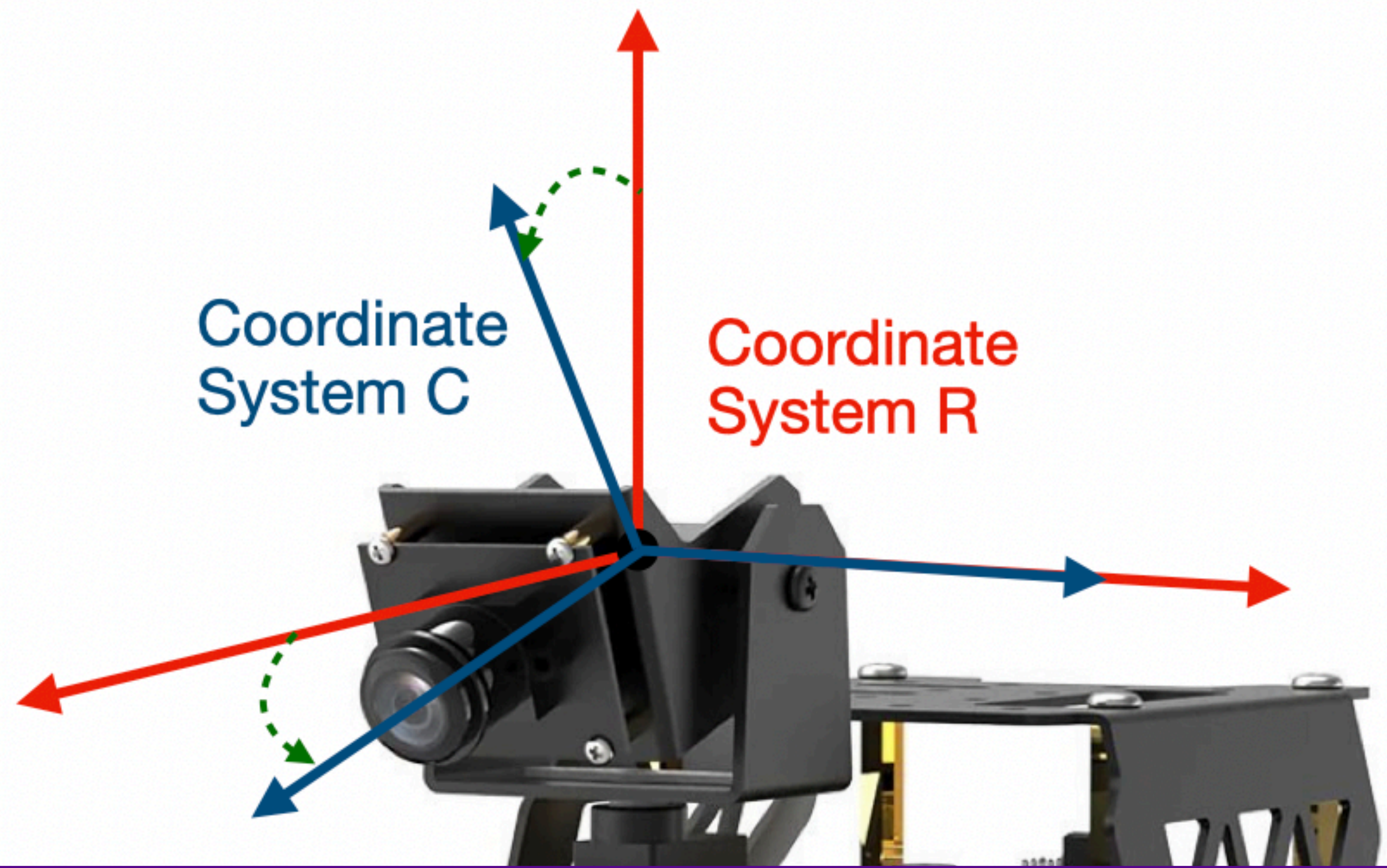
ROB-UY 2004

Robotic Manipulation & Locomotion

Lecture 03A - Rotations



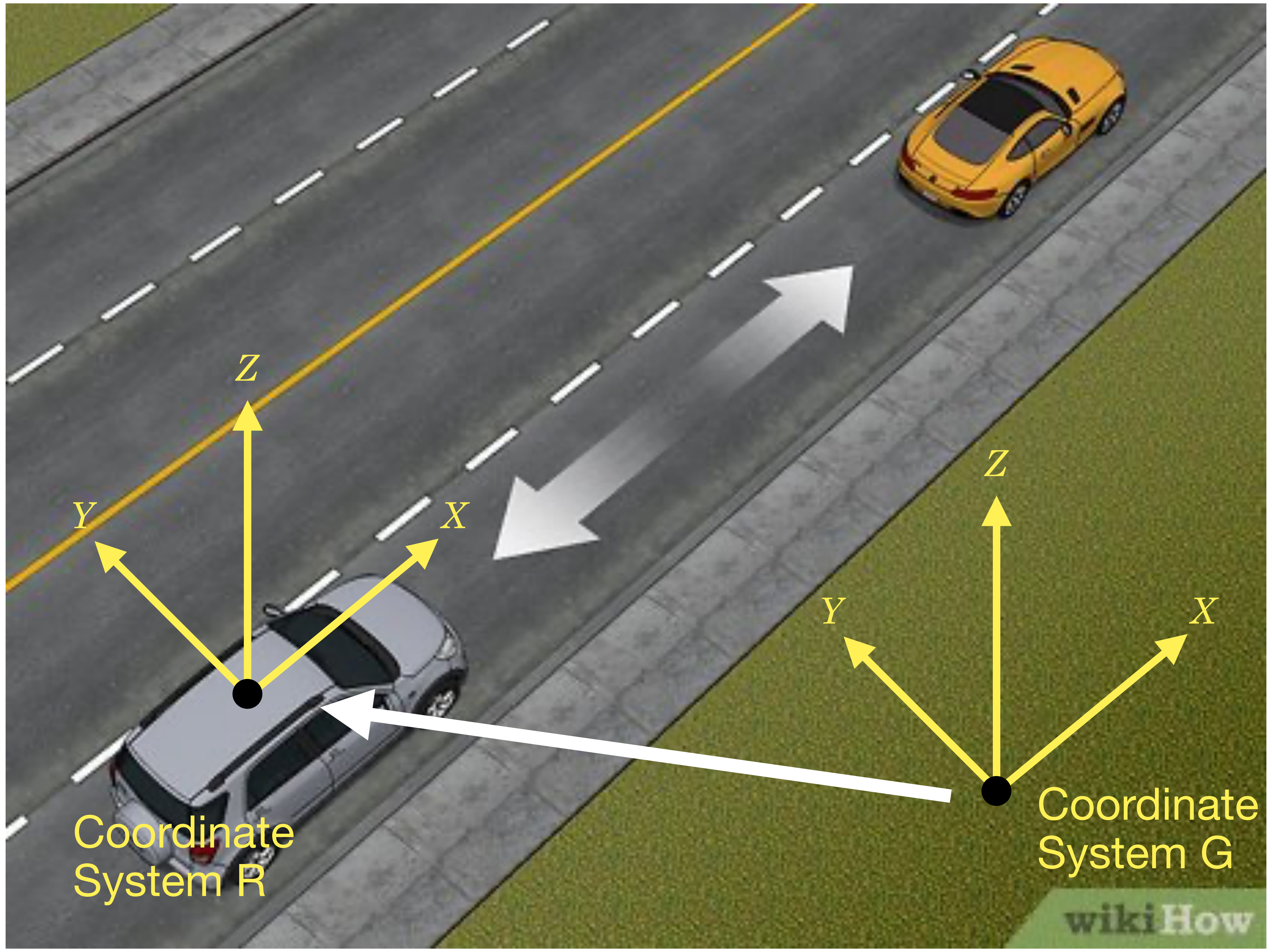
Agenda

1. Motivation
2. 2D Rotations and 2D Coordinate Frames
3. 3D Rotations and 3D Coordinate Frames
4. Properties of Rotations

Agenda

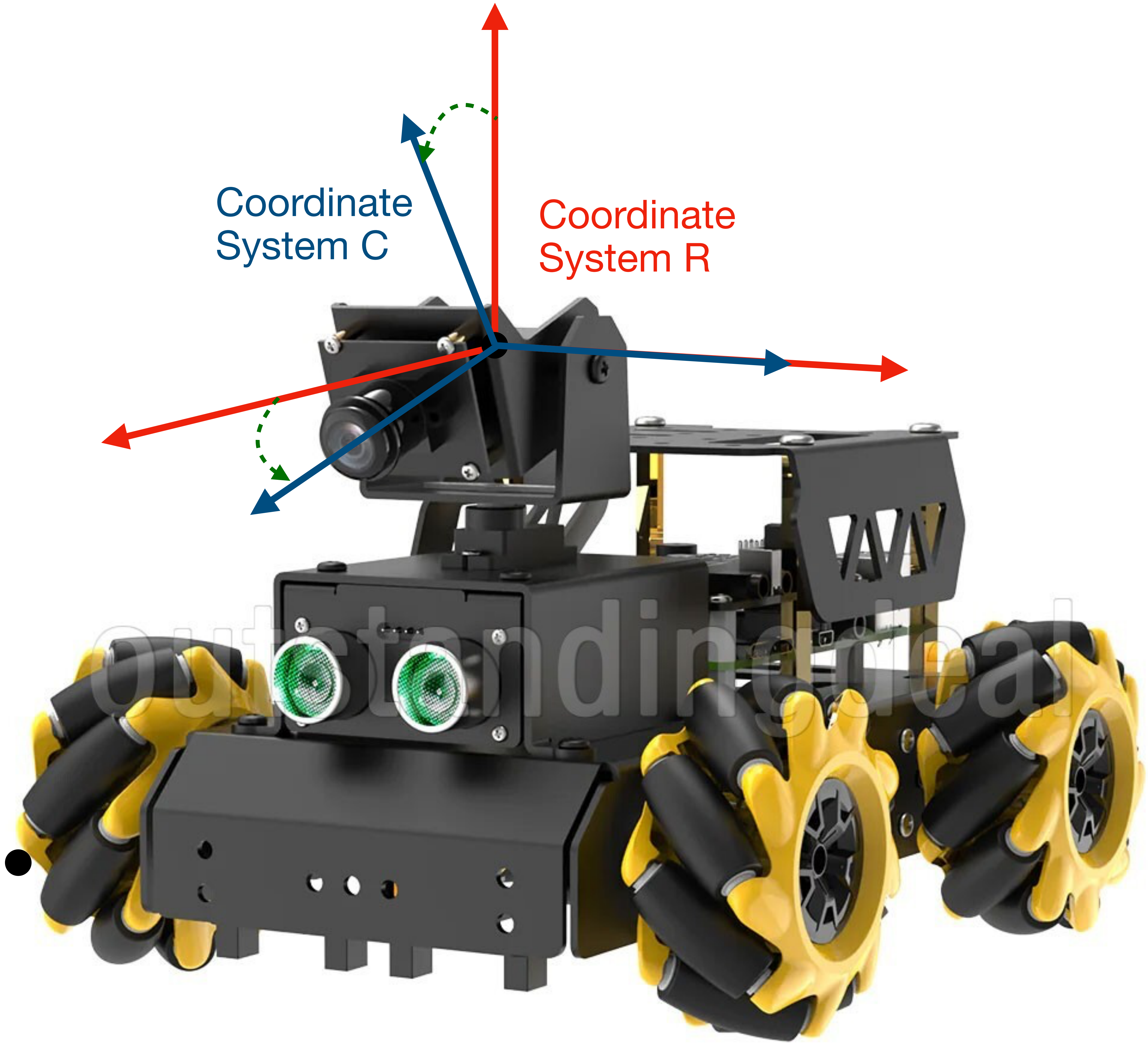
Coordinate Systems

- In robotics, the frame is attached to something physical (e.g. a robot or the ground).
- It is helpful to be able to:
  - Describe the **relative position** of one frame with respect to another frame



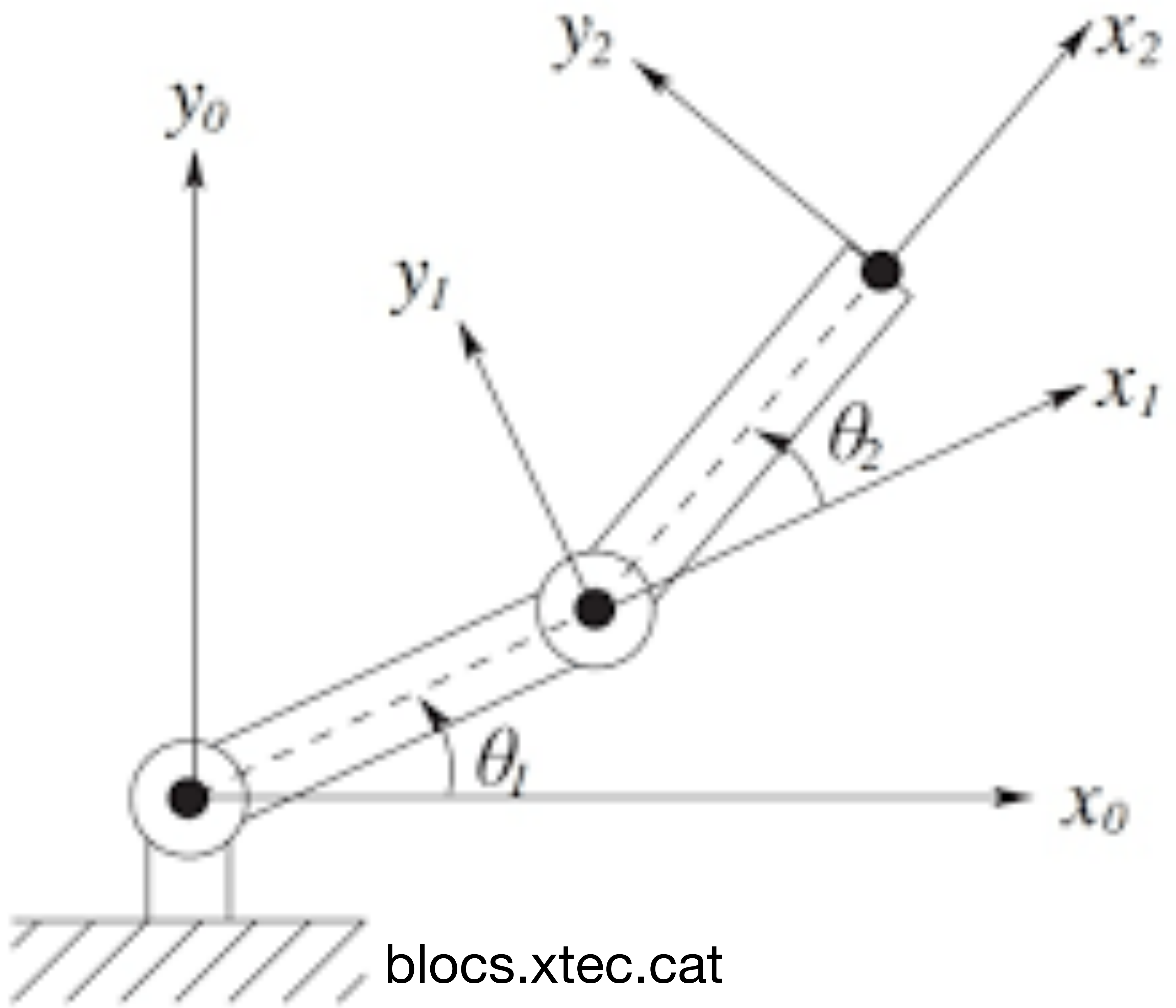
Coordinate Systems

- In robotics, the frame is attached to something physical (e.g. a robot or the ground).
- It is helpful to be able to:
  - Describe the relative position of one frame with respect to another frame
  - Describe the **relative rotation** of one frame with respect to another frame



Coordinate Systems

- In robotics, the frame is attached to something physical (e.g. a robot or the ground).
- It is helpful to be able to:
  - Describe the **relative position** and **relative rotation** of one frame with respect to another frame



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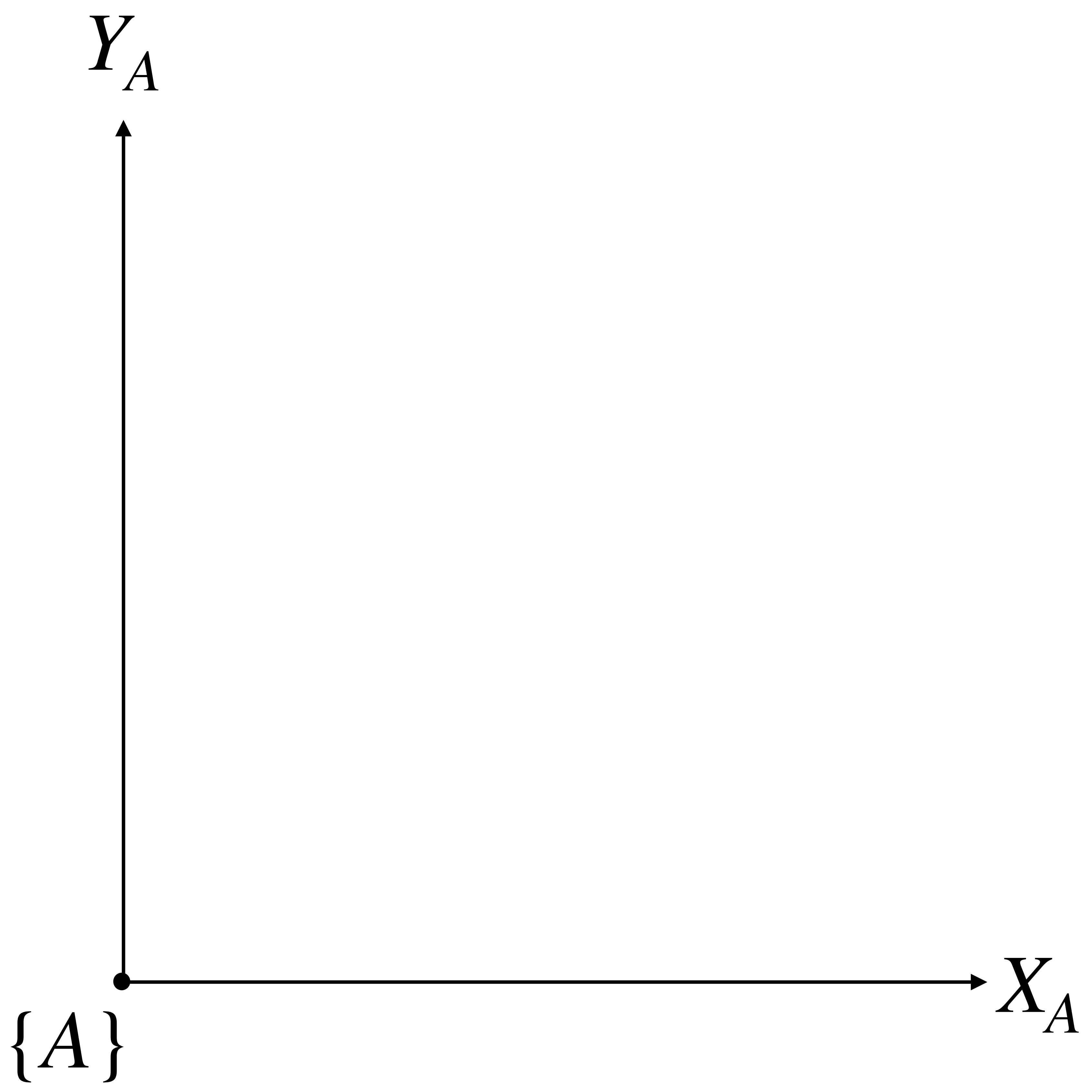
Agenda

1. Motivation
2. **2D Rotations and 2D Coordinate Frames**
3. 3D Rotations and 3D Coordinate Frames
4. Properties of Rotations

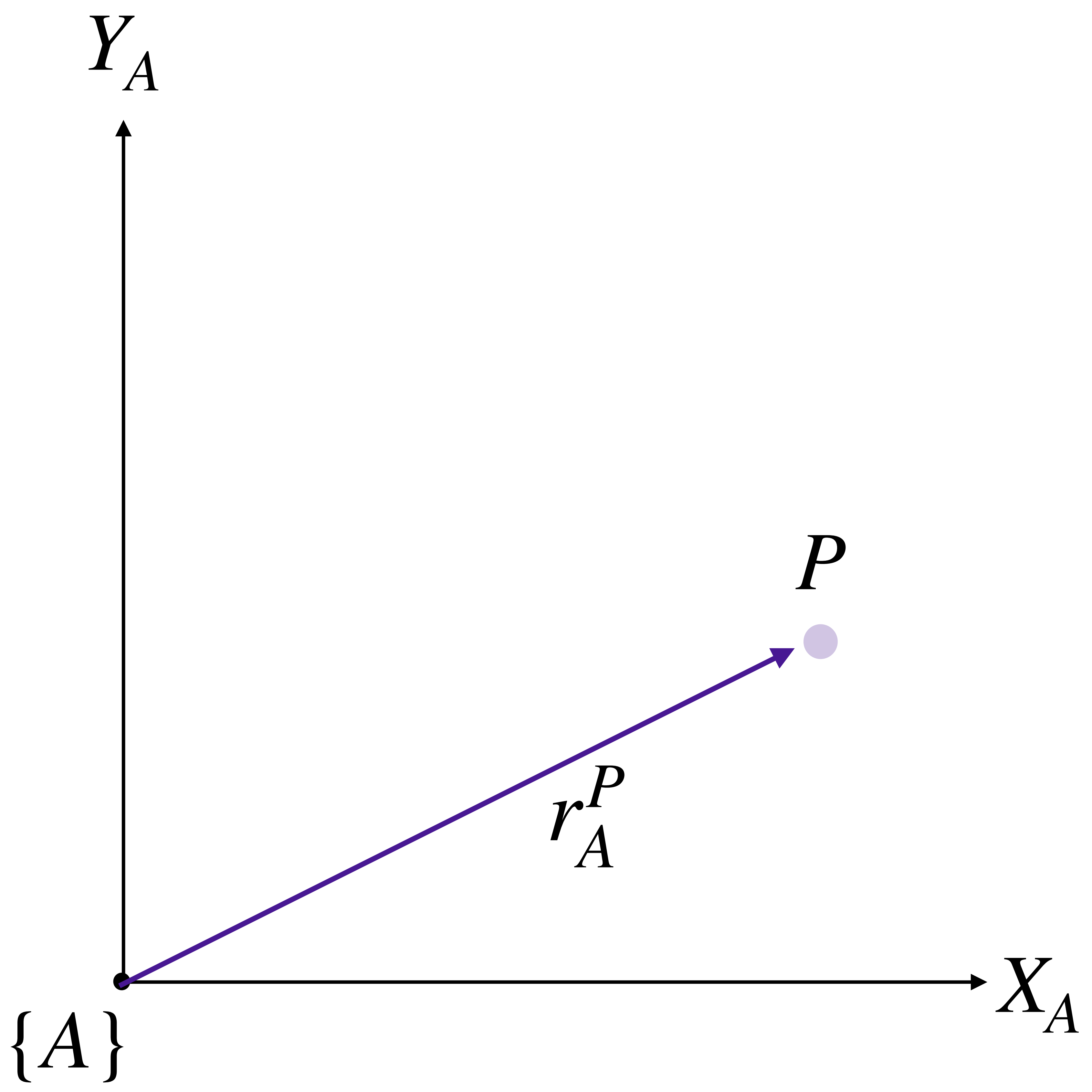
2D Rotations

- Assume we have a coordinate frame A.



