Introduction

We use it to identify potential lurking variables

Before: **Simple** regression model: the relationship between a response variable y and ONE predictor variable x:

$$Y = \beta_0 + \beta_1 X + \epsilon \tag{1}$$

Now: **Multiple** regression model: the relationship between a response variable y and MORE THAN ONE predictor variables:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_{p-1} X_{p-1} + \epsilon \tag{2}$$

The model assumes that the slope for a particular explanatory variable is identical for all fixed values of the other explanatory variables

In multiple regression, a slope describes the effect of an explanatory variable while controlling effects of the other explanatory variables in the model

Introduction

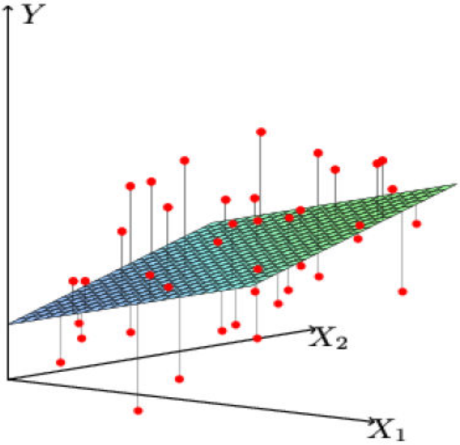
- The good news!– Everything we learned about the simple linear regression model extends (with minor modification) to multiple linear regression.
- The even better news!– We will learn new stuff! Because now we have a more complicated model. We need to consider not only the relation between predictors and response, but also relations among predictors.

Example

Suppose that a researcher is studying factors that might affect blood pressures for women aged 45 to 65 years old. The Y-variable is blood pressure. Suppose that two predictor variables (X-variables) of interest are age and body mass index (calculated as $weight/height^2$). The general structure of a linear multiple regression model for this situation would be

$$\text{Blood Pressure} = \beta_0 + \beta_1 \text{Age} + \beta_2 \text{Body Mass} + \text{Error}$$

- The equation $\beta_0 + \beta_1 \text{Age} + \beta_2 \text{Body Mass}$ describes the mean value of Blood Pressure for specific values of Age and Body Mass.
- The Error term describes the characteristics of the differences between individual values of blood pressure and the mean blood pressure$= \beta_0 + \beta_1 \text{Age} + \beta_2 \text{Body Mass}$.


Y
X2
X1

Notation for Multiple Regression Model

Model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_{p-1} X_{p-1} + \epsilon.$$

or $$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_{p-1} x_{i,p-1} + \epsilon_i.$$

Assumption: The linear model above is reasonable, and ϵ_i are independent and follows $N(0, \sigma^2)$ distribution.

Note: The subscript i refers to the ith individual or unit. In the notation for the x-variables, the subscript following i indicates which x-variable it is. e.g. x_{i2} represents the 2nd x-variable for subject i.

Matrix Notation for Multiple Regression Model

Goal:
represent the above regression models in terms of **matrices, vectors and their operations**.

Reasons:

1. Neat, clean.
2. Statistical softwares basically realize all the calculations for estimations etc using matrix/vector representations.

Matrix Notation for Multiple Regression Model

Consider the multiple regression models for all the individuals in the sample (size n), with intercept and $p-1$ predictor variables:

$$
\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_{1,1} + \beta_2 x_{1,2} + \ldots + \beta_{p-1} x_{1,p-1} + \epsilon_1 \\
y_2 &= \beta_0 + \beta_1 x_{2,1} + \beta_2 x_{2,2} + \ldots + \beta_{p-1} x_{2,p-1} + \epsilon_2 \\
&\ldots \\
y_i &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \ldots + \beta_{p-1} x_{i,p-1} + \epsilon_i \\
&\ldots \\
y_n &= \beta_0 + \beta_1 x_{n,1} + \beta_2 x_{n,2} + \ldots + \beta_{p-1} x_{n,p-1} + \epsilon_n
\end{aligned}
$$

