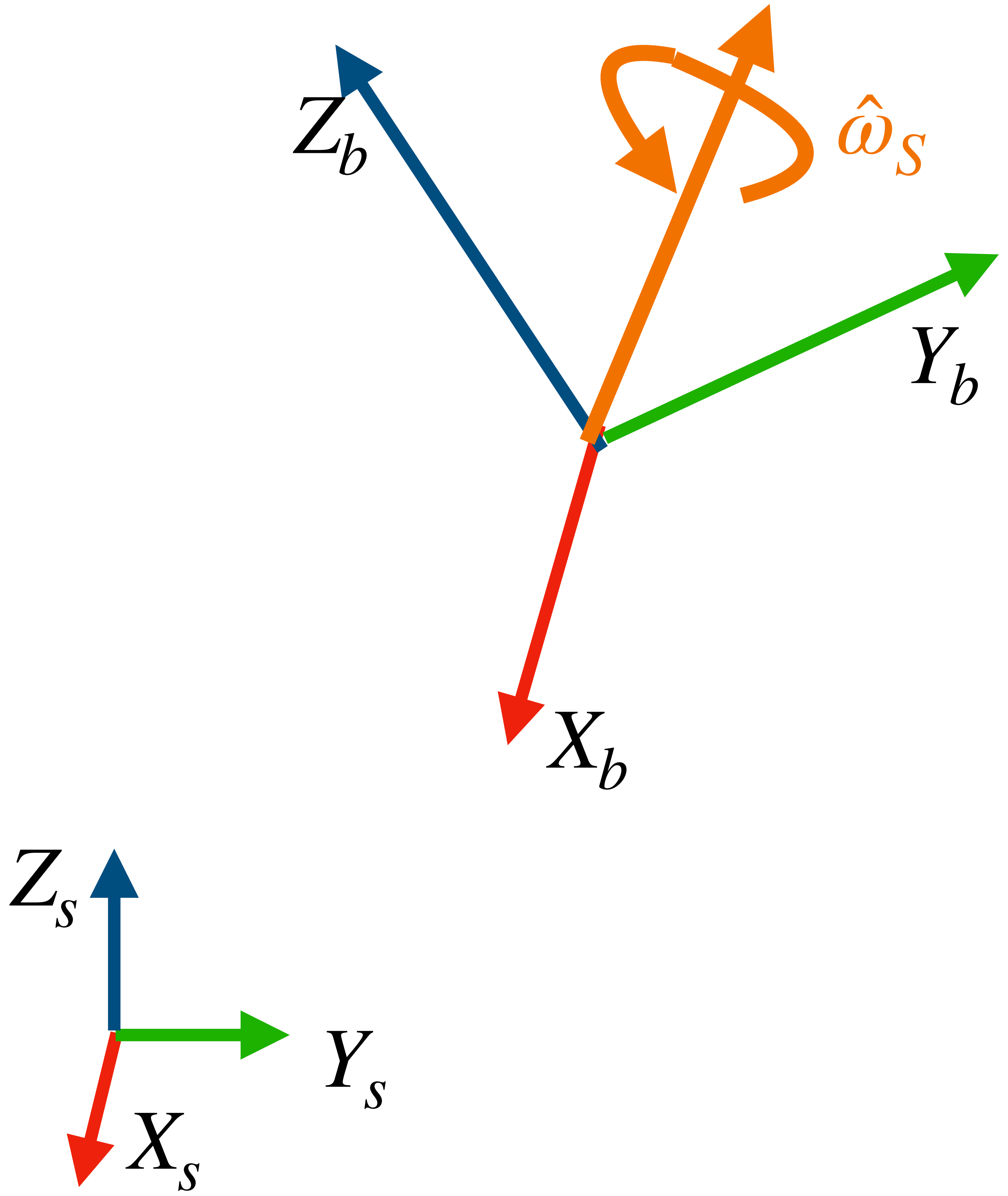
$$\dot{R}_{SB} = ?$$

$$[\omega_S] = ?$$

$$[\omega_B] = ?$$

Object Motion

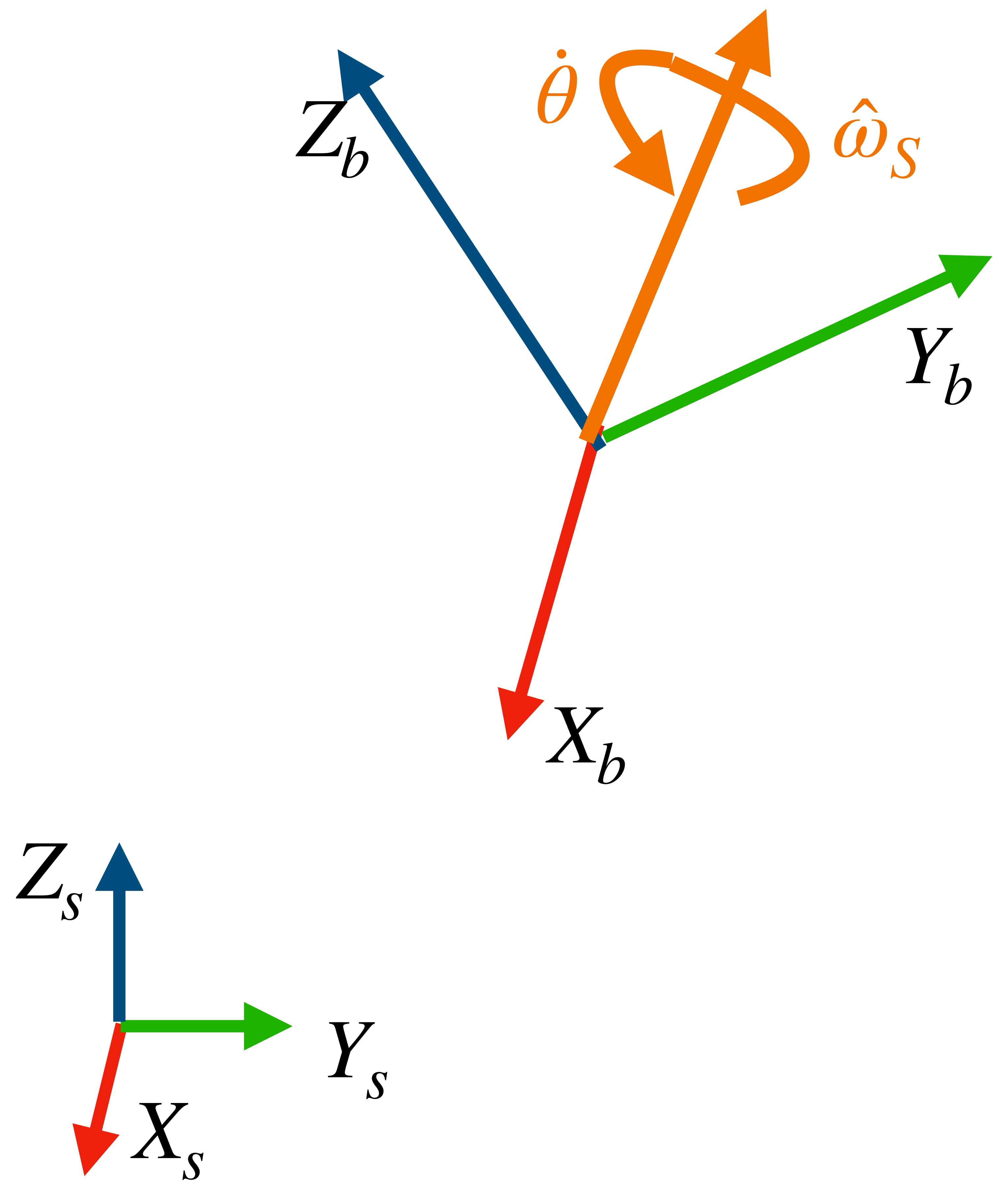
- We want to describe the **rotation** rate of the object in a frame



Object Motion

