SAPIENZA UNIVERSITÀ DI ROMA
DIPARTIMENTO DI INFORMATICA

FOUNDATIONS OF COMPUTER SCIENCE
LECTURE 3: Regular expressions

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Expressions for denoting regular languages

- Regular languages are closed under union, concatenation and star
- Here, we prove that, given an alphabet Σ, all regular languages over it can be expressed just by these three operations
- This suggests that we can use expressions built from characters of Σ and from the three regular operations to denote regular languages
- We shall also prove that all three operations are required for obtaining these languages
- This is another characterization of such languages, that are:
  - Those accepted by a DFA
  - Those accepted by a NFA
  - Those corresponding to a regular expression
- In the next class, we shall see a last characterization, based on so called *regular grammars*

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Regular expressions

DEFINITION

Say that R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

- The expression ε represents the language containing a single string (the empty one) whereas ∅ represents the language that doesn't contain any string
- The star operation is done first, followed by concatenation, and finally union, unless parentheses change this order
  → EX.: $a \cup b \circ c^*$ actually denotes $a \cup (b \circ (c^*))$
- We shall usually omit $\circ$ and write bc instead of $b \circ c$; furthermore, sometimes ∪ is denoted by + (so write $a+bc^*$ instead of $a \cup b \circ c^*$)
- R^+ is a shorthand for RR^* (hence, $R^* = R^+ \cup \varepsilon$)
- R^k denotes the concatenation of k copies of R's

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Language of a regular expression

It is straightforward to associate a language to a regular expression:

$L(a) = \{a\}$ $\quad L(\varepsilon) = \{\varepsilon\}$ $\quad L(\emptyset) = \emptyset$

$L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$ $\quad L(R_1 \circ R_2) = L(R_1) \circ L(R_2)$ $\quad L(R^*) = (\, L(R)\,)^*$

EXAMPLES: Take $\Sigma = \{0,1\}$. With abuse of notation, we write Σ to denote the r.e. $(0 \cup 1)$

- $0^*10^* = \{w \mid w \text{ contains a single } 1\}$
- $\Sigma^*1\Sigma^* = \{w \mid w \text{ has at least one } 1\}$
- $\Sigma^*001\Sigma^* = \{w \mid w \text{ contains the string } 001 \text{ as a substring}\}$
- $(\Sigma\Sigma)^* = \{w \mid w \text{ is a string of even length}\}$
- $(\Sigma\Sigma\Sigma)^* = \{w \mid \text{the length of } w \text{ is a multiple of } 3\}$
- $(0 \cup \varepsilon)(1 \cup \varepsilon) = \{\varepsilon, 0, 1, 01\}$
  This distributivity law recalls distributivity of + over · in arithmetics; for this reason, union of regular expressions is usually denoted with +
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1 = \{w \mid w \text{ starts and ends with the same symbol}\}$

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A few algebraic laws

- $(R_1 \cup R_2)\, R_3 = R_1R_3 \cup R_2R_3$ $\qquad R_1\,(R_2 \cup R_3) = R_1R_2 \cup R_1R_3$
  → *distributivity* of $\cup$ over $\circ$
- $(R_1 \cup R_2) \cup R_3 = R_1 \cup (R_2 \cup R_3)$ $\qquad (R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$
  → *associativity* of $\cup$ and $\circ$
- $R \cup \emptyset = \emptyset \cup R = R$
  → $\emptyset$ is the *neutral element* of $\cup$
- $R \circ \varepsilon = \varepsilon \circ R = R$
  → ε is the *neutral element* of $\circ$
- $R \circ \emptyset = R \circ \emptyset = \emptyset$
  → $\emptyset$ is the *annihilator element* for concatenation
- $R_1 \cup R_2 = R_2 \cup R_1$
  → commutativity of $\cup$

These laws are natural, if you think at union as sum and concatenation as product in the algebra of reals → ATTENTION: $R_1 \circ R_2 \neq R_2 \circ R_1$

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An example: syntax of numbers

- Regular expressions are useful tools in the design of compilers for programming languages to describe basic objects such as variable names, constants, …
- For example, a numerical constant that may include a fractional part and/or a sign may be described by the regular expression

$$(+ \cup - \cup \varepsilon)\,(D^+ \cup D^+ . D^* \cup D^* . D^+)$$

where $D = \{0,1,2,3,4,5,6,7,8,9\}$ is the alphabet of decimal digits.