Matrix Notation for Multiple Regression Model

And the corresponding regression equations:

$$
\begin{aligned}
E(y_1) &= \beta_0 + \beta_1 x_{1,1} + \beta_2 x_{1,2} + \ldots + \beta_{p-1} x_{1,p-1} \\
E(y_2) &= \beta_0 + \beta_1 x_{2,1} + \beta_2 x_{2,2} + \ldots + \beta_{p-1} x_{2,p-1} \\
&\ldots \\
E(y_i) &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \ldots + \beta_{p-1} x_{i,p-1} \\
&\ldots \\
E(y_n) &= \beta_0 + \beta_1 x_{n,1} + \beta_2 x_{n,2} + \ldots + \beta_{p-1} x_{n,p-1}
\end{aligned}
$$

Matrix Notation for Multiple Regression Model

Then, we have:

$$
\begin{aligned}
y_1 &= E(y_1) + \epsilon_1 \\
y_2 &= E(y_2) + \epsilon_2 \\
&\dots \\
y_i &= E(y_i) + \epsilon_i \\
&\dots \\
y_n &= E(y_n) + \epsilon_n
\end{aligned}
$$

Matrix Notation for Multiple Regression Model

We can denote all the observed response as a vector of length n, all the error terms as a vector of length n, all the mean responses as a vector of length n. Then, we have:

$$\mathrm{Y} = \mathrm{E}(\mathrm{Y}) + \boldsymbol{\epsilon}$$

where $\mathrm{Y} = \begin{pmatrix} y_1 \\ y_2 \\ . \\ y_i \\ . \\ y_n \end{pmatrix}$, $\mathrm{E}(\mathrm{Y}) = \begin{pmatrix} E(y_1) \\ E(y_2) \\ .. \\ E(y_i) \\ .. \\ E(y_n) \end{pmatrix}$, $\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \\ \epsilon_i \\ \\ \epsilon_n \end{pmatrix}$. Hence, we only need to focus on the matrix representation of $\mathrm{E}(Y)$.

Matrix Notation for Multiple Regression Model

Now we write E(Y) as a design matrix (or say X matrix) X postmultiplied by a vector of $\boldsymbol{\beta}$ coefficients. Notice that for each subject i,

$$
\begin{aligned}
E(y_i) &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + ... + \beta_{p-1} x_{i,p-1} \\
&= 1 \cdot \beta_0 + x_{i,1} \cdot \beta_1 + x_{i,2} \cdot \beta_2 + ... + x_{i,p-1} \cdot \beta_{p-1} \\
&= (1 \; x_{i,1} \; x_{i,2} ... x_{i,p-1}) \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ .. \\ \beta_{p-1} \end{pmatrix}.
\end{aligned}
$$

Matrix Notation for Multiple Regression Model

$$E(y_1) = 1 \cdot \beta_0 + x_{1,1} \cdot \beta_1 + x_{1,2} \cdot \beta_2 + ... + x_{1,p-1} \cdot \beta_{p-1}$$

$$\begin{pmatrix} E(y_1) \\ .. \\ .. \\ .. \\ .. \end{pmatrix} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & ... & x_{1,p-1} \\ & . & . & & .. \\ & . & . & & .. \\ & . & . & & .. \\ & . & . & & .. \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ .. \\ \beta_{p-1} \end{pmatrix}$$

Matrix Notation for Multiple Regression Model

$$E(y_i) = 1 \cdot \beta_0 + x_{i,1} \cdot \beta_1 + x_{i,2} \cdot \beta_2 + ... + x_{i,p-1} \cdot \beta_{p-1}; i = 1,2$$

$$\begin{pmatrix} E(y_1) \\ E(y_2) \\ .. \\ .. \\ .. \end{pmatrix} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & ... & x_{1,p-1} \\ 1 & x_{2,1} & x_{2,2} & ... & x_{2,p-1} \\ & . & . & & .. \\ & . & . & & .. \\ & . & . & & .. \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ .. \\ \beta_{p-1} \end{pmatrix}$$

Matrix Notation for Multiple Regression Model

$$E(y_i) = 1 \cdot \beta_0 + x_{i,1} \cdot \beta_1 + x_{i,2} \cdot \beta_2 + ... + x_{i,p-1} \cdot \beta_{p-1}; i = 1, 2, 3$$

$$\begin{pmatrix} E(y_1) \\ E(y_2) \\ E(y_3) \\ .. \\ .. \end{pmatrix} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & ... & x_{1,p-1} \\ 1 & x_{2,1} & x_{2,2} & ... & x_{2,p-1} \\ 1 & x_{3,1} & x_{3,2} & ... & x_{3,p-1} \\ & . & . & & .. \\ & . & . & & .. \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ .. \\ \beta_{p-1} \end{pmatrix}$$