- We want to describe the **rotation** rate of the object in a frame

$$\omega_S = \hat{\omega}_S \dot{\theta}$$



Object Motion

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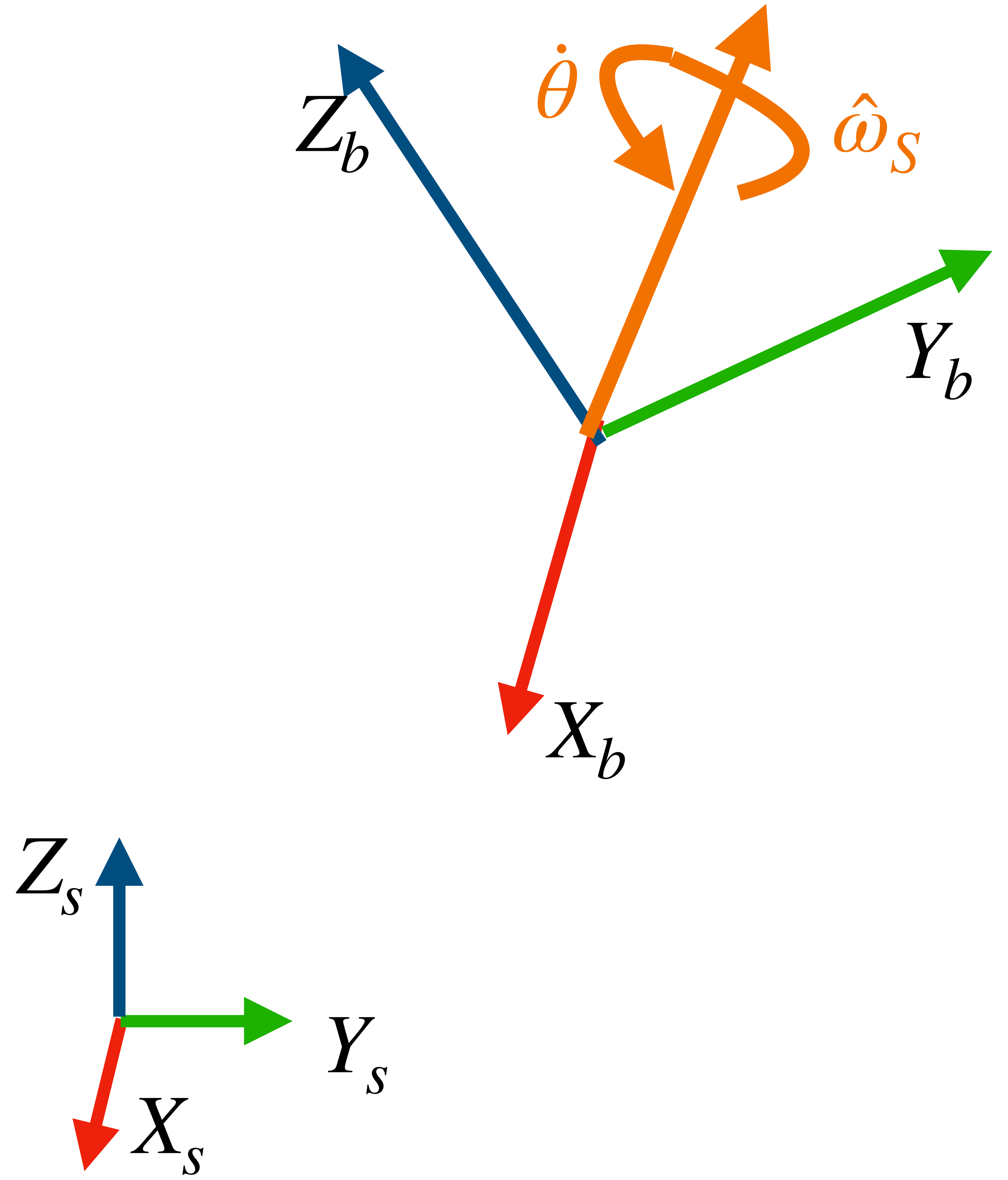
$$\omega_S = \hat{\omega}_S \dot{\theta}$$

