New Bounds for $n \times n$ square free words

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$3 \times n$ square free words

The lower bound for the number of letters required to form a $3 \times n$ square free words is 7.

Proposition : A $3 \times n$ square free word cannot be formed using 6 letters.

Proof. It is well known that a $1 \times n$ square free word requires at least 3 letters. WLOG, we can fill in the first two layers of the $3 \times n$ matrix with $\{a, b, c\}$ and $\{d, e, f\}$ We cannot have letters $\{d, e, f\}$ in the final layer as squares will form because of the same letters present in the second layer. So, only $\{a, b, c\}$ can be used in the final layer. However, we know that we can fix the parity of only one letter with respect to 3 when constructing a 3 letter square free word. WLOG, assume that we fix the parity of a. This means that b and c will occur in more than one parity and the difference in their positions won't be a factor of 3, forming a 2 letter square. □

Proposition : A $3 \times n$ square free word can be formed using 7 letters.

Proof. If we were to construct a 4 letter square free word using $\{a, b, c, d\}$ then we can do so by fixing the parity of a, b as 0 (mod 2) and parity of c, d as 1 (mod 2).
It can be constructed as follows:

a b c d ...
e f g ...
a b c d ...

Letters with same parity will always have an even difference between them and thus a square won't form because of the first layer and the third layer. □

$4 \times n$ square free words

7 letters

Proposition : We cannot form a $4 \times n$ word using 7 letters

Case I : All layers with 3 letters each

Proof. Let's fill the first two layers with $\{a, b, c\}$ and $\{d, e, f\}$ and WLOG, let's fix the parity of the letter a. This will only allow us to have the 7^{th} letter g and the letter a in the third layer. □

Case II : 4-3 letters in the first two layers.

Proof. We begin by filling in letters $\{a, b, c, d\}$ in the first layer and $\{e, f, g\}$ in the second layer (Parity of e fixed). Now we can use either 3 or 4 letters from $\{a, b, c, d\}$. Supposedly we pick 3 letters, say $\{a, b, c\}$, then the fourth layer would only allow $\{d, e\}$. □

8 letters

Proposition I : We cannot have all layers with 3 letters each.

Proof. We begin by filling in the first two layers with $\{a, b, c\}$ and $\{d, e, f\}$ and fixing the parity of a and d (mod 2). This allows for letters $\{a, g, h\}$ in the third layer. Now, the final layer will only allow for $\{b, c, d\}$ as $\{a, g, h\}$ are in the adjacent layer and parity of $\{e, f\}$ (mod 2) in the second layer is not fixed. We can show that $\{b, c\}$ cannot be used in the final layer as well as their parity (mod 3) cannot be fixed. □

Remark : Any layer cannot have ≥ 6 letters as the adjacent layer will only allow for ≤ 2 letters for which a square free word is not possible in that particular layer.

Proposition II : No layer can have 5 letters.

Proof. We first begin by checking the layers where 5 letters can be allowed. Let's say we filled the second layer with five letters $\{a, b, c, d, e\}$. Now both the adjacent layers will require the remaining three letters $\{f, g, h\}$. This is impossible as we can fix the parity (mod 2) of only one of the letters from $\{f, g, h\}$ which will lead to formation of a 2 letter square.

Let's say we filled the first layer with five letters $\{a, b, c, d, e\}$. The second layer contains$\{f, g, h\}$. Now for the third layer, 3 letters from $\{a, b, c, d, e\}$ must be chosen such that all of their parities (mod 2) is fixed. This is impossible. The symmetric nature of the problem also allows us to conclude that we cannot have 5 letters in the last layer as well. □

Proposition III : No layer can have 3 letters.

Proof. We have already established that a layer can only have 3 or 4 letters. Since the problem is symmetric, if we prove that both the first and the second layer must have 4 letters each, then we are done. We have already proven in **P1** that we cannot have 3 letters in both of the first and the second layers.

Assume that the first layer has 4 letters $\{a, b, c, d\}$ and the second layer has $\{e, f, g\}$. For the third layer, we can use the letters $\{a, b\}$ (their parity is fixed (mod 2)) and the remaining letter $\{h\}$. However the word present in the third layer would require the parity of both $\{a, b\}$ to be fixed which is not possible as we know that only one letter's parity can be fixed in a word formed by 3 letters.

Assume that the first layer has 3 letters $\{a, b, c\}$ and the second layer has 4 letters $\{d, e, f, g\}$. Now for the third layer only one of $\{a, b, c\}$ can be used as we cannot have more then two letters with their parity (mod 2) fixed and we can use the remaining letter $\{h\}$. This restricts the letters allowed in the third layer to 2 and a square free word cannot be filled in that layer. □