Fisher matrix for gravitational wave forecasting

XIII ET symposium hackathon

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2023-05-09

When we build Einstein Telescope, how many compact binary signals will it be able to detect? How well will it localize them in the sky? How well will it measure their parameters, such as the radii of neutron stars?

Gravitational wave data analysis

Gravitational wave data is modelled as signal plus additive noise:
$d(t) = n(t) + h_\theta(t)$.

We can estimate the parameters θ by exploring the posterior distribution

$$p(\theta|d) = \mathcal{L}(d|\theta)\pi(\theta) = \mathcal{N}\exp\left((d|h_\theta) - \frac{1}{2}(h_\theta|h_\theta)\right)\pi(\theta)\,,$$

where

$$(a|b) = 4\Re\int_0^\infty \frac{a(f)b^*(f)}{S_n(f)}\mathsf{d}f\,,$$

while $\pi(\theta)$ is our prior distribution on the parameters.

The posterior is explored stochastically, and we can compute summary statistics:

- mean $\langle \theta_i \rangle$,
- variance $\sigma_i^2 = \langle (\theta_i - \langle \theta_i \rangle)^2 \rangle$,
- covariance $\mathcal{C}_{ij} = \langle (\theta_i - \langle \theta_i \rangle)(\theta_j - \langle \theta_j \rangle) \rangle$.

At this stage, we are not making any approximation, and the covariance matrix is just a summary tool.

Signal-to-noise ratio

The signal-to-noise ratio (SNR) statistic is $\rho = (d|h)/\sqrt{(h|h)}$.

With the expected noise realization,

$$\rho \approx \sqrt{(h|h)} = 2\sqrt{\int_0^\infty \frac{|h(f)|^2}{S_n(f)} \mathsf{d}f}\,.$$

SNR thresholds

Without time shifts nor non-Gaussianities, the SNR would simply follow a χ^2 distribution with two degrees of freedom: “five σ” significance with a threshold of $\rho = 5.5$.

In real data, typically:

$$\mathsf{FAR} = \mathsf{FAR}_8 \exp\left(-\frac{\rho - 8}{\alpha}\right) .$$

For BNS in O1: $\alpha = 0.13$ and $\mathsf{FAR}_8 = 30000\mathsf{yr}^{-1}$.

Parameter dependence of CBC signals

A discussion of the parameters a BNS signal depends on, with relative error (σ_x/x) values computed from the parameter estimation of GW170817.

Intrinsic parameters

- **masses** m_1 and m_2: $\sigma_x/x \sim 10\%$,
- **chirp mass** $\mathcal{M}$: $\sigma_x/x \sim 0.1\%$.

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

- **mass ratio** $q = m_1/m_2$: $\sigma_x/x \sim 20\%$.

We are measuring the *detector-frame* mass:

$$\mathcal{M} = \mathcal{M}_{\text{source}}(1 + z)$$

Alternative parametrization:

- **symmetric mass ratio** $\nu = \mu/M = q/(1+q)^2$: $\sigma_x/x \approx 4\%$
- **total mass** $M = m_1 + m_2$: $\sigma_x/x \approx 3\%$

- **aligned spin**: χ_{1z} and χ_{2z}: $\sigma_x/x \sim 3$ and 10 respectively,
- **effective aligned spin** $\chi_{\mathrm{eff}} = (m_1\chi_{1z} + m_2\chi_{2z})/(m_1 + m_2)$: $\sigma_x/x \sim 1$ (compatible with zero)
- **precessing spin** χ_p: compatible with zero,

- **tidal polarizability** Λ_1 and Λ_2: $\sigma_x/x \sim 1.5$,
- **effective tidal parameter** $\widetilde{\Lambda}$: $\sigma_x/x \sim 0.6$.

$$\Lambda_i = \frac{2}{3}\kappa_2 \left(\frac{R_i c^2}{G m_i}\right)^5$$

Extrinsic parameters

- **distance** d_L $\sigma_x/x \sim 20\%$,
- degeneracy with the **inclination** of the source, ι: $\sigma_x/x \sim 10\%$,
- **arrival time** at geocenter $t_\oplus$,
- **phase** ϕ,
- **polarization** angle ψ: $\sigma_x \sim 0.3$rad,
- **sky position** $(\mathrm{ra}, \mathrm{dec})$: $\sigma_x \sim 2$deg and 9deg.

Sky area:

$$\Delta\Omega_{90\%} \approx -2\pi \log(1-0.9) |\cos(\mathsf{dec})| \sqrt{\sigma_{\mathsf{ra}}^2 \sigma_{\mathsf{dec}}^2 - \mathsf{cov}_{\mathsf{ra,\ dec}}^2} \times \left(\frac{180\ \mathsf{deg}}{\pi\ \mathsf{rad}}\right)^2$$

For GW170817, using the posterior covariance matrix, this approximation yields $28\mathsf{deg}^2$.

