How Probability Quantifies Randomness

Random Phenomena

random experiment (or phenomena) = the outcome is uncertain

Probability

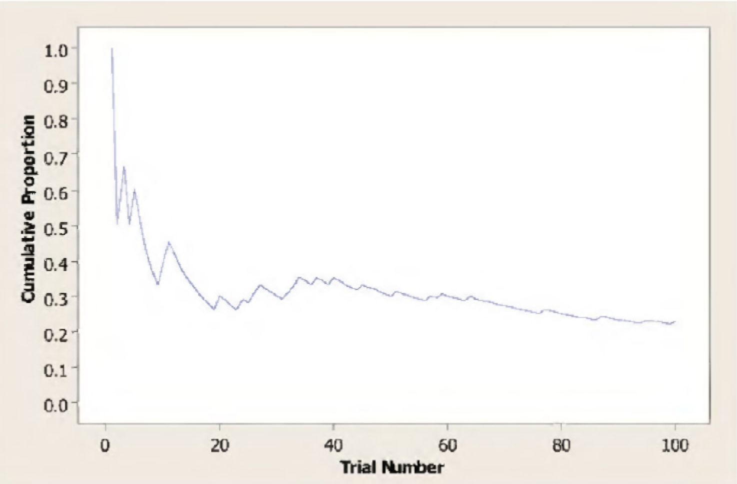
- In the short-run, the proportion of times that something happens is highly random (unpredictable).
- In the long-run, the proportion of times that something happens becomes very predictable.

Probability quantifies long-run randomness

Law of Large Numbers

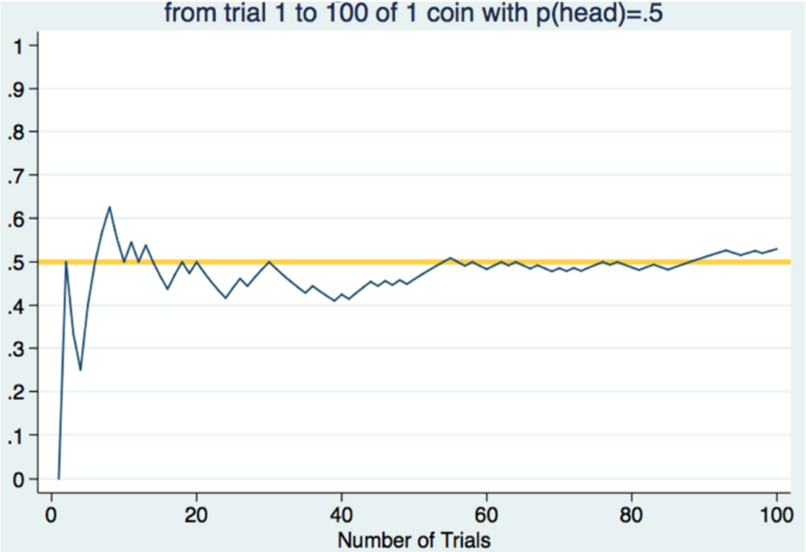
- As the number of trials increase, the proportion of occurrences of any given outcome approaches a particular number "in the long run".
- For example, as one tosses a (perfect) die, in the long run $1/6$ of the observations will be a 6.

Trial	6 Occurs?	Cumulative Proportion of 6s	
1	yes	1/1	= 1.0
2	no	1/2	= 0.500
3	yes	2/3	= 0.667
4	no	2/4	= 0.500
5	yes	3/5	= 0.600
6	no	3/6	= 0.500
7	no	3/7	= 0.429
8	no	3/8	= 0.375
⋮	⋮	⋮	⋮



In the long run the cumulative proportion of 6s approaches $1/6 \approx 0.167$.