Matrix Notation for Multiple Regression Model

$$E(y_i) = 1 \cdot \beta_0 + x_{i,1} \cdot \beta_1 + x_{i,2} \cdot \beta_2 + ... + x_{i,p-1} \cdot \beta_{p-1}; i = 1, 2, 3, ..., n$$

$$\mathrm{E}(Y) = \begin{pmatrix} E(y_1) \\ E(y_2) \\ E(y_3) \\ .. \\ E(y_n) \end{pmatrix} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & ... & x_{1,p-1} \\ 1 & x_{2,1} & x_{2,2} & ... & x_{2,p-1} \\ 1 & x_{3,1} & x_{3,2} & ... & x_{3,p-1} \\ & . & . & & .. \\ 1 & x_{n,1} & x_{n,2} & ... & x_{n,p-1} \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ .. \\ \beta_{p-1} \end{pmatrix}$$

Matrix Notation for Multiple Regression Model [IMPORTANT]

Regression equation: $E(Y) = X\boldsymbol{\beta}$;
Regression model: $Y = E(Y) + \boldsymbol{\epsilon}$ or $Y = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ y_n \end{pmatrix}, X = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & \dots & x_{1,p-1} \\ 1 & x_{2,1} & x_{2,2} & \dots & x_{2,p-1} \\ 1 & x_{3,1} & x_{3,2} & \dots & x_{3,p-1} \\ & . & . & & .. \\ 1 & x_{n,1} & x_{n,2} & \dots & x_{n,p-1} \end{pmatrix},$$

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ .. \\ \beta_{p-1} \end{pmatrix}, \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \\ \epsilon_n \end{pmatrix}$$

Matrix Notation for Multiple Regression Model

X is called design matrix. Each row of X corresponds to one subject/unit, each column (except the 1st) corresponds to a predictor. The first column is the intercept. And, its form is the same as the way we input data in R.

Matrix Notation for Multiple Regression Model: Example

Consider a toy example with a data set of size 4, we have response Y and two predictors X_1, X_2 and consider fitting a multiple regression model. Based on the data given in the following table, how can we represent the models using matrix notations?

Y	12	17	15	11
X_1	3	5	4	2
X_2	1	1	2	2

In other words, for the matrix notation representation $\mathrm{Y} = \mathrm{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, what are the exact forms of the design matrix and vectors based on the data?

How to estimate the coefficients $\beta_0, \beta_1, \ldots, \beta_{p-1}$, or in other words, the coefficient vector β?

Estimates (Least Square Criterion)

- The estimates of the β coefficients minimize the sum of squared distances from the observation points to the regression hyperplane.

$$\min \sum_i (y_i - \beta_0 - \beta_1 x_{i,1} - ... - \beta_{p-1} x_{i,p-1})^2$$

- The letter b is used to represent a sample estimate of a β coefficient.

$$(b_0, b_1, ..., b_{p-1}) = argmin_{\beta_0,...,\beta_{p-1}} \sum_i (y_i - \beta_0 - ... - \beta_{p-1} x_{i,p-1})^2$$

Interpretation

$$b = \begin{pmatrix} b_0 \\ b_1 \\ .. \\ .. \\ b_{p-1} \end{pmatrix}.$$

- b_0: The estimated mean y when **all x variables are 0**.
- $b_j,\ j = 1, \ldots, p-1$: The estimated mean change in y when x_j increases one unit **when other x-variables remain the same.**

Estimation (in matrix form)

- In matrix form:

$$
\begin{aligned}
\boldsymbol{b} \quad & \text{minimize} \quad \sum_i (y_i - \beta_0 - \beta_1 x_{i,1} - \ldots - \beta_{p-1} x_{i,p-1})^2 \\
\Leftrightarrow \quad & \text{minimize} \quad (\mathrm{Y} - \mathrm{X}\boldsymbol{\beta})^{\mathsf{T}} (\mathrm{Y} - \mathrm{X}\boldsymbol{\beta})
\end{aligned}
$$

- Solution:

$$
\mathrm{b} = \begin{pmatrix} b_0 \\ b_1 \\ .. \\ .. \\ b_{p-1} \end{pmatrix} = (\mathrm{X}^{\mathsf{T}}\mathrm{X})^{-1}\mathrm{X}^{\mathsf{T}}\mathrm{Y}.
$$

Sampling Distribution of b

Recall: How do we get a sampling distribution.

