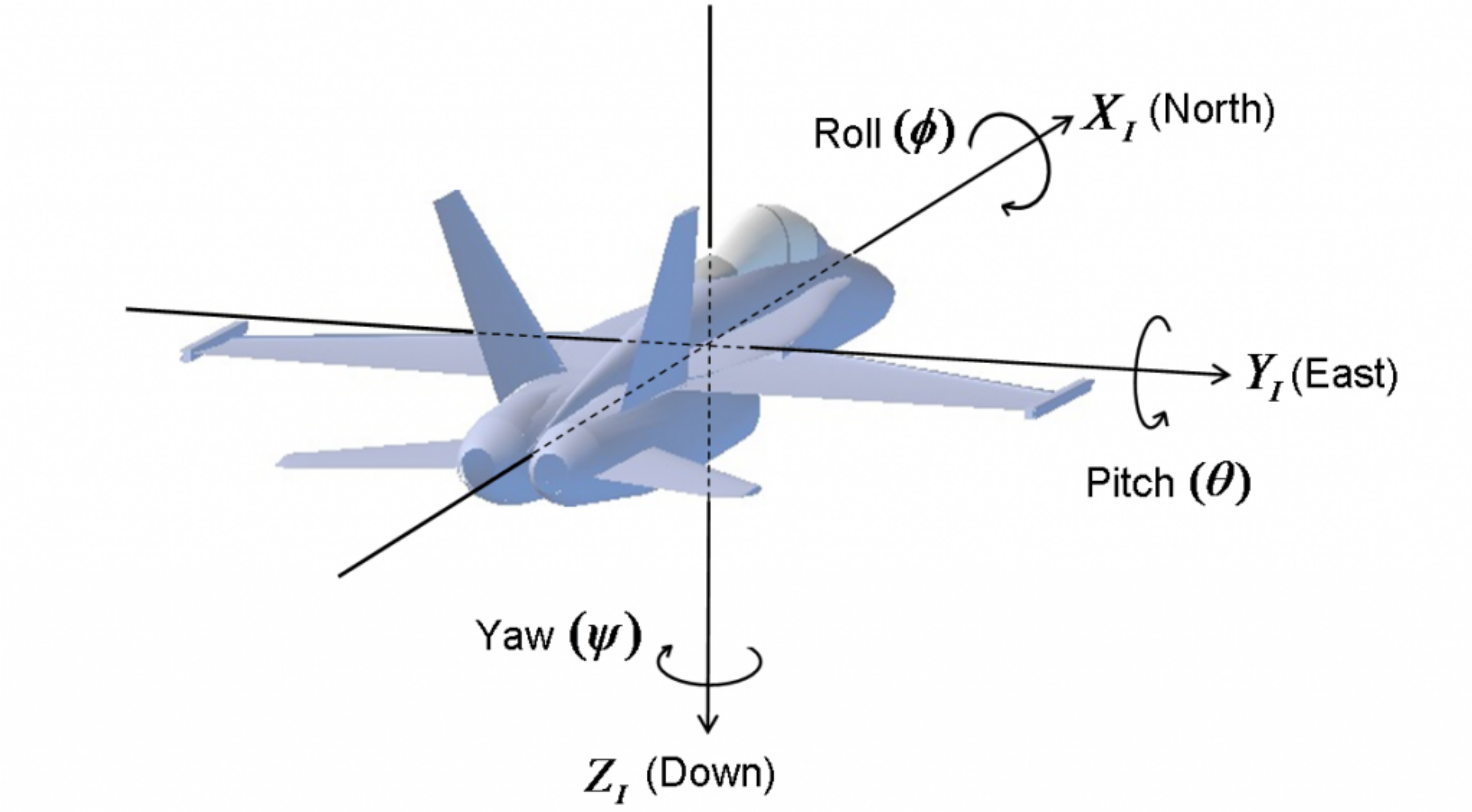
ROB-UY 2004

Robotic Manipulation & Locomotion

Lecture 06B - More Rotations



Chromatics.com

Agenda

1. Euler Angles
2. Roll-Pitch-Yaw
3. Axis-Angle
4. Quaternions

Agenda

1. **Euler Angles**
2. Roll-Pitch-Yaw
3. Axis-Angle
4. Quaternions

Rotation Matrices

In 2D

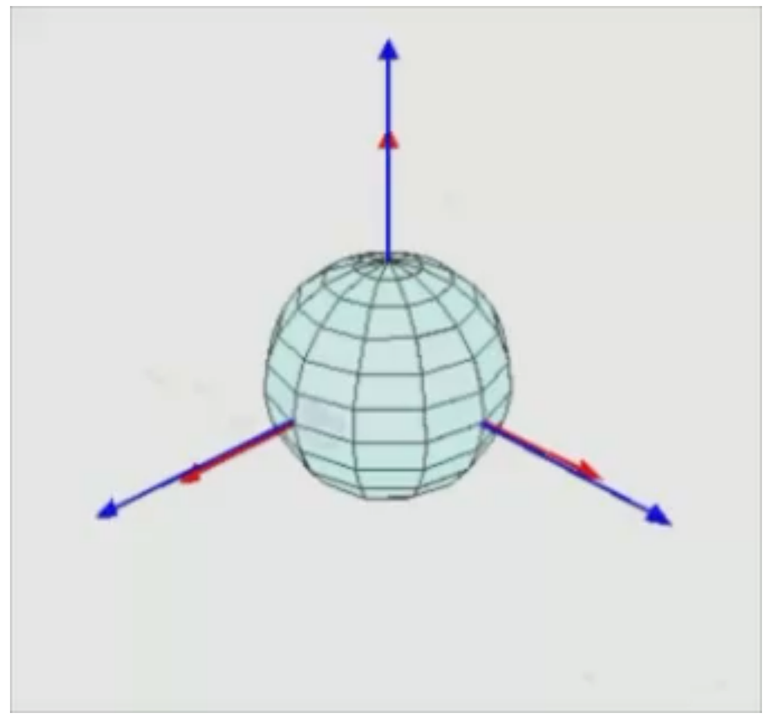
$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In 3D

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Rotation Matrices & Euler Angles