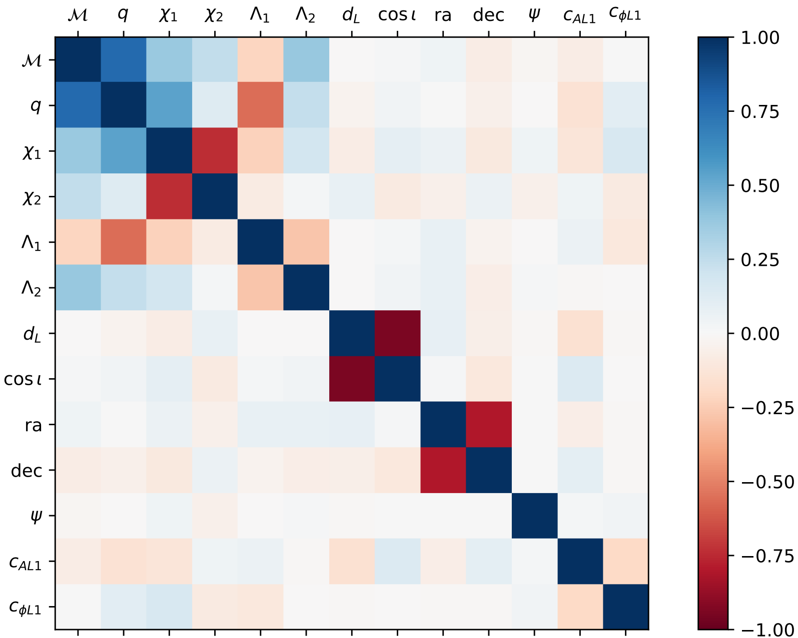
GW150914 comparison

- $\sigma_{\mathcal{M}}/\mathcal{M} = \sigma_M/M \approx 3\%$: not so many cycles
- two-detector event: sky area was 600deg^2, but the Gaussian approximation gives 1800deg^2.

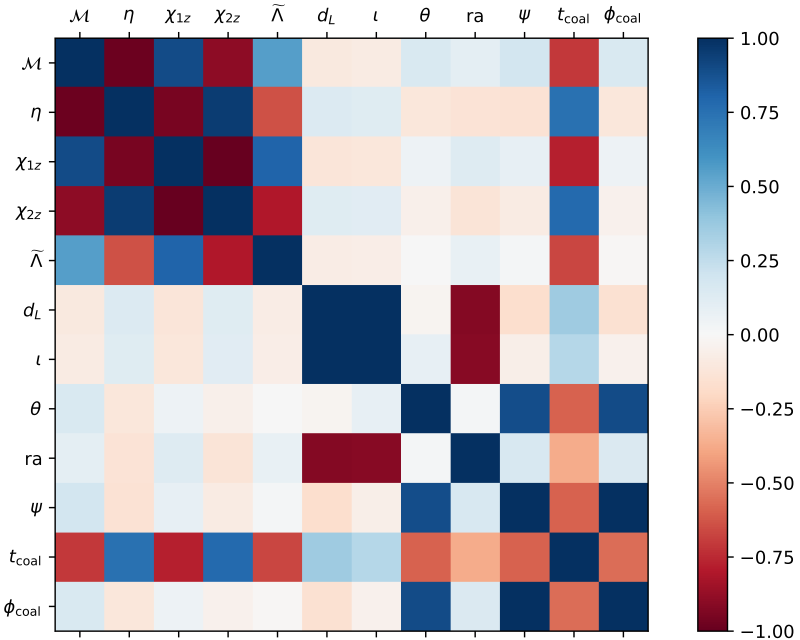
Correlation structure

We can compute Pearson correlation coefficients:

$$\rho_{ij} = \frac{\mathrm{cov}(\theta_i, \theta_j)}{\sigma_i \sigma_j}$$



$\mathcal{M}$
q
χ_1
χ_2
Λ_1
Λ_2
d_L
$\cos\iota$
ra
dec
ψ
c_{AL1}
$c_{\phi L1}$
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00

