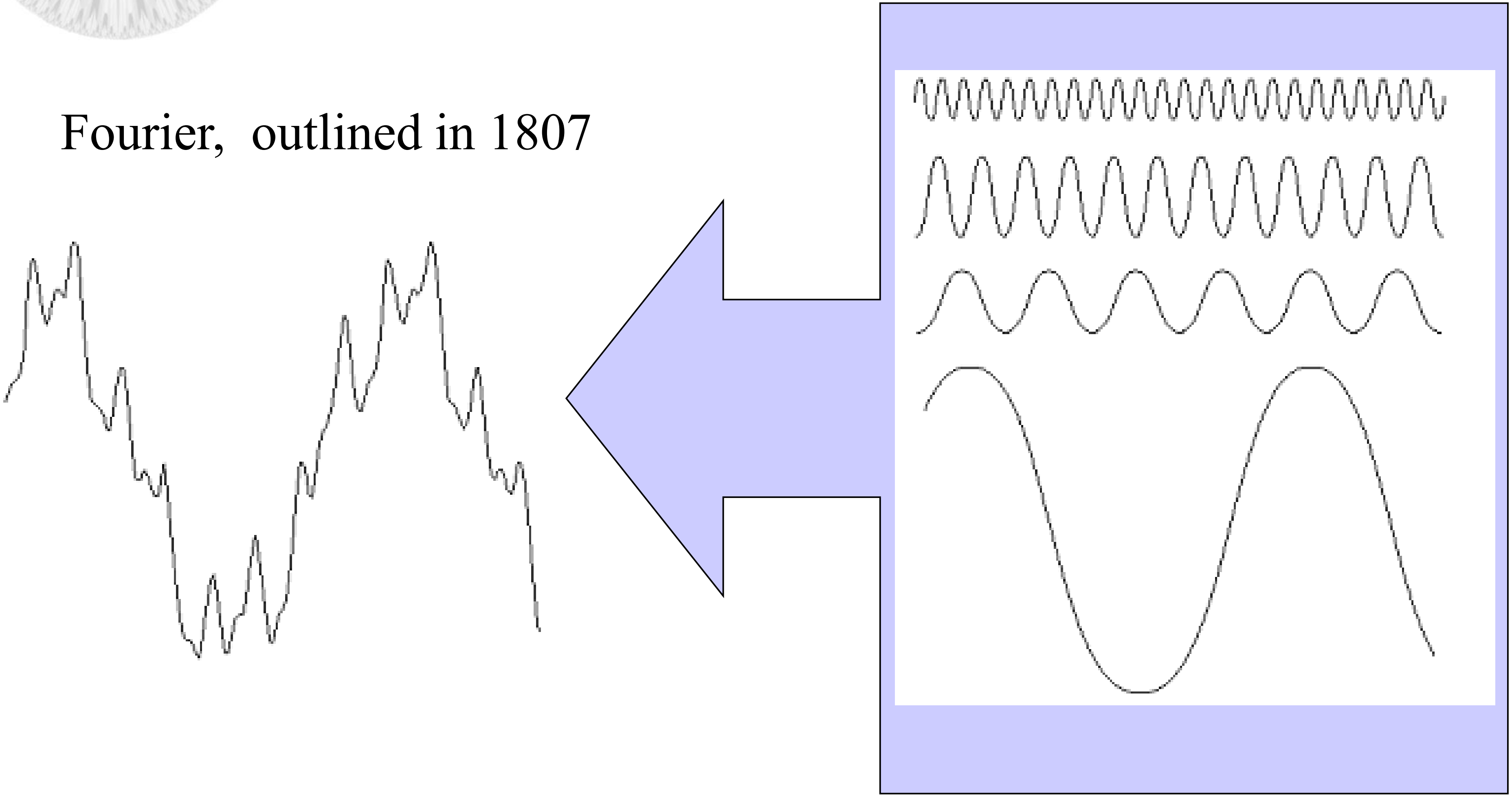
Chapter 4
Image Enhancement in the Frequency Domain