- **Euler angles** are a 3 angles (introduced by Leonard Euler) used to describe the orientation of a rigid body relative to a fixed coordinate frame
- **Any rotation** be described by 3 successive Euler angle rotations



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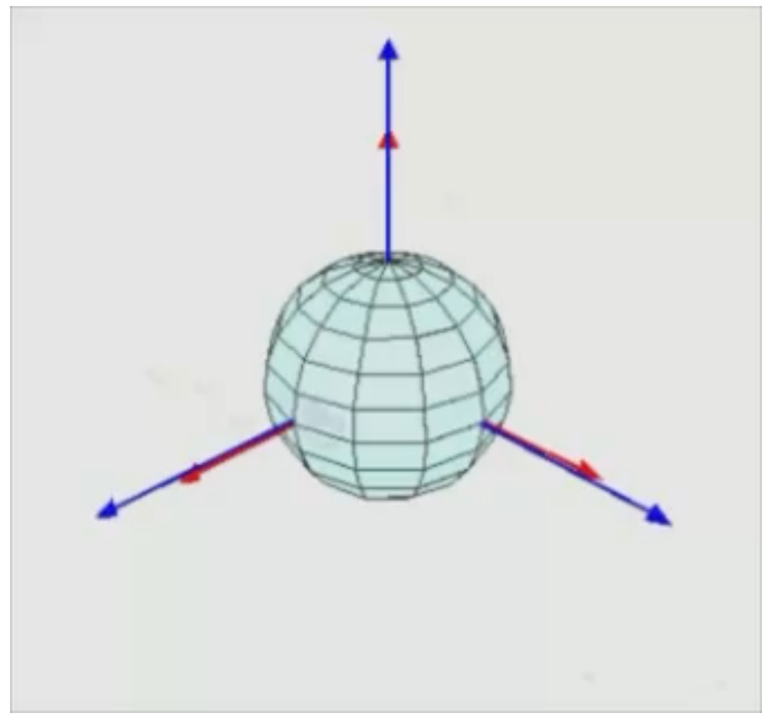

ZXZ - Wikipedia

Rotation Matrices & Euler Angles

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- **Any rotation** be described by 3 successive Euler angle rotations
- One example succession is ZYX

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \qquad R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}.$$