Cumulative proportion of heads



- **Probability**. With random phenomena, the *probability* of a particular outcome is the proportion of times (relative frequency) that the outcome would occur in a long run of observations (independent trials).
  Example. When rolling a die, the outcome of "6" has probability $= 1/6$. In other words, the proportion of times that a 6 would occur in a long run of observations is $1/6$.
- **Independent Trials**. Different trials of a random phenomenon are independent if the outcome of any one trial is not affected by the outcome of any other trial.
  Example. If you have 20 flips of a coin in a row that are " "heads", you are not "due" a "tail" - the probability of a tail on your next flip is still $1/2$. The trial of flipping a coin is independent of previous flips. The coin does not have memory.

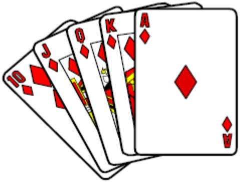
Finding Probabilities

Sample Space

The sample space is the set of all possible outcomes.

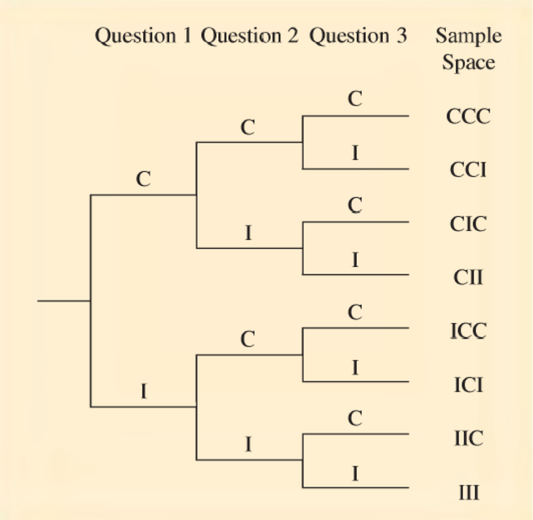
Example.

Experiment: draw a card form a bridge deck.



The sample space is the set of 52 cards.

Experiment: Three questions. Students can answer Correctly or Incorrectly.



Event

- An event is a subset of the sample space
- An event corresponds to a particular outcome or a group of possible outcomes.

In the previous examples:
Deck

- Event C: a red card (13 possibles outcomes)
- Event D: a queen (4 possibles outcomes)

Exam

- Event A: student answers all 3 questions correctly = (CCC)
- Event B: student passes (at least 2 correct) = (CCI, CIC, ICC, CCC)

Finding Probabilities of Events

By defining the probability as the relative frequency in the long run. We deduce that each outcome in a sample space has a probability such that

- the probability of each individual outcome is between 0 and 1
- the sum of all the individual probabilities equals 1

From the above two properties, it follows that:

Classical rule. If the individual outcomes are equally likely to occur, i.e. they have the same probability, then the probability is $1/n$ ($n =$ numbers of elements in the sample space).

Example: rolling a die

$$\left.\begin{array}{l} P(1) = P(2) = \ldots = P(6) = p \\ P(1) + P(2) + \ldots + P(6) = 1 \end{array}\right\} \Rightarrow p = 1/6$$

Probability of an event. The probability of an event A, denoted by P(A), is obtained by adding the probabilities of the individual outcomes in the event.

Classical Rule. When all outcomes are equally likely, then

$$P(A) = \frac{\text{number of outcomes in A}}{\text{number of possible outcomes}}$$

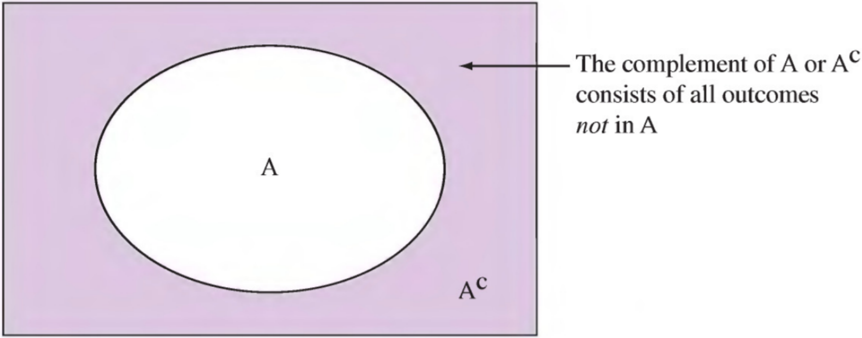
Examples

Experiment: draw at random a card from a bridge deck.