4.1 Fundamentals Related to Fourier Transform

Fourier, outlined in 1807



Fourier Series

Given a function $f(t)$ of a continuous variable t, with period T

頻段要夠高才能作轉換

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \qquad (4.2\text{-}6)$$

因此從負無窮大到正無窮大

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots \qquad (4.2\text{-}7)$$

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

1D and 2D Fourier Transform: Continuous Case

1D case

音訊

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux}\, du.$$

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

影像

2D case

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)}\, du\, dv.$$

Textbook
Section 4.2.4

Fourier Transform
-- function of one continuous variable

$$\Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt \qquad (4.2\text{-}15)$$

t被積掉了，只剩u

Let $\Im\{f(t)\} = F(\mu)$

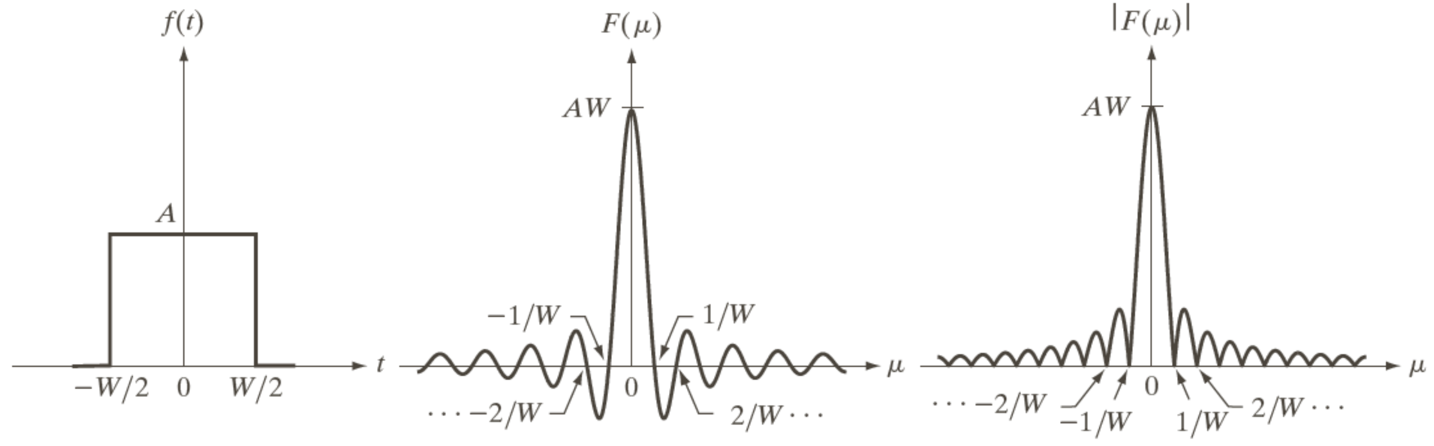
在頻率domain,不同頻率強度

$$\boxed{F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt \qquad (4.2\text{-}16)}$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t)\big[\cos(2\pi\mu t) - j\sin(2\pi\mu t)\big]\, dt \qquad (4.2\text{-}18)$$

Then

$$\boxed{f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{j2\pi\mu t}\, d\mu \qquad (4.2\text{-}17)}$$



a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

考試重點

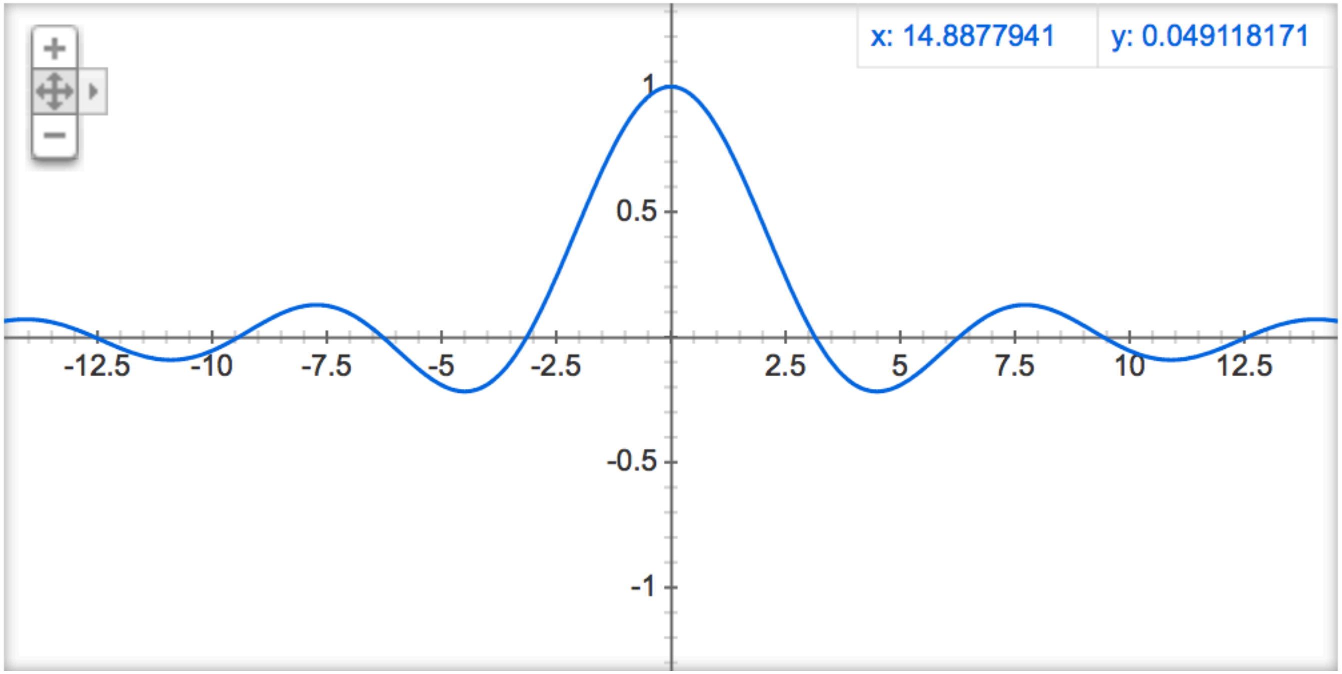
$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

$$= \frac{-A}{j2\pi\mu}\Big[e^{-j2\pi\mu t}\Big]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\Big[e^{-j\pi\mu W} - e^{j\pi\mu W}\Big]$$

$$= \frac{A}{j2\pi\mu}\Big[e^{j\pi\mu W} - e^{-j\pi\mu W}\Big]$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$ = sin(x)/x ->趨近於一

sinx/x



$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

$$= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right]$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$

$$= AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

Let $\mathrm{sinc}(m) = \dfrac{\sin(\pi m)}{(\pi m)}$

Then, $|F(\mu)| = AT\left|\dfrac{\sin(\pi\mu W)}{(\pi\mu W)}\right|$

Textbook
Section 4.2.3

Impulses and Their Sifting Property

A unit impulse of a continuous variable t located at $t = 0$

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \tag{4.2-8a}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{4.2-8b}$$

Sifting property （過濾、篩選）

篩選0

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0) \tag{4.2-9}$$

篩選'Xo'

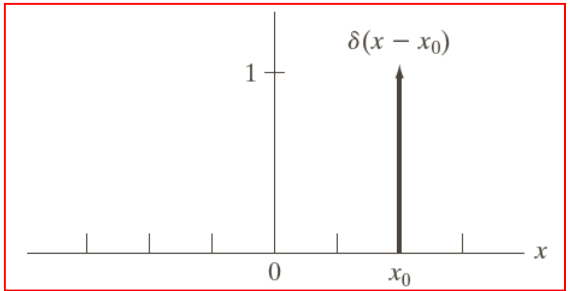
$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = f(t_0) \tag{4.2-10}$$

Impulses and Their Sifting Property (discrete case)

- Unit impulse

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad (4.2\text{-}11a)$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1 \qquad (4.2\text{-}11b)$$



- Sifting property （過濾、篩選）

篩選0

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0) \qquad (4.2\text{-}12)$$

篩選'Xo'

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0) \qquad (4.2\text{-}13)$$

Fourier Transform of an Impulse (continuous case)

對unit impulse 作傅立葉

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi\mu t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-j2\pi\mu t}\, \delta(t)\, dt \\
&= e^{-j2\pi\mu 0} = e^{0} \\
&= 1
\end{aligned}
$$

sift to 'to'.

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j2\pi\mu t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-j2\pi\mu t}\, \delta(t - t_0)\, dt \\
&= e^{-j2\pi\mu t_0} \\
&= \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)
\end{aligned}
$$

Impulse Train and Its Fourier Transform

- Impulse train with period ΔT

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) \tag{4.2-14}$$

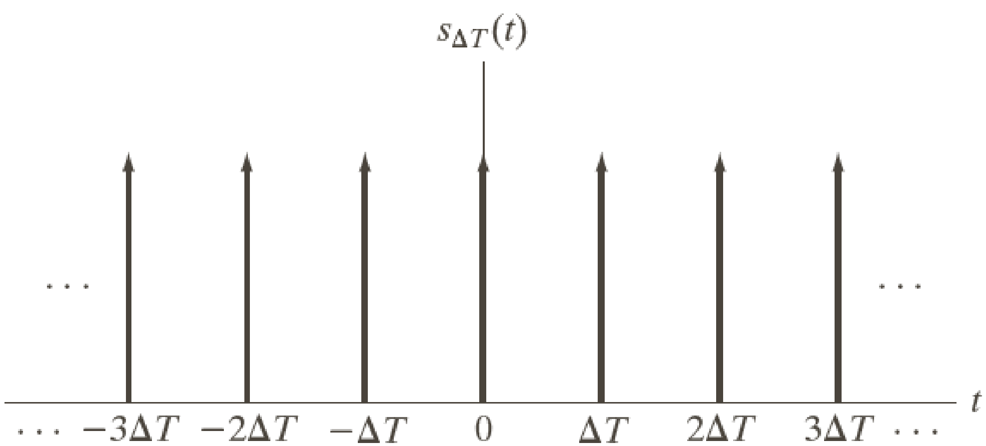


FIGURE 4.3 An impulse train.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \quad (4.2\text{-}6)$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots \quad (4.2\text{-}7)$$

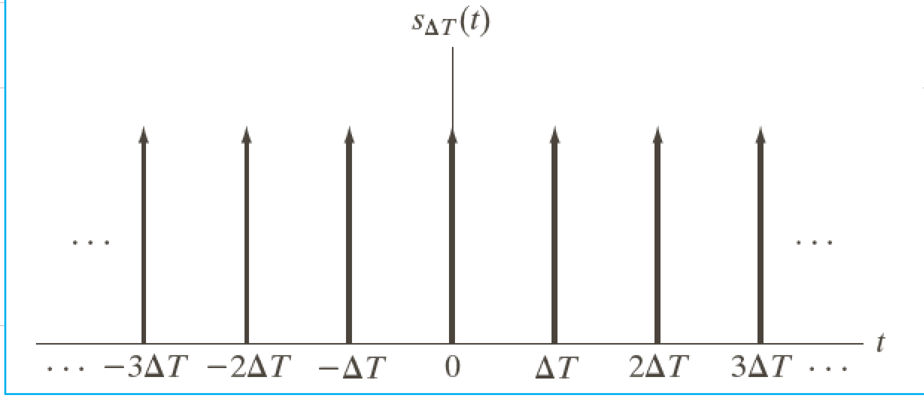
- An impulse train with period ΔT can be expressed as a Fourier series

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t)\, e^{-j\frac{2\pi n}{\Delta T}t}\, dt$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t)\, e^{-j\frac{2\pi n}{\Delta T}t}\, dt$$



$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t)\, e^{-j\frac{2\pi n}{\Delta T}t}\, dt$$

$$= \frac{1}{\Delta T} e^0$$

$$= \frac{1}{\Delta T}$$

The Fourier series expansion becomes

$$s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

Fourier Transform of an Impulse Train

$$S(\mu) = \Im\{s_{\Delta T}(t)\}$$

$$= \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$= \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

$$\Im\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

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Section 4.2.5

Convolution

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau)\, d\tau \qquad (4.2\text{-}20)$$

- Fourier transform of the convolution of two functions *in the space domain* = Product *in the frequency domain* of the Fourier transforms of the two functions

$$\Im\{f(t) \star h(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau)\, d\tau \right] e^{-j2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi\mu t}\, dt \right] d\tau$$

$$\Im\left\{f(t) \star h(t)\right\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau\right] e^{-j2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi\mu t}\, dt\right] d\tau$$

$$\Im\{h(t-\tau)\} = H(\mu)\mathrm{e}^{-j2\pi\mu\tau}$$

$$\Im\left\{f(t) \star h(t)\right\} = \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)\, e^{-j2\pi\mu\tau}\right] d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)\, e^{-j2\pi\mu\tau}\, d\tau$$

$$= H(\mu)\, F(\mu)$$

Convolution Theorem

$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu) \qquad (4.2\text{-}21)$$

$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu) \qquad (4.2\text{-}22)$$

Textbook
Section 4.3

Sampling and Fourier Transform of Sampled Fuctions

4.3.1 Sampling

4.3.2 Fourier Transform of Sampled Function

4.3.3 Sampling Theorem

4.3.4 Aliasing

4.3.5 Function Reconstruction from Sampled Data

Textbook
Section 4.3

Sampled function

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T)$$

Sampled value

$$f_k = \int_{-\infty}^{\infty} f(t)\,\delta(t - k\Delta T)\,dt$$
$$= f(k\Delta T)$$

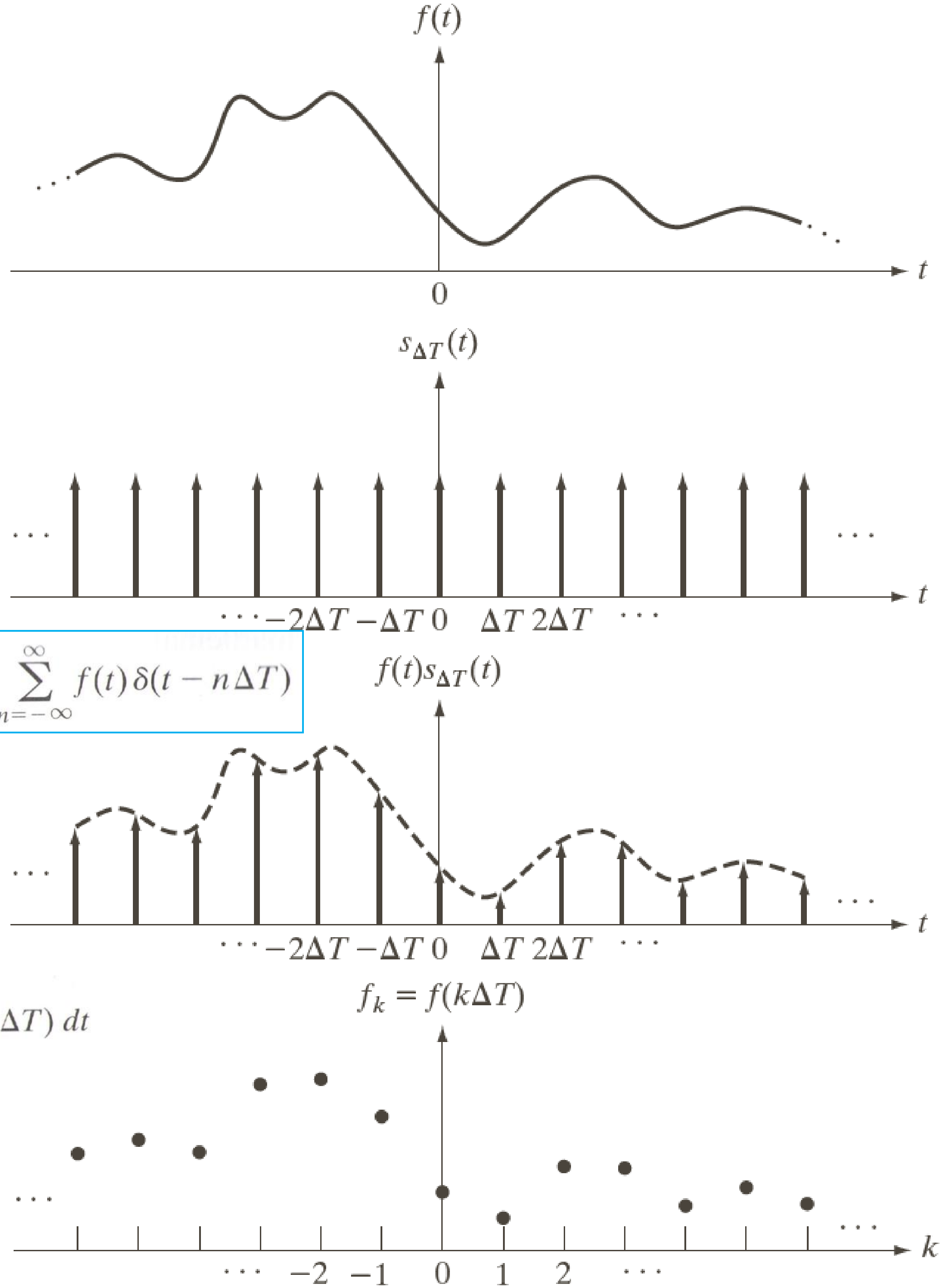


FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

Fourier Transform of Sampled Functions

$$\tilde{F}(\mu) = \Im\{\tilde{f}(t)\}$$
$$= \Im\{f(t)s_{\Delta T}(t)\} \qquad (4.3\text{-}3)$$
$$= F(\mu) \star S(\mu)$$

where the Fourier transform of an impulse train has been shown to be

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right) \qquad (4.3\text{-}4)$$

$$\begin{aligned}\tilde{F}(\mu) &= \Im\{\tilde{f}(t)\} \\ &= \Im\{f(t)s_{\Delta T}(t)\} \qquad (4.3\text{-}3) \\ &= F(\mu) \star S(\mu)\end{aligned}$$

$$\begin{aligned}\tilde{F}(\mu) &= F(\mu) \star S(\mu) \\ &= \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau)\, d\tau \\ &= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \qquad (4.3\text{-}5) \\ &= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\, \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\ &= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)\end{aligned}$$

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

-- an *infinite, periodic* sequence of copies of $F(\mu)$

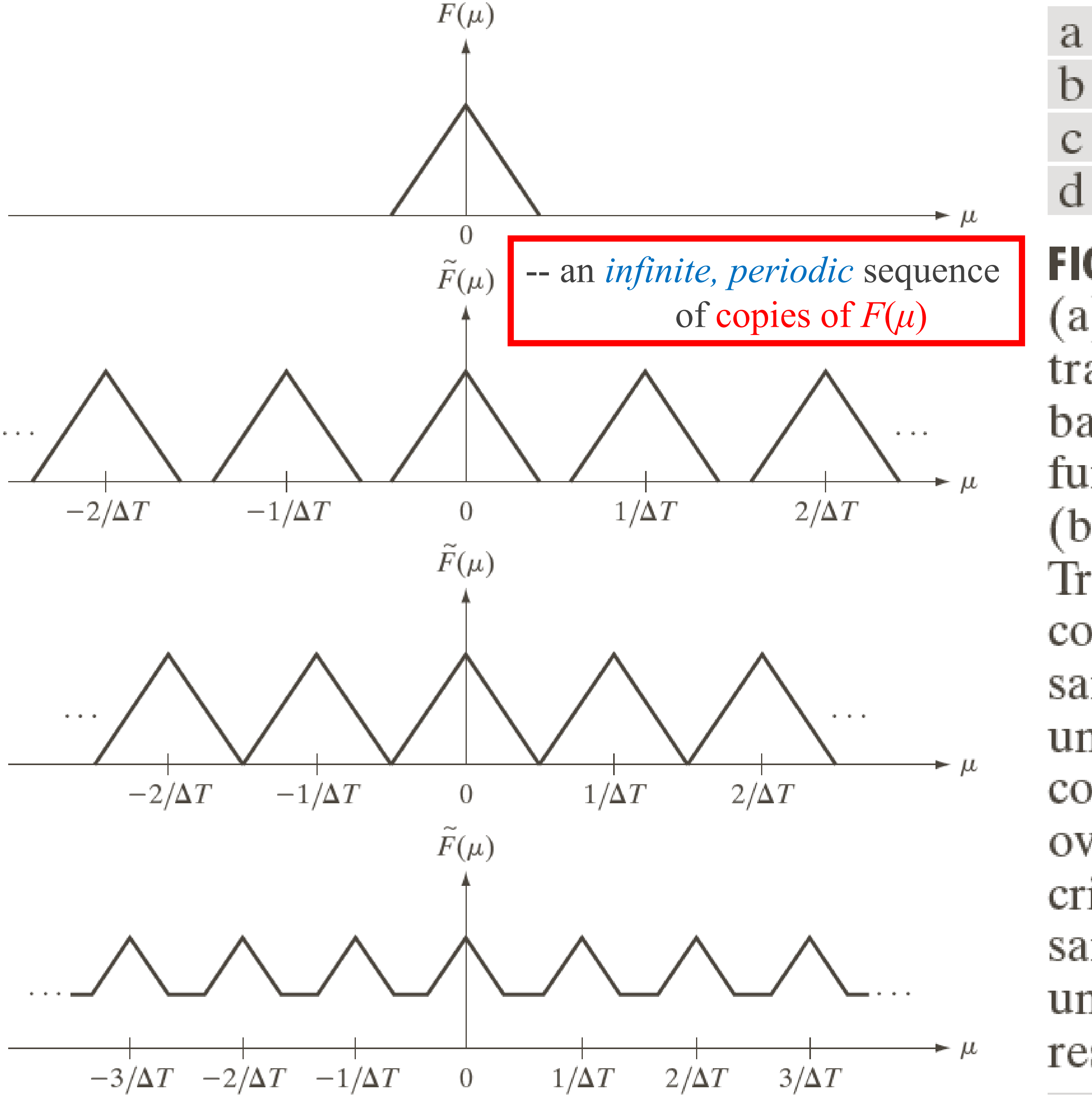
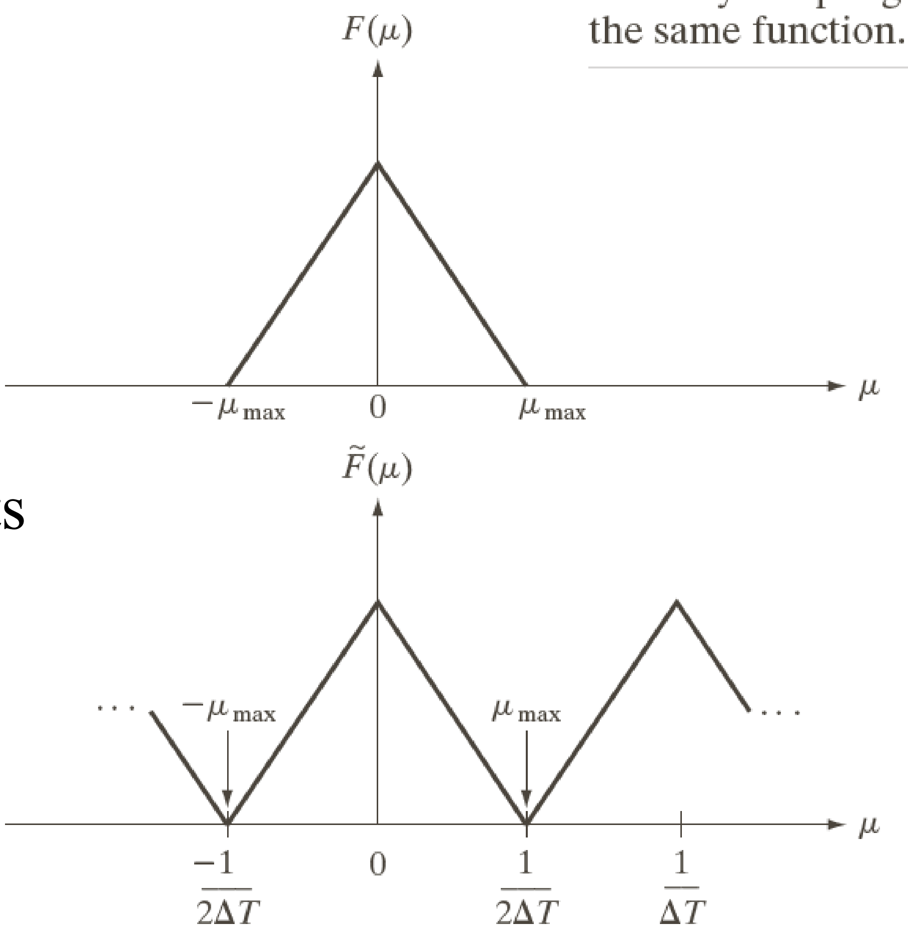