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$$R = \begin{bmatrix} cos\alpha cos\beta & cos\alpha sin\beta sin\gamma - sin\alpha cos\gamma & cos\alpha sin\beta cos\gamma + sin\alpha sin\gamma \\ sin\alpha cos\beta & sin\alpha sin\beta sin\gamma + cos\alpha cos\gamma & sin\alpha sin\beta cos\gamma - cos\alpha sin\gamma \\ -sin\beta & cos\beta sin\gamma & cos\beta cos\gamma \end{bmatrix}$$

Rotation Matrices & Euler Angles

- Unfortunately, this rotation can become limiting …

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Rotation Matrices & Euler Angles

- Unfortunately, this rotation can become limiting
- If $\beta = \pi/2...$ (and then $sin\beta = 1$ and $cos\beta = 0$)

$$R(\beta = \pi/2) = \begin{bmatrix} 0 & cos\alpha sin\gamma - sin\alpha cos\gamma & cos\alpha cos\gamma + sin\alpha sin\gamma \\ 0 & sin\alpha sin\gamma + cos\alpha cos\gamma & sin\alpha cos\gamma - cos\alpha sin\gamma \\ -1 & 0 & 0 \end{bmatrix}$$

Rotation Matrices & Euler Angles

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- If $\beta = \pi/2 \ldots$ (and then $sin\beta = 1$ and $cos\beta = 0$)

$$R(\beta = \pi/2) = \begin{bmatrix} 0 & -sin(\alpha - \gamma) & cos(\alpha - \gamma) \\ 0 & cos(\alpha - \gamma) & sin(\alpha - \gamma) \\ -1 & 0 & 0 \end{bmatrix}$$

Rotation Matrices & Euler Angles

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It doesn't matter what $\alpha - \gamma$ is, there are **two rotations only**: β and $\alpha - \gamma$

This is called a “singularity” of the Euler angle representation.

https://www.youtube.com/watch?v=zc8b2Jo7mno

Euler Angle Representations

- While ZYX is defined by 3 rotations $R(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_x(\gamma)$
- There are many types of Euler angles ZXY, XYX, …

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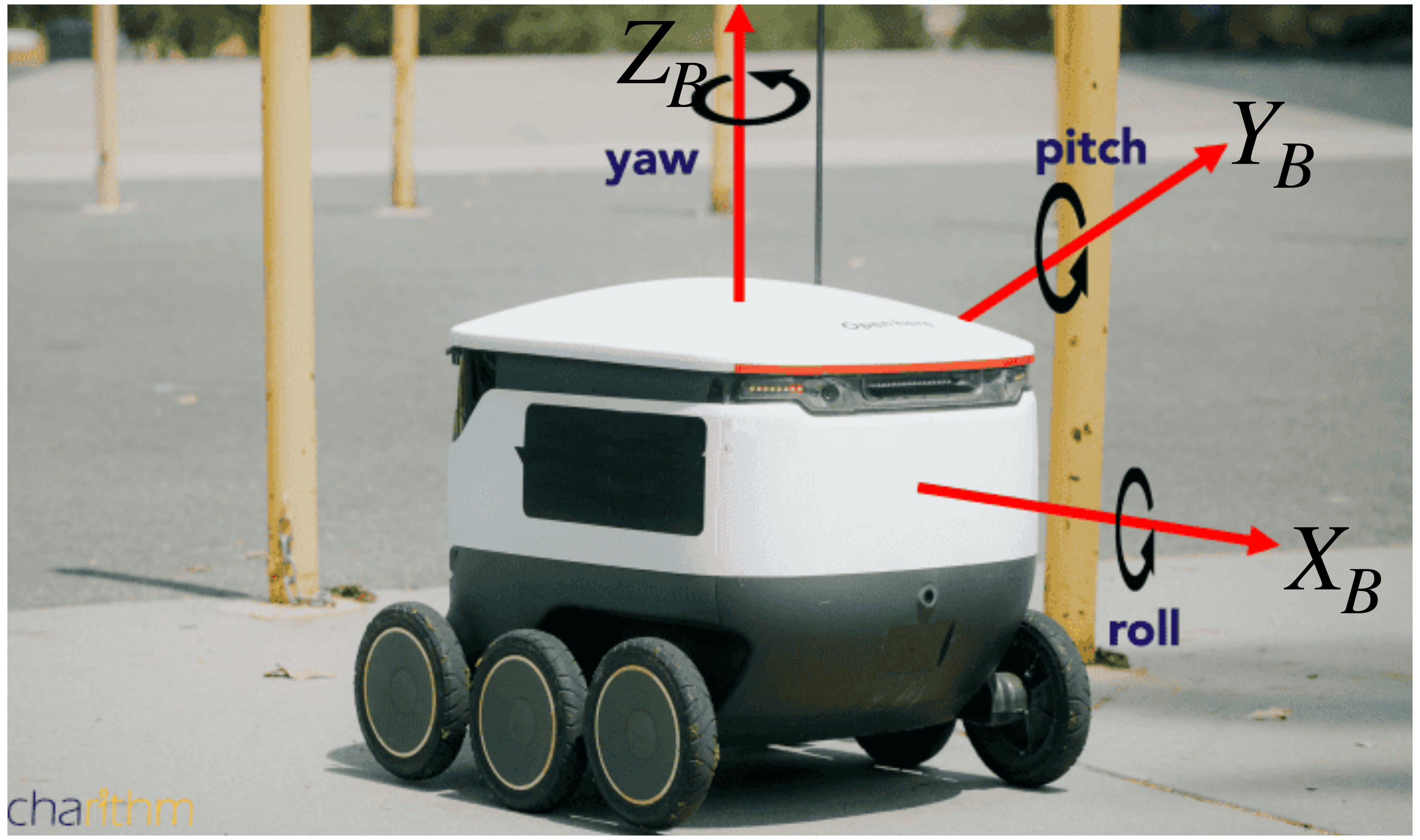
- There are many types of Euler angles ZXY, XYX, …
- These are easy to visualize, understand, debug
- But, ALL Euler angle rotations will have the singularity though !