- P(diamond) $= 13/52$
- P(ace) $= 4/52$
- P(red) $= 26/52 = 1/2$
- P(queen) $= 4/52$

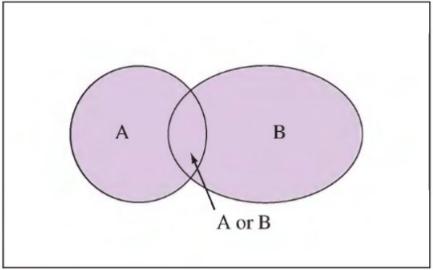
Basic rules for finding probabilities about a pair of events

- The complement of an event A consists of all outcomes in the sample space that are not in A.
  - The probabilities of A and of A^c add to 1
  - $P(A^c) = 1 - P(A)$



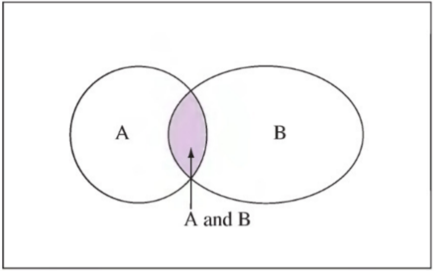
- The union of A and B consists of outcomes that are in A and/or B.
- When all the possible outcomes are equally likely

$$P(\text{A or B}) = P(A \cup B) = \frac{\text{number of outcomes in A or B}}{\text{total number of individual outcomes}}$$



- The intersection of A and B consists of outcomes that are in both A and B.
- When all the possible outcomes are equally likely

$$\text{P(A and B)} = P(A \cap B) = \frac{\text{number of outcomes in A and B}}{\text{total number of individual outcomes}}$$



Probability of the union of two events

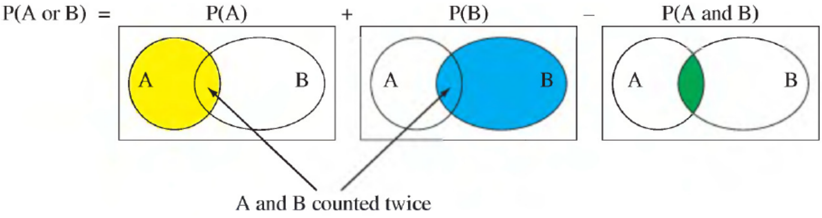
Addition Rule

For the *union* of two events,

$$P(AorB) = P(A) + P(B) - P(AandB)$$

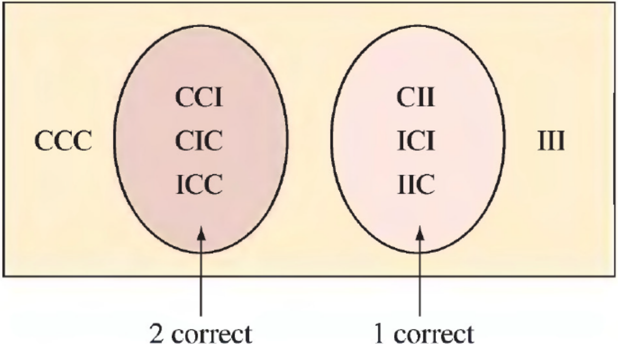
If the events are *disjoint*, P(A and B) = 0, so

$$P(AorB) = P(A) + P(B)$$



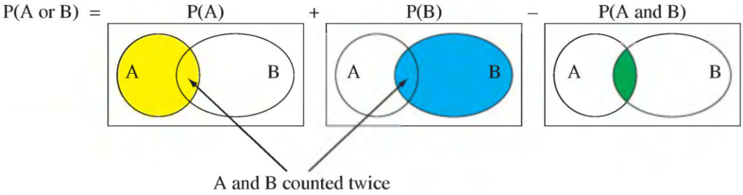
- Two events, A and B, are disjoint if they do not have any common outcomes (for example, M and F are disjoint events).