Rotation vector — Unit vector — Rotation rate

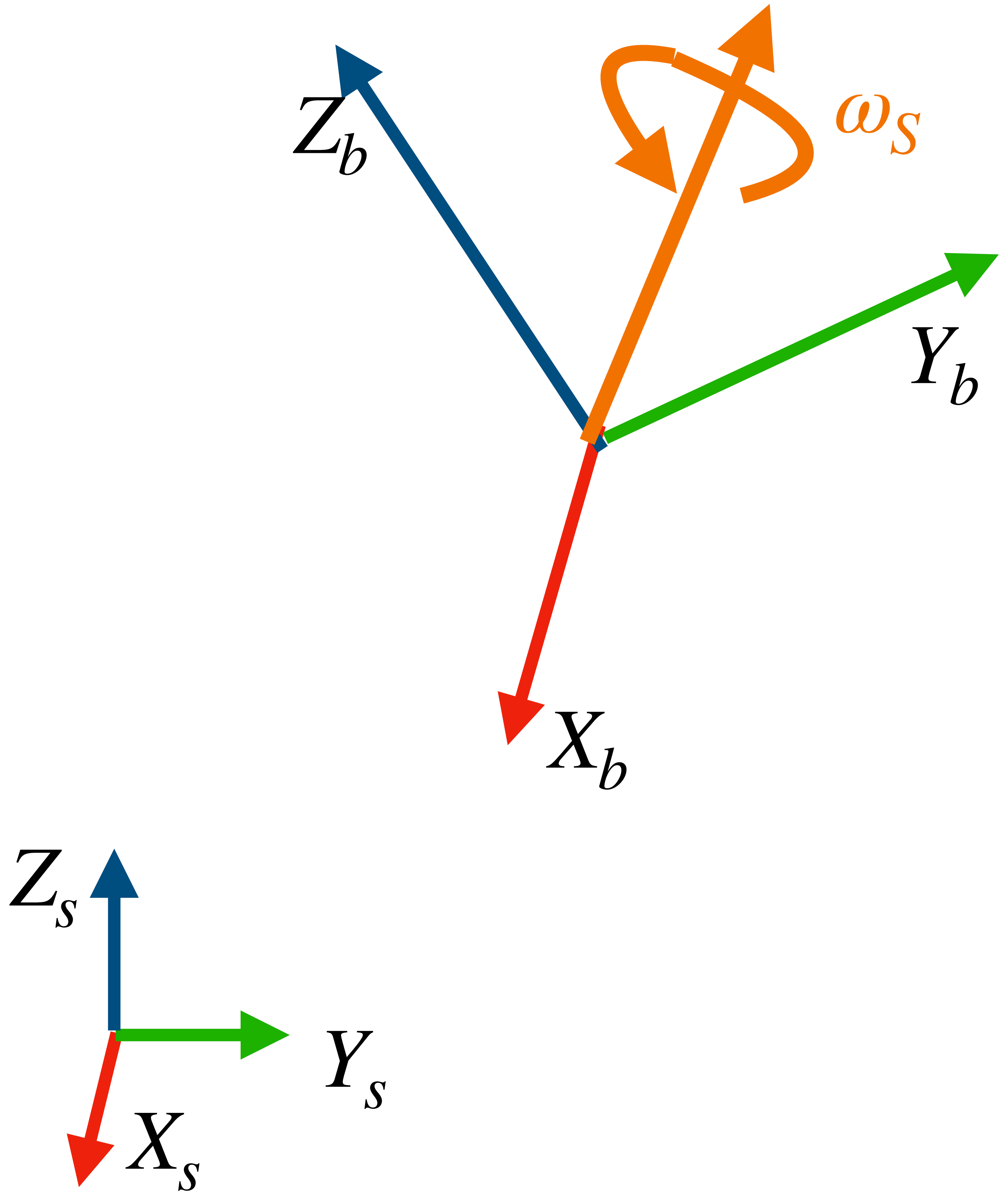


Object Motion

- We want to describe the **rotation** rate of the object in a frame

$$\omega_S = \hat{\omega}_S \dot{\theta}$$

$$\omega_B = R_{SB}^T \omega_S$$



$$\omega_S = \hat{\omega}_S \dot{\theta}$$

$$\omega_B = R_{SB}^T \omega_S$$

$$\dot{R}_{SB} = ?$$

$$[\omega_S] = ?$$

$$[\omega_B] = ?$$

$$R_{SB} = \begin{bmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{bmatrix}$$

$$\dot{R}_{SB} = \frac{d}{dt}\begin{bmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{dt}\hat{x}_B & \frac{d}{dt}\hat{y}_B & \frac{d}{dt}\hat{z}_B \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\hat{x}}_B & \dot{\hat{y}}_B & \dot{\hat{z}}_B \end{bmatrix}$$