Agenda

1. Euler Angles
2. **Roll-Pitch-Yaw**
3. Axis-Angle
4. Quaternions

Roll-Pitch-Yaw

- Often use ZYX Euler angles for **spatial** frame axis S
- In sequence, rotate around:
    1. X_S axis (Roll)
    2. Y_S axis (Pitch)
    3. Z_S axis (Yaw)



Roll-Pitch-Yaw

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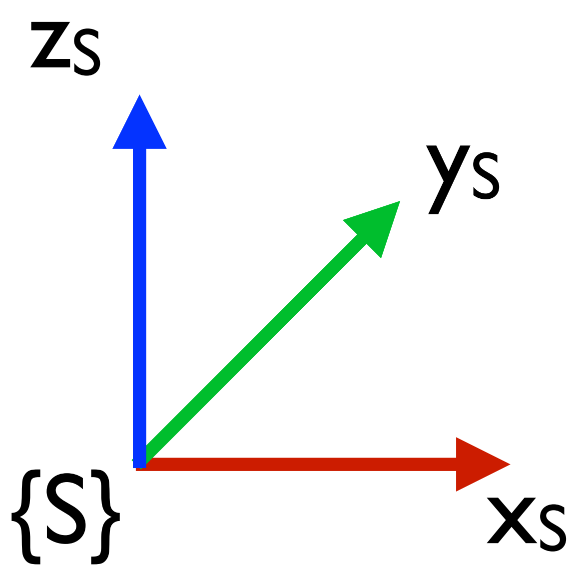
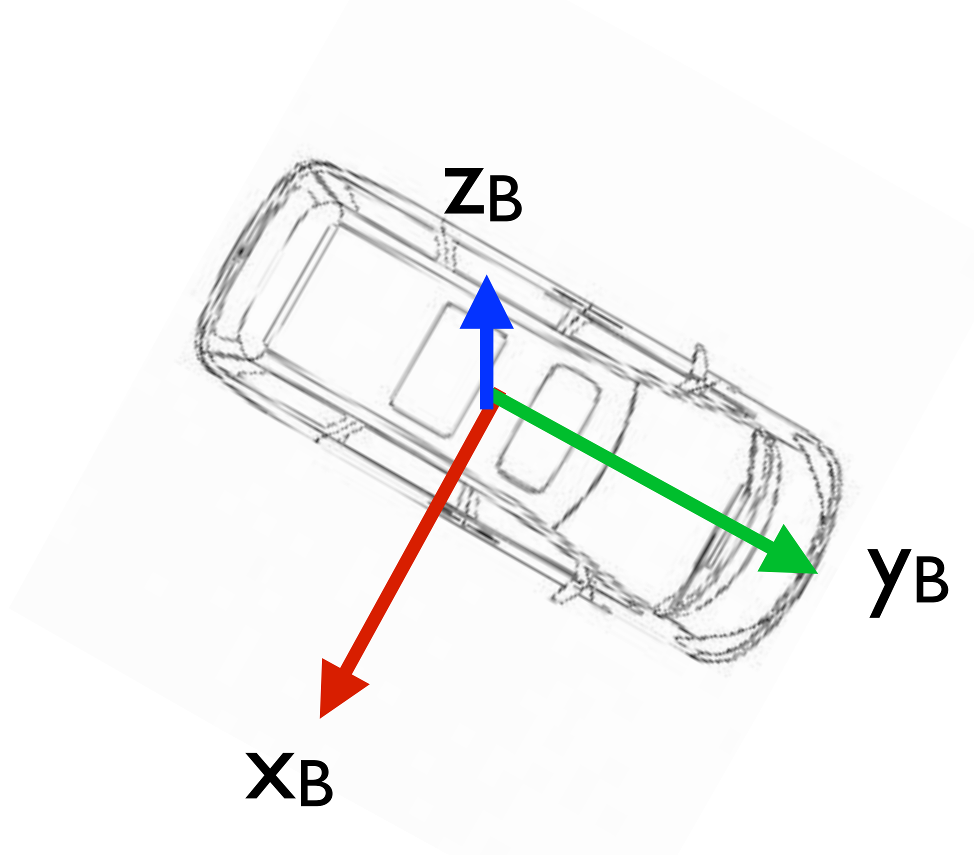
$$R = R_z(\alpha) \cdot R_y(\beta) \cdot R_x(\gamma)$$