Example. Experiment: Three questions. Students can answer Correctly or Incorrectly. Are the two events "1 correct answer" and "2 correct answers" disjoint?



Probability Properties

1. $0 \leq P(A) \leq 1$
2. $P(\bar{A}) = 1 - P(A)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

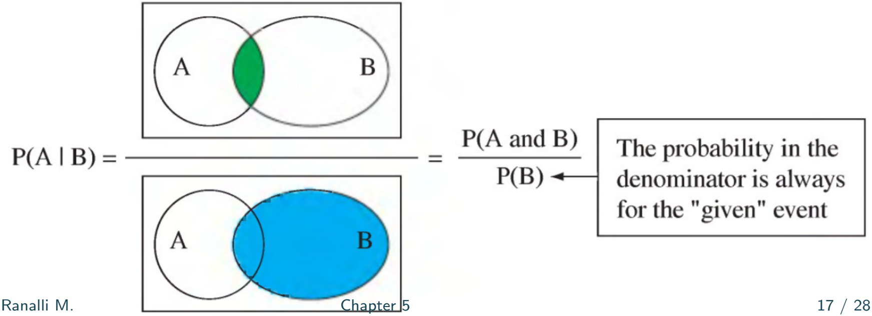


Conditional Probability

- For events A and B, the conditional probability of event A, given that event B has occurred, is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- $P(A \mid B)$ is interpreted as the "Probability event A happens given that event B has happened". Of the times that B occurs, $P(A|B)$ is the proportion of times that A also occurs.



Example

Experiment: draw at random a person from the following population

		Smoker		
		Yes	No	
Gender	**M**	100	60	160
	F	10	30	40
		110	90	200

- $P(YES \mid M) = P(YES \cap M)/P(M) = (100/200) \times (200/160) = 100/160$
- $P(NO \mid M) = P(NO \cap M)/P(M) = (60/200) \times (200/160) = 60/160$

Example

A study of 5282 pregnant women aged 35 or over analyzed the Triple Blood Test to test its accuracy. The results are

TABLE 5.5: Contingency Table for Triple Blood Test of Down Syndrome

	Blood Test Result		
Down Syndrome Status	POS	NEG	**Total**
D (Down)	48	6	**54**
D^c (unaffected)	1307	3921	**5228**
Total	**1355**	**3927**	**5282**

- A positive test result states that the condition is present
- A negative test result states that the condition is not present
- False Positive: Test states the condition is present, but it is actually absent
- False Negative: Test states the condition is absent, but it is actually present

Assuming the sample is representative of the population, find the estimated probability of a positive test for a randomly chosen pregnant woman 35 years or older

$$P(POS) = 1355/5282 = 0.257$$

Given that the diagnostic test result is positive, find the estimated probability that Down syndrome truly is present

$$P(\mathrm{D} \mid \mathrm{POS}) = \frac{P(\mathrm{D}andPOS)}{P(\mathrm{POS})} = \frac{48/5282}{1355/5282} = \frac{0.009}{0.257} = 0.035$$

Summary: Of the women who tested positive, fewer than 4% actually had fetuses with Down syndrome

Multiplication rule for finding P(A and B)

For events A and B, the probability that A and B both occur equals:

$$\mathrm{P(A\ and\ B)} = \mathrm{P(A|}\ B) \times \mathrm{P(B)} also \mathrm{P(A\ and\ B)} = \mathrm{P(B|}\ A) \times \mathrm{P(A)}$$

Example

Two cards are drawn at random from a deck (without replacement). Compute the probabilities:
1) ($1\heartsuit$, $3\diamondsuit$); 2) I card $4\spadesuit$; 3) II card $4\spadesuit$; 4) P(II $4\spadesuit$ | I $3\diamondsuit$).