- Examples of strings in the language generated by this expression are:
  72 3.14159 +7. -.01

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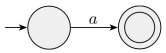
Equivalence with NFA (1)

Thm.1: for every regular expression R there exists a NFA N such that $L(R) = L(N)$

PROOF Let's convert R into an NFA N. We consider the six cases in the formal definition of regular expressions.

1. $R = a$ for some $a \in \Sigma$. Then $L(R) = \{a\}$, and the following NFA recognizes $L(R)$.



2. $R = \varepsilon$. Then $L(R) = \{\varepsilon\}$, and the following NFA recognizes $L(R)$.

3. $R = \emptyset$. Then $L(R) = \emptyset$, and the following NFA recognizes $L(R)$.

4. $R = R_1 \cup R_2$.
5. $R = R_1 \circ R_2$.
6. $R = R_1^*$.

we use the constructions given in the proofs that the class of regular languages is closed under the regular operations.

Q.E.D.

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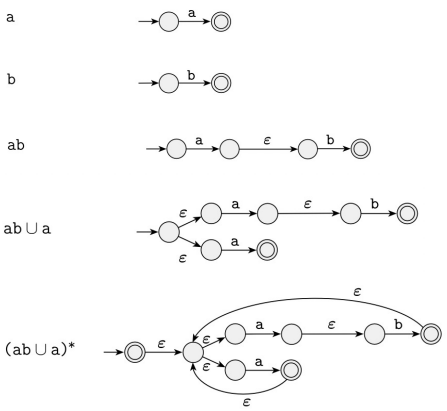
From a Reg. Expr. To a NFA: an example

Let's consider the R.E. (ab ∪ a)*



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Equivalence with NFA (2)

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Thm.2: for every DFA M there exists a regular expression R such that $L(R) = L(M)$

To prove this result, we consider ***generalized nondeterministic finite automata*** (GNFA):

- GNFA are simply NFA wherein the transition arrows may have any regular expressions as labels, instead of only members of the alphabet or ε.
- So, a GNFA reads blocks of symbols from the input, not necessarily just one symbol at a time.
- The GNFA moves along a transition arrow connecting two states by reading a block of symbols from the input, which constitute a string described by the regular expression on that arrow.

First we show how to convert DFAs into GNFAs, and then GNFAs into regular expr's.

For convenience, we require that GNFAs always have a special form:

- The start state has transition arrows going to every other state but no arrows coming in;
- There is only a single accept state (different from the starting one) and it has arrows coming in from other states but no arrows going to any other state;
- For all other states, one arrow goes to every other state (including itself).

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Equivalence with NFA (3)

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DEFINITION

A ***generalized nondeterministic finite automaton*** is a 5-tuple, $(Q, \Sigma, \delta, q_{\text{start}}, \{q_{\text{accept}}\})$ where

1. Q is the finite set of states,
2. Σ is the input alphabet,
3. $\delta : (Q - \{q_{\text{accept}}\}) \times (Q - \{q_{\text{start}}\}) \longrightarrow \mathcal{R}$ is the transition function,
4. q_{start} is the start state, and
5. q_{accept} is the accept state.

The symbol $\mathcal{R}$ is the collection of all regular expressions over the alphabet Σ.

A GNFA accepts a string w in Σ^* if $w = w_1 w_2 \cdots w_k$, where each w_i is in Σ^* and a sequence of states $q_0, q_1, \ldots, q_k$ exists such that

1. $q_0 = q_{\text{start}}$ is the start state,
2. $q_k = q_{\text{accept}}$ is the accept state, and
3. for each i, we have $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$; in other words, R_i is the expression on the arrow from q_{i-1} to q_i.

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Equivalence with NFA (4)

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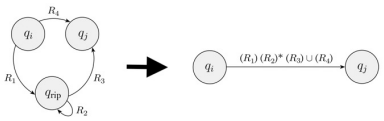
Algorithm for converting a GNFA G into a regular expression R:

CONVERT(G):

1. Let k be the number of states of G.
2. If $k = 2$, then G must consist of a start state, an accept state, and a single arrow connecting them and labeled with a regular expression R. Return the expression R.
3. If $k > 2$, we select any state $q_{\text{rip}} \in Q$ different from q_{start} and q_{accept} and let G' be the GNFA $(Q', \Sigma, \delta', q_{\text{start}}, q_{\text{accept}})$, where
$$Q' = Q - \{q_{\text{rip}}\},$$
and for any $q_i \in Q' - \{q_{\text{accept}}\}$ and any $q_j \in Q' - \{q_{\text{start}}\}$, let
$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4),$$
for $R_1 = \delta(q_i, q_{\text{rip}})$, $R_2 = \delta(q_{\text{rip}}, q_{\text{rip}})$, $R_3 = \delta(q_{\text{rip}}, q_j)$, and $R_4 = \delta(q_i, q_j)$.
4. Compute CONVERT(G') and return this value.