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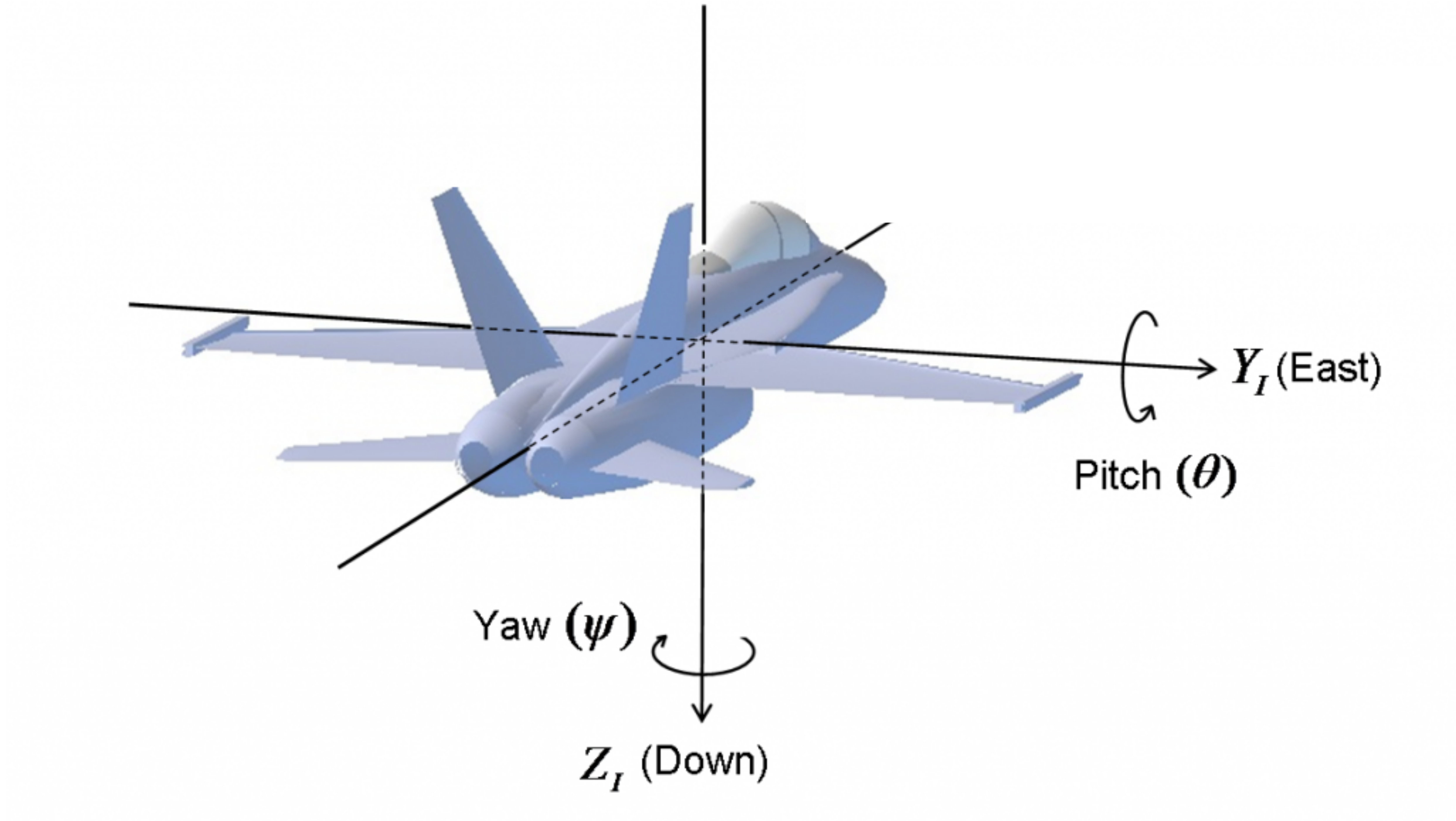
Roll - Pitch - Yaw