$$\omega_S = \hat{\omega}_S \dot{\theta}$$

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Object Motion

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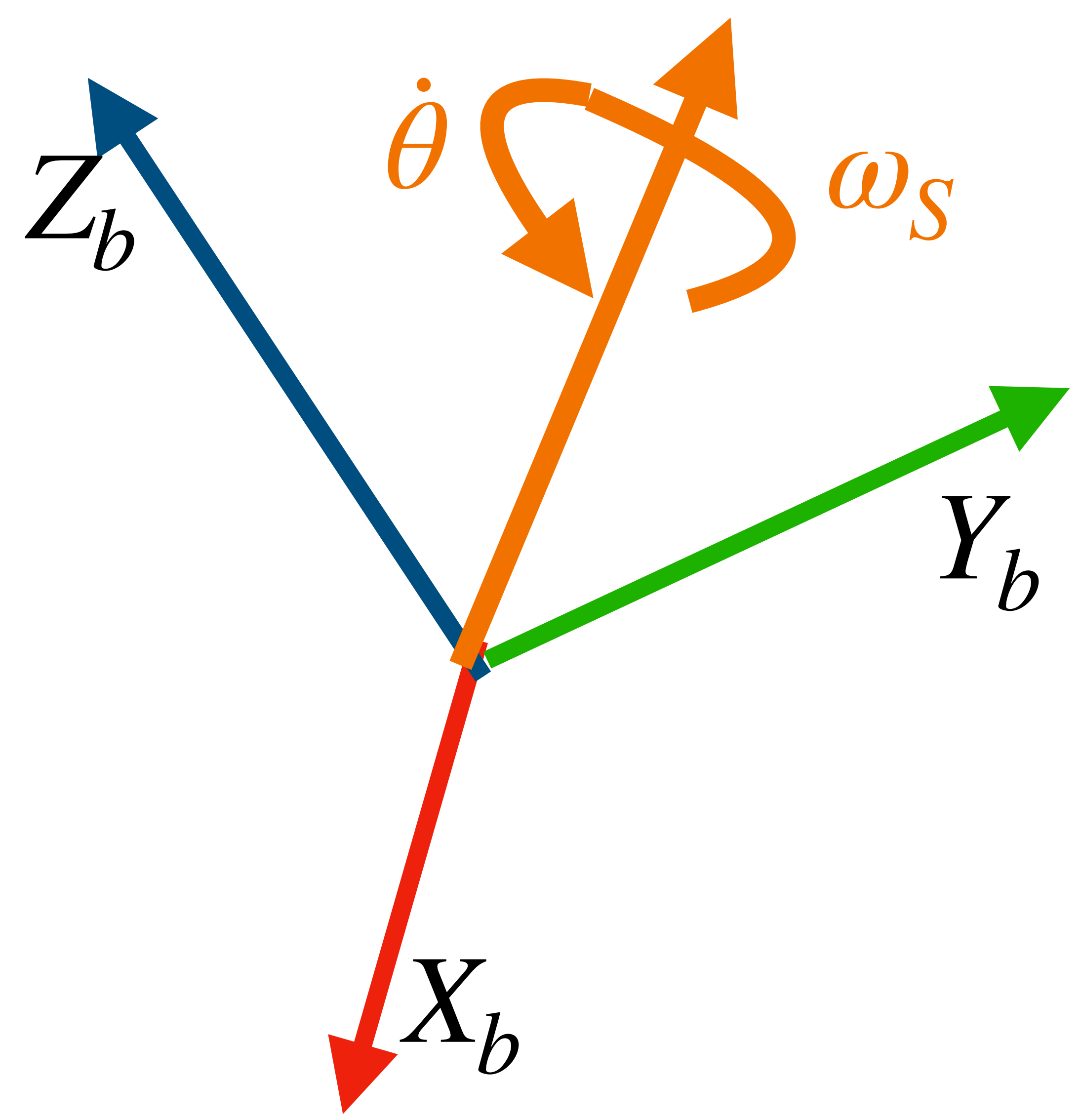
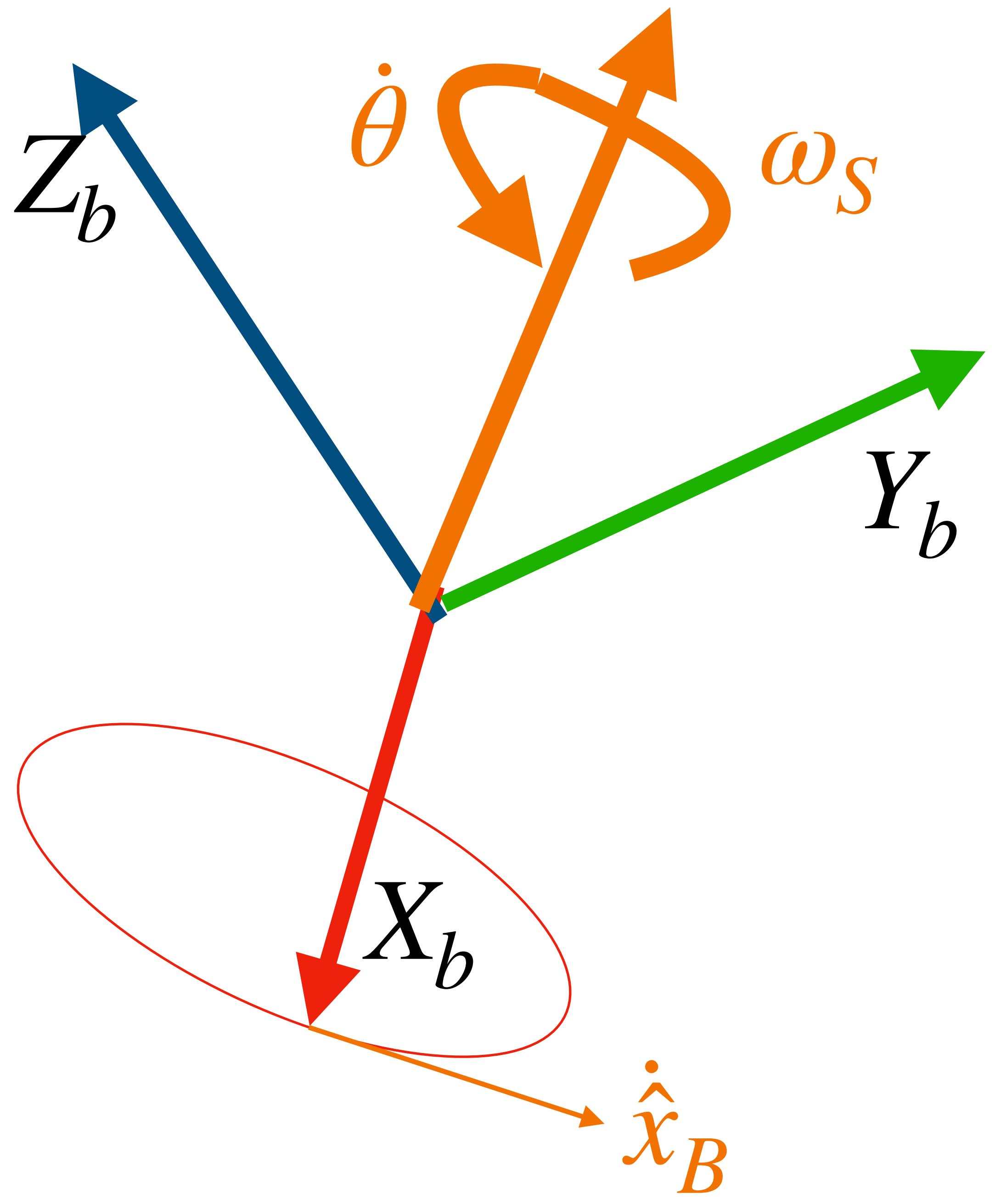