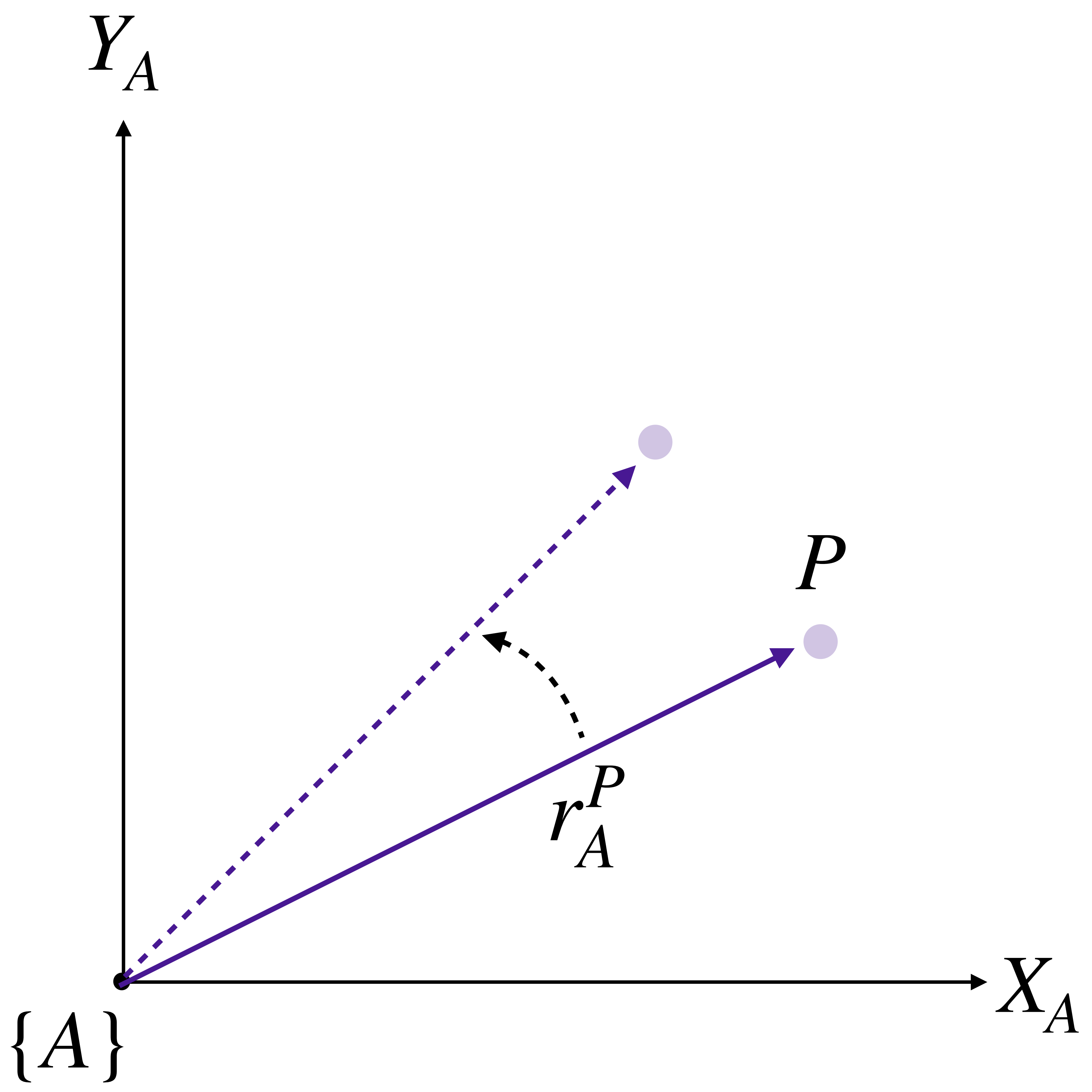
2D Rotations

- Assume we have a coordinate frame A.
- There is a point P in space, and a vector that defines P relative to coordinate frame A.



2D Rotations

- Assume we have a coordinate frame A.
- There is a point P in space, and a vector that defines P relative to coordinate frame A.
- Let us rotate point P about an axis of rotation that is perpendicular to the 2D plane and passes through the frame A's origin.



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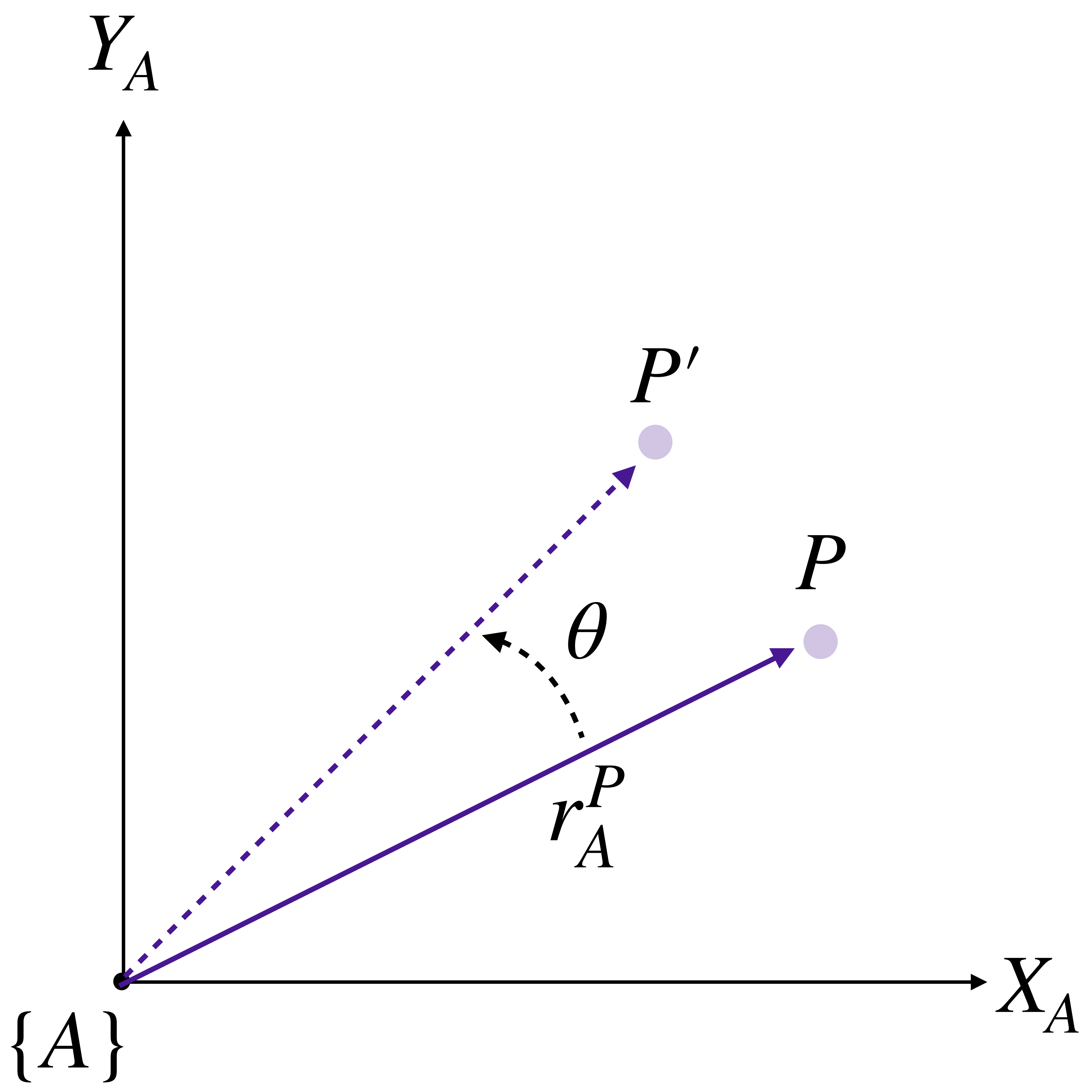

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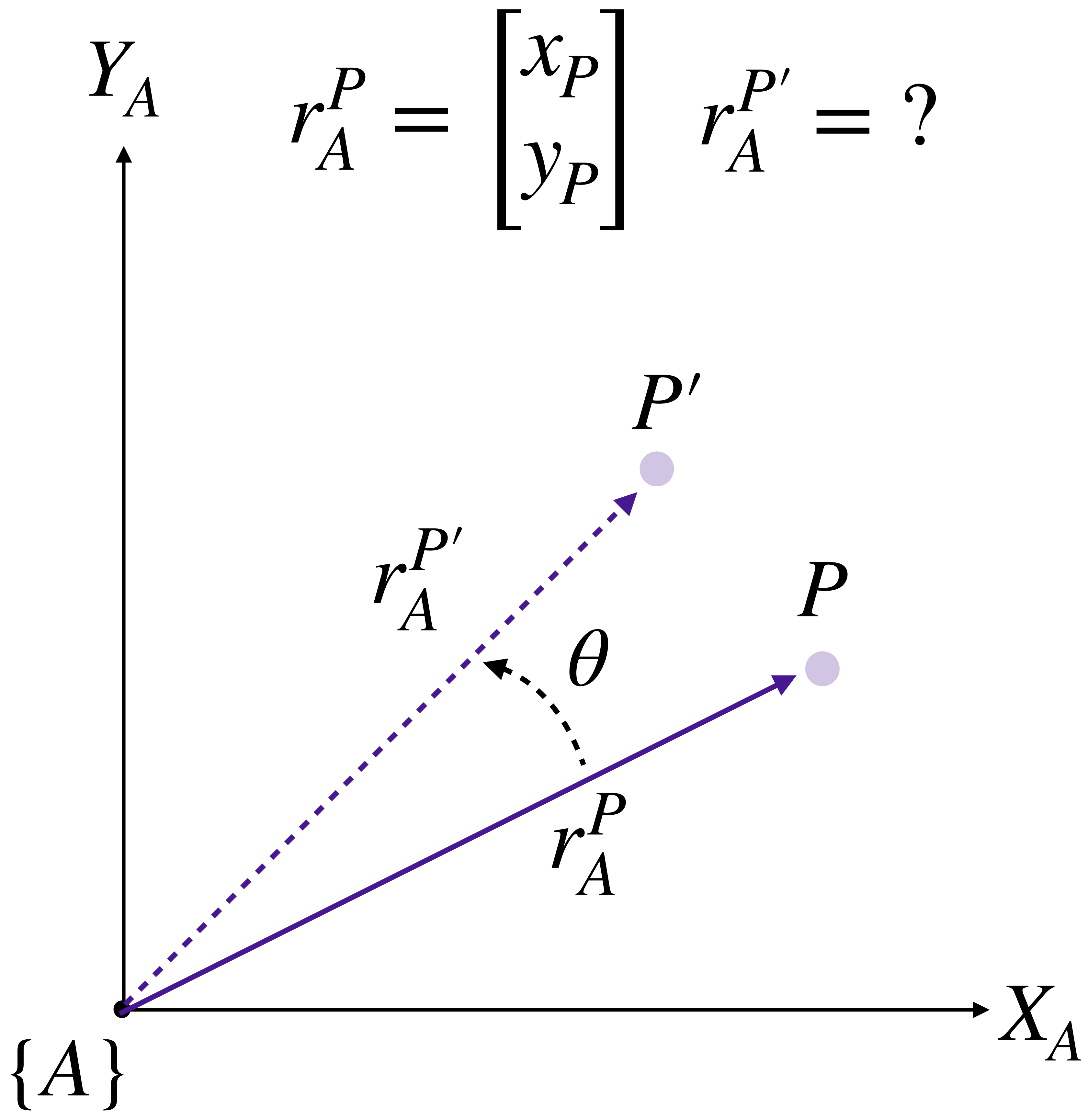
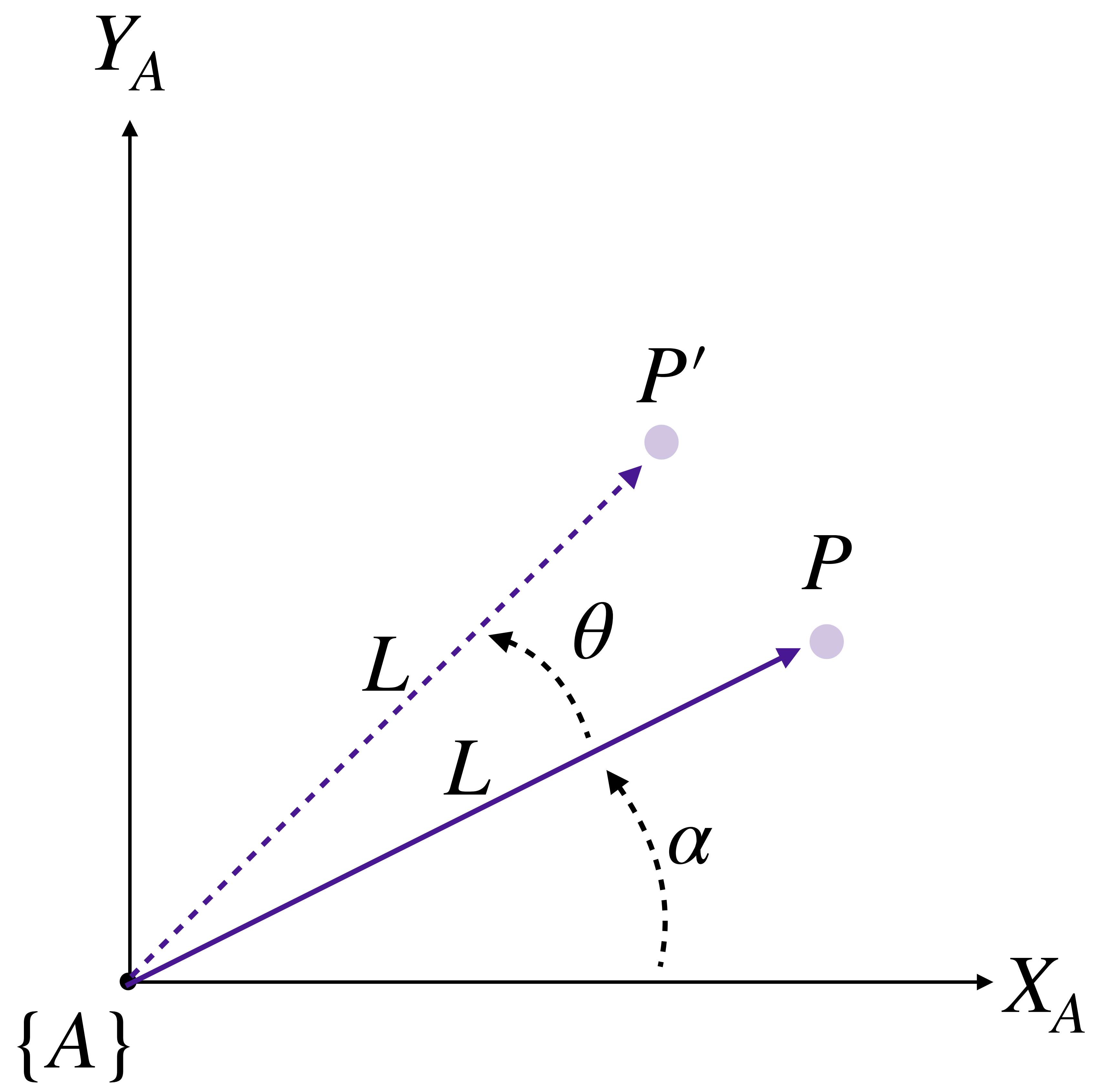
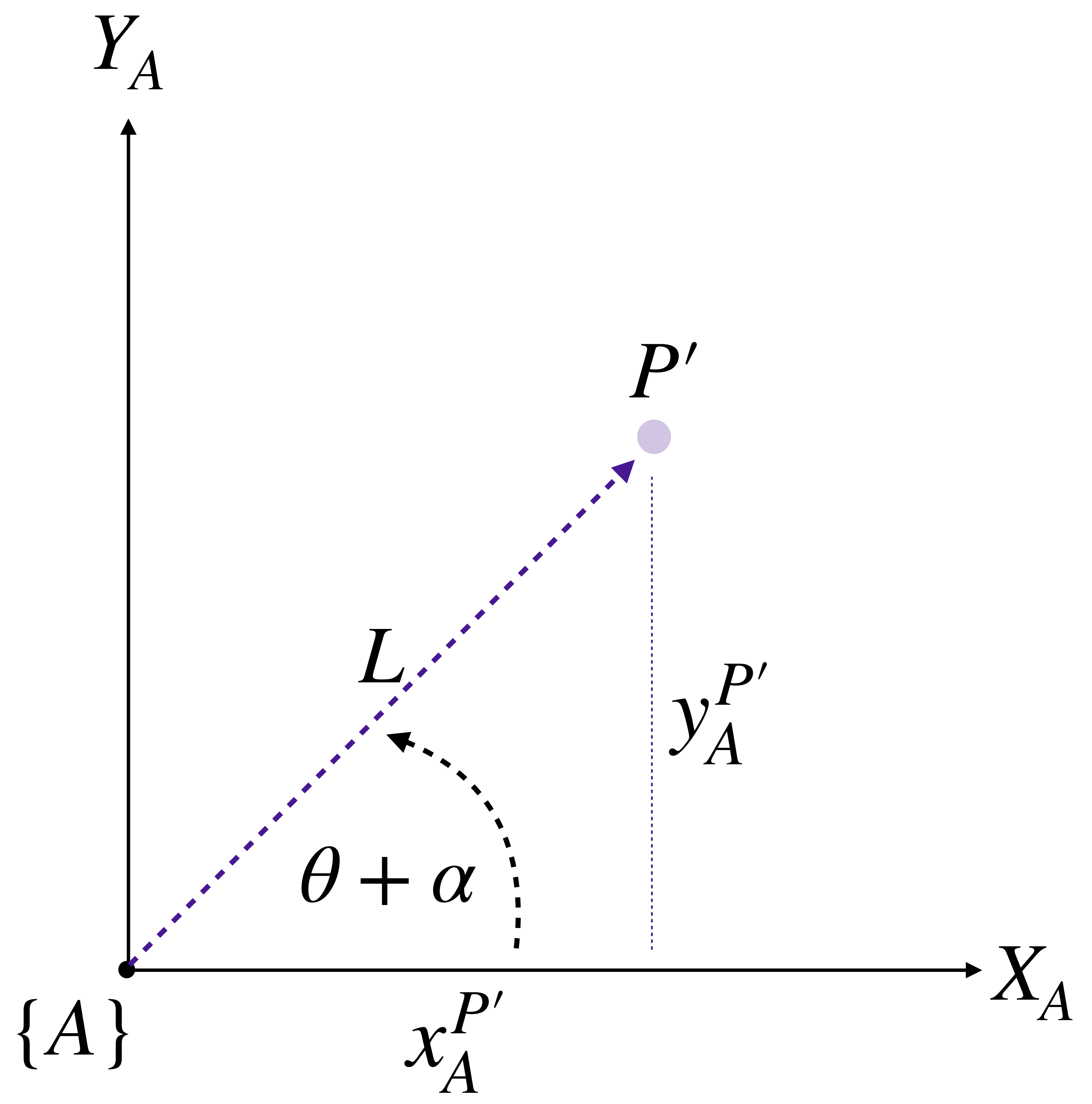



