線性代數 2016 期中考 〔請依題號順序作答，否則扣 5%；若該題沒有解答，請回答"無解"；除了回答 Yes or No 或 Dep. or Ind.的題目外，應寫出計算的主要過程，否則不計分。〕

1 Definition or description (20%) 請回答時不必抄題目:

1.1 Two systems of equations involving the same variables are said to be equivalent. p.**20** if they have same solution sets.

1.2 An $n\times n$ matrix is said to be singular. p.**70** If It doe not have a multiplicative inverse.

1.3 An $n\times n$ matrix A is said to be symmetric. p.**57** If $A^T=A$.

1.4 $\mathbb{R}^{m\times n}$ p**131** The set of all $m\times n$ matrices with real entries

1.5 The nullspace of a $m\times n$ matrix A. p.**138** The set of all solutions to the homogeneous system $A\mathbf{x}=\mathbf{0}$.

2 Given a 3×3 linear system of equations $\begin{aligned} 2x_1+x_2-4x_3&=1\\ x_2+3x_3&=5\\ \alpha x_3&=\beta-9\end{aligned}$, where $\alpha, \beta \in R$.

2.1 Let the matrix equation of the 3×3 linear system of equations be $A\mathbf{x} = \mathbf{b}$. Find A, $\mathbf{x}$ and $\mathbf{b}$. (5%) Q2

Ans: $A=\begin{bmatrix}2&1&-4\\0&1&3\\0&0&\alpha\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}1\\5\\\beta-9\end{bmatrix}$

2.2 Is the system in *strict triangular form* ? Please answer **Yes** or **No**. (5%) Q1

Ans: Yes

2.3 For what values of α and β will the system have infinitely many solutions ? (5%) Q1

Ans: $\alpha = 0$, $\beta = 9$ so that a free variable exists in a consistent system.

3 Let $A\mathbf{x}=\mathbf{b}$ be a linear system whose augmented matrix $(A\,|\,\mathbf{b})$ has reduced row echelon form $\left[\begin{array}{ccccc|c}1&-1&0&3&-6&3\\0&0&1&-2&4&5\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right]$. Let $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5)$, where $\mathbf{a}_1=\begin{bmatrix}5\\0\\1\\3\end{bmatrix}$ and $\mathbf{a}_3=\begin{bmatrix}7\\1\\0\\4\end{bmatrix}$.

3.1 Write $\mathbf{a}_4$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_3$. (5%)

Ans: The augmented matrix $(A\,|\,\mathbf{0})$ has reduced row echelon form $\left[\begin{array}{ccccc|c}1&-1&0&3&-6&0\\0&0&1&-2&4&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right]$

Let $\mathbf{x}=(x_1,x_2,x_3,x_4,x_5)^T$ be a solution to $A\mathbf{x}=\mathbf{0}$.

⇨ x_1 and x_3 are lead (pivot) variables and the others are free variables.

⇨ Let $x_2 = \alpha$, $x_4 = \beta$, $x_5 = \gamma$.

⇨ The solution $\mathbf{x}=(\alpha-3\beta+6\gamma,\alpha,2\beta-4\gamma,\beta,\gamma)^T$, where $\alpha, \beta, \gamma \in R$. ------ (*)

Substitute $\alpha = \gamma = 0$, $\beta = 1$ into (*), we have $\mathbf{x} = (-3, 0, 2, 1, 0)^T$ is a solution to $A\mathbf{x}=\mathbf{0}$.

⇨ $x_1\mathbf{a}_1+x_2\mathbf{a}_2+x_3\mathbf{a}_3+x_4\mathbf{a}_4+x_5\mathbf{a}_5 = -3\mathbf{a}_1+2\mathbf{a}_3+\mathbf{a}_4=\mathbf{0}$

⇨ $\mathbf{a}_4 = 3\mathbf{a}_1 - 2\mathbf{a}_3$ -----(**)

3.2 Determine $\mathbf{a}_4$. (5%) M2015

Ans: By (**), $\mathbf{a}_4 = 3\mathbf{a}_1 - 2\mathbf{a}_3 = (1, -2, 3, 1)^T$

4 Let $P=\begin{bmatrix} I & O \\ B & I \end{bmatrix}$ and $Q=\begin{bmatrix} I & O \\ -B & I \end{bmatrix}$, where I is the $n\times n$ identity matrix, O is the $n\times n$ zero matrix, and B is a symmetric $n\times n$ matrix.