Object Motion

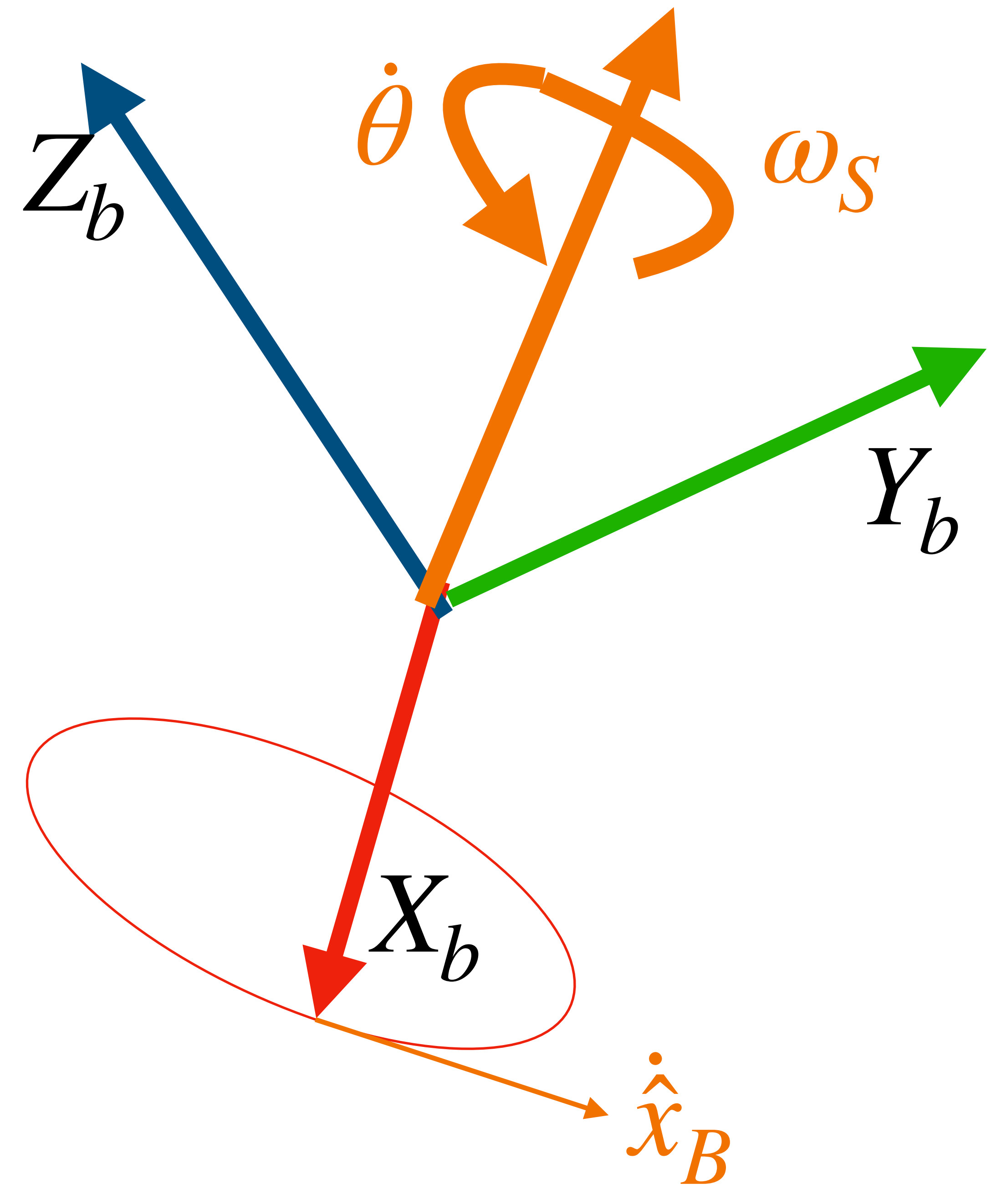
- We want to describe the **rotation** rate of the object in a frame