Y_A
P'
P
L
θ
L
α
$\{A\}$
X_A

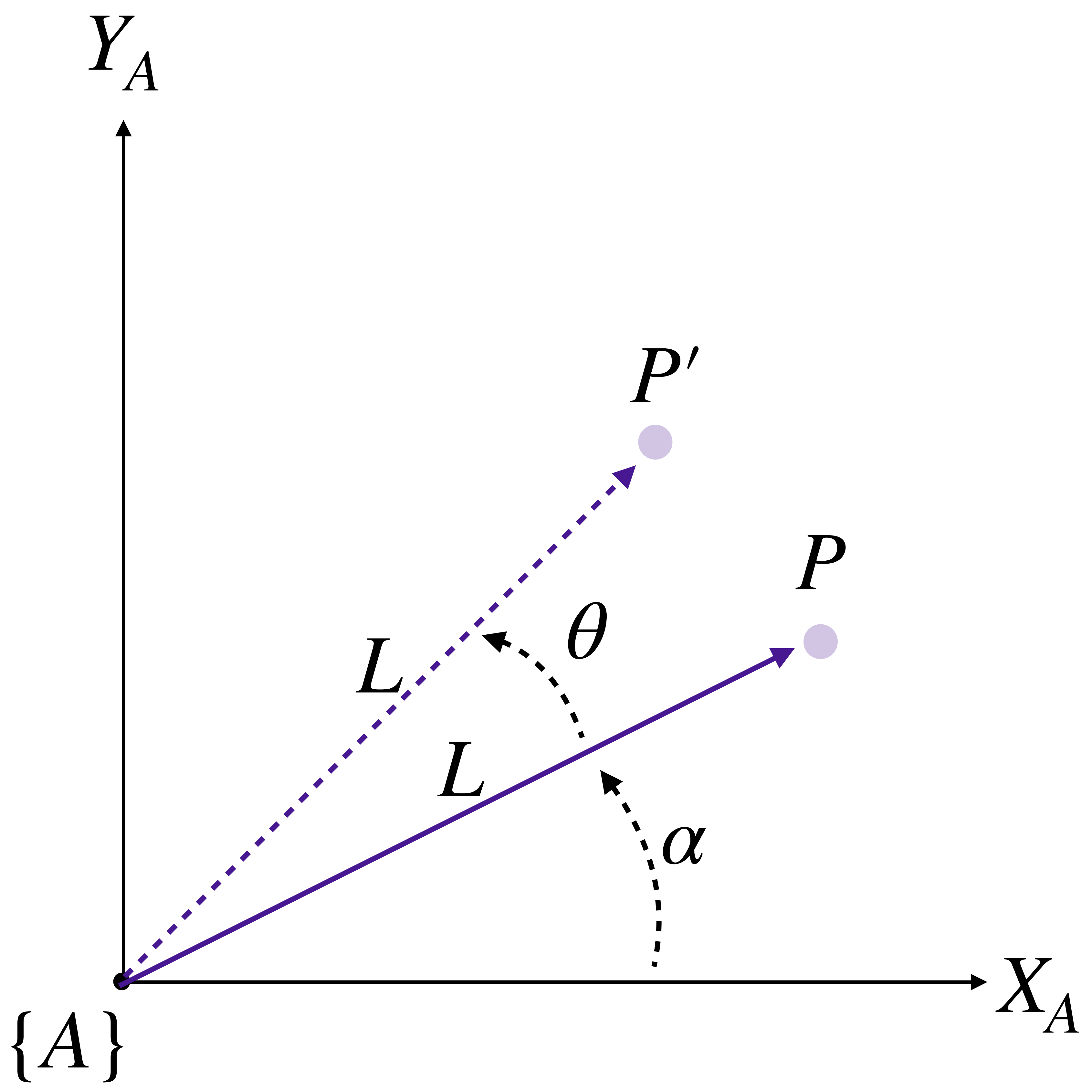
$$x_A^{P'} = Lcos(\alpha + \theta)$$

$$y_A^{P'} = Lsin(\alpha + \theta)$$



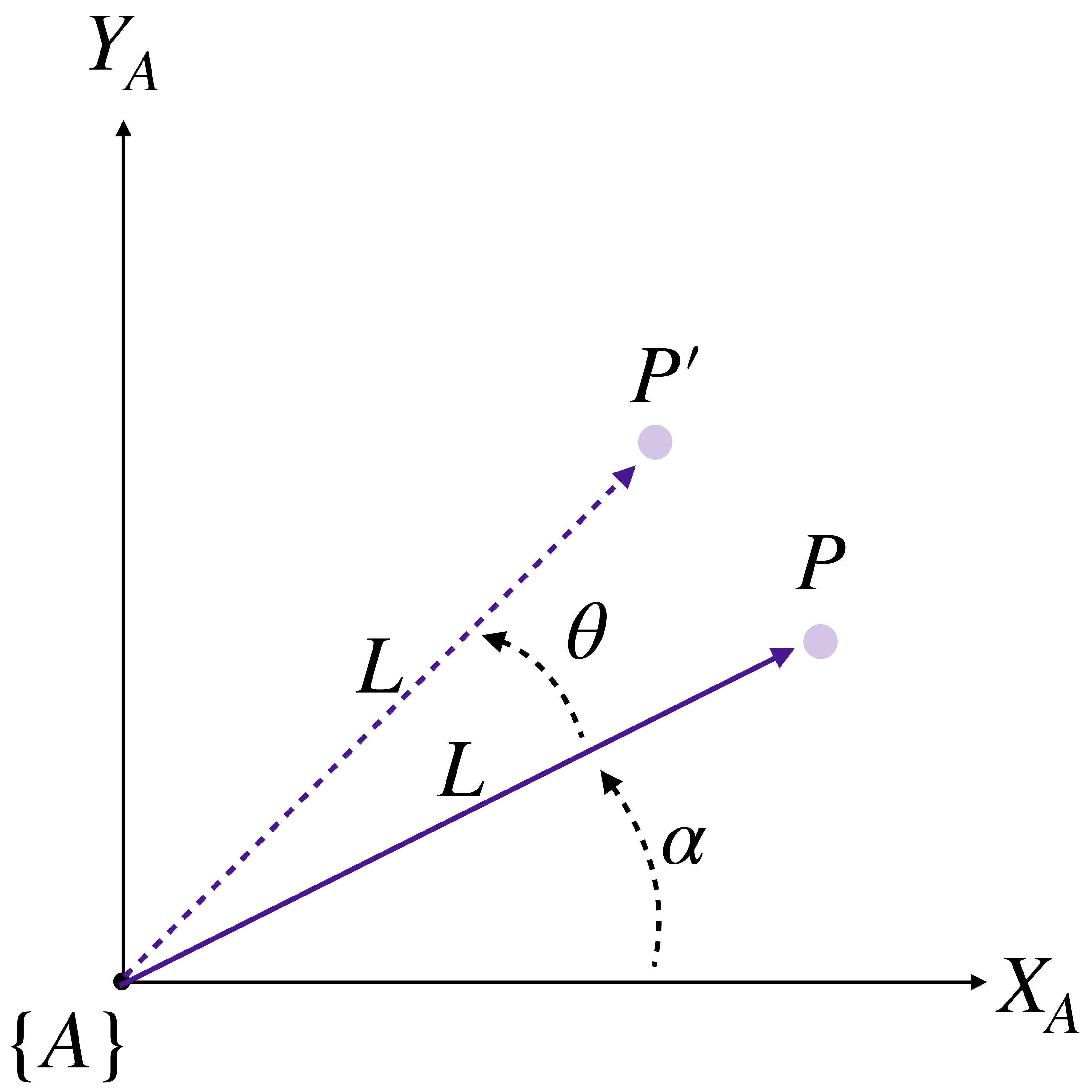
$$x_A^{P'} = Lcos(\alpha + \theta)$$
$$= Lcos(\alpha)cos(\theta) - Lsin(\alpha)sin(\theta)$$

$$y_A^{P'} = Lsin(\alpha + \theta)$$
$$= Lcos(\alpha)sin(\theta) + Lsin(\alpha)cos(\theta)$$



$$
\begin{aligned}
x_A^{P'} &= Lcos(\alpha + \theta) \\
&= Lcos(\alpha)cos(\theta) - Lsin(\alpha)sin(\theta) \\
&= x_A^P cos(\theta) - y_A^P sin(\theta)
\end{aligned}
$$

$$
\begin{aligned}
y_A^{P'} &= Lsin(\alpha + \theta) \\
&= Lcos(\alpha)sin(\theta) + Lsin(\alpha)cos(\theta) \\
&= x_A^P sin(\theta) + y_A^P cos(\theta)
\end{aligned}
$$



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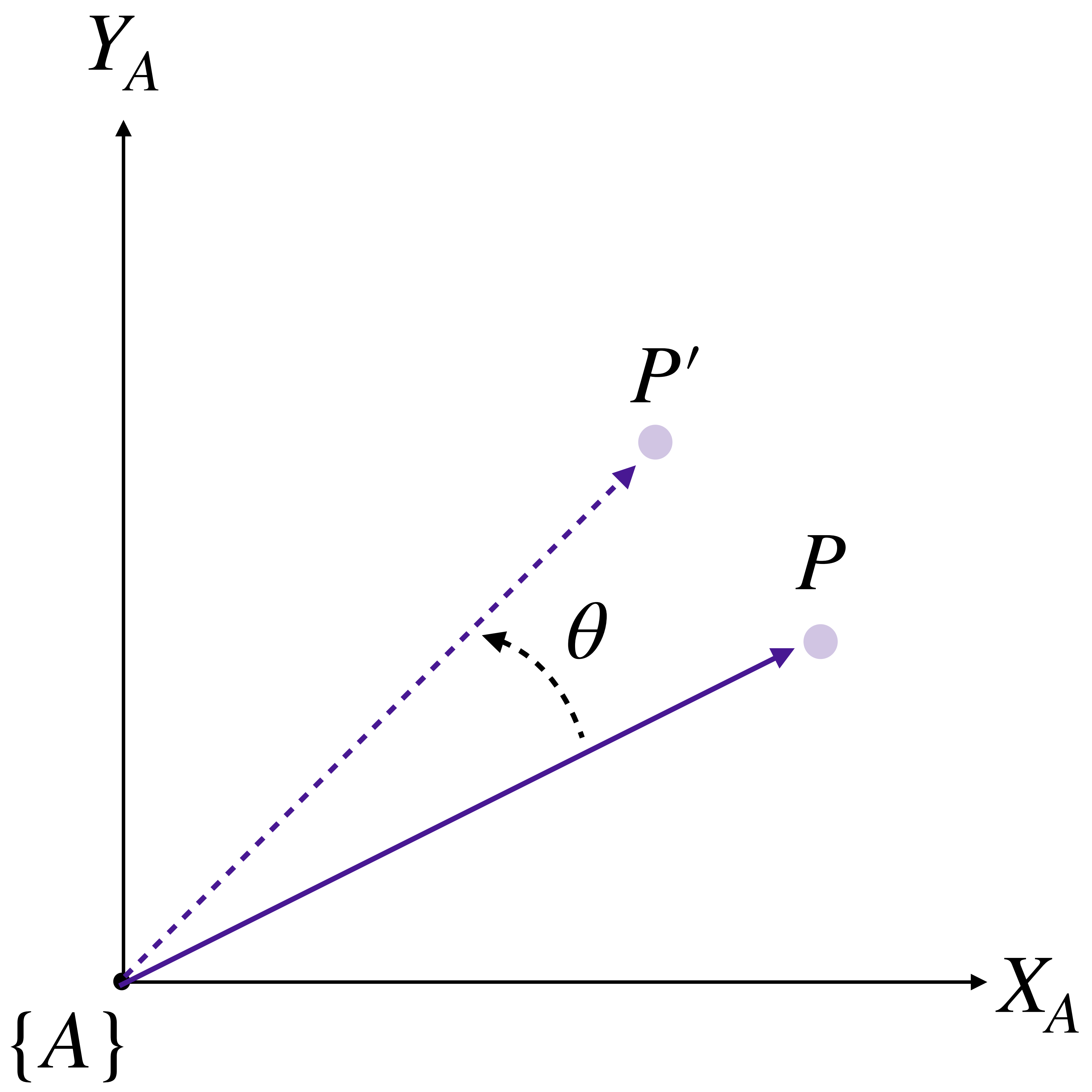
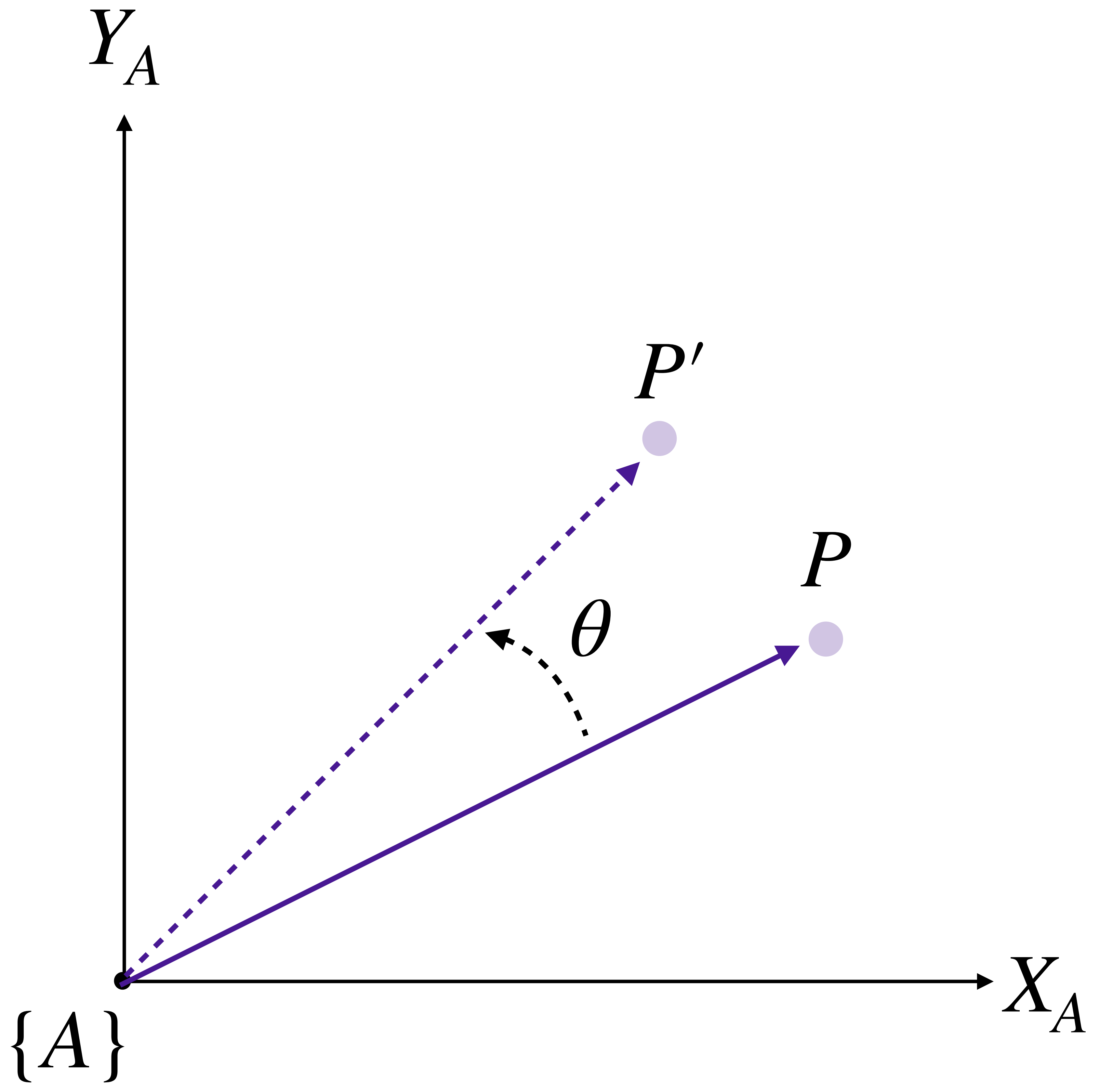


$$x_A^{P'} = x_A^P cos(\theta) - y_A^P sin(\theta)$$

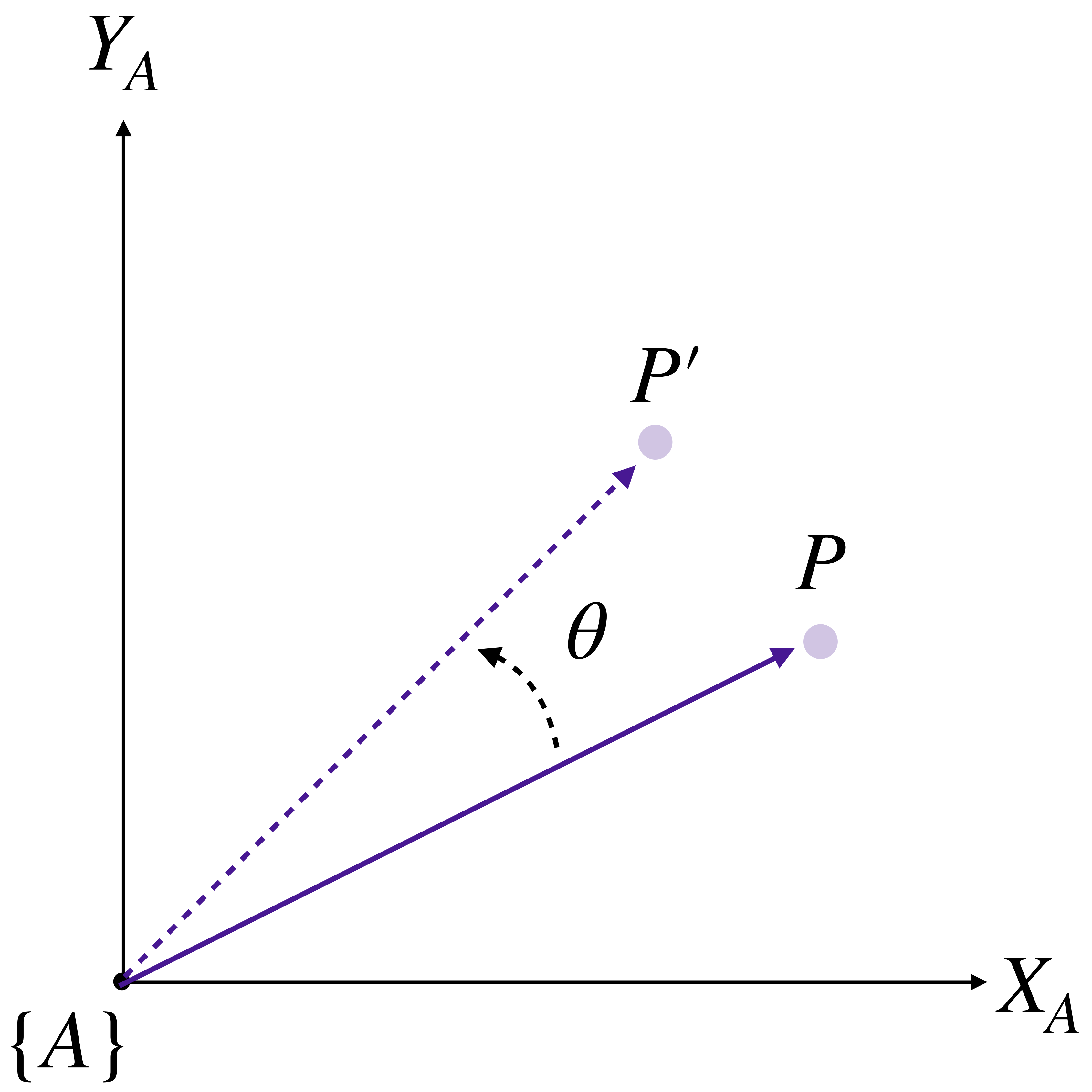
$$y_A^{P'} = x_A^P sin(\theta) + y_A^P cos(\theta)$$

$$\downarrow$$

$$\begin{bmatrix} x_A^{P'} \\ y_A^{P'} \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x_A^P \\ y_A^P \end{bmatrix}$$

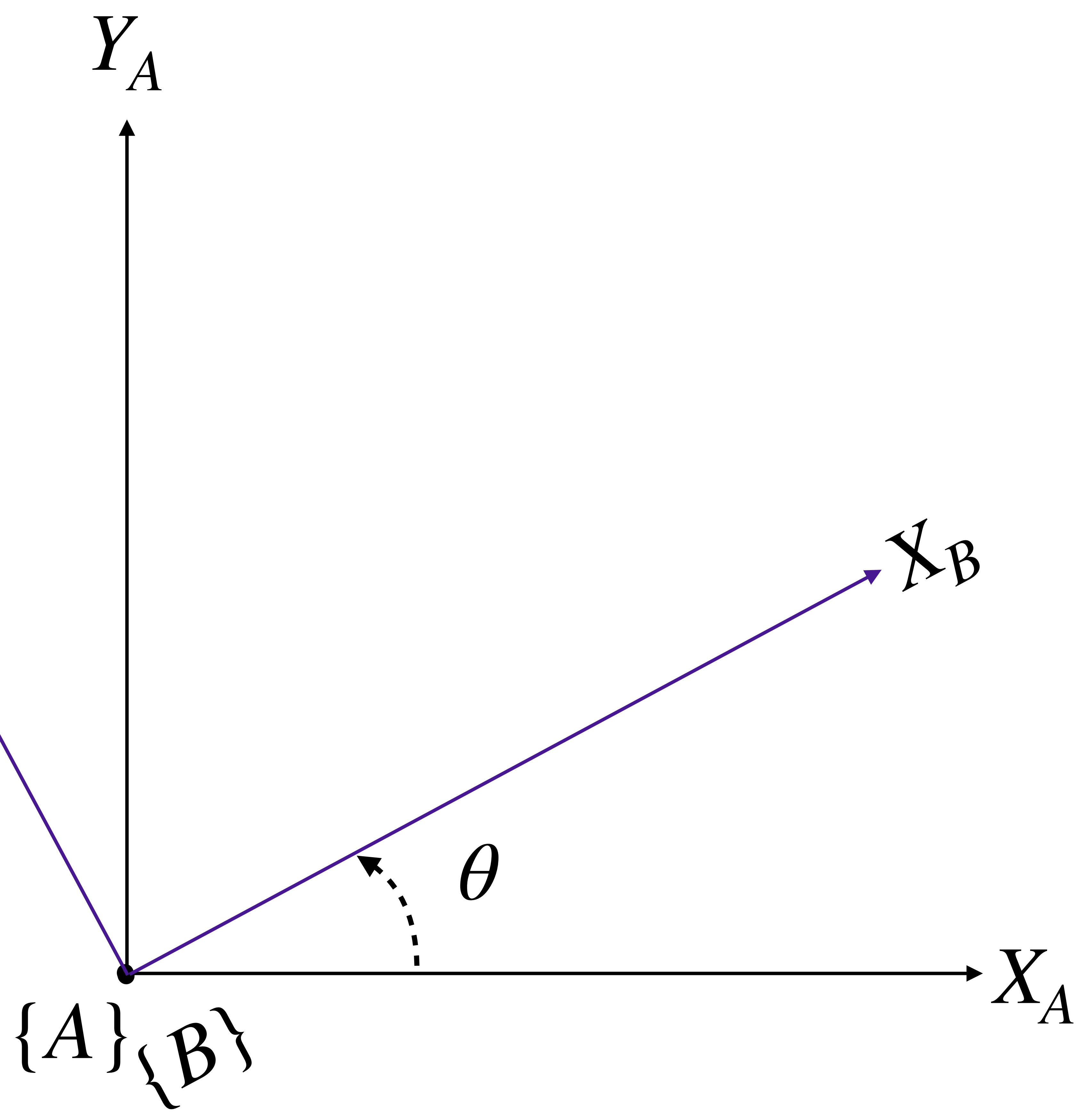


$$r_A^{P'} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} r_A^{P} = R(\theta) r_A^{P}$$



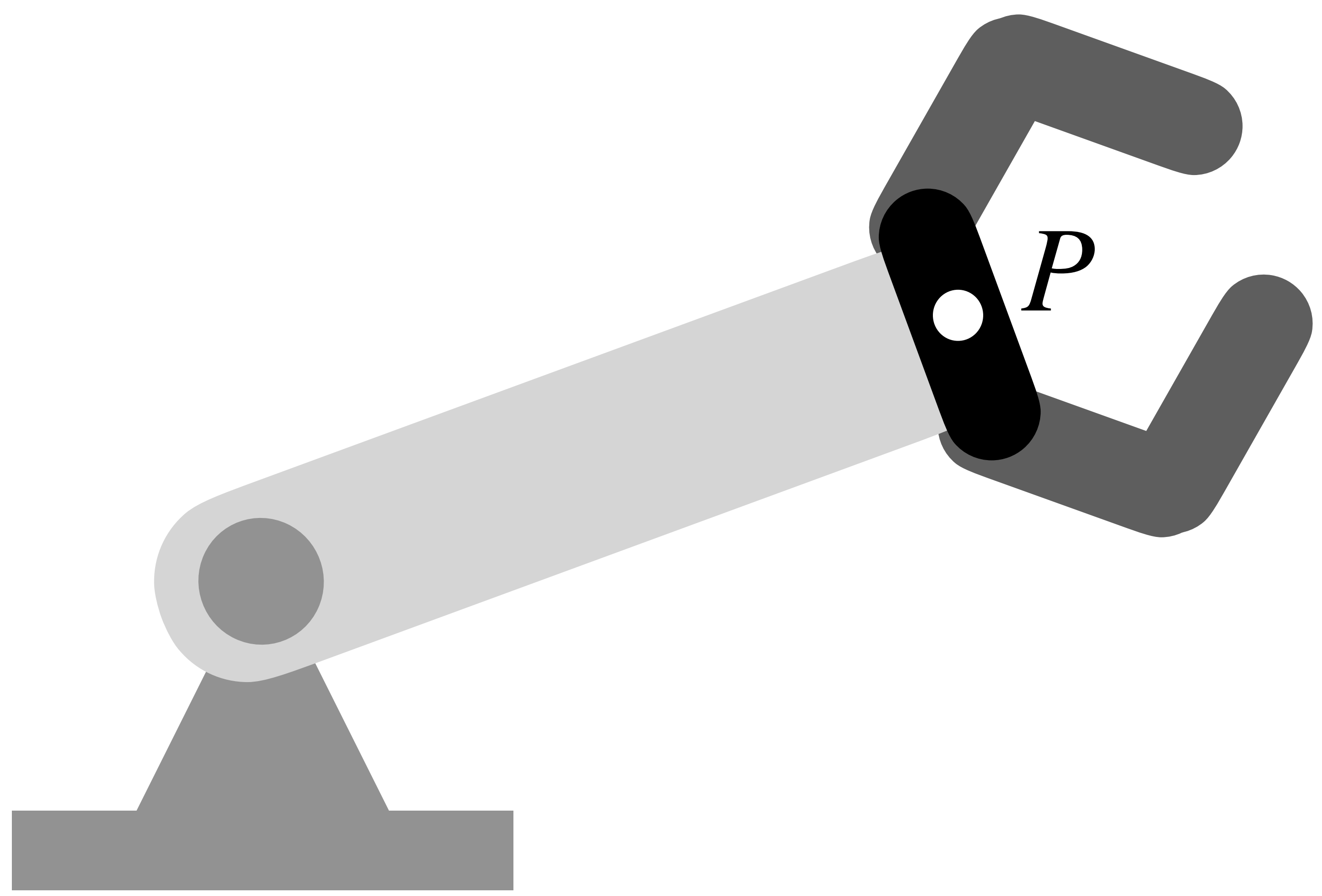
2D Rotations

- We can rotate any point P about an axis with this rotation matrix, therefore we can rotate points between two coordinate frames.