4.1 Explain why Q is the (multiplicative) inverse of P. (5%)

Ans: Since $PQ=\begin{bmatrix} I & O \\ B & I \end{bmatrix}\begin{bmatrix} I & O \\ -B & I \end{bmatrix}=\begin{bmatrix} I & O \\ O & I \end{bmatrix}=I_{2n\times 2n}$

and $QP=\begin{bmatrix} I & O \\ -B & I \end{bmatrix}\begin{bmatrix} I & O \\ B & I \end{bmatrix}=\begin{bmatrix} I & O \\ O & I \end{bmatrix}=I_{2n\times 2n}$

4.2 Compute $2P^T-PQ$ (5%) (The result should be shown by blocks I, O and B) Q2

Ans: $2P^T-PQ=2P^T-I_{2n\times 2n}=2\begin{bmatrix} I & B \\ O & I \end{bmatrix}-\begin{bmatrix} I & O \\ O & I \end{bmatrix}=\begin{bmatrix} I & 2B \\ O & I \end{bmatrix}$.

5 Given $A=(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 3 \end{bmatrix}$, $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

5.1 Compute the LDU factorization of A, where L is a 3×3 unit lower triangular matrix, D is a diagonal matrix, and U is a 3×3 unit upper triangular matrix. (5%) Q3

Ans: $A\xrightarrow{R_3\to R_3-2R_1}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=DU$. Since $l_{31}=2$, $L=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

5.2 Express A as a product of elementary matrices (i.e., Find E_1, E_2, E_3 such that $A=E_1E_2E_3$) (5%) Q3

Ans: Since $A=LDU$, where L, D, U are elementary matrices.

⇨ $E_1=L$, $E_2=D$, and $E_1=U$.

5.3 Find the value of det(−2A). (5%) Q4

Ans: $\det(A)=\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 3 \end{vmatrix}=3$, (or $\det(A)=\det(L)\cdot\det(D)\cdot\det(U)=1\cdot 3\cdot 1=3$)

⇨ $\det(-2A)=(-2)^3\cdot\det(A)=(-8)\cdot 3=-24$

5.4 Find the (1,2) entry of A^{-1} by computing a quotient of two determinants. (5%) Q4

Ans: $(A^{-1})_{12}=\dfrac{1}{\det(A)}\cdot A_{21}=\dfrac{1}{\det(A)}\cdot(-1)^{2+1}\cdot\det(M_{21})=\dfrac{1}{3}(-1)\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}=1$

5.5 Use Cramer's rule to solve $A\mathbf{x}=\mathbf{b}$ for x_1. (5%) Q4

Ans: $x_1=\dfrac{1}{\det(A)}\begin{vmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 3 \end{vmatrix}=\dfrac{1}{3}(6)=2$

5.6 Find the nullspace of A. (5%) Q5

Ans: Since $\det(A)=3\neq 0$, A is nonsingular and $A\mathbf{x}=\mathbf{0}$ has only the trivial solution $\mathbf{0}$.

⇨ $N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

5.7 Is the set { $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$} a spanning set for $\mathbb{R}^3$? (a) Please answer Yes or No. (5%) Q5

Ans: Yes, since A is nonsingular and $A\mathbf{x}=\mathbf{b}$ has unique solution for any $\mathbf{b} \in \mathbb{R}^3$.

6 Does the set $\{(x_1, x_2, x_3)^T \mid x_1 = 0 \text{ and } x_2 = 1\}$ form a subspace of $\mathbb{R}^3$? Please answer Yes or No. (5%) Q5

Ans: No. Let $\mathbf{x} = (0, 1, a)^T$. Then $\mathbf{x}$ belongs to the set.

Observe $2\mathbf{x} = (0, 2, 2a)^T$ does not belong to the set, since $x_2=2\neq 1$.

⇨ This set is not closed under scalar multiplication and does not form a subspace.

7 Find all values of λ for which the determinant $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$ will equal to 0. (5%) Q4

Ans: $\det(A) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ -\lambda^2+2\lambda & \lambda & 0 \end{vmatrix} = (+1)\cdot\begin{vmatrix} \lambda & -\lambda \\ -\lambda^2+2\lambda & \lambda \end{vmatrix} = (+1)\cdot\lambda\cdot\lambda\begin{vmatrix} 1 & -1 \\ -\lambda+2 & 1 \end{vmatrix} = \lambda^2(3-\lambda) = 0$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + (\lambda - 1)R_1$

Cofactor expansion along the 3rd column

$R_1 \to \frac{1}{\lambda}R_1$
$R_2 \to \frac{1}{\lambda}R_2$

⇨ $\lambda = 0, 3.$