Object Motion

- We want to describe the **rotation** rate of the object in a frame

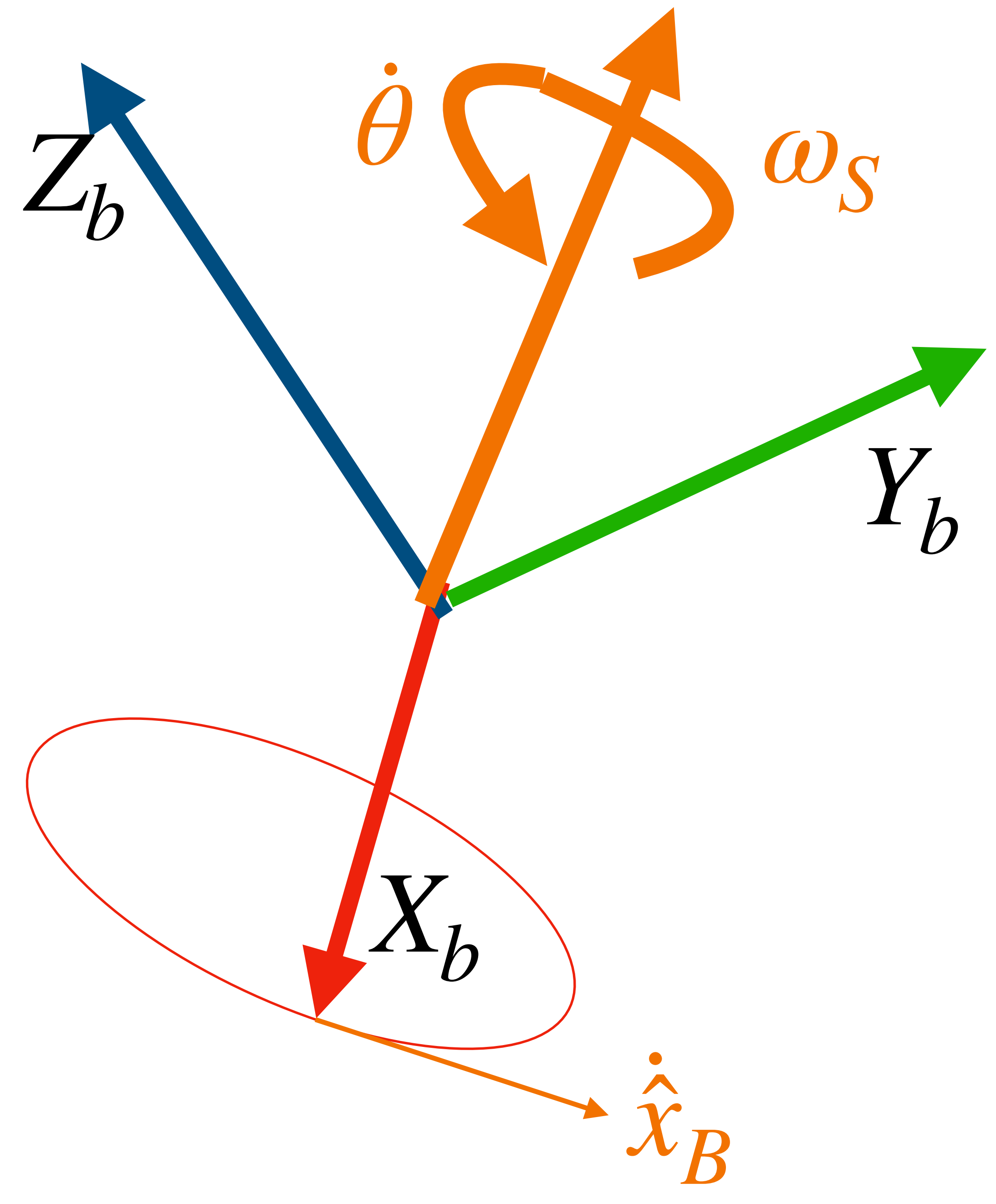
$$\dot{\hat{x}}_B = \omega_S \times \hat{x}_B$$



