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DIPARTIMENTO DI INFORMATICA

FOUNDATIONS OF COMPUTER SCIENCE
LECTURE 15: NPC

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NP completeness

For certain **NP** problems their individual complexity is related to that of the entire class

→ If a polynomial time algorithm exists for any of these problems,
all problems in **NP** would be polynomial time solvable (hence, **P** = **NP**)

Important both theoretically and practically:

- If any problem in **NP** requires more than polynomial time, every **NPC** one does. Furthermore, to prove that **P** equals **NP**, you 'only' need to find a polynomial time algorithm for one **NPC** problem.
- The phenomenon of **NP**-completeness may prevent wasting time searching for a (non-existent) polynomial time algorithm to solve a particular problem

  → We believe that **P** is different from **NP**

  → Proving that a problem is **NPC** is a strong evidence of its non-polynomiality

To define **NPC**, we need to define what is

1. a «reduction» from one problem to another
2. when such a reduction is «efficient»

Polynomial reducibility

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DEFINITION

A function $f: \Sigma^* \longrightarrow \Sigma^*$ is a ***polynomial time computable function*** if some polynomial time Turing machine M exists that halts with just $f(w)$ on its tape, when started on any input w.

DEFINITION

Language A is ***polynomial time mapping reducible***, or simply ***polynomial time reducible***, to language B, written $A \leq_P B$, if a polynomial time computable function $f: \Sigma^* \longrightarrow \Sigma^*$ exists, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***polynomial time reduction*** of A to B.

Prop.: If $A \leq_P B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.

Proof

Let M be the polynomial time decider for B and M' the polynomial time TM for the reduction f of A to B.

To decide A on w, we first run M' (to compute $f(w)$), then run M on its output and finally return whatever M returns. Q.E.D.

NPC

DEFINITION

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time reducible to B.

Cor.: If $B \in \mathbf{NPC}$ and $B \in \mathbf{P}$, then $\mathbf{P} = \mathbf{NP}$.

Prop.: If $B \in \mathbf{NPC}$ and $B \leq_P C \in \mathbf{NP}$, then $C \in \mathbf{NPC}$.

Proof

By hypothesis

- every $A \in \mathbf{NP}$ polynomially reduces to B (because $B \in \mathbf{NPC}$)
- B polynomially reduces to C.

Since the combination of polynomial reductions is a polynomial reduction, we have that every $A \in \mathbf{NP}$ polynomially reduces to C.

Since by hypothesis $C \in \mathbf{NP}$, we can conclude.

Q.E.D.

NPC proofs

To exploit the power of **NP**-completeness, we have two tasks to carry out:

1. Find a problem that is in **NPC**
   → this is very difficult !!

2. After this, to prove that a problem is in **NPC** it suffices to
   - Show that it is in **NP**
     → This is usually very easy (show that it can be polynomially verified)
   - Find a **NPC** problem that is polynomially reducible to it
     → this can be easy or not (but surely easier than 1.)

The first NPC problem (1)

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In boolean algebra, variables can take values in {0,1}.

The ***Boolean operations*** AND, OR, and NOT (represented by the symbols ∧, ∨, and $\overline{\cdot}$) are defined as:

$$
\begin{array}{lll}
0 \wedge 0 = 0 & 0 \vee 0 = 0 & \overline{0} = 1 \\
0 \wedge 1 = 0 & 0 \vee 1 = 1 & \overline{1} = 0 \\
1 \wedge 0 = 0 & 1 \vee 0 = 1 & \\
1 \wedge 1 = 1 & 1 \vee 1 = 1 &
\end{array}
$$

A ***formula*** is an expression involving constants, variables and operations.

A formula is ***satisfiable*** if some assignment of 0s and 1s to its variables makes the formula evaluate to 1.

The ***satisfiability problem*** is to test whether a Boolean formula is satisfiable or not.

As a language, we define

$$SAT = \{ \langle \varphi \rangle \mid \varphi \text{ is a satisfiable Boolean formula}\}$$

The first NPC problem (2)

<u>Thm</u> (Cook & Levin, 1971): *SAT* ∈ **NPC**.

