# Contents

# Equations

**Objective:** *Recall equations of angular motion and compare them with equations of linear motion.*

## Acceleration in Circular Motion

### Concepts on acceleration

- Acceleration is change in velocity of a body. Velocity as a vector quantity has direction and magnitude.
- In 2-D plane acceleration can be resolved in two components.
  - The parallel component $a_{\|}$.
    - The parallel component is responsible for change in magnitude of velocity of a particle.
  - The perpendicular component $a_{\perp}$.
    - The perpendicular component is responsible for changing the direction of a particle.
- The acceleration can be expressed as

### Uniform Circular Motion

- In uniform circular motion the **magnitude** of **velocity** particle is always **constant**.
- The **direction** changes continuously.
- There is **acceleration** as there is change in velocity.
- There is only **perpendicular** component of acceleration because the motion is *uniform* .
- The perpendicular component of acceleration can be related with **linear velocity** of the particle.
- A particle is moving in a **circle** with a *uniform* speed.
- At a point $P$ the **velocity** of the particle is $\vec{v_1}$
- After a short time $\triangle t$ the **velocity** is $\vec{v_2}$

- A angle $\phi$ is made between $\vec{v_1}$ and $\vec{v_2}$ inside the circle.
  In the similar triangles formed by the **velocity vectors** and their displacement in circular arc of radius $R$,

$\triangle s$ = Displacement in circular arc
Differentiating $(i)$ w.r.t. $\triangle t$

$$\vec{\triangle v} = \vec{v_2} - \vec{v_1}$$
$$\frac{\triangle v}{\triangle s} = \frac{v_1}{\vec{R}} \longrightarrow (i)$$
$$\frac{d\vec{\triangle v}}{d\triangle t} = \frac{v_1 \times d\triangle s}{R \times d\triangle t}$$
$$a_{rad} = \frac{v^2}{R}$$

$$a_{rad} = \frac{v^2}{R}$$

**Non Uniform Circular Motion**

- There is also a parallel component in **acceleration** in non uniform circular motion.
- The **parallel** component is $a_{\|} \ = \frac{d|v|}{dt}$

## Angular Kinematics

### Angular Displacement ($\theta$)

It is the magnitude of rotation of a particle in a circle of fixed radius $R$.

$$\theta = \frac{s}{R}$$

### Angular velocity ($\omega$)

It is rate of change of angular displacement.

$$\omega = \frac{\theta}{t}$$

**Angular acceleration ($\alpha$)**

It is rate of change of angular velocity.

$$\alpha = \frac{\omega}{t}$$

**Equations of angular kinematics**

$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha s$$

### Relation among angular and linear motion

Differentiating above equations with respect to $t$,

$$v = \omega \times R$$

$$a_{\parallel} = \alpha \times R$$

## Rotational $K.E.$

**Objective:** *Derive the expression for rotational kinetic energy*

**Rotational Kinetic Eergy** of a particle is energy due to it's rotation.

- The kinetic energy of a body in translation motion is:

$$K.E. = \frac{1}{2}mv^2$$
$$K.E. = \frac{1}{2}m(\omega r)^2$$
$$K.E. = \frac{1}{2}m\omega^2 r^2$$
$$K.E. = \frac{1}{2}I\omega^2$$

The **rotational kinetic energy** is given by:

$$K.E = \frac{1}{2}I\omega^2$$

## Moment of Inertia $I$

**Objective:** *Describe the term moment of inertia and radius of gyration*

### Moment of inertia $I$

- **Moment of inertia** of a body is it's tendency to resist angular acceleration.
- It is the sum of product of **mass** of each *particle* with the **square** of **distance** of *particle* from the axis.