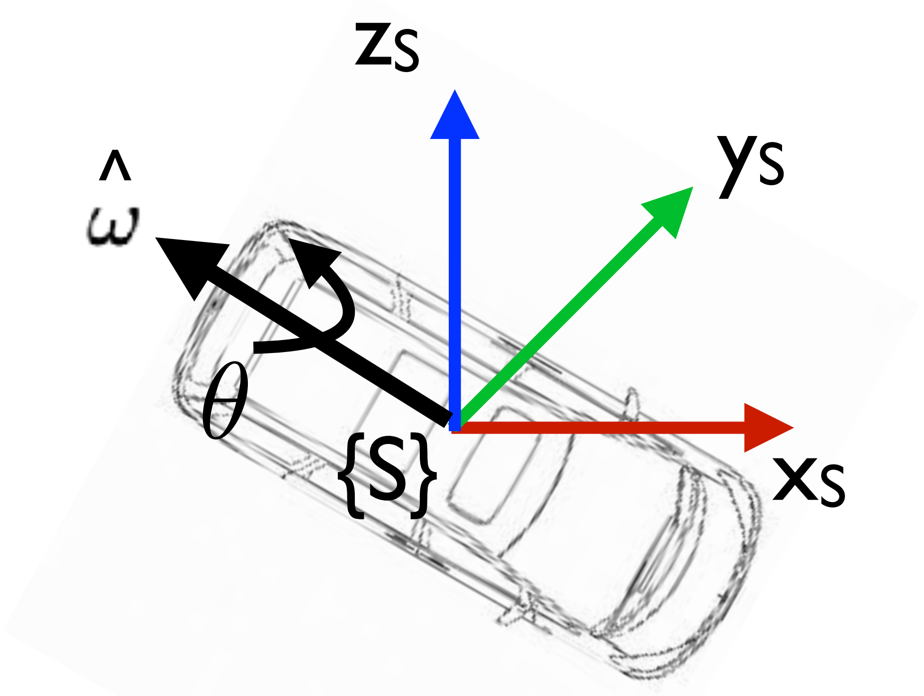

Y_I (East)
Pitch (θ)
Yaw (ψ)
Z_I (Down)

Agenda

1. Euler Angles
2. Roll-Pitch-Yaw
3. **Axis-Angle**
4. Quaternions

Axis Angle

- Instead of using 3 Euler angles, we define one 3D vector $\hat{\omega}$ as the axis to rotate around
- The magnitude of rotation θ is bounded between 0 and π
- The axis $\hat{\omega}$ is a unit norm vector



$$\theta \in [0, \pi] \qquad ||\hat{w}|| = 1$$

Axis Angle

- How to we go from axis angle to rotation matrix and back again?

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\updownarrow$$

$$\hat{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Axis Angle

- How to we go from axis angle to rotation matrix and back again?

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Bracket Notation

Axis Angle

- How to we go from axis angle to rotation matrix and back again?