Solutions

1. $P(1\heartsuit, 3\diamondsuit) = P(\text{II}3\diamondsuit \mid \text{I}1\heartsuit)P(\text{I}1\heartsuit) = \dfrac{1}{51}\dfrac{1}{52}$
2. $P(\text{I}4\spadesuit) = \dfrac{1}{52}$
3. $P(\text{II}4\spadesuit) = \dfrac{51}{51 \times 52} = \dfrac{1}{52}$
4. $P(\text{II}4\spadesuit \mid \text{I}3\diamondsuit) = \dfrac{1}{51}$

Independent events defined using conditional probability

- Two events A and B are independent if the probability that one occurs is not affected by whether or not the other event occurs
- Events A and B are independent if: $P(A \mid B) = P(A)$, or equivalently, $P(B \mid A) = P(B)$
- If events A and B are independent then: $P(A \cap B) = P(A) \times P(B)$

For any given probabilities for events A and B, the events are independent if any **ONE** of the following are true

1. $P(A \cap B) = P(A) \times P(B)$
2. $P(A \mid B) = P(A)$
3. $P(B \mid A) = P(B)$

Probability of the intersection of two independent events

From the definition of independent events we deduce the following

Multiplication Rule

For the intersection of two **independent** events, A and B,

$$P(A \cap B) = P(A) \times P(B)$$

Example Test

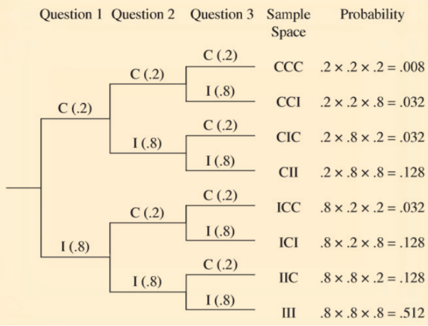
1. What is the probability of getting 3 questions correct by guessing?
2. What is the probability of getting 2 questions correct by guessing?
3. What is the probability of getting at least 2 questions correct by guessing?

Probability of guessing correctly is 0.2

Example

Knowing that probability of guessing correctly is 0.2 and assuming that each answer is independent of the other...

1. What is the probability of getting 3 questions correct by guessing? **0.008**
2. What is the probability of getting 2 questions correct by guessing? $\mathbf{0.032 + 0.032 + 0.032 = 0.096}$
3. What is the probability of getting at least 2 questions correct by guessing?$\mathbf{0.032 + 0.032 + 0.032 + 0.008 = 0.104}$



Example

Experiment: draw at random a person from the following population

		Smoker		
		Yes	**No**	
Gender	**M**	100	60	160
	F	10	30	40
		110	90	200

Are the events M and YES independent?

- $P(YES \cap M) = 100/200 = 0.5 \neq 0.44 = 110/200 \times 160/200 = P(YES) \times P(M)$
- $P(YES \mid M) = 100/160 \neq 110/200 = P(YES)$

They are not independent

How do we find probabilities?

- Calculate theoretical probabilities based on assumptions about the random phenomena.
  For example, it is often reasonable to assume that outcomes are equally likely such as when flipping a coin, rolling a die or drawing at random a person from a population.
- Observe many trials of the random phenomenon and use the sample proportion of the number of times the outcome occurs as its probability.
  This is merely an estimate (approximation) of the actual probability. In the third part, we will see how to use this method.

Types of (Definitions of) Probabilities: Relative Frequency and Subjective

- The relative frequency definition of probability is the long run proportion of times that the outcome occurs in a very large number of trials - not always helpful/possible.
- When a long run of trials is not feasible, you must rely on subjective information. In this case, the subjective definition of the probability of an outcome is your degree of belief that the outcome will occur based on the information available.

  Bayesian statistics is a branch of statistics that uses subjective probability as its foundation