2D Rotations

- One point that we often care about in robotics, is the base position of the "end effector"
- Let us track that point using coordinate frames
- And look at the rotations between coordinate frames



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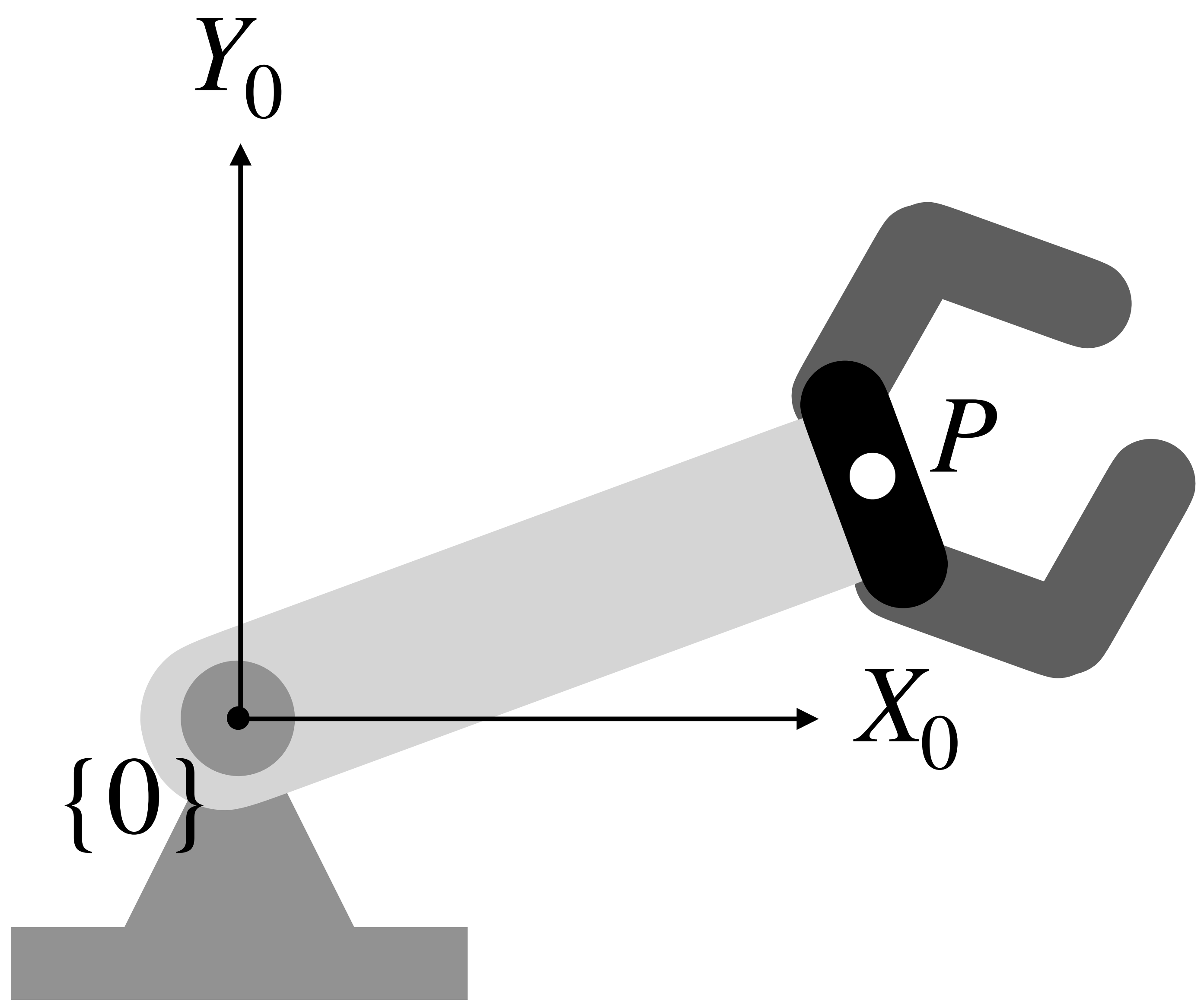

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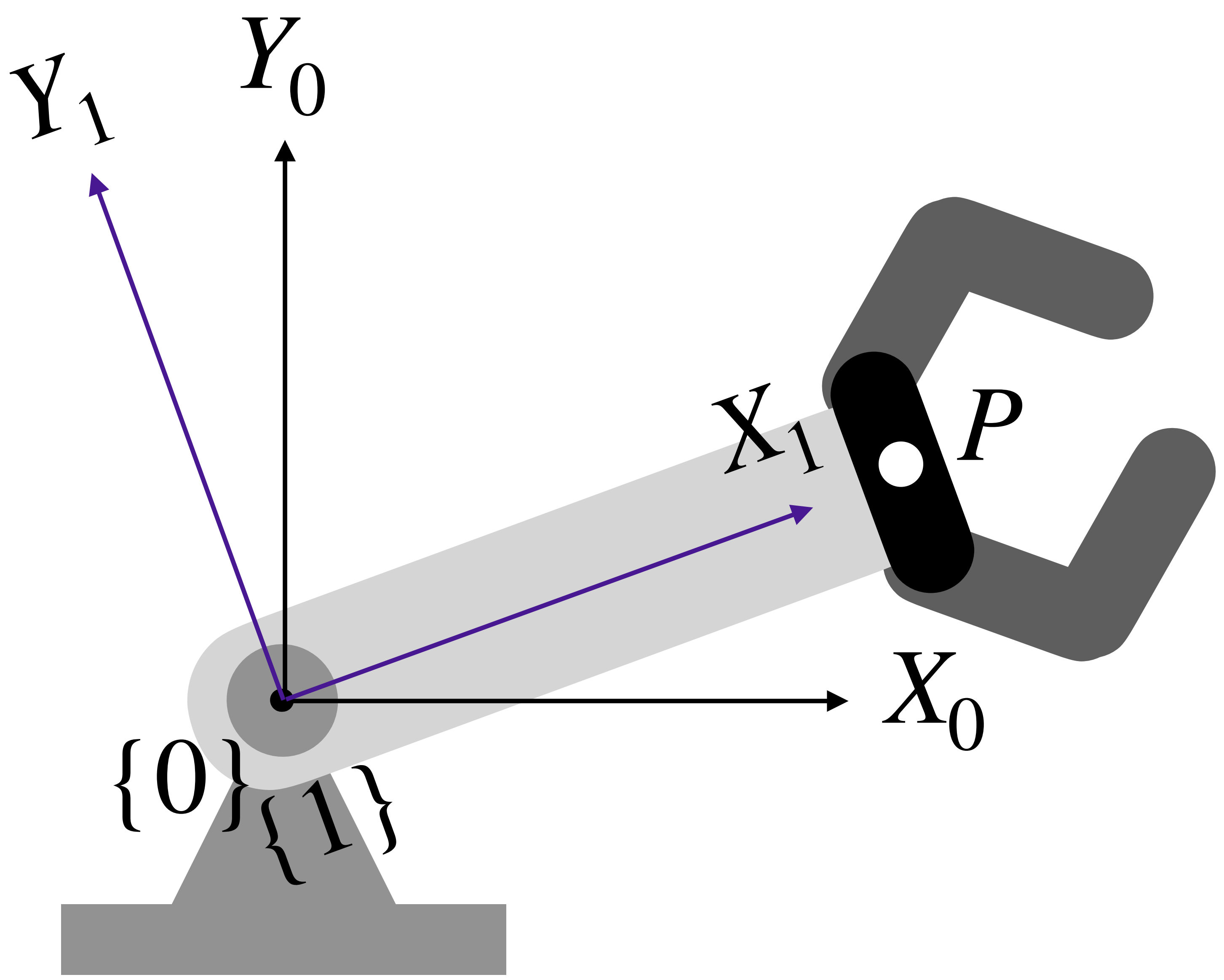

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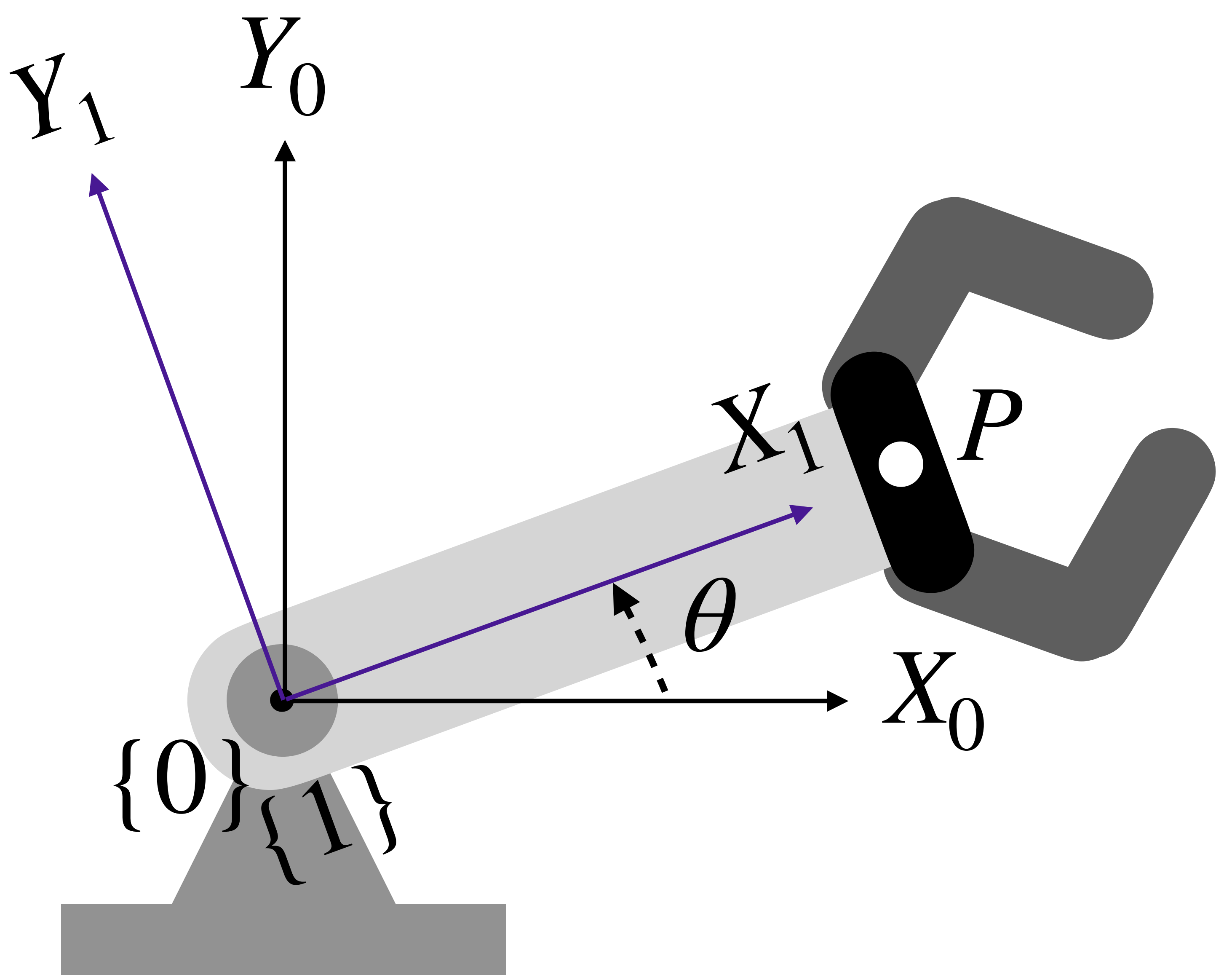
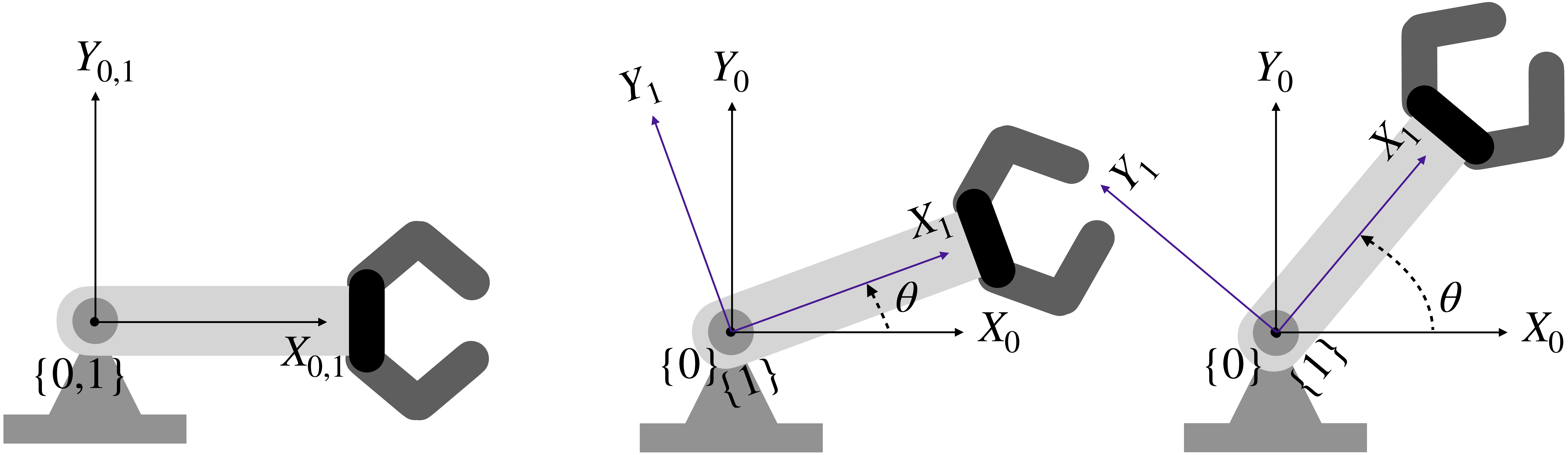
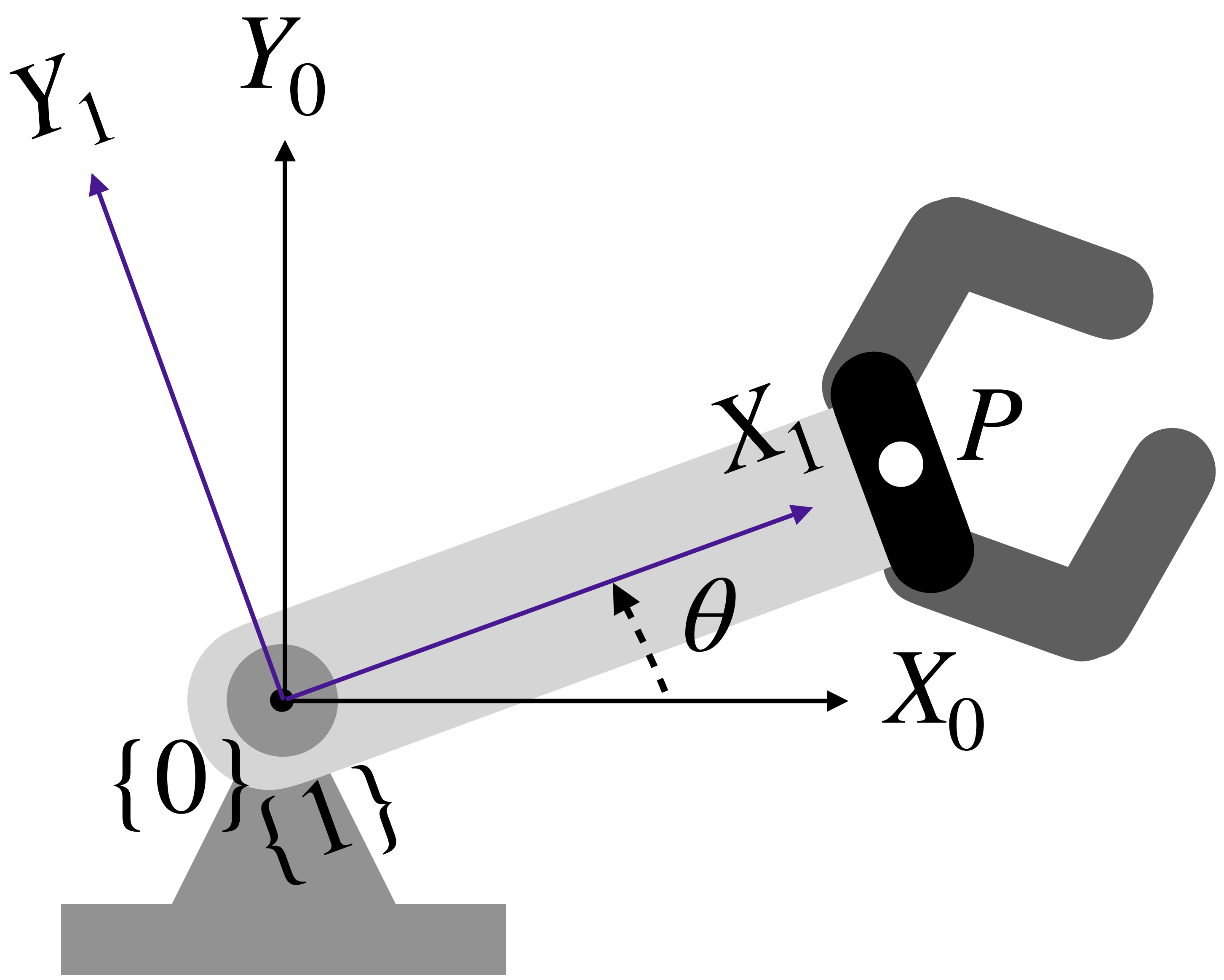