Now

- Under normality assumption about random errors, we have:

$$\mathrm{b} = (\mathrm{X}^{\mathsf{T}}\mathrm{X})^{-1}\mathrm{X}^{\mathsf{T}}\mathrm{Y} \sim \mathrm{MVN}(\boldsymbol{\beta}, \sigma^2(\mathrm{X}^{\mathsf{T}}\mathrm{X})^{-1})$$

multivariate normal

- Each b_i follows a normal distribution with mean β_i and variance the ith diagonal elements of $(\mathrm{X}^{\mathsf{T}}\mathrm{X})^{-1}$

Estimation

$$b = (X^T X)^{-1} X^T Y.$$

- Keep in mind: Whenever some of the columns in design matrix X are highly correlated (i.e. some of the predictor variables are linear related), there is no unique solution for inverse of $X^T X$. That is, in such a case, $(X^T X)^{-1}$ does not exist uniquely. This is the problem of multicollinearity.

Consequences of **high** multicollinearity

- Often confusing and misleading results: interpretation.
- Unstable estimates

Model fitting

- **Predicted/Fitted value**:

$$\hat{y}_i = b_0 + b_1 x_{i,1} + b_2 x_{i,2} + ... + b_{p-1} x_{i,p-1}, \;\; i = 1, \dots, n;$$

In matrix notation: $\hat{\mathrm{Y}} = \mathrm{Xb}$.

- **Residual**:

$$e_i = y_i - \hat{y}_i, \;\; i = 1, \dots, n;$$

In matrix notation: $\mathrm{e} = \mathrm{Y} - \hat{\mathrm{Y}}$, where $\mathrm{e} = (\mathrm{e}_1, \dots, \mathrm{e}_\mathrm{n})^\mathrm{T}$.

- **SSE and MSE**:

$$SSE = \sum_i (y_i - b_0 - b_1 x_{i,1} - ... - b_{p-1} x_{i,p-1})^2$$

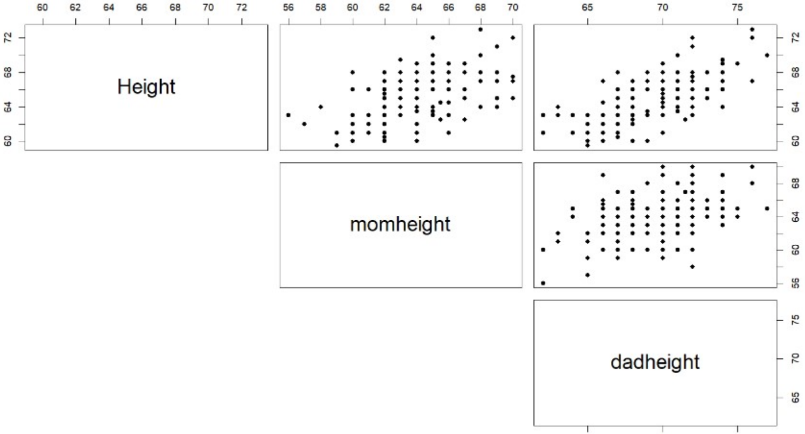
In matrix notation: $SSE = (\mathrm{Y} - \mathrm{Xb})^\mathrm{T}(\mathrm{Y} - \mathrm{Xb}) = \mathrm{e}^\mathrm{T}\mathrm{e}$. And $MSE = \frac{SSE}{n-p}$ estimates σ^2, $S = \sqrt{MSE}$ estimate σ.

Validate Model Assumptions

- **Residual plot (versus the fit)**:
  Still a 2-D plot (e_i versus $\hat{y}_i$)! Ideally should resemble a horizontal random band.
- **Normal probability plot**:
  Expect the normal probability plot to be a straight line and R-J test to have p-value larger than 0.05.