Proof

Easily, *SAT* ∈ **NP**:

- Given a formula φ with *n* variables, a certificate is a sequence of *n* bits
- We verify that the given sequence is a satisfying assignment by:
  1. Replacing within φ every x_i with the i-th bit of the sequence
  2. Computing the final value and comparing it to 1
- Both operations are linear in the size of the formula (i.e., in the number of its variables and operators)

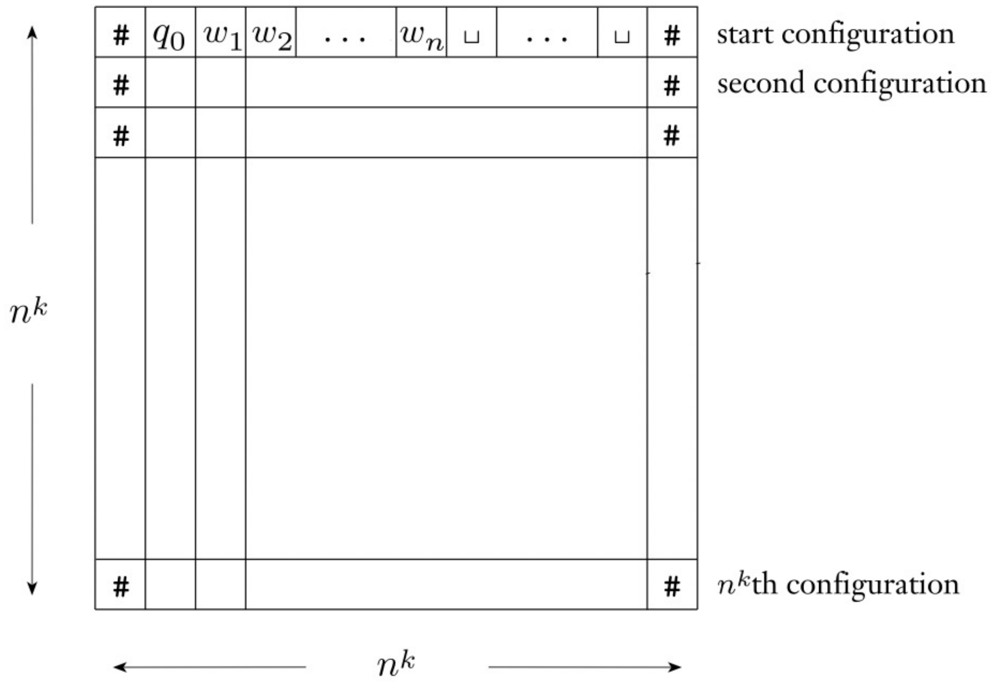
To show that every problem *A* ∈ **NP** polynomially reduces to *SAT*, we

- Consider a **NP** decider *N* for *A,* and
- For every *w*, we polynomially build a φ such that *N* accepts *w* IFF ⟨φ⟩ ∈ *SAT*.

Fix an *A* ∈ **NP** and a *w* ∈Σ*; let |*w*| = *n* and n^k–3 be (an upper bound on) the time spent by *N* to accept *w*.

The first NPC problem (3)

- A ***tableau*** for N on w is an $n^k \times n^k$ matrix whose rows are the configurations of a branch of the computation of N on input w.
- The first row of the tableau is the starting configuration of N on w.
- Each row follows the previous one according to N's transition function.
- For convenience, we assume that each configuration starts and ends with a # symbol; therefore, the first and last columns of a tableau are all #s.



- A tableau is ***accepting*** if any row of the tableau is an accepting configuration.
- N accepts w IFF there exists an accepting tableau for N on w.

The first NPC problem (4)

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Variables of formula φ:

- Say that Q and Γ are the states and tape alphabet of N
- Let $C = Q \cup \Gamma \cup \{\#\}$
- Each of the $(n^k)^2$ entries of a tableau is called a ***cell***.
- The cell in row i and column j is denoted $cell[i, j]$ and contains a symbol from C.
- For each $i, j \in \{1,\dots, n^k\}$ and $s \in C$, we have a variable $x_{i, j, s}$.
- We let $x_{i, j, s} = 1$ if and only if $cell[i, j]$ contains s.

The formula φ is the AND of four parts:

$$\varphi_{\text{cell}} \wedge \varphi_{\text{start}} \wedge \varphi_{\text{move}} \wedge \varphi_{\text{accept}}\,.$$

The first NPC problem (5)

φ_{cell} ensures that the assignment gives 1 to exactly one variable for each cell.

It is defined as
$$\bigwedge_{1\leq i,j\leq n^k} \bigvee_{s\in C}\left(x_{i,j,s} \wedge \bigwedge_{t\in C\setminus s} \overline{x_{i,j,t}} \right)$$

This formula should be understood as:

for all *i, j* there exists an *s* such that

1. $x_{i,j,s} = 1$ (i.e., *cell*[*i, j*] contains symbol *s*), and
2. for every $t \neq s$, we have $x_{i,j,t} = 0$ (i.e., *cell*[*i, j*] doesn't contain symbol *t*)

Indeed, a conjunction acts as a universal quantification: to satisfy $\varphi_1 \wedge \ldots \wedge \varphi_k$, all the φ_i must evaluate to 1.

Similarly, a disjunction acts as an existential quantification: to satisfy $\varphi_1 \vee \ldots \vee \varphi_k$, at least one of the φ_i must evaluate to 1.