$Y_{0,1}$
$X_{0,1}$
{0,1}
Y_1
Y_0
X_1
θ
X_0
{0}
{1}
Y_0
X_1
Y_1
θ
X_0
{0}
{1}

2D Rotations

- In coordinate frame 1, point P might be located at:

$$r_1^P = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



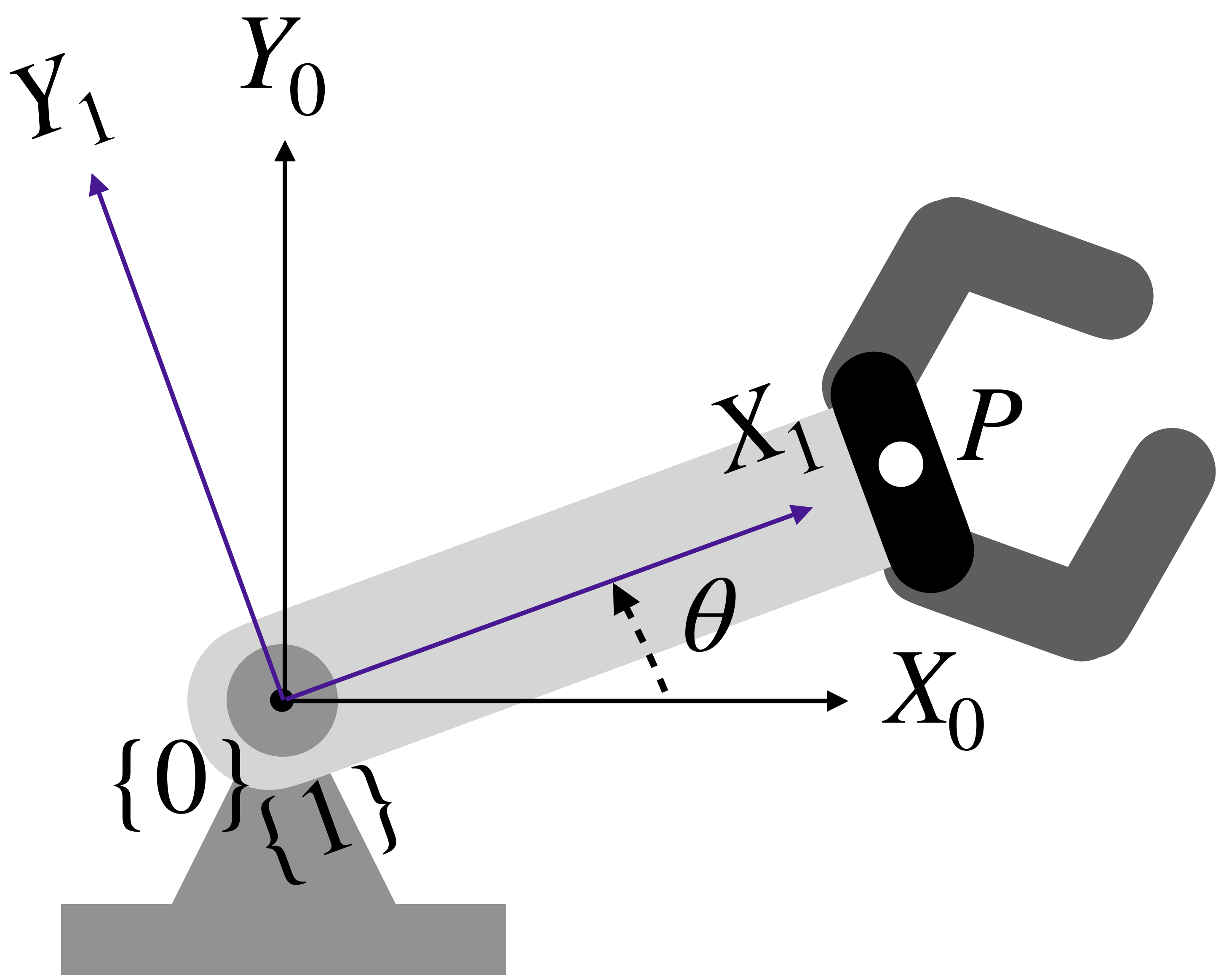
2D Rotations

- In coordinate frame 1, point P might be located at:

$$r_1^P = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

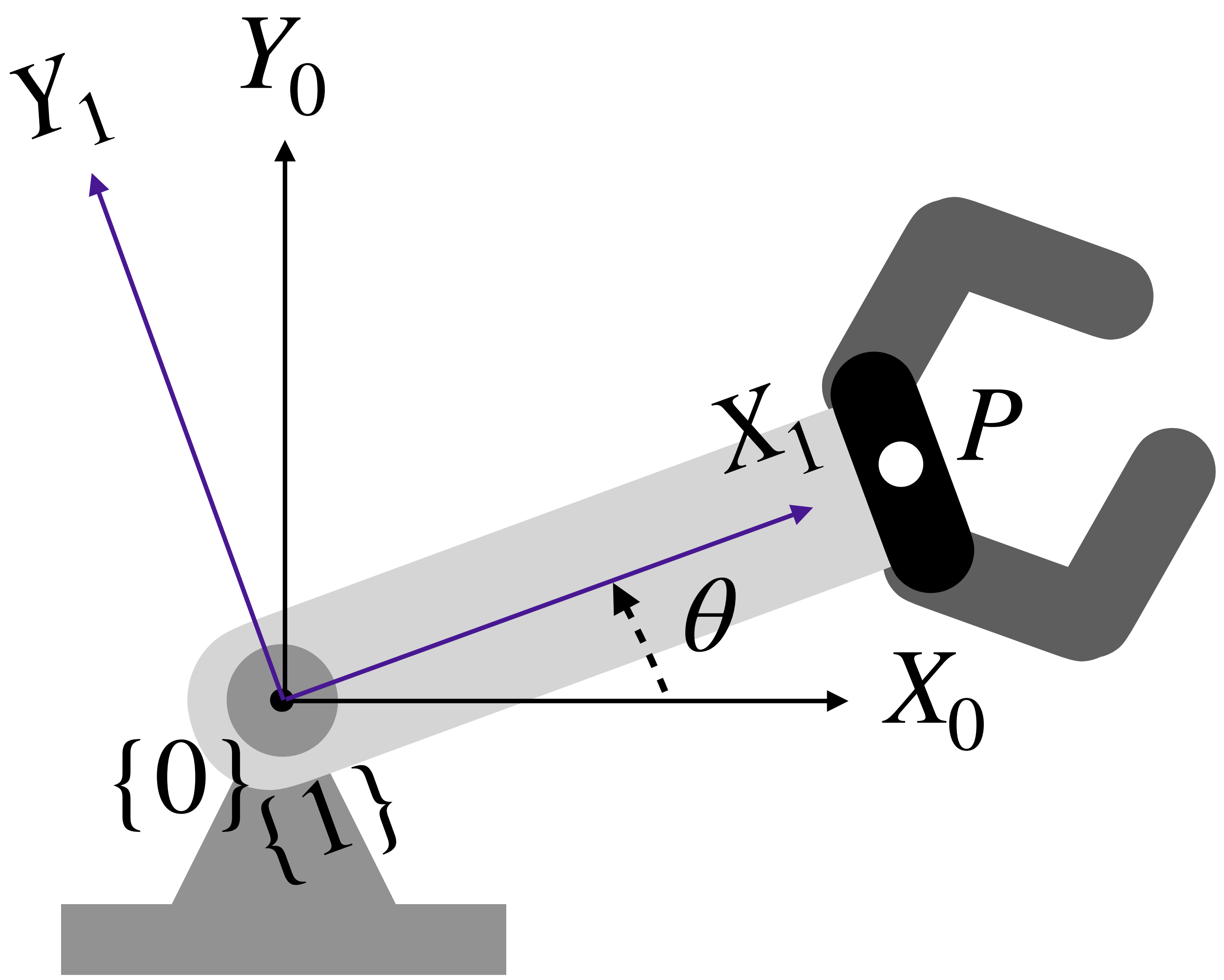
- But in coordinate frame 2, point P would be located at:

$$r_0^P = R_{01}(\theta) r_1^P$$



2D Rotations

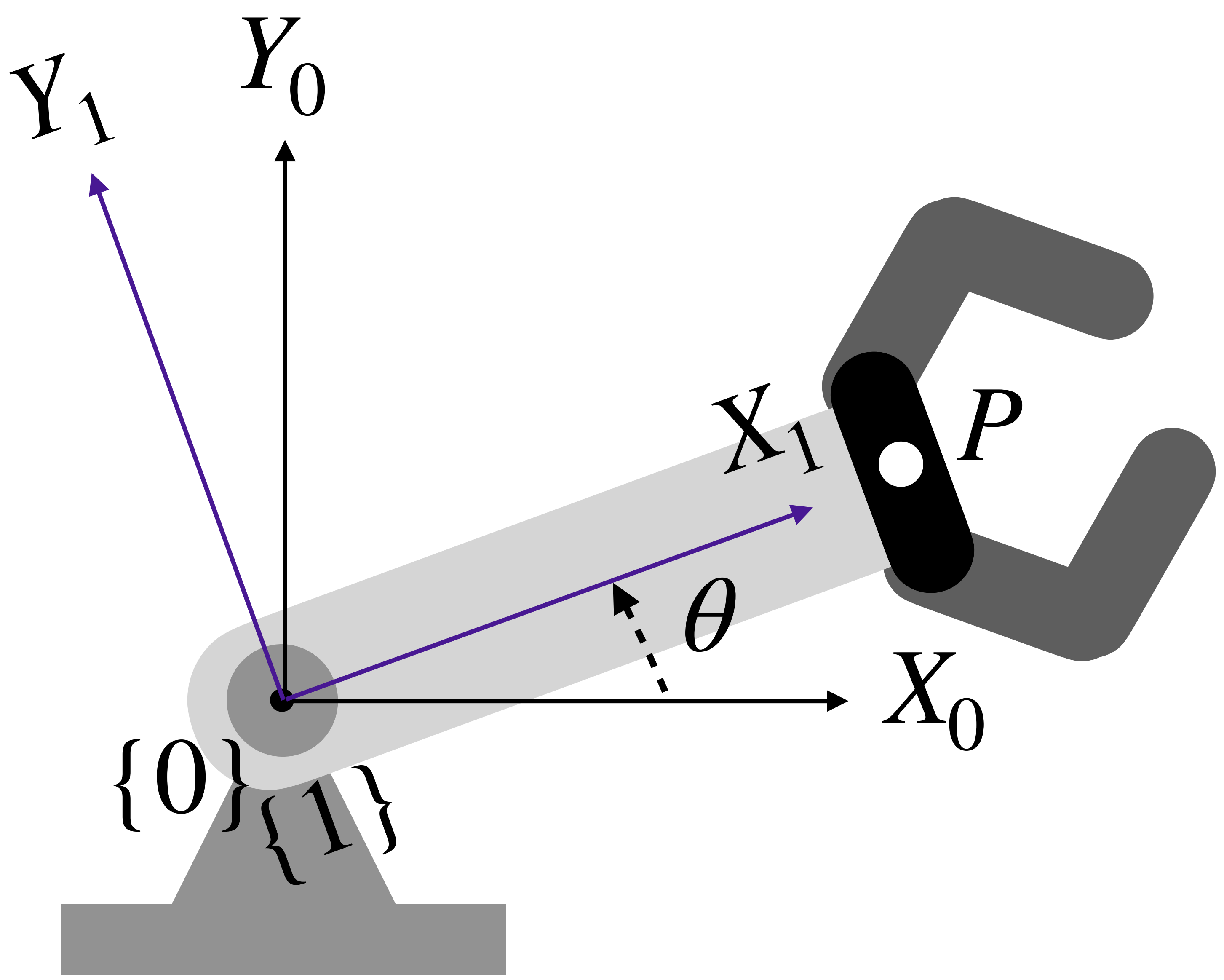
$$r_0^P = R_{01}(\theta) r_1^P$$



2D Rotations

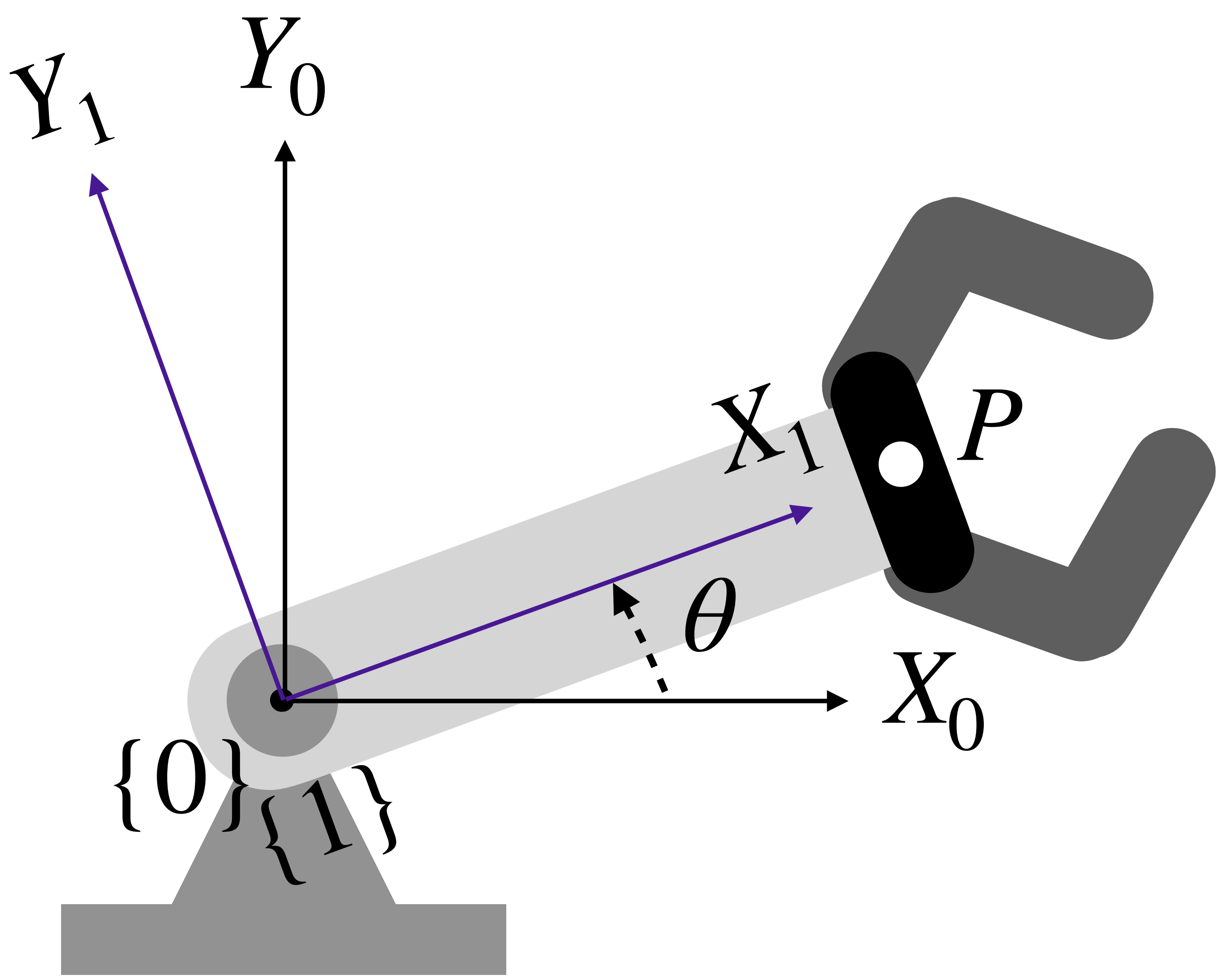
$$r_0^P = R_{01}(\theta) r_1^P$$

$$= \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} r_1^P$$



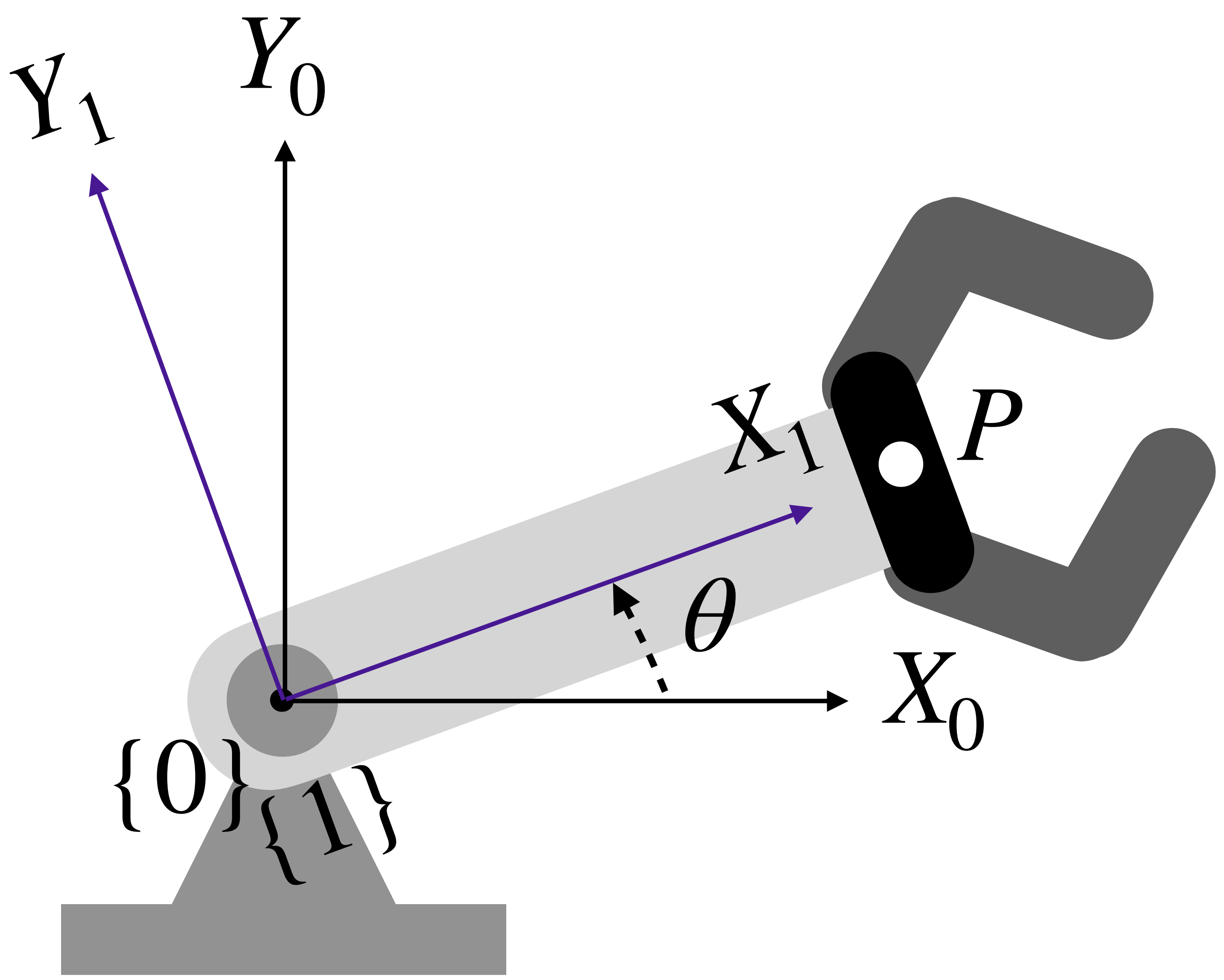
2D Rotations

$$
\begin{aligned}
r_0^P &= R_{01}(\theta) r_1^P \\
&= \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} r_1^P \\
&= \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2cos(\theta) \\ 2sin(\theta) \end{bmatrix}
\end{aligned}
$$



2D Rotations

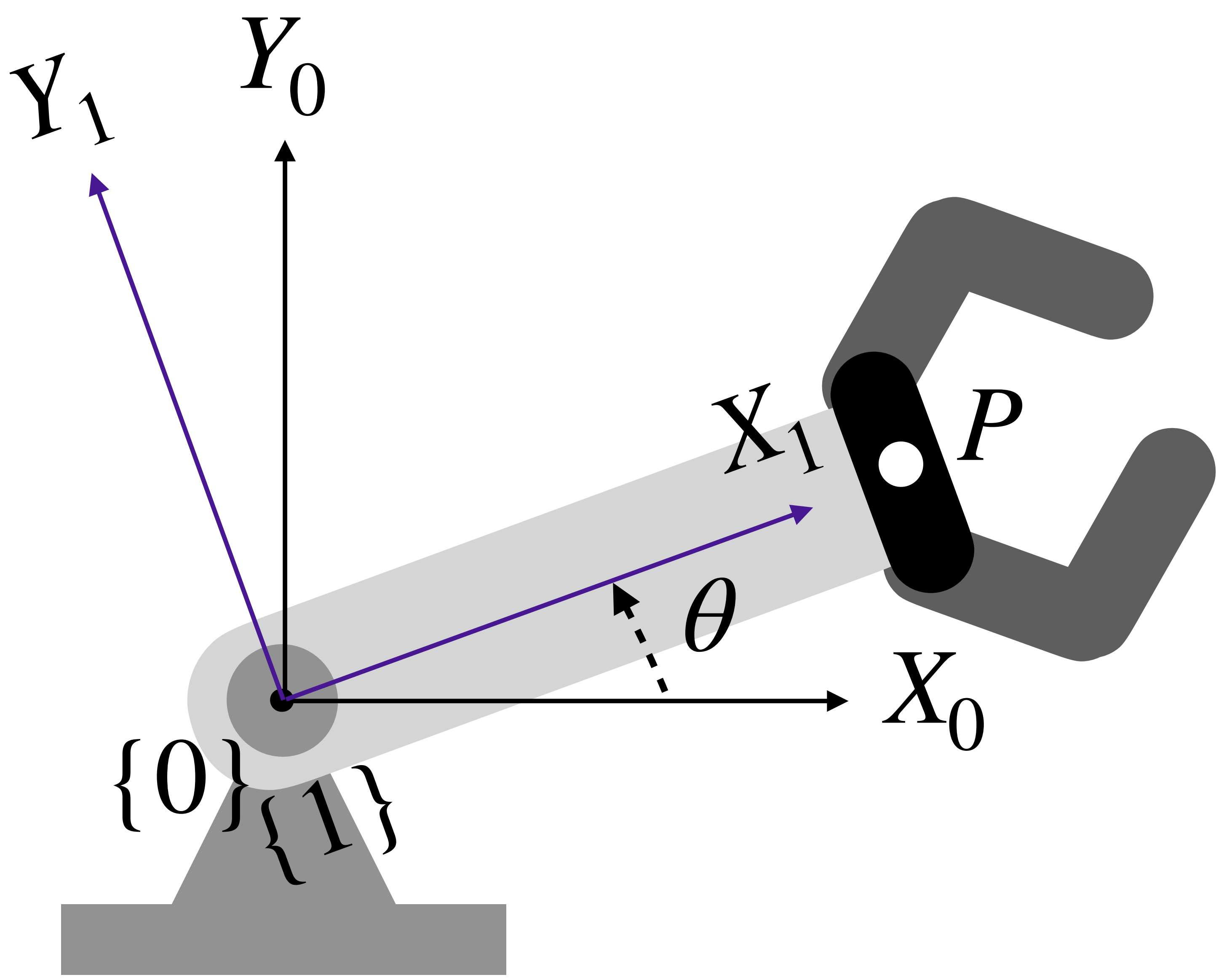
Question 1: What is the location of P in the reference frame 1 if θ is 45 degrees?



2D Rotations

Question 1: What is the location of P in the reference frame 1 if θ is 45 degrees?

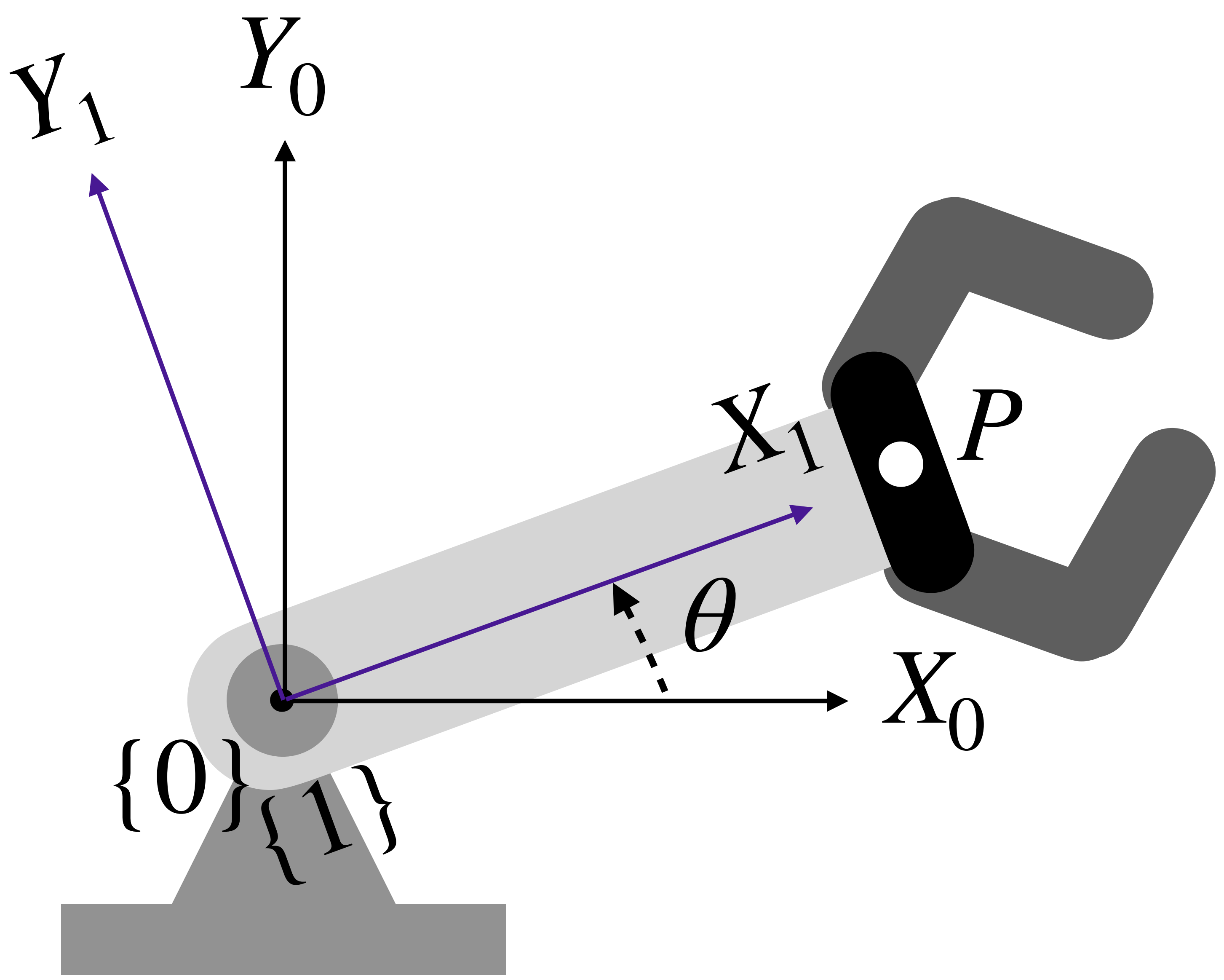
Question 2: What is the location of P in the reference frame 0 if θ is 45 degrees?