$$I = \sum_{i=1}^{n} m_i r_i^2$$

$$I = \int x^2 dm$$

### Radius of gyration $R$

- **Radius of gyration** is the radial distance from the axis of rotation at which the **moment of inertia** of the *body* is equivalent to the product of *mass* of the body with the **radial distance.**

$$M = \sum_{i=1}^{n} m_i$$

- $R$ = Radius of gyration

$$I = MR^2$$

## Application of $I$

**Objective:** *Find the moment of inertia of thin uniform rod rotating about it's centre and it's one end*

### Moment of inertia of thin uniform rod from a distance $h$

- Length of the rod = $L$
- Length of a small mass $dm$ = $dx$
- Distance of axis from the origin = $h$
- Remaining length of the rod = $L - h$

$$\frac{M}{dm} = \frac{L}{dx}$$
$$dm = \frac{M}{L}dx$$
$$I = \int x^2 dm$$
$$I = \frac{M}{L}\int_{-h}^{L-h} x^2 dx$$
$$I = [\frac{M}{L}\frac{x^3}{3}]_{-h}^{L-h}$$
$$I = \frac{M}{3L}[(L^3 - 3L^2h + 3Lh^2 - h^3) + h^3]$$
$$I = \frac{1}{3}M(L^2 - 3Lh + 3h^2)$$

- **Moment of inertia** of a *thin* uniform rod with it's axis from the **origin** in a distance $h$ is :

$$I = \frac{1}{3}M(L^2 - 3Lh + 3h^2)$$

### Rotation from center

$$h = \frac{L}{2}$$
$$I = \frac{1}{3}M(L^2 - 3\frac{L^2}{2} + 3\frac{L^2}{4})$$
$$I = \frac{1}{3}M\frac{L^2}{4}$$

- The moment of inertia of a thin uniform rod rotating about the **center** as it's axis is:

$$I = \frac{1}{12}ML^2$$

#### Rotation from one end

$$h = 0$$
$$I = \frac{1}{3}M(L^2 - 3 \times L \times 0 + 3 \times 0^2$$

- The moment of inertia of a thin uniform rod rotating about the **it's one end** as it's axis is:

$$I = \frac{1}{3}ML^2$$

## Torque ($\tau$)

**Objective:** *Establish the relation between torque and angular acceleration.*

- **Torque** is a quantity of cause of rotation of a body.
- It is the **product** of **force** with it's **perpendicular distance** from the axis of rotation.

$$\vec{\tau} = \vec{F} \times \vec{r}$$

- **Torque** is rate of change of angular momentum of a body.

$$\tau = \frac{d\vec{L}}{dt}$$

#### Relation between $\tau$ and $\alpha$

$$\tau = ma_{||}r$$
$$\tau = m(r\alpha)r$$
$$\tau = mr^2\alpha$$

- **Torque** is the product of **moment of inertia** of a body with it's *angular acceleration* .

$$\tau = I\alpha$$

## Work and Power

**Objective:** *Describe the work and power in rotational motion with expression.*

### Work $W$

- **Work** in rotational motion is the quantity expressing the product of **torque** with **angular displacement** .

$$W = Fs$$
$$W = FR\triangle\theta$$

- **Work** is expressed as:

$$W = \tau\triangle\theta$$

### Power

- **Power** in rotational motion is the quantity expressing the product of **torque** with angular velocity.
- **Power** is work done per unit time.

Differentiating both sides with respect to t,

$$\frac{dW}{dt} = \tau\frac{d\triangle\theta}{dt}$$

- **Power** is expressed as:

$$P = \tau\omega$$

## Angular momentum

**Objective:** *Define angular momentum and prove the principle of conservation of angular momentum.*

- **Angular momentum** is the *quantity* of rotational motion *contained* in a body.

$$\vec{L} = \vec{p} \times \vec{r}$$
$$\vec{L} = m\vec{v} \times \vec{r}$$
$$\vec{L} = m(\vec{\omega} \times \vec{r}) \times \vec{r}$$
$$L_z = mr^2\omega_z$$

- **Angular momentum** is expressed as:

$$L_z = I\omega_z$$

## Conservation of angular momentum

- When the net external **torque** applied on a system is **zero** , the **angular momentum** of the system is **constant.**
- A body $A$ applies torque to body $B$ equivalent to $\tau_{AonB}$
- A body $B$ applies torque to body $A$ equivalent to $\tau_{BonA}$

From Newton's **third law of motion,**

$$\begin{aligned} \tau_{AonB} &= -\tau BonA \\ \frac{d\vec{L_A}}{dt} + \frac{d\vec{L_B}}{dt} &= 0 \\ \vec{L_A} + \vec{L_B} &= 0 \\ \vec{L} &= \vec{L_A} + \vec{L_B} \end{aligned}$$

The total *constant* **angular momentum** is expressed as:

$$\vec{L} = 0$$