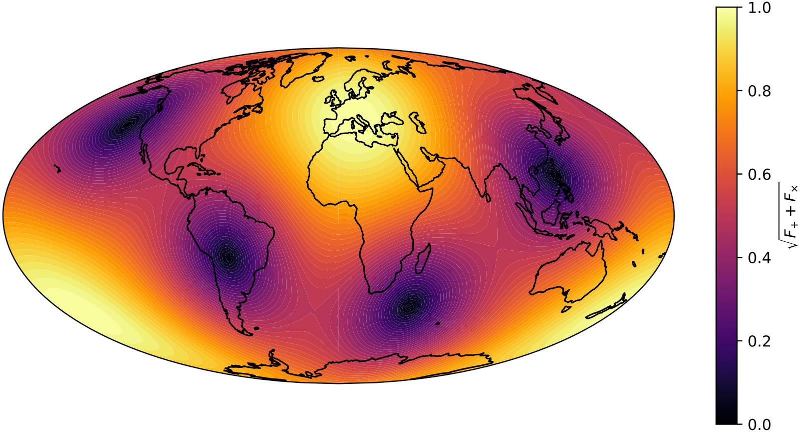


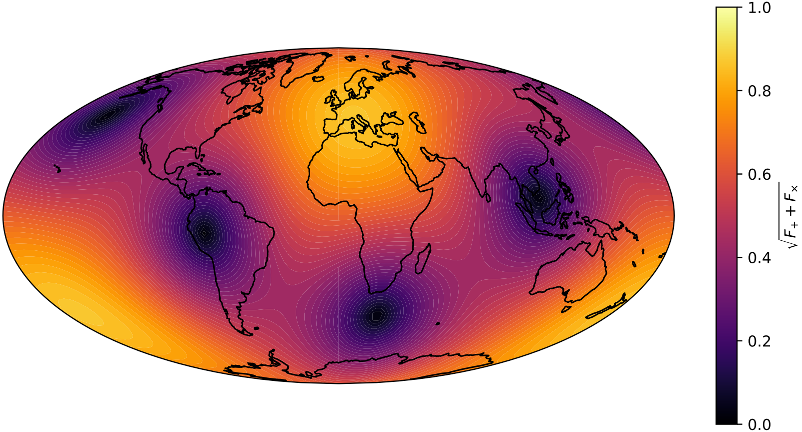
$\mathcal{M}$
η
χ_{1z}
χ_{2z}
$\widetilde{\Lambda}$
d_L
ι
θ
ra
ψ
$t_{\rm coal}$
$\phi_{\rm coal}$
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00

Antenna pattern

The strain at the detector depends on the antenna pattern:

$$h(t) = h_{ij}(t) D_{ij}(t) = h_+(t) F_+(t) + h_\times(t) F_\times(t)\,.$$


1.0
0.8
0.6
0.4
0.2
0.0
$\sqrt{F_+ + F_\times}$



1.0
0.8
0.6
0.4
0.2
0.0
$\sqrt{F_+ + F_\times}$

Fisher matrix

In the Fisher matrix approximation, we are approximating the likelihood as

$$\mathcal{L}(d|\theta) \approx \mathcal{N} \exp\left(-\frac{1}{2}\Delta\theta^i \mathcal{F}_{ij} \Delta\theta^j\right)$$

where $\Delta\theta^i = \theta^i - \langle\theta^i\rangle$.

A **multivariate normal distribution**, with covariance matrix $\mathcal{C}_{ij} = \mathcal{F}_{ij}^{-1}$. This is a good approximation in the high-SNR limit.

The Fisher matrix $\mathcal{F}_{ij}$ can be computed as the scalar product of the derivatives of waveforms:

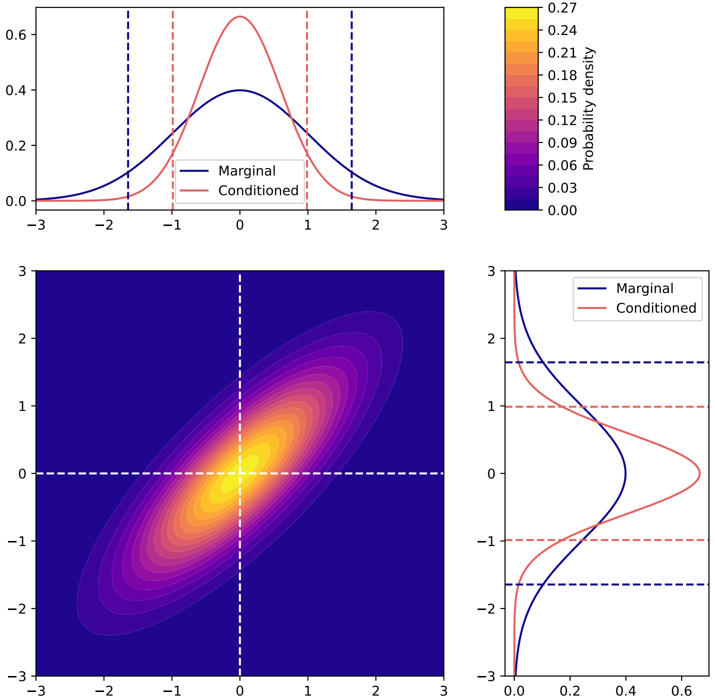
$$\mathcal{F}_{ij} = \left\langle \partial_i \partial_j \mathcal{L} \right\rangle \Big|_{\theta = \langle \theta \rangle} = (\partial_i h | \partial_j h) = 4\Re \int_0^\infty \frac{1}{S_n(f)} \frac{\partial h}{\partial \theta_i} \frac{\partial h^*}{\partial \theta_j} \mathsf{d}f \,.$$

For N detectors,

$$\mathcal{F}_{ij} = \sum_{k=1}^{N} \mathcal{F}_{ij}^{(k)}$$

The covariance matrix can be evaluated in seconds, while full parameter estimation takes hours to weeks.

Tricky step computationally: inverting $\mathcal{F}_{ij}$ to get $\mathcal{C}_{ij}$.


Marginal
Conditioned
Probability density
0.27
0.24
0.21
0.18
0.15
0.12
0.09
0.06
0.03
0.00
Marginal
Conditioned