IDEA:



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Equivalence with NFA (5)

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Lemma: For any GNFA G, CONVERT(G) is (language) equivalent to G.

Proof (by induction on k, the number of states of G)

Base (k=2): trivial, by step 2 of the algorithm

Induction step (k>2: assume the claim for $k-1$ states and prove the claim for k states):

Consider the first step performed by the algorithm, that reduces G to some G' by erasing some q_{rip}.

1. $w \in L(G)$ implies $w \in L(G')$:

- $w \in L(G)$ means that exists an accepting branch of G $q_{\text{start}}\, q_1\, q_2\, q_3 \ldots q_{\text{accept}}$ and w belongs to the regular expression obtained from the concatenation of the reg.expr's labeling the transitions
- If $q_{\text{rip}} \notin \{q_1\, q_2\, q_3 \ldots\}$, clearly G' also accepts w (the new regular expressions labeling the arrows of G' contains the old regular expression as part of a union)
- Otherwise, consider every occurrence of q_{rip}, say $q_{\text{rip}} = q_i$, and let q_h and q_j be the closest preceeding and following states in the sequence different from q_{rip}.
- Then, the reg.expr. labeling the arrow from q_h to q_j in G' contains in its union the concatenation of the reg.exp. from q_h to q_{rip}, from q_{rip} to itself (as many times as needed), and from q_{rip} to q_j.
- By repeating this for every occurrence of q_{rip} in $q_1\, q_2\, q_3 \ldots$, we show that w leads G' from q_{start} to q_{accept} (without passing from q_{rip}).

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Equivalence with NFA (6)

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2. $w \in L(G')$ implies $w \in L(G)$:
   - each arrow between any q_i and q_j in G' describes the collection of strings taking q_i to q_j in G, either directly or via q_{rip}
   - hence, also G accepts w.
3. Points 1 and 2 prove that $L(G) = L(G')$
4. Since G' has k–1 states, by inductive hyp. $L(G') = L(\text{CONVERT}(G'))$.
5. By definition of the algorithm, $\text{CONVERT}(G) = \text{CONVERT}(G')$.
6. So, $L(G) = L(\text{CONVERT}(G))$. Q.E.D.

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Equivalence with NFA (7)

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Proof of Thm.2:

Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA.

Let q_{start} , $q_{accept} \notin Q$. We build the GNFA $G = (Q \cup \{q_{start}, q_{accept}\}, \Sigma, \delta', q_{start}, \{q_{accept}\})$ where δ' associates the transitions

- from q_{start} to q_0 with reg.exp. ε
- from q_{start} to every $q \in Q' - \{q_0, q_{start}\}$ with reg.exp. $\emptyset$
- from every $q \in F$ to q_{accept} with reg.exp. ε
- from every $q \in Q \backslash F$ to q_{accept} with reg.exp. $\emptyset$
- from every $q \in Q$ to every $q' \in Q$ such that $\exists a.\delta(q, a) = q'$ with reg.exp. a
- from every $q \in Q$ to every $q' \in Q$ such that $\nexists a.\delta(q, a) = q'$ with reg.exp. $\emptyset$

Trivially, $L(M) = L(G)$ and, by the Lemma, $L(G) = L(\text{CONVERT}(G))$.

Q.E.D.

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From a DFA to a Reg. Expr.: an example

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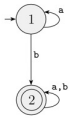
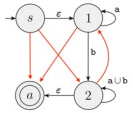
Consider the DFA:

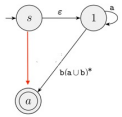


The associated GNFA is (red arrows are those labeled with ∅):



If we now remove state 2, we have three reg.expr.'s:

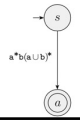
- One from s to a: $\emptyset(a \cup b)^* \varepsilon \cup \emptyset = \emptyset$
- One from s to 1: $\emptyset(a \cup b)^* \emptyset \cup \varepsilon = \varepsilon$
- One from 1 to a: $b(a \cup b)^* \varepsilon \cup \emptyset = b(a \cup b)^*$



Finally, if we remove state 1, the only reg.expr. left is

$\varepsilon\,(a)^*\, b(a \cup b)^* \cup \emptyset \;=\; a^* b(a \cup b)^*$

that generates the language of the original DFA.



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