R Example

The sample is from $n = 124$ girls at University of California at Davis. y =student's self-reported height, x_1 =her mother's height, and x_2 =her father's height. All heights are in inches. The following are scatter plots of between each pair of variables.



R Example

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 14.30733    4.69649   3.046  0.00284 **
momheight    0.26191    0.07133   3.672  0.00036 ***
dadheight    0.48875    0.06602   7.404 1.94e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.002 on 121 degrees of freedom
Multiple R-squared:  0.5001,	Adjusted R-squared:  0.4918
F-statistic: 60.51 on 2 and 121 DF,  p-value: < 2.2e-16
```

- We can interpret the "slopes" in the same way as the simple linear model, but we have to add the constraint that values of other variables remain constant.
  1. **When father's height is held constant**, the *average* student height increases 0.262 inches with one-inch increase in mother's height.
  2. **When mother's height is held constant**, the *average* student height increases 0.489 inches with one-inch increase in father's height.

R Example

R^2 = (sum(y-y_bar)^2 - sum(y-y_hat)^2)/sum(y-y_bar)^2

y_bar is sample mean, y_hat is prediction, y is given

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 14.30733    4.69649   3.046  0.00284 **
momheight    0.26191    0.07133   3.672  0.00036 ***
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---
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F-statistic: 60.51 on 2 and 121 DF,  p-value: < 2.2e-16
```
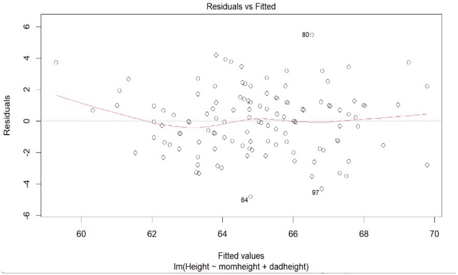

- The sample regression equation is
  Fitted/Predicted student height $=$
  $14.3 + 0.262 \times$ Mother's height $+ 0.489 \times$ Father's height
- The value of $R^2 = 50.0\%$ means that the model (the two x-variables) explains 50.0% of the observed variation in student heights (...but look at the Adjusted R^2).
- The value $S = 2.002$ is the estimated standard deviation of the errors. Roughly, it is the average absolute size of a residual. $S = \sqrt{MSE}$.

Previous R Example

Residual Plot

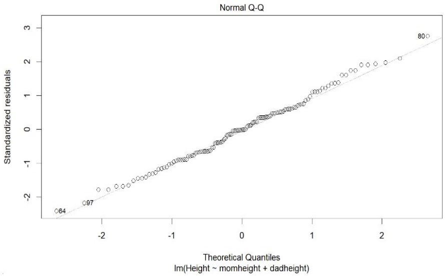


Just as in simple regression, we can use a plot of residuals versus fits to evaluate the validity of assumptions. The residual plot for these data is shown above. Roughly, it looks about as it should, although there may be a bit of increasing (vertical) variance as we move across.

Previous R Example

Normal Probability Plot of the residuals and Shapiro test

The straight line pattern indicates normality as does the *p*-value of the test ($p - value = 0.82 > 0.05$). This means that it's reasonable to assume that the errors follow a normal distribution.




```
> shapiro.test(out$residuals)

        Shapiro-Wilk normality test

data:  out$residuals
W = 0.99327, p-value = 0.8196
```

Inference: Testing Significance of Each β Coefficient

We may want to assess whether a particular x-variable is making a useful contribution to the model. That is, given the presence of the other x-variables in the model, does a particular x-variable help us to explain more about the y-variable?

As an example, suppose that we have three x-variables in the model. The general structure of the model could be

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \epsilon$$

The null hypothesis of the Shapiro-Wilk test is that the data is normally distributed

If the p-value of the test is less than the chosen significance level (usually 0.05), then the null hypothesis is rejected, and the data is considered to not be normally distributed.

Inference: Testing Significance of Each β Coefficient

To determine whether variable x_1 is a useful predictor variable in this model, we could test

$$\begin{aligned} H_0 &: \beta_1 = 0 \\ H_1 &: \beta_1 \neq 0 \end{aligned}$$

If the null hypothesis above were the case, y and x_1 are not significantly related, or x_1 is not important **when x_2 and x_3 are in the model**.

