HW3-6: The Infinity Ward

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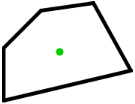
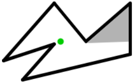
Preliminaries

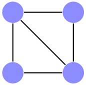


(a) One Infinity Ward is enough.

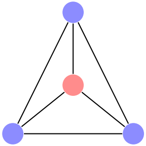


(b) One Infinity Ward is not enough, since the gray region is not visible from the Infinity Ward.

Black lines form the polygon P, green dots are the set S of Infinity Wards.

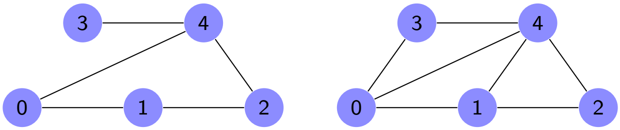


(a) An outerplanar graph.



(b) Not an outerplanar graph, because the red vertex is surrounded by three blue vertices.

Examples of an outerplanar graph and a non-outerplanar graph.



(a) Not a maximal outerplanar graph. (b) A maximal outerplanar graph.

Examples of a non-maximal outerplanar graph and a maximal outerplanar graph.

Subproblem (a)

Please prove the following statement:

Outerplanar graph with n vertices has at most $2n-3$ edges.

Hint: $V-E+F=2$ holds for any planar graph, where V is the number of vertices, E is the number of edges, and F is the number of faces (regions), including the exterior face. As an illustration, K_2 has one face, K_3 has two faces, and K_4 has four faces.

Solution (a)

We have

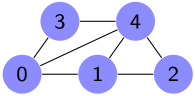
$$V-E+F=2 \text{ and } 3(F-1)+V \leq 2E. \tag{1}$$

Then,

$$3(1-V+E)+V=3-2V+3E \leq 2E. \tag{2}$$

Finally,

$$E \leq 2V-3. \tag{3}$$



Subproblem (b)

Please prove the following statement:

Outerplanar graph is 3-colorable.

That is, 3 colors are sufficient to color all vertices of the graph such that no neighboring vertices have the same color.
Hint: You can directly employ the statement in subproblem (a) in your proof.

Solution (b)

From subproblem (a), we notice that there are at least one vertex with degree less than 3. Thus, we can use mathematical induction to prove the statement.

Subproblem (c)

Please write down the pseudo-code of a fastest algorithm that can color outerplanar graphs in 3 colors.

Solution (c)

The solution is a recursive linear-time greedy coloring algorithm. Following solution (b), we remove a vertex with degree less than 3, color the remaining graph recursively, then add back the removed vertex with a color different from its two neighbors.

Subproblem (d)

Please prove the time complexity of your algorithm in subproblem (c).

Solution (d)

Linear time.

Subproblem (e) and Solution

Subproblem (e)

Derive the answer to the original problem and prove it.
Hint: The answer is $\left\lfloor \frac{n}{3} \right\rfloor$.

Solution (e)

1. Model the vertices of the polygon as vertices of a graph, which will be a outerplanar graph.
2. Transform the outerplanar graph into a maximal outerplanar graph by adding edges, so that every face is a triangle, which implies that any vertex in the triangle can cover the whole triangle.
3. Since outerplanar graph is 3-colorable, color it with 3 colors.
4. Choose the color that are least used.
5. Thus, the answer is $\left\lfloor \frac{n}{3} \right\rfloor$.

Subproblem (f)

Draw an actual polygon to show that your answer in subproblem (e) is the tighest upper bound.

Solution (f)