2D Rotations

- This is a very compact, but powerful form:

$$r_0^P = R_{01}(\theta) r_1^P$$



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- This is a very compact, but powerful form:

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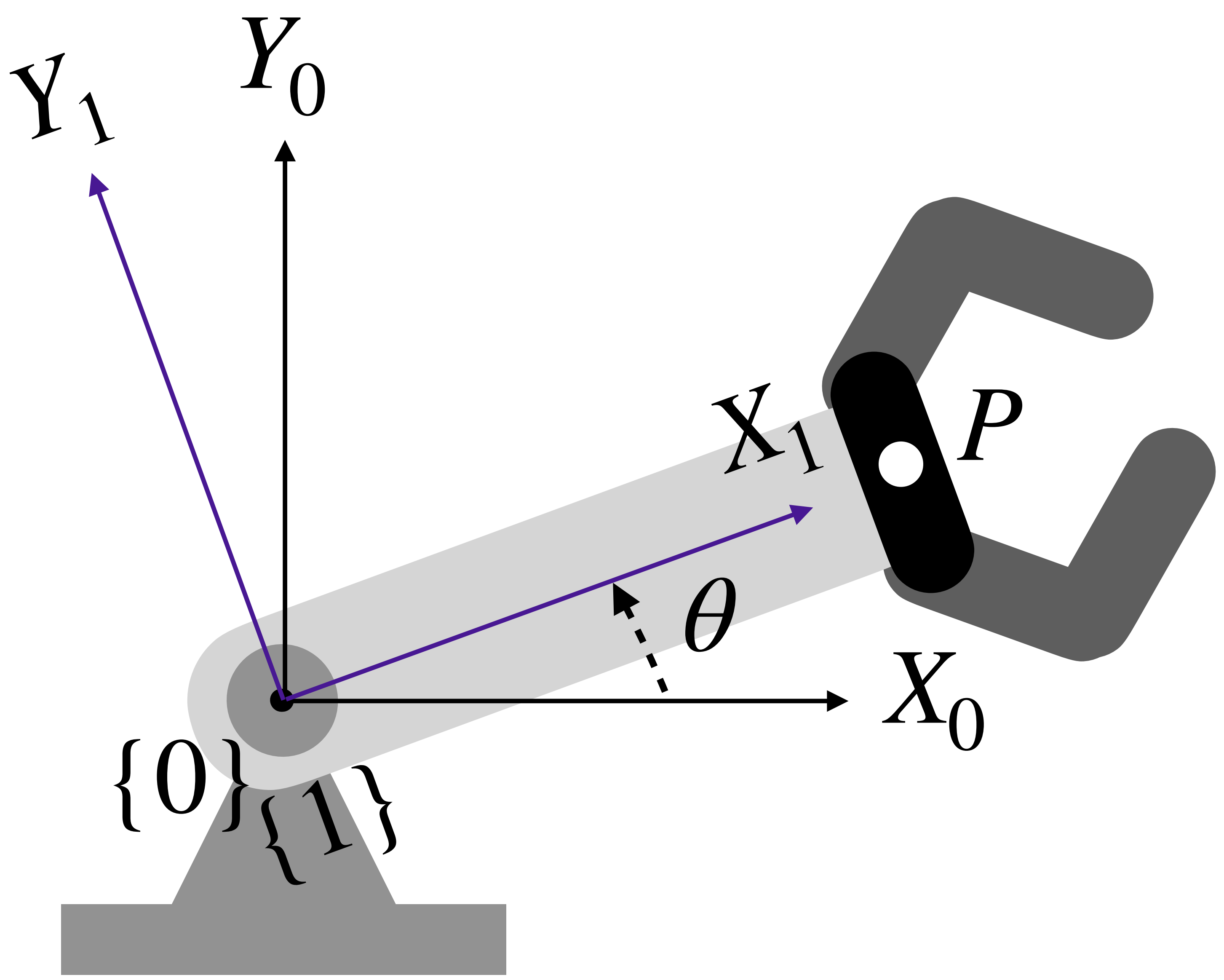
- **Note 1:** It is reversible:

$$r_1^P = R_{10}(\theta) r_0^P$$



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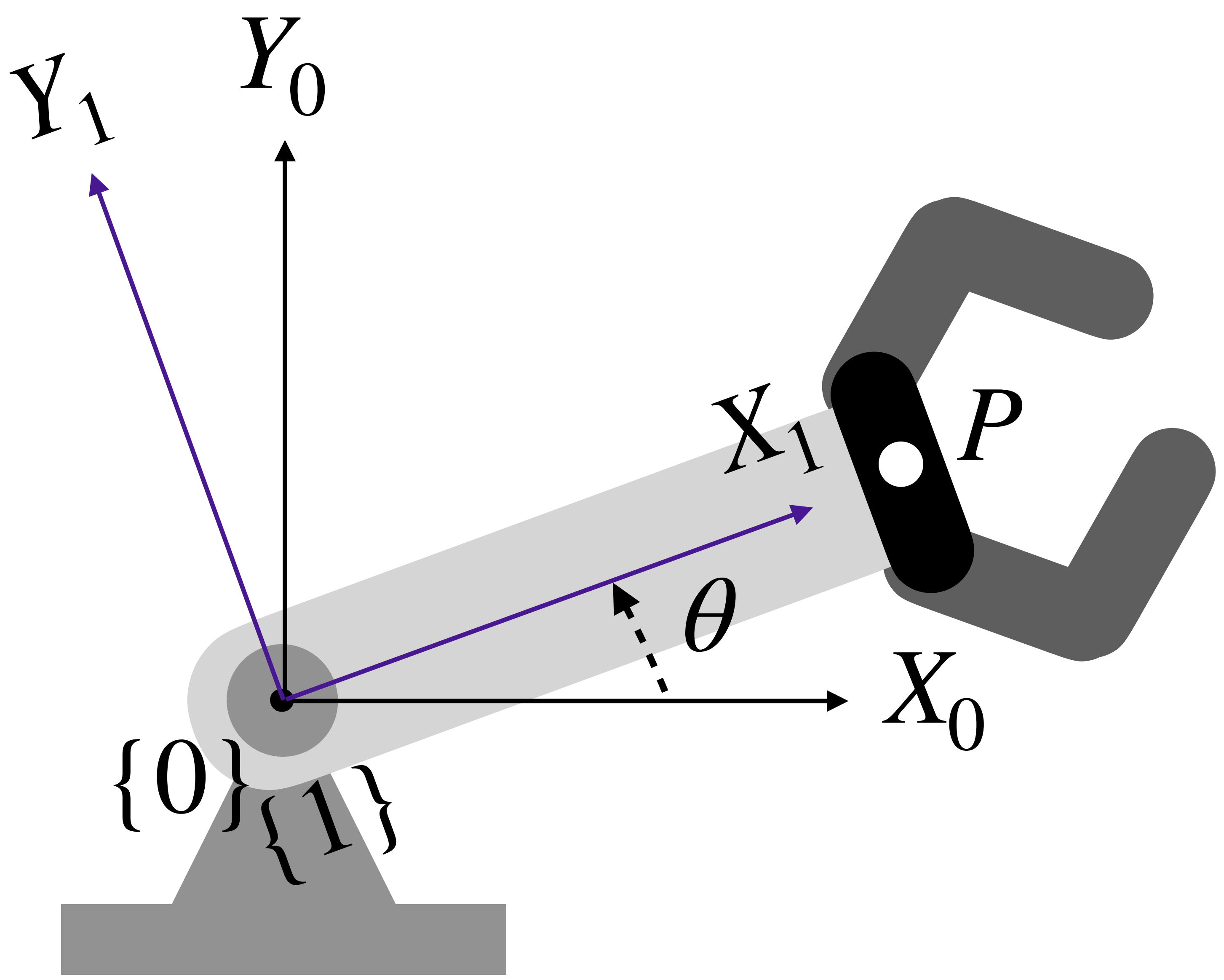
$$R_{10} = R_{01}^{-1}$$



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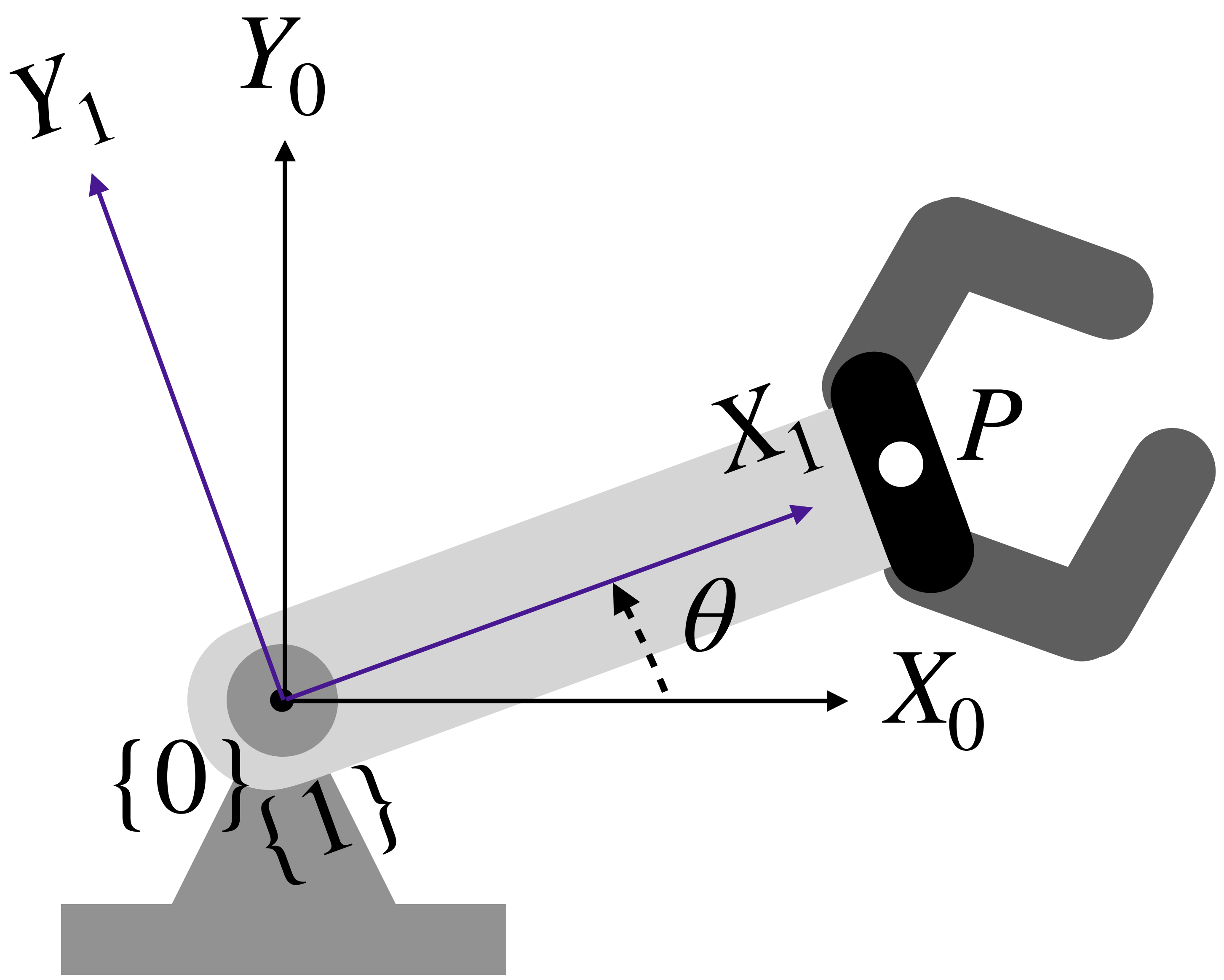
$$r_1^P = R_{10}(\theta) r_0^P$$

$$R_{10} = R_{01}^{-1} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$$



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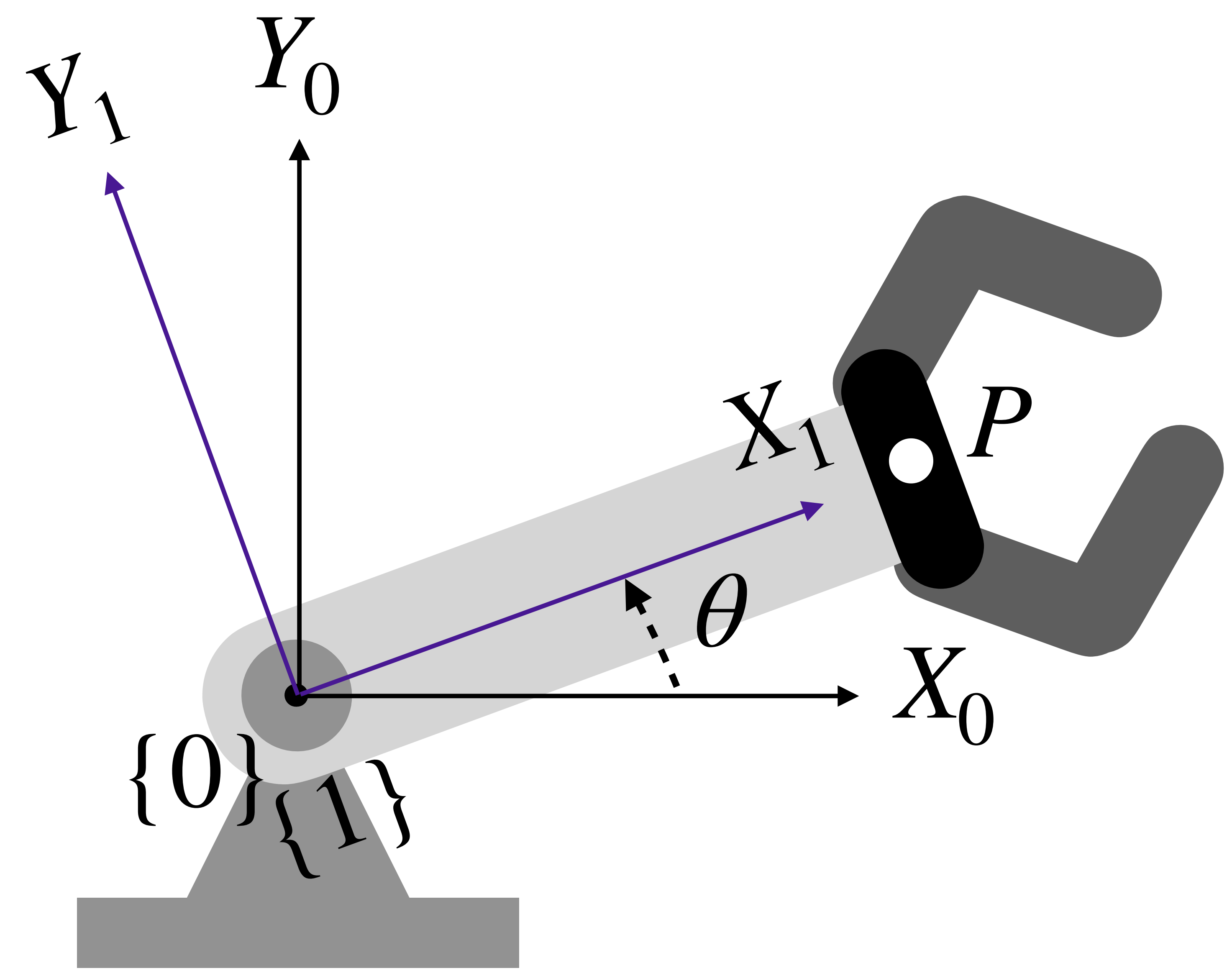


$$R_{10} = R_{01}^{-1} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} = R_{01}^T$$

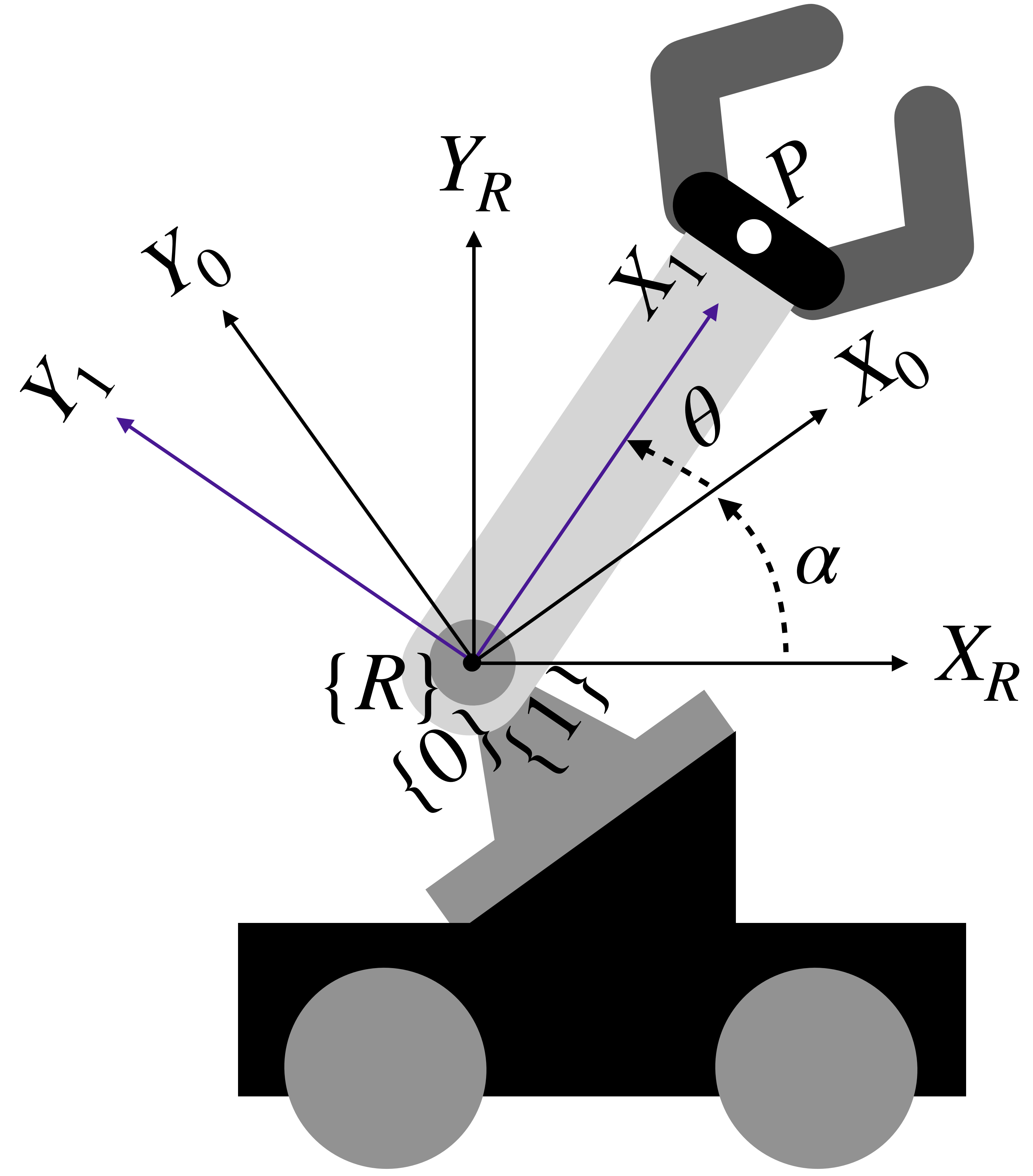
2D Rotations

- This is a very compact, but powerful form:

$$r_0^P = R_{01}(\theta) r_1^P$$

- **Note 2:** We can chain rotations:

$$r_R^P = R_{R0}(\alpha) r_0^P$$



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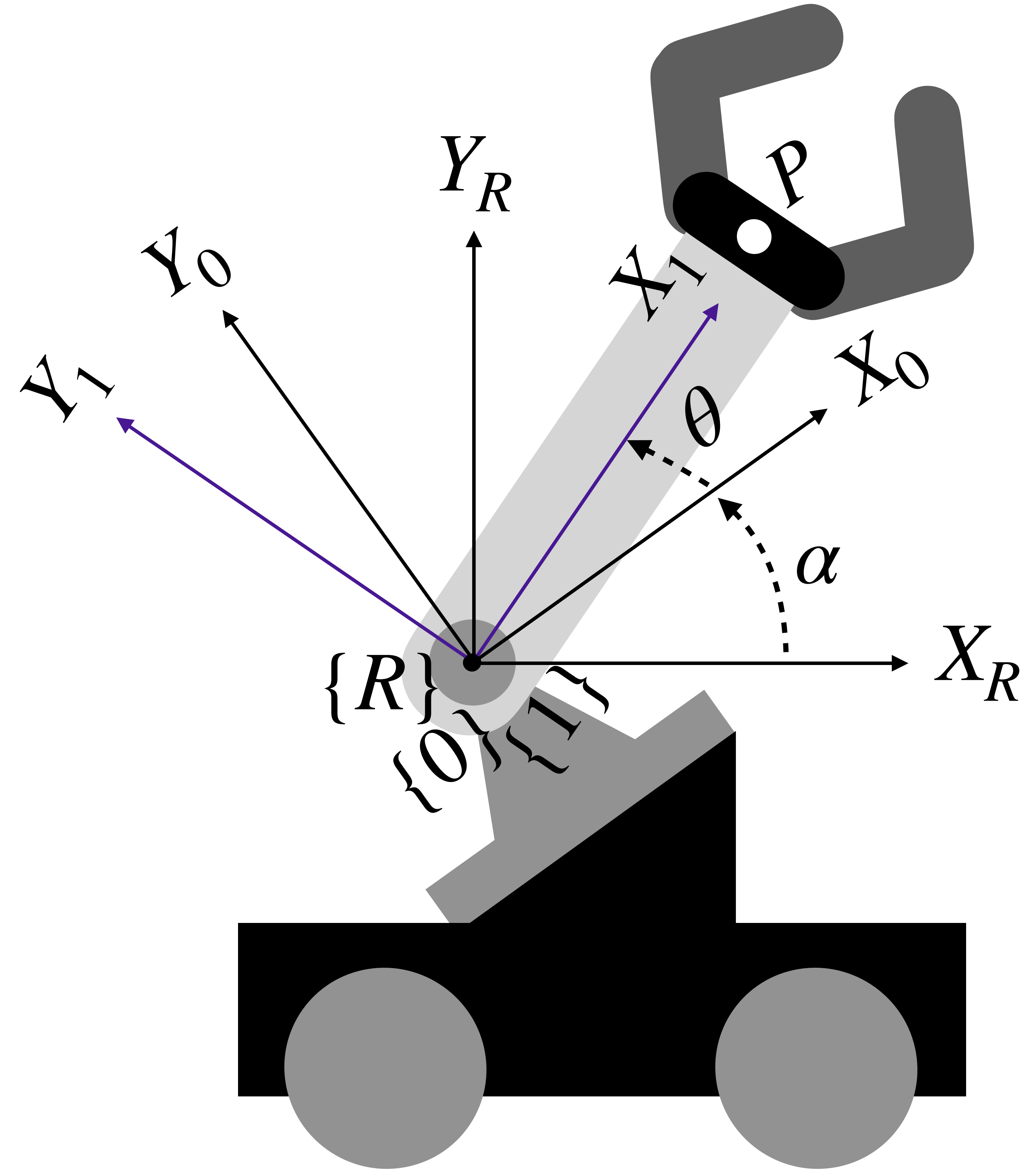
$$= R_{R0}(\alpha) R_{01}(\theta) r_1^P$$