NOTE: If the null were true, we would still be left with variables x_2 and x_3 being present in the model. So when we cannot reject the null hypothesis above, we should say that do not need variable x_1 in the model given that variables x_2 and x_3 will remain in the model.

Inference: Testing Significance of Each β Coefficient

Carry out the test:

$$T = \frac{\text{sample coefficient}}{\text{standard error of the coefficient}} \sim t(n-p) \text{ under } H_0.$$

1. **critical value approach:** if $|T_0| > t_{\alpha/2}(n-p)$, reject H_0.
2. **p-value approach:** if p-value $= P(|T| > |T_0|) < \alpha$, reject H_0.

where T_0 is the observed value of T using the given sample.

Previous R Example

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 14.30733    4.69649   3.046  0.00284 **
momheight    0.26191    0.07133   3.672  0.00036 ***
dadheight    0.48875    0.06602   7.404 1.94e-11 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.002 on 121 degrees of freedom
Multiple R-squared:  0.5001,    Adjusted R-squared:  0.4918
F-statistic: 60.51 on 2 and 121 DF,  p-value: < 2.2e-16
```

- The p-values given for the two x-variables tell us that student height is significantly related to both predictors.

Previous R Example

Does father's height has more impact on the average heights of children than mother's height?

Other inference tools

- Confidence interval for E(Y) given $x = (x_1, x_2, ..., x_{p-1})$

- Prediction interval for y given $x_h = (x_{h,1}, x_{h,2}, ..., x_{h,p-1})$

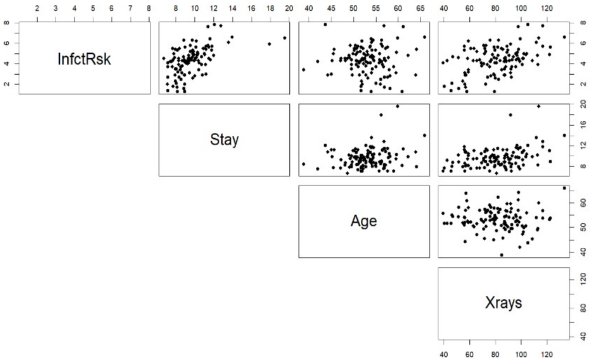
R Example

Data from n $= 113$ hospitals in the United States are used to assess factors related to the likelihood that a hospital patients acquires an infection while hospitalized. The variables here are $y =$infection risk, $x_1 =$average length of patient stay, $x_2 =$average patient age, $x_3 =$measure of how many x-rays are given in the hospital.

Note: sample size $n = 113$, number of predictors$= 3$ (so, if we include all of them, the regression model will have 4 β coefficients, i.e. $p = 4$).

R Example

Step 1. Check individual scatter plots of y versus x_i, $i = 1, 2, 3$.



residual std dev = sqrt(sum(residual)^2 / n - (parameters in regression))

It seems that y has obvious linear relations with both x_1 and x_3 but mild linear relation with x_2. And relation between x_1, x_2, and x_3 are not significantly patterned.

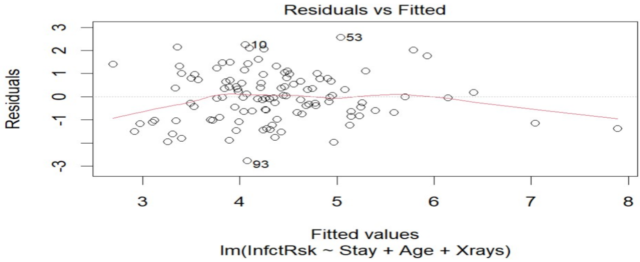
R Example

Step 2. Now, simply include all of the three predictor variables into the model and fit the multiple regression model.