Actually, this formula ensures that exactly one variable $x_{i,j}$ holds 1:

→ if $x_{i,j,s} = 1$, then $x_{i,j,t} = 0$ for $t \neq s$; so, every conjunct that starts with $x_{i,j,t}$ will be false.

The first NPC problem (6)

φ_{start} ensures that the first row of the table is the starting configuration of N on w:

$$x_{1,1,\#} \wedge x_{1,2,q_0} \wedge$$
$$x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \ldots \wedge x_{1,n+2,w_n} \wedge$$
$$x_{1,n+3,\sqcup} \wedge \ldots \wedge x_{1,n^k-1,\sqcup} \wedge x_{1,n^k,\#}$$

φ_{accept} guarantees that an accepting configuration occurs in the tableau

→ it ensures that q_{accept} appears in one of the cells:

$$\bigvee_{1 \leq i,j \leq n^k} x_{i,j,q_{accept}}$$

The first NPC problem (7)

φ_{move} guarantees that each row of the tableau corresponds to a configuration that legally follows the preceding row's configuration according to N's rules.

It does so by ensuring that each 2×3 window of cells is legal, where a 2×3 window is ***legal*** if that window does not violate the actions specified by N's transition function.

→ very heavy

EXAMPLE: Assume that $\delta(q_1, a) = \{(q_1, b, R)\}$ and $\delta(q_1, b) = \{(q_2, c, L), (q_2, a, R)\}$.

Then, the following are examples of legal windows:

a	q_1	b
q_2	a	c

a	q_1	b
a	a	q_2

a	a	q_1
a	a	b

#	b	a
#	b	a

a	b	a
a	b	q_2

b	b	b
c	b	b

whereas the following are examples of not legal windows:

a	b	a
a	a	a

a	q_1	b
q_2	a	a

b	q_1	b
q_2	b	q_2

The first NPC problem (8)

According to the transition function of N, we have many 2×3 windows that are legal.

We number windows by the upmost central position:

	j-1	j	j+1
i			
i+1			

Hence, formula φ_{move} must require that every position i, j contains a 6-tuple of elements of $C = Q \cup \Gamma \cup \{\#\}$ that form a legal window.

So, formula φ_{move} is

$$\bigwedge_{1 \le i < n^k,\ 1 < j < n^k} \ \bigvee_{\substack{a_1,\ldots,a_6 \\ \text{legal window}}} \left(x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6} \right)$$

The first NPC problem (9)

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Time complexity:

1. Number of variables:

- the tableau is an $n^k \times n^k$ table, so it contains n^{2k} cells.
- Each cell has h variables associated with it, where $h = |C|$.
- Because h depends only on N and not on n, the number of variables is $O(n^{2k})$.
- REMARK: each variable has indices that range up to n^k; so they require $O(\log n)$ symbols to be codified into the formula

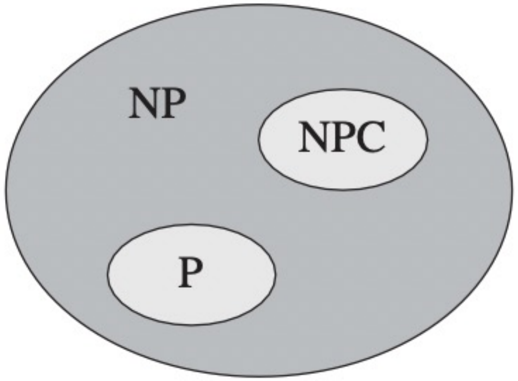
  → but $O(\log n) < O(n)$, so this factor does not influence polynomiality of the reduction

2. Size of the formula:

- φ_{cell} contains a fixed-size fragment of the formula for each cell of the tableau, so its size is $O(n^{2k})$.
- Formula φ_{start} has a variable for each cell in the top row, so its size is $O(n^k)$.
- Formulas φ_{move} and φ_{accept} each contain a fixed-size fragment of the formula for each cell of the tableau, so their size is $O(n^{2k})$.
- Thus, φ's total size is $O(n^{2k})$. <u>Q.E.D.</u>

NPC and P vs NP

As usual, the scenario is not known, but the most widely believed conjecture is



The typical NP problem that is not believed to be NPC is ***graph isomorphism***:

- Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are ***isomorphic*** if there exists a bijection f: $V_1 \longrightarrow V_2$ such that $(u, v) \in E_1$ if and only if $(f(u), f(v))$ E_2.
- *GRAPH-ISOM* is the problem of saying whether two graphs are isomorphic or not.
- *GRAPH-ISOM* is trivially in **NP** (a certificate is the bijection among the vertices)
- Nobody has been able to prove its **NP**-completeness
- By contrast, a related **NPC** problem is ***sub-graph isomorphism***:

  SUBG-ISOM: given two graphs, say whether the first one is isomorphic to a subgraph of the second one