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Bracket Notation

$$R = I + [\hat{\omega}]sin(\theta) + [\hat{\omega}]^2(1 - cos(\theta))$$

Rodrigues' Formula

Axis Angle

- How to we go from axis angle to rotation matrix and back again?

$$\theta = cos^{-1}\frac{Tr(R) - 1}{2}$$

$$[\hat{\omega}] = \frac{1}{2sin\theta}(R - R^T)$$

Any 3-parameter representation has necessarily a singularity

Then let's use 4 numbers to represent rotations?

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1. Euler Angles
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4. **Quaternions**

Quaternions

- Another popular representation is to **quaternions**

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- A quaternion is a 4D generalization of **complex** numbers

$$q = q_0 + iq_1$$

$$i^2 = -1$$

Quaternions

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- A quaternion is a 4D generalization of **complex** numbers

$$q = q_0 + iq_1 + jq_2 + kq_3$$

$$i^2 = j^2 = k^2 = ijk = -1$$

Quaternions

- Another popular representation is to **quaternions**
- A quaternion is a 4D generalization of **complex** numbers
- To represent rotations, we use **unit quaternions**

$$q = q_0 + iq_1 + jq_2 + kq_3$$

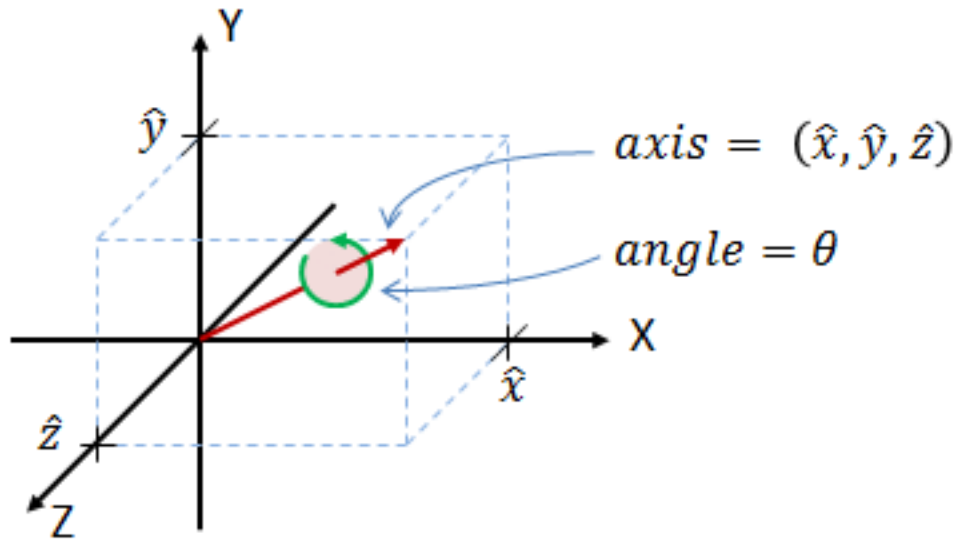
$$i^2 = j^2 = k^2 = ijk = -1$$

$$||q|| = 1$$

Quaternions

- We will often write a quaternion as a vector
- But always remember this is a 4D **complex** numbers

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$



Quaternions

- We will often write a quaternion as a vector
- But always remember this is a 4D **complex** numbers

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$q_0 \longrightarrow \theta$ Angle information

$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \longrightarrow \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ Axis of rotation information +