```
Call:
lm(formula = InfctRsk ~ Stay + Age + Xrays, data = Senic)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.77320 -0.73779 -0.03345  0.73308  2.56331 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept)  1.001162   1.314724   0.761 0.448003    
Stay         0.308181   0.059396   5.189 9.88e-07 ***
Age         -0.023005   0.023516  -0.978 0.330098    
Xrays        0.019661   0.005759   3.414 0.000899 ***
---
Signif. codes:  
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.085 on 109 degrees of freedom
Multiple R-squared:  0.363,	Adjusted R-squared:  0.3455 
F-statistic:  20.7 on 3 and 109 DF,  p-value: 1.087e-10
```

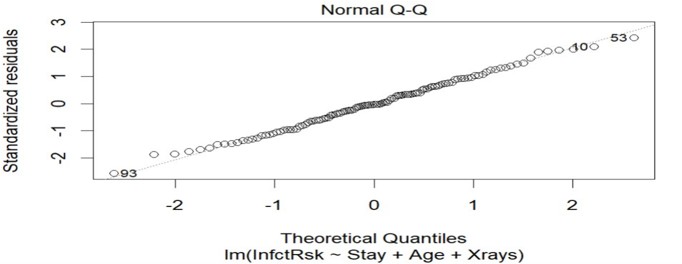
R Example

Step 3. Check validity of assumptions for multiple regression model.



R Example

Step 3. Check validity of assumptions for multiple regression model.




```
        Shapiro-Wilk normality test

data:  out$residuals
W = 0.9939, p-value = 0.9037
```

R Example

Step 4. Inference. The hypothesis testing for β_1:

$$\begin{aligned} H_0 : \beta_1 &= 0 \\ H_1 : \beta_1 &\neq 0 \end{aligned}$$

- $T_0 = \frac{b_1}{s.e.(b_1)} = 5.189 \Rightarrow |T_0| > t_{0.025}(113-4) = 1.98$, so reject H_0.
- p-value$= Prob(|T| \geq 5.189) \approx 0$, so reject H_0.

R Example

Step 4. Inference. The hypothesis testing for β_2:

$$\begin{aligned} H_0 : \beta_2 &= 0 \\ H_1 : \beta_2 &\neq 0 \end{aligned}$$

- $T_0 = \frac{b_2}{s.e.(b_2)} = -0.978 \Rightarrow |T_0| < t_{0.025}(113-4) = 1.98$, so fail to reject H_0.
- p-value$= Prob(|T| \geq 0.98) \approx 0.33$, so fail to reject H_0.

Thus we cannot reject the null hypothesis $H_0 : \beta_2 = 0$, so “Age” is not a useful predictor **within this model** (i.e. **given the presence of the other two predictors**).

R Example

Step 4. Inference. The hypothesis testing for β_3:

$$\begin{aligned} H_0 : \beta_3 &= 0 \\ H_1 : \beta_3 &\neq 0 \end{aligned}$$

In a similar fashion, we can also obtain the p-value$= 0.001 < 0.05$. Therefore, X-rays is useful for predicting y with the other two variables in the model.

Note: Usually, we don't worry about the p-value for "'Constant'. It only refer to the "intercept" of the model and doesn't give us information about how changing an x-variables might change the mean of response y.

Multiple Regression: Analysis of Variance (ANOVA) Table

F = MSR / MSE

Recall: In simple linear regression, we can obtain two types of information in a ANOVA table

- A decomposition of variance (SSTO=SSR+SSE)
- A significance test (F-test)

For multiple regression, it is very similar.

1. A decomposition of variance

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ANOVA table displays quantities that measure how much of the variability in the y-variable is explained and how much is not explained by the y-variables relationship with the x-variables.

Recall (**Basic Idea**):

$$\begin{array}{lcll} \text{Overall variation in y} & = & \begin{array}{l}\text{variation explained}\\ \text{by regression}\end{array} & + \text{error variation} \\ SST & = & SSR & + SSE \end{array}$$

CI for a multiple regression beta = estimated slope +- t0.025(se)

df = n - number of parameters in regression

1. A decomposition of variance

- **Sums of squares for total**: $SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$.
  - Total degrees of freedom$= n - 1$.
  - SST is a measure of the overall variation in the y-variables.

- **Sums of squared errors**: $SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$.
  - Error degrees of freedom$= n - p$.
    $p = \#\beta$ coefficients in the model (including) β_0.
  - $MSE = \frac{SSE}{n-p}$ is the mean squared error.

- **Sums of squares due to Regression**: $SSR = SST - SSE$.
  - Regression degrees of freedom
    $=$ total df $-$ error df
    $= (n-1) - (n-p) = p - 1$.
  - $MSR = \frac{SSR}{p-1}$ is the mean square for the regression.

1. A decomposition of variance

Souce	DF	SS	MS	F
Regression	$p-1$	SST-SSE	MSR	MSR/MSE
Error	$n-p$	$\sum_{i=1}^{n}(y_i-\hat{y}_i)^2$	MSE	
Total	$n-1$	$\sum_{i=1}^{n}(y_i-\bar{y})^2$		

The computation of the table is identical with the simple regression model.

2. Significance test: F test

- The F statistic in the analysis of variance can be used to test whether the y-variable is related to at least one x-variables in the model. Specifically,

$$\begin{aligned} H_0 &: \beta_1 = \beta_2 = ... = \beta_{p-1} = 0 \\ H_a &: \text{at least one of the } \beta_i \neq 0, \text{ for } i = 1, ..., p-1. \end{aligned}$$

1. The null hypothesis means that the y-**variable is not related to any of the** x-**variables in the model**.
   The alternative hypothesis means that the y-**variable is related to one or more of the** x-**variables in the model**.

2. Significance test: F test

2. $F = MSR/MSE \sim F(p-1, n-p)$ under H_0.
   If $F_0 > F_\alpha(p-1, n-p)$, reject H_0.
3. Statistical software reports a p-value for this test statistic: p-value$= P(F > F_0)$. Usually, if $p - value \leq 0.05$, reject the null hypothesis, and conclude that y is related to **at least one** of the x-variables in the model.
4. **Cautious! In multiple linear regression, the T-test and F-test are different!**
   T-test tests linear relation between y and a certain x_i while all other x-variables are in the model, while F-test test the linear relation between y and all x-variables together!

Other Uses of ANOVA Table: MSE and R^2

- MSE is the estimate of the error variance. Thus $S = \sqrt{MSE}$ estimates the standard deviation of the errors.
- The Total and Error lines give the SS values used in the calculation of $R^2 = \frac{SST-SSE}{SST} = \frac{SSR}{SSE}$.

R Example

Example For a sample of individuals, we have measurements of y = body fat, x_1 = triceps skinfold thickness, x_2 = thigh circumference, and x_3 = midarm circumference. Some results for a multiple regression with these variables are as follows:

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  117.085     99.782   1.173    0.258
Triceps        4.334      3.016   1.437    0.170
Thigh         -2.857      2.582  -1.106    0.285
Midarm        -2.186      1.595  -1.370    0.190

Residual standard error: 2.48 on 16 degrees of freedom
Multiple R-squared:  0.8014,	Adjusted R-squared:  0.7641
F-statistic: 21.52 on 3 and 16 DF,  p-value: 7.343e-06
```

```
> anova(out)
Analysis of Variance Table

Response: Bodyfat
          Df Sum Sq Mean Sq F value    Pr(>F)
Triceps    1 352.27  352.27 57.2768 1.131e-06 ***
Thigh      1  33.17   33.17  5.3931   0.03373 *
Midarm     1  11.55   11.55  1.8773   0.18956
Residuals 16  98.40    6.15
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
> 
```