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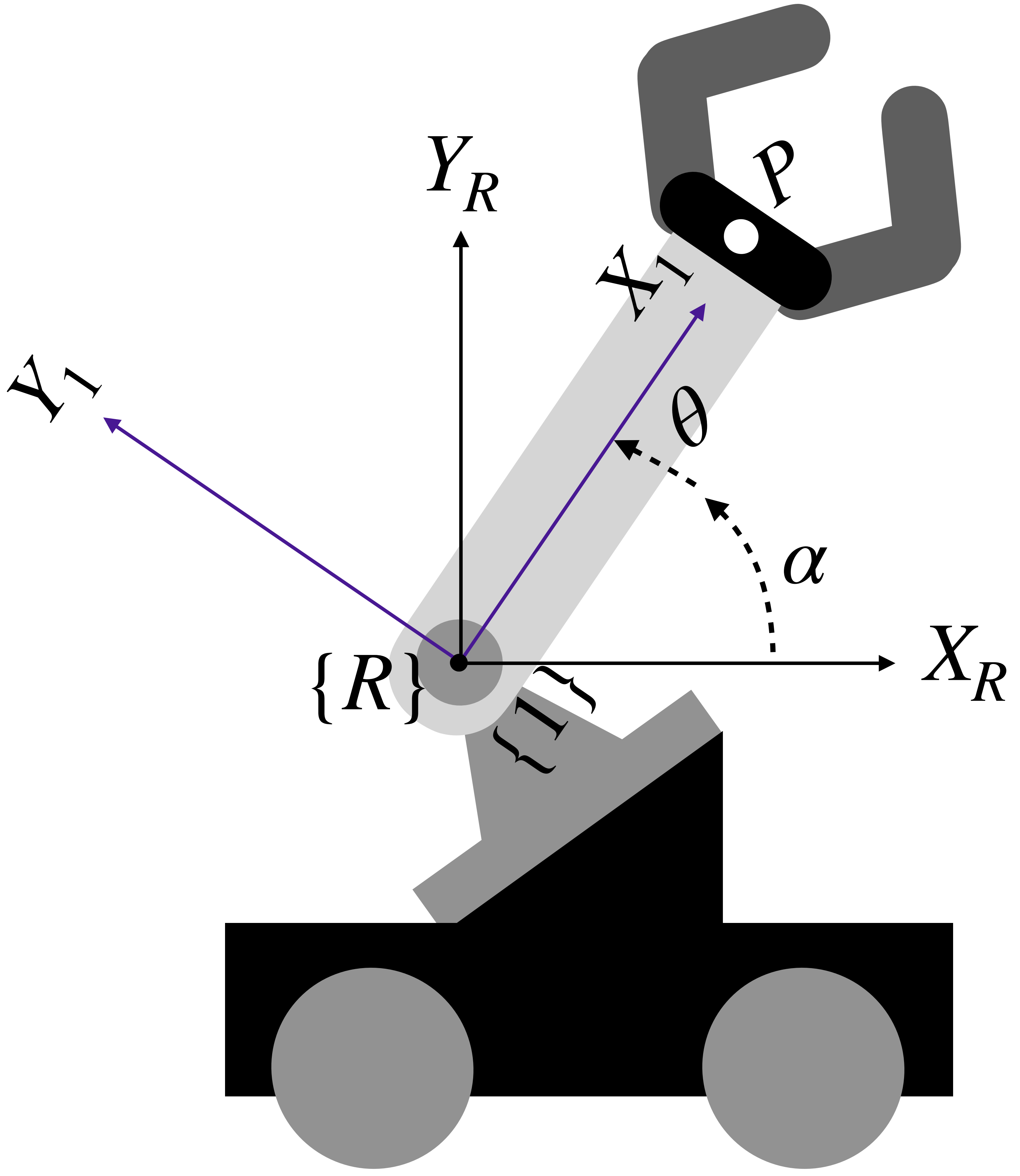

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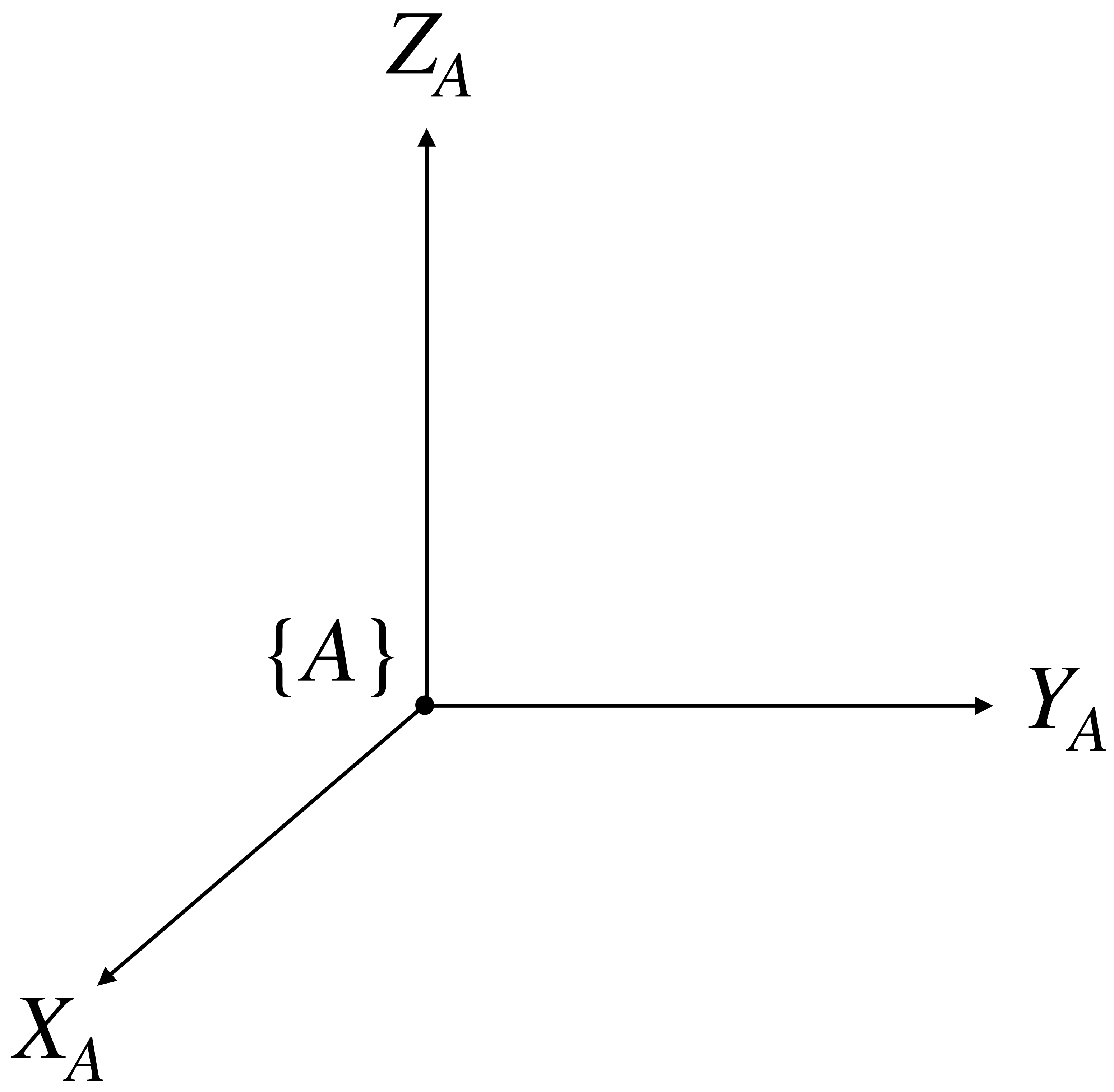
3D Rotations

- Assume we have a coordinate frame A.
- There is a point P in space, and a vector that defines P relative to coordinate frame A.
- Let us rotate point P **about the Z axis** of rotation that is perpendicular to the 2D X-Y plane and passes through the frame A's origin
- If we rotate P to a new point P', and the amount of rotation is θ radians, what are the coordinates of $r_A^{P'}$?



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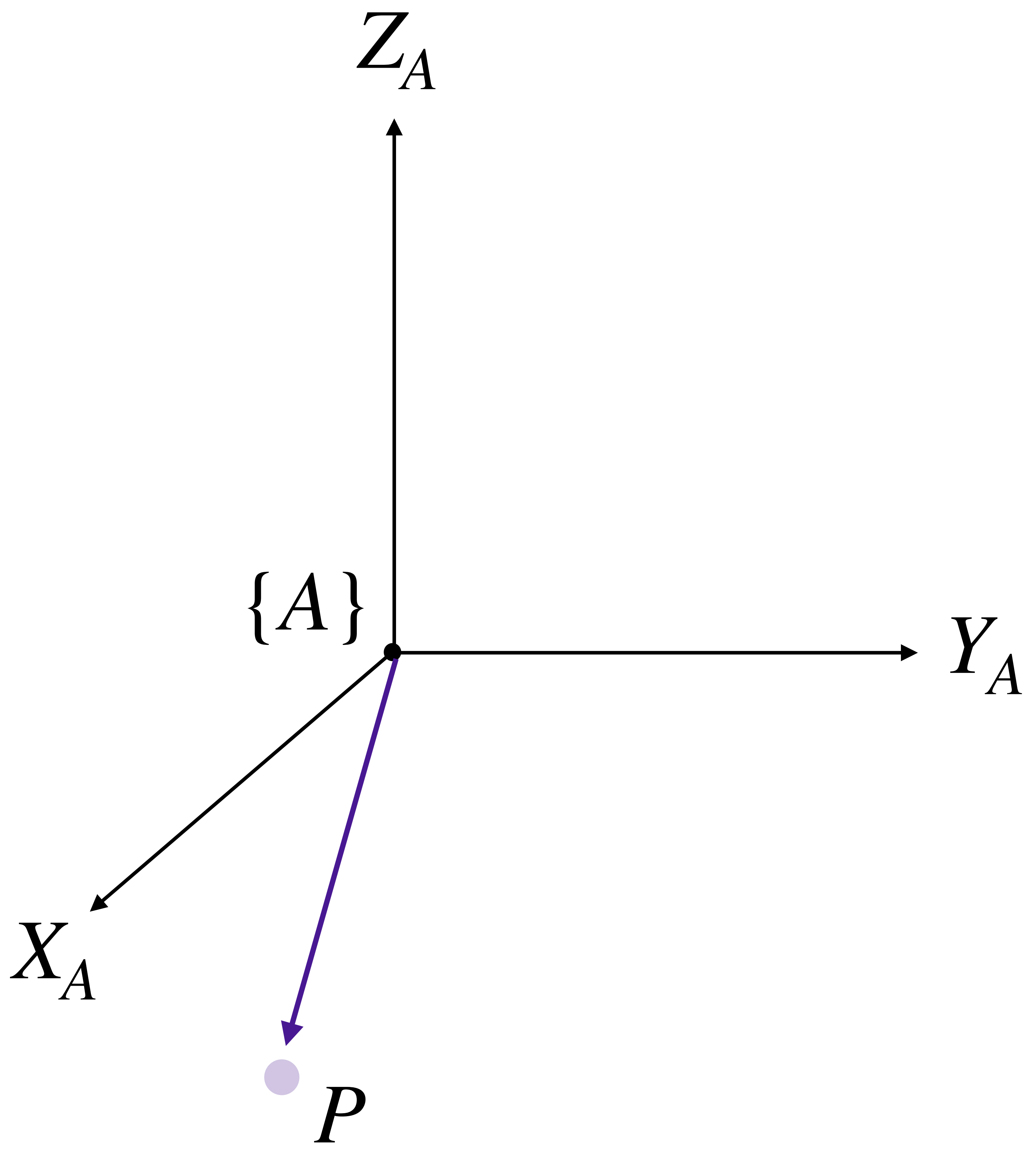

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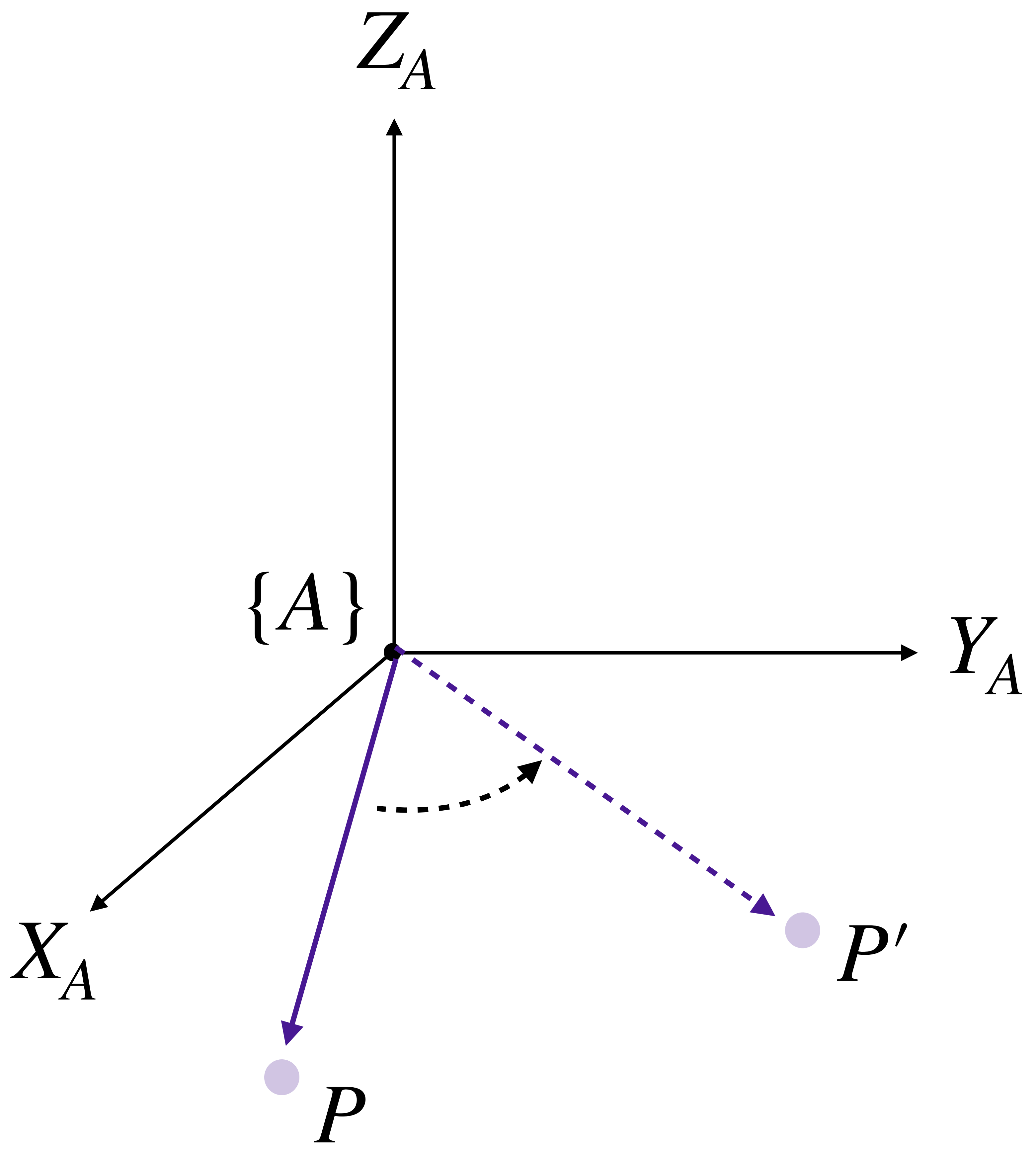

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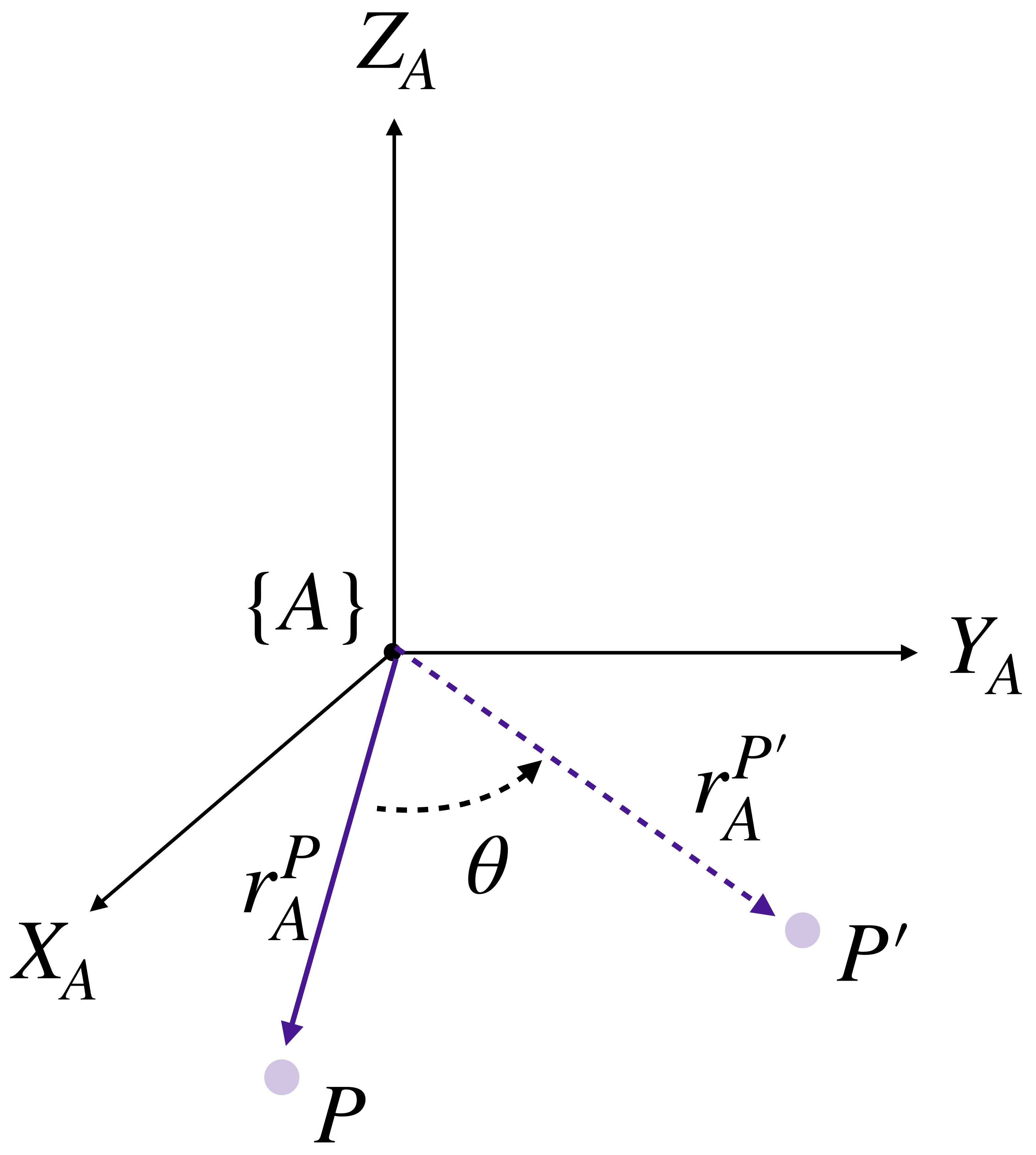
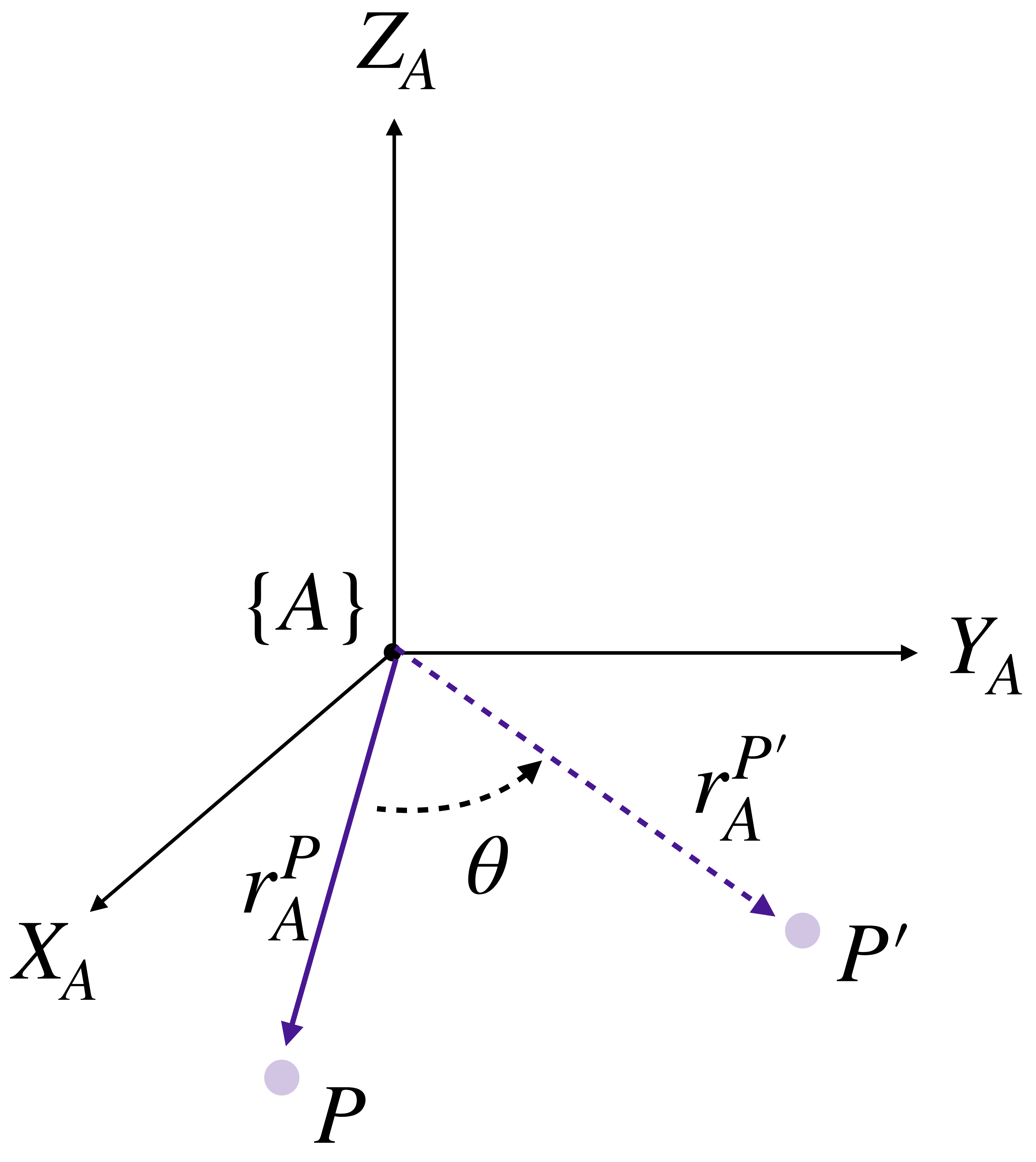


2D Rotations

$$r_0^P = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} r_1^P$$

3D Rotations (about Z)

$$r_0^P = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} r_1^P$$



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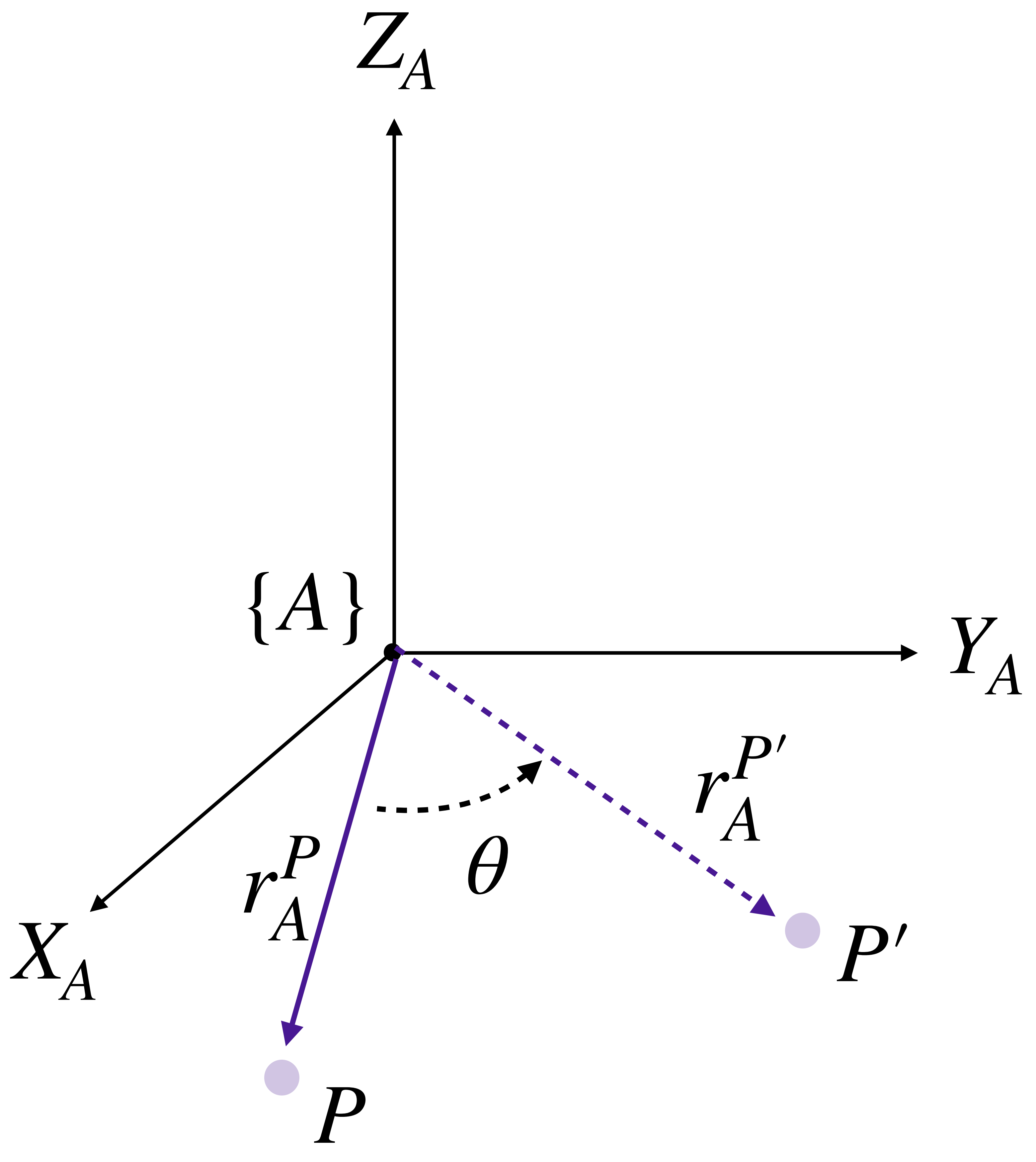

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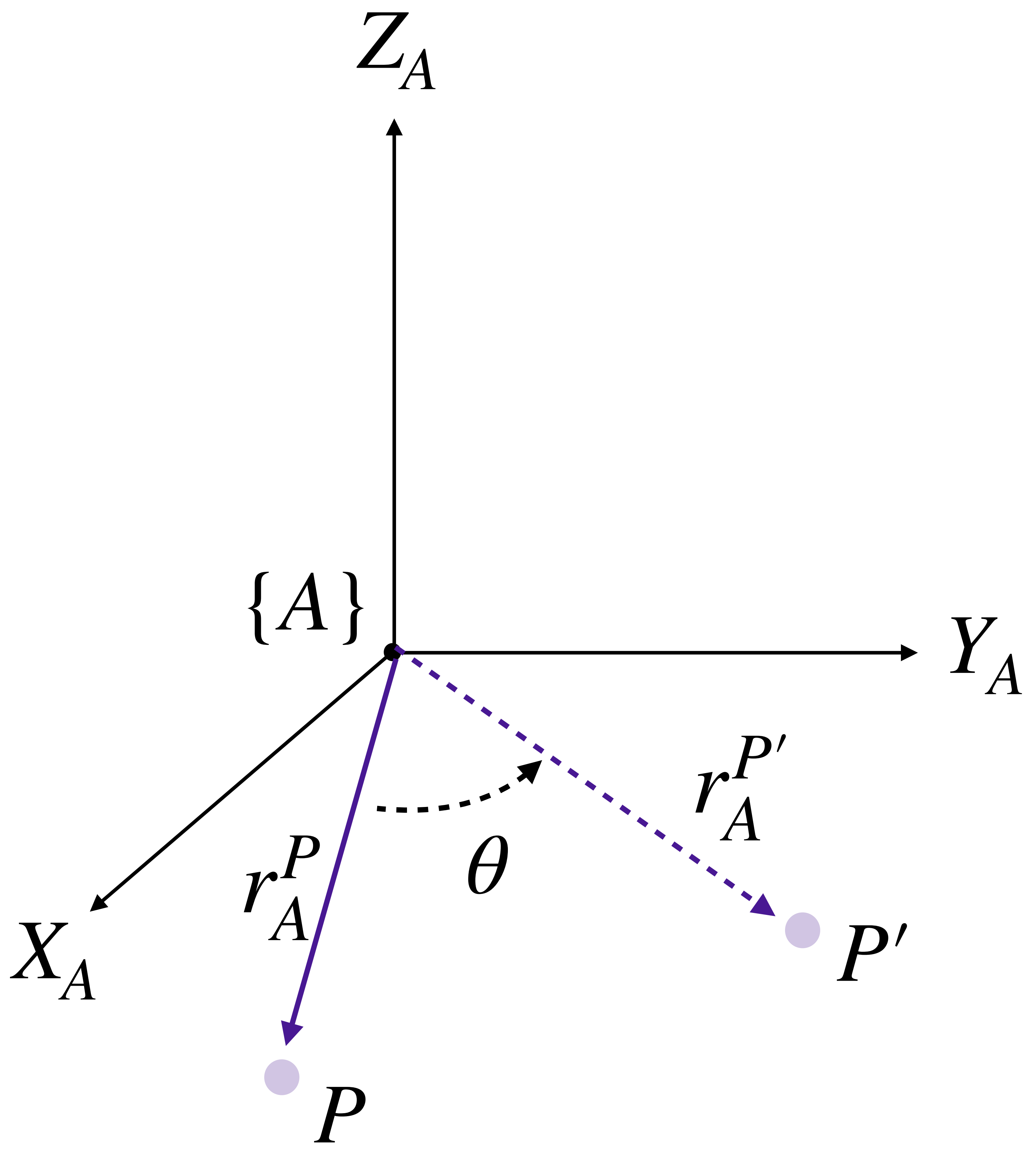
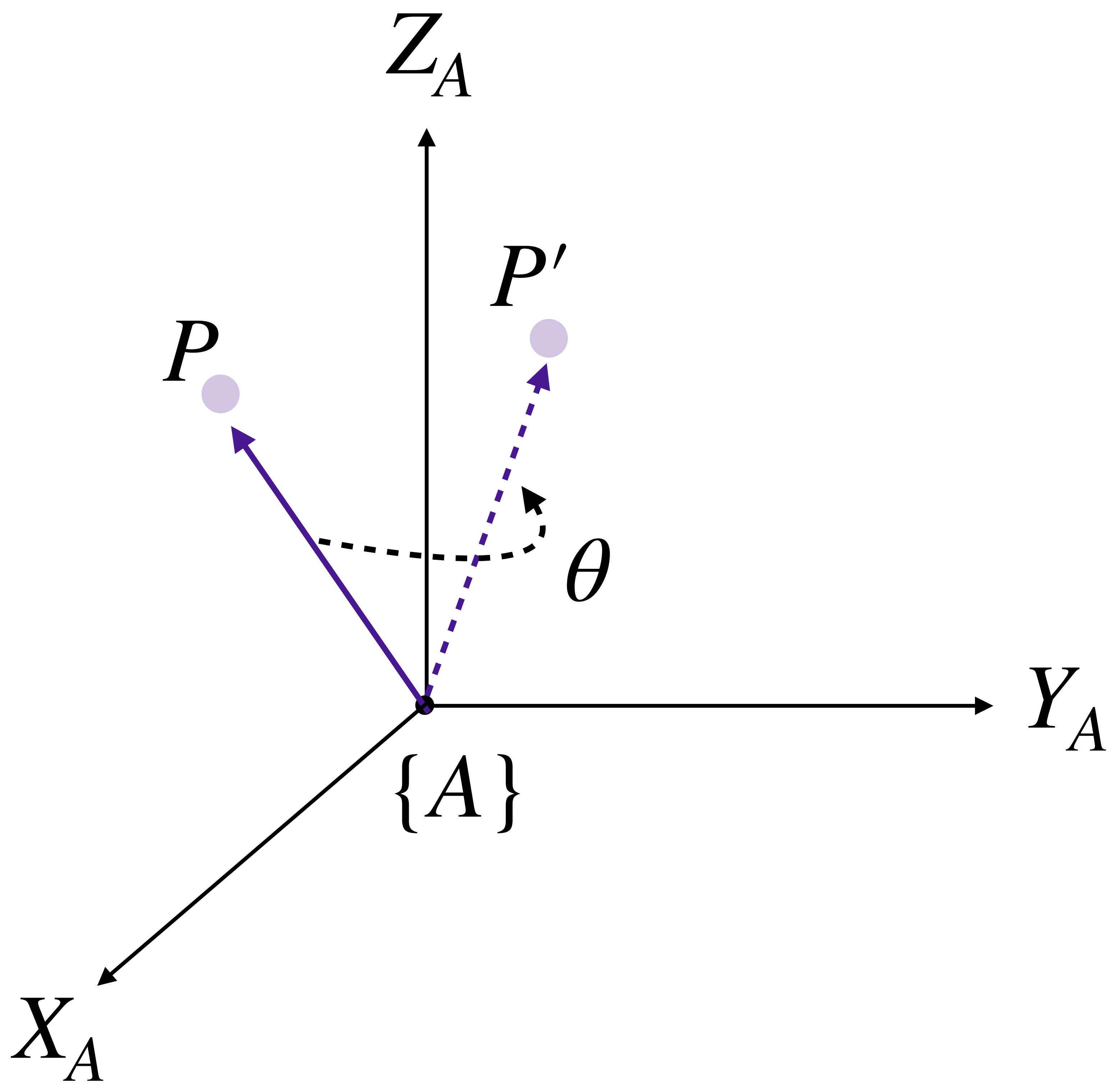


3D Rotations

- Assume we have a coordinate frame A.
- There is a point P in space, and a vector that defines P relative to coordinate frame A.
- Let us rotate point P **about the Z axis** of rotation that is perpendicular to the 2D X-Y plane and passes through the frame A's origin
- If we rotate P to a new point P', and the amount of rotation is θ radians, what are the coordinates of $r_A^{P'}$?



3D Rotations About Each Axis

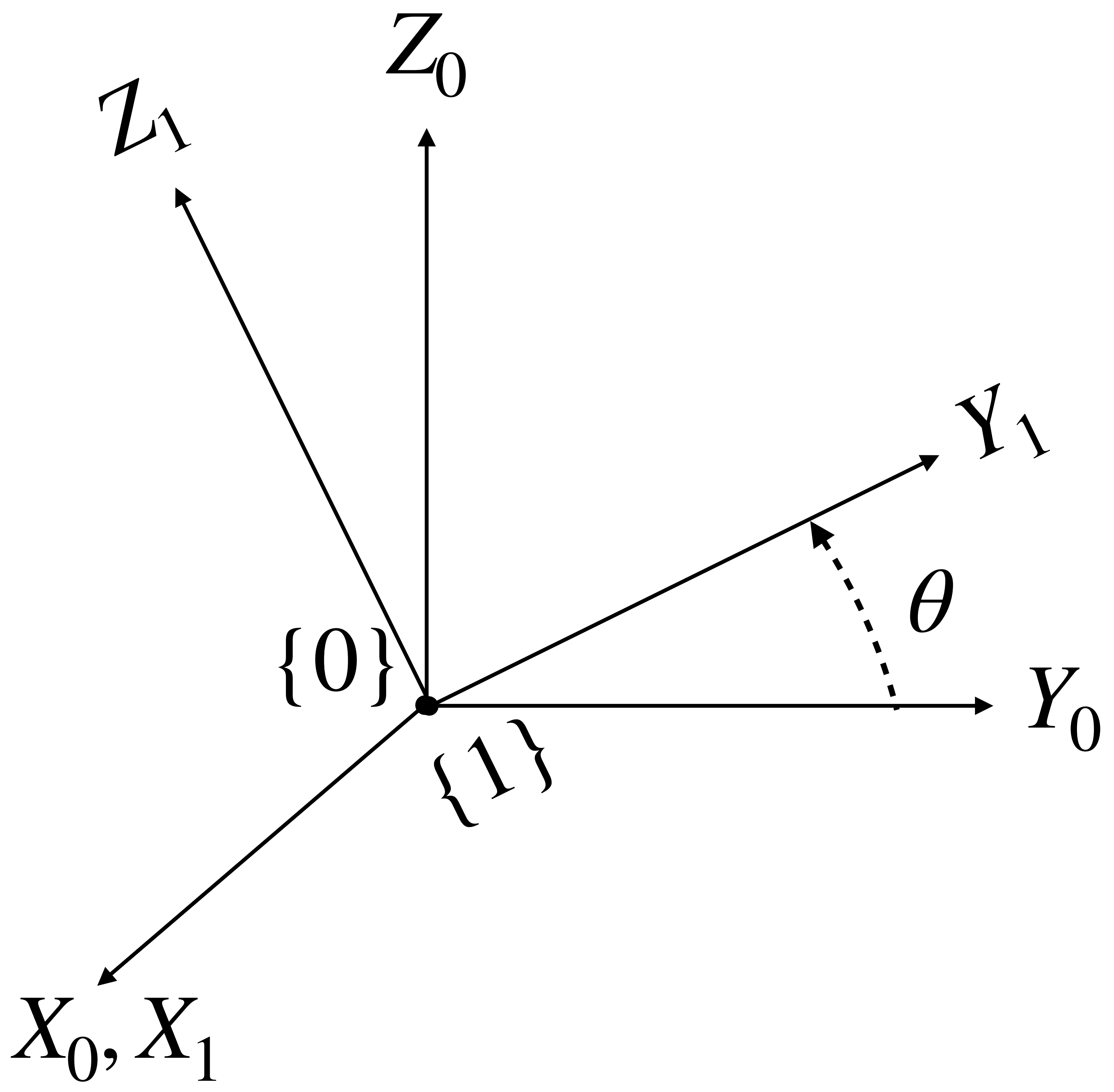
$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

$$R_Y(\theta) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix}$$

$$R_Z(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3D Rotations for Reference Frames

- Similar to 2D, we can attach reference frames to objects and robots in 3D space, and understand their relative rotations using 3D rotation matrices.
- The image right shows reference frame {1} is obtained by rotating frame {0} by θ radians about the X axis.



Agenda

1. Motivation
2. 2D Rotations and 2D Coordinate Frames
3. 3D Rotations and 3D Coordinate Frames
4. **Properties of Rotations**

Rotation Matrix Properties

- The rotation matrix R_{01} which describes the orientation of frame 1 with respect to frame 0 has the form:

$$R_{01}(\theta) = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}$$

- The columns correspond to the coordinates of the basis vectors of frame 1 in the coordinate frame 0 (i.e. x_1 is a column vector containing the coordinates of the X axis of frame 1 expressed in the coordinates of frame 0)

Rotation Matrix Properties

- A rotation matrix is orthogonal, i.e. columns (and rows) are orthogonal to each other.
- $R^T R = RR^T = I$
- $R^T = R^{-1}$
- det(R) = 1
- The product of two rotation matrices is a rotation matrix
- Identity matrix is a rotation matrix

$$R_Z(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$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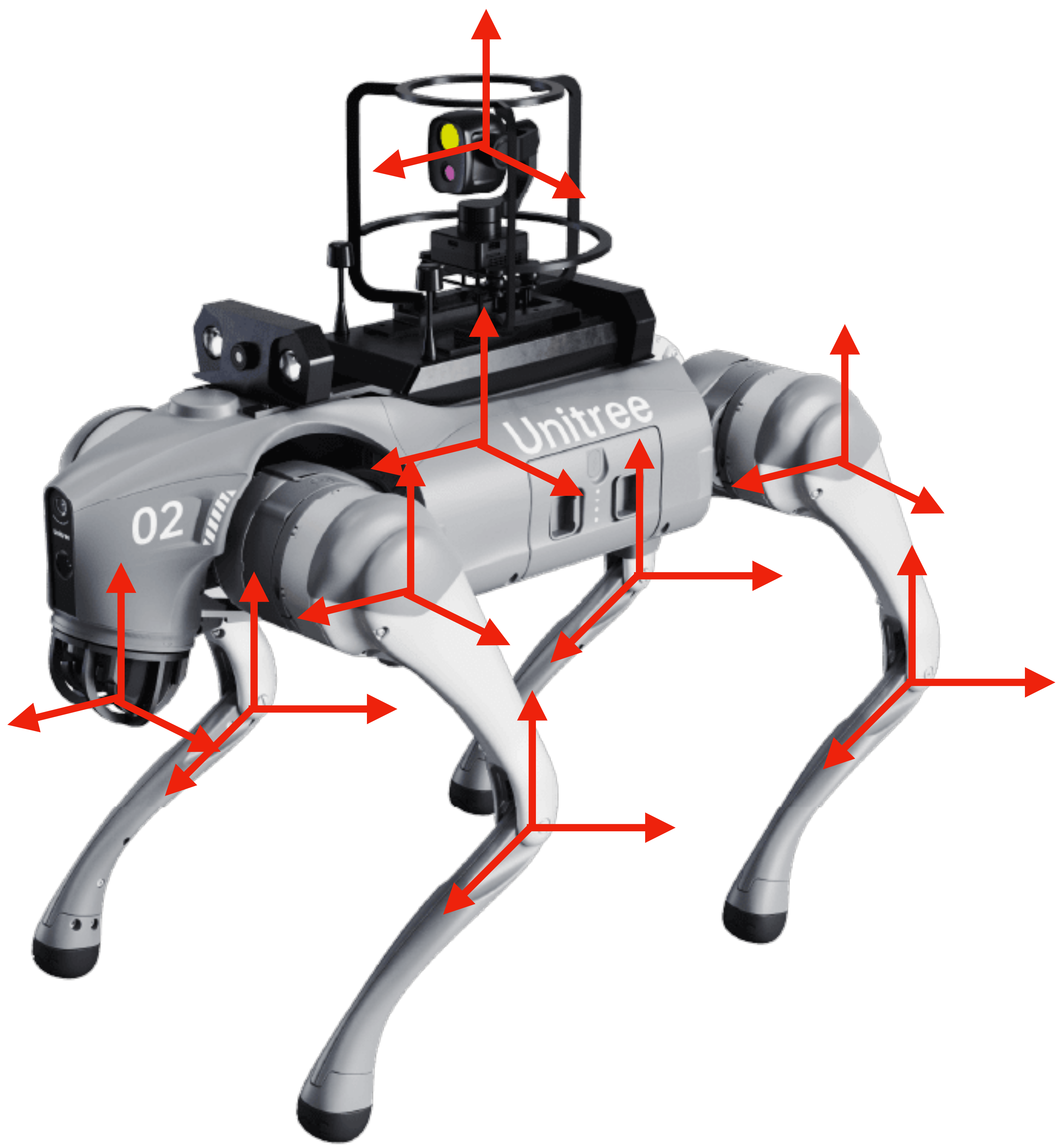
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