Quaternions

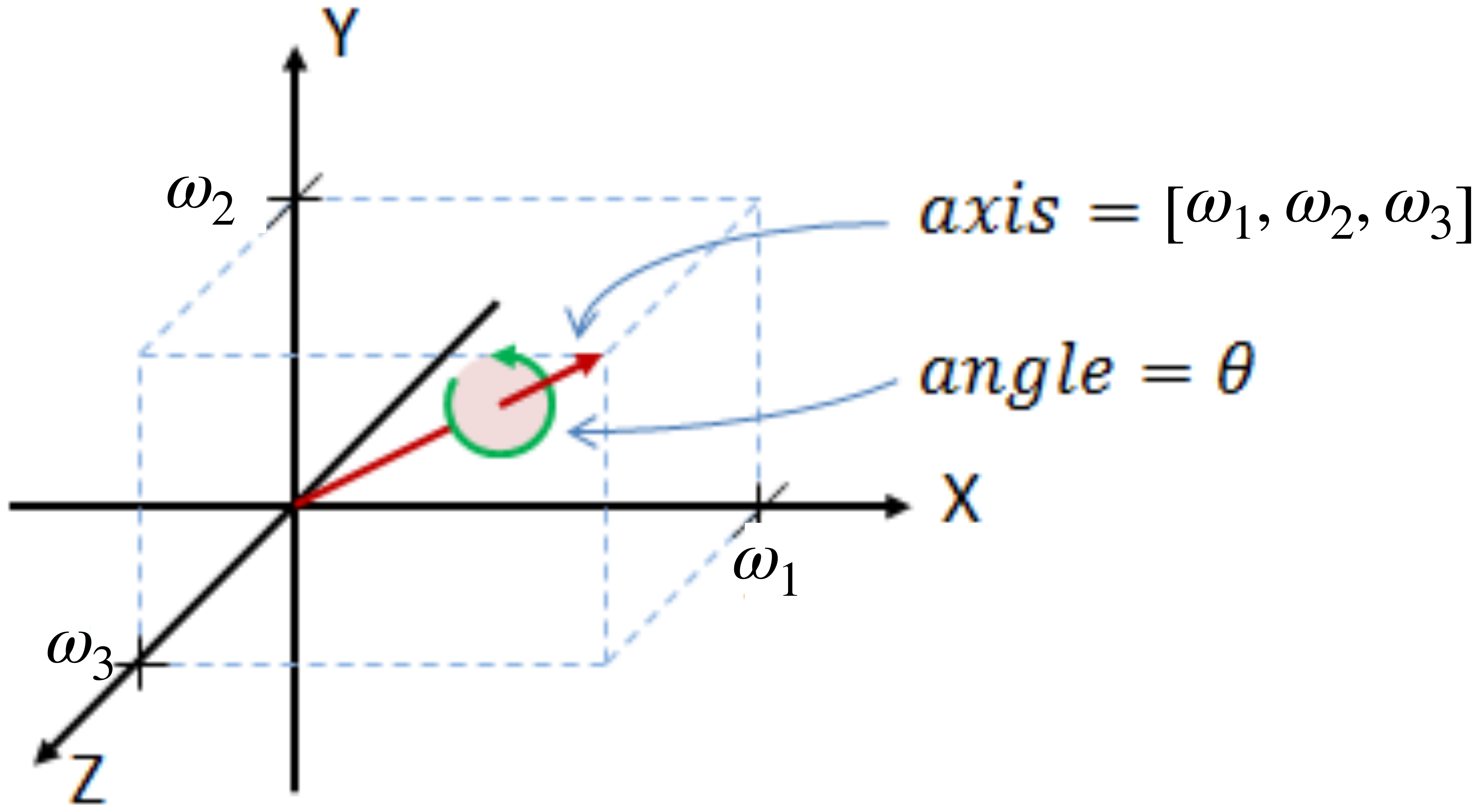
- Determining the q_i values

$$q_0 = cos(\frac{\theta}{2})$$

$$q_1 = \omega_1 sin(\frac{\theta}{2})$$

$$q_2 = \omega_2 sin(\frac{\theta}{2})$$

$$q_3 = \omega_3 sin(\frac{\theta}{2})$$



Quaternions

- How do we map quaternions to rotation matrices? And back?

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \longleftrightarrow R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Quaternions

- How do we map quaternions to rotation matrices?

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_0 q_2 + q_1 q_3) \\ 2(q_0 q_3 + q_1 q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_0 q_1 + q_2 q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Quaternions

- How do we map rotation matrices to quaternions?

$$
\begin{aligned}
q_0 &= \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}, \\
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} &= \frac{1}{4q_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix}.
\end{aligned}
$$

Quaternions

- Composition, or multiplication of quaternions is non-trivial
    - Called the Hamilton product

$$q \cdot p = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_0p_1 + p_0q_1 + q_2p_3 - q_3p_2 \\ q_0p_2 + p_0q_2 - q_1p_3 + q_3p_1 \\ q_0p_3 + p_0q_3 + q_1p_2 - q_2p_1 \end{bmatrix}$$

Rotations Summary

- There are many ways to represent rotations
  - Rotation Matrices
  - Euler Angles
  - Roll, Pitch, Yaw
  - Axis-Angle
  - Quaternions
  - …
- 3 dimension vectors are not sufficient representations if you want to avoid singularities
- Euler angles are more intuitive, good for visualization, debugging, …
- Unit quaternions are often preferred for calculations
- We can go back and forth between representations