R Example

- The number of beta coefficients in the model is $p = 4$.
- The value of F involved in this test was

$$F = 21.52 = MSR/MSE = 132.33/6.15,$$

  and the p-value is given as 0.000. This means that at least one of the three x-variables is a useful predictor of the y-variable.
- The value of $R^2 = \frac{SST - SSE}{SST} = \frac{495.39 - 94.40}{495.39} = 0.801$, or 80.1%. The model explains 80.1% of the observed variation in body fat.
- The estimated standard deviation of the errors is $S = \sqrt{MSE} = 2.48$.

Practice

1. Suppose we are interested in students' final score for Statistics course by considering two predictors: average hours of studying per day and average score of two midterms. By fitting a multiple linear regression model, we have the following fitted model:

$$\text{(estimated) Final Score} = b_0 + b_1 \cdot \text{Study Hour} + b_2 \cdot \text{Midterm}$$

where b_0, b_1, b_2 are the estimated coefficients (based on Least SSE criterion) with some positive real values. Please write down the interpretation of each of the three estimated coefficients in this context.

Practice

2. Complete the following ANOVA table for a multiple linear regression model if the number of x-variables is 3, and the number of observations in the sample is 35.

Source	DF	SS	MS	F
Regression		322.1		
Error				NA
Total		547.6	NA	NA