Object Motion

- We want to describe the **rotation** rate of the object in a frame

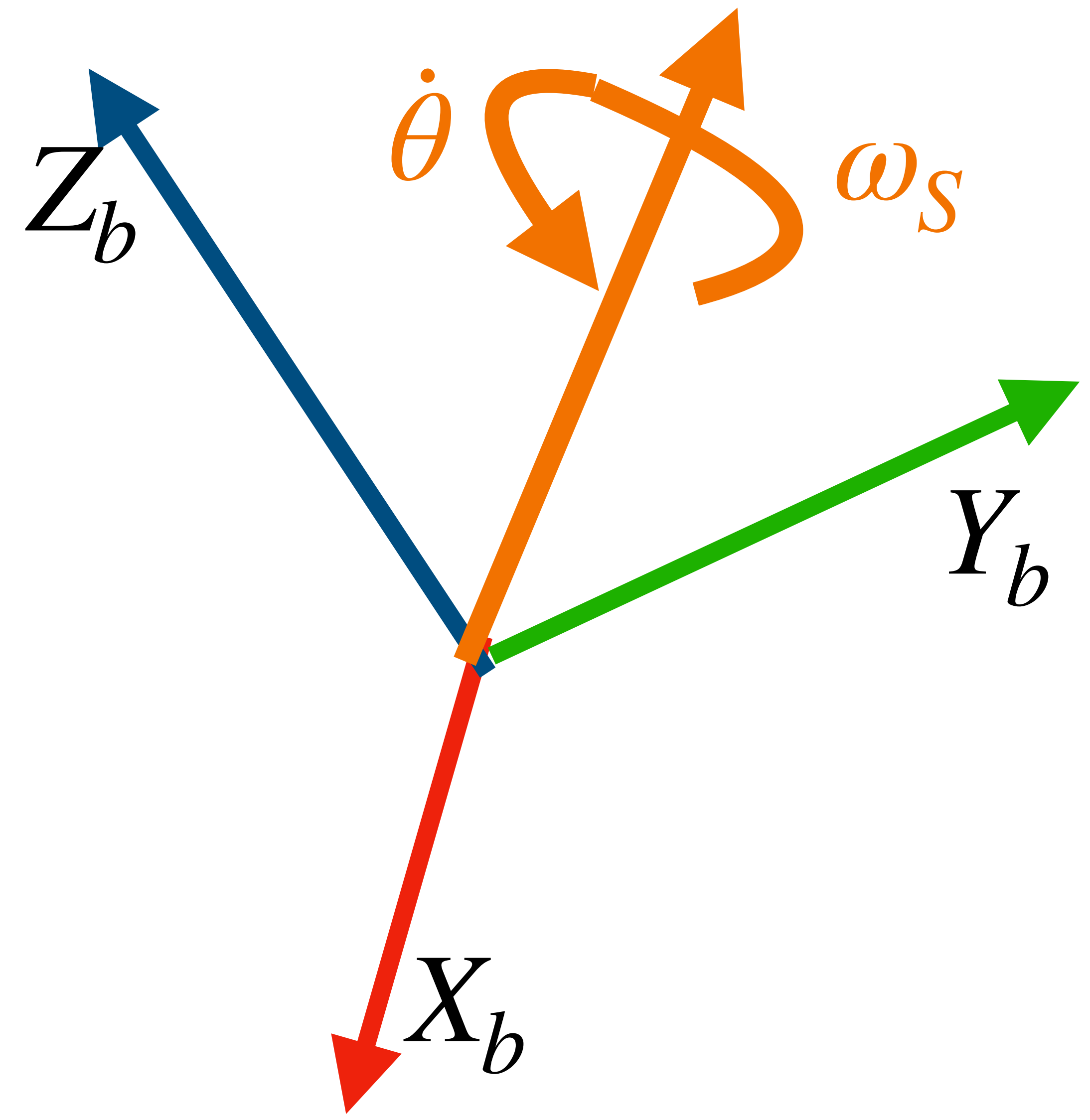
$$\dot{\hat{x}}_B = \omega_S \times \hat{x}_B$$
$$= [\omega_S]\hat{x}_B$$



Object Motion

- We want to describe the **rotation** rate of the object in a frame

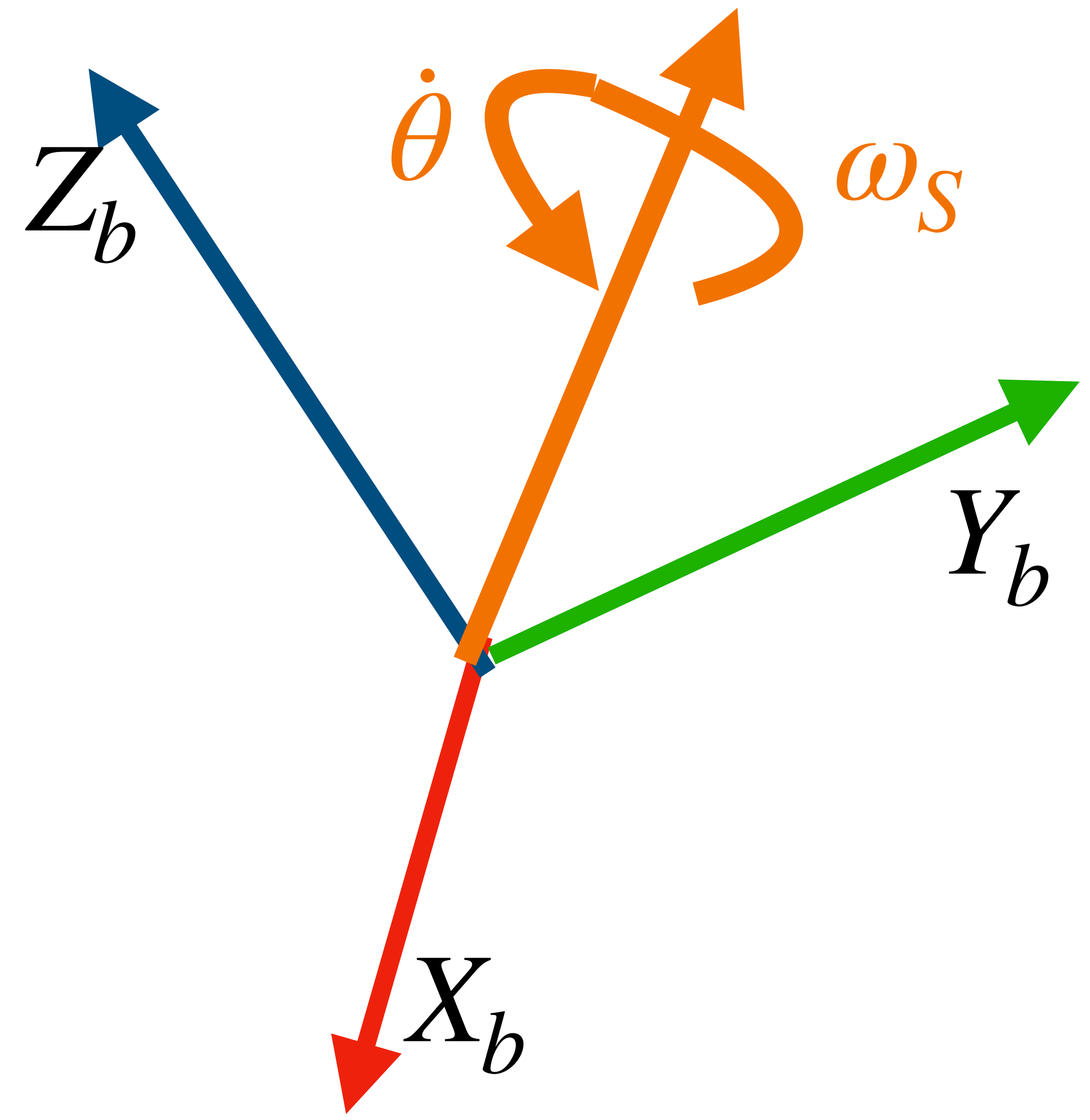
$$\dot{\hat{x}}_B = [\omega_S]\hat{x}_B$$



Object Motion

- We want to describe the **rotation** rate of the object in a frame

$$\dot{\hat{x}}_B = [\omega_S]\hat{x}_B$$
$$\dot{\hat{y}}_B = [\omega_S]\hat{y}_B$$
$$\dot{\hat{z}}_B = [\omega_S]\hat{z}_B$$



$$\dot{R}_{SB} = \begin{bmatrix} \dot{\hat{x}}_B & \dot{\hat{y}}_B & \dot{\hat{z}}_B \end{bmatrix}$$

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$$= \begin{bmatrix} [\omega_S]\hat{x}_B & [\omega_S]\hat{y}_B & [\omega_S]\hat{z}_B \end{bmatrix}$$

$$
\begin{aligned}
\dot{R}_{SB} &= \begin{bmatrix} \dot{\hat{x}}_B & \dot{\hat{y}}_B & \dot{\hat{z}}_B \end{bmatrix} \\
&= \begin{bmatrix} [\omega_S]\hat{x}_B & [\omega_S]\hat{y}_B & [\omega_S]\hat{z}_B \end{bmatrix} \\
&= [\omega_S]\begin{bmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\dot{R}_{SB} &= \begin{bmatrix} \dot{\hat{x}}_B & \dot{\hat{y}}_B & \dot{\hat{z}}_B \end{bmatrix} \\
&= \begin{bmatrix} [\omega_S]\hat{x}_B & [\omega_S]\hat{y}_B & [\omega_S]\hat{z}_B \end{bmatrix} \\
&= [\omega_S]\begin{bmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{bmatrix} \\
&= [\omega_S]R_{SB}
\end{aligned}
$$

$$\dot{R}_{SB} = \begin{bmatrix} \dot{\hat{x}}_B & \dot{\hat{y}}_B & \dot{\hat{z}}_B \end{bmatrix}$$

$$= \begin{bmatrix} [\omega_S]\hat{x}_B & [\omega_S]\hat{y}_B & [\omega_S]\hat{z}_B \end{bmatrix}$$

$$= [\omega_S]\begin{bmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{bmatrix}$$

$$= [\omega_S]R_{SB}$$

$$\downarrow$$

$$[\omega_S] = \dot{R}_{SB}R_{SB}^T$$

$$\omega_S = \hat{\omega}_S \dot{\theta}$$

$$\omega_B = R_{SB}^T \omega_S$$

$$\dot{R}_{SB} = \begin{bmatrix} \dot{\hat{x}}_B & \dot{\hat{y}}_B & \dot{\hat{z}}_B \end{bmatrix}$$

$$[\omega_S] = \dot{R}_{SB} R_{SB}^T$$

$$[\omega_B] = ?$$

$$[\omega_B] = \,?$$

$$\omega_B = R_{SB}^T \omega_S$$

$$\omega_B = R_{SB}^T \omega_S$$
$$[\omega_B] = [R_{SB}^T \omega_S]$$

$$\begin{aligned} \omega_B &= R_{SB}^T \omega_S \\ [\omega_B] &= [R_{SB}^T \omega_S] \\ &= R_{SB}^T [\omega_S] R_{SB} \end{aligned}$$

$$
\begin{aligned}
\omega_B &= R_{SB}^T \omega_S \\
[\omega_B] &= [R_{SB}^T \omega_S] \\
&= R_{SB}^T [\omega_S] R_{SB} \\
&= R_{SB}^T (\dot{R}_{SB} R_{SB}^T) R_{SB}
\end{aligned}
$$

$$
\begin{aligned}
\omega_B &= R_{SB}^T \omega_S \\
[\omega_B] &= [R_{SB}^T \omega_S] \\
&= R_{SB}^T [\omega_S] R_{SB} \\
&= R_{SB}^T (\dot{R}_{SB} R_{SB}^T) R_{SB} \\
&= R_{SB}^T \dot{R}_{SB}
\end{aligned}
$$

$$\omega_S = \hat{\omega}_S \dot{\theta}$$

$$\omega_B = R_{SB}^T \omega_S$$

$$\dot{R}_{SB} = \begin{bmatrix} \dot{\hat{x}}_B & \dot{\hat{y}}_B & \dot{\hat{z}}_B \end{bmatrix}$$

$$[\omega_S] = \dot{R}_{SB} R_{SB}^T$$

$$[\omega_B] = R_{SB}^T \dot{R}_{SB}$$

These 5 representations of rotational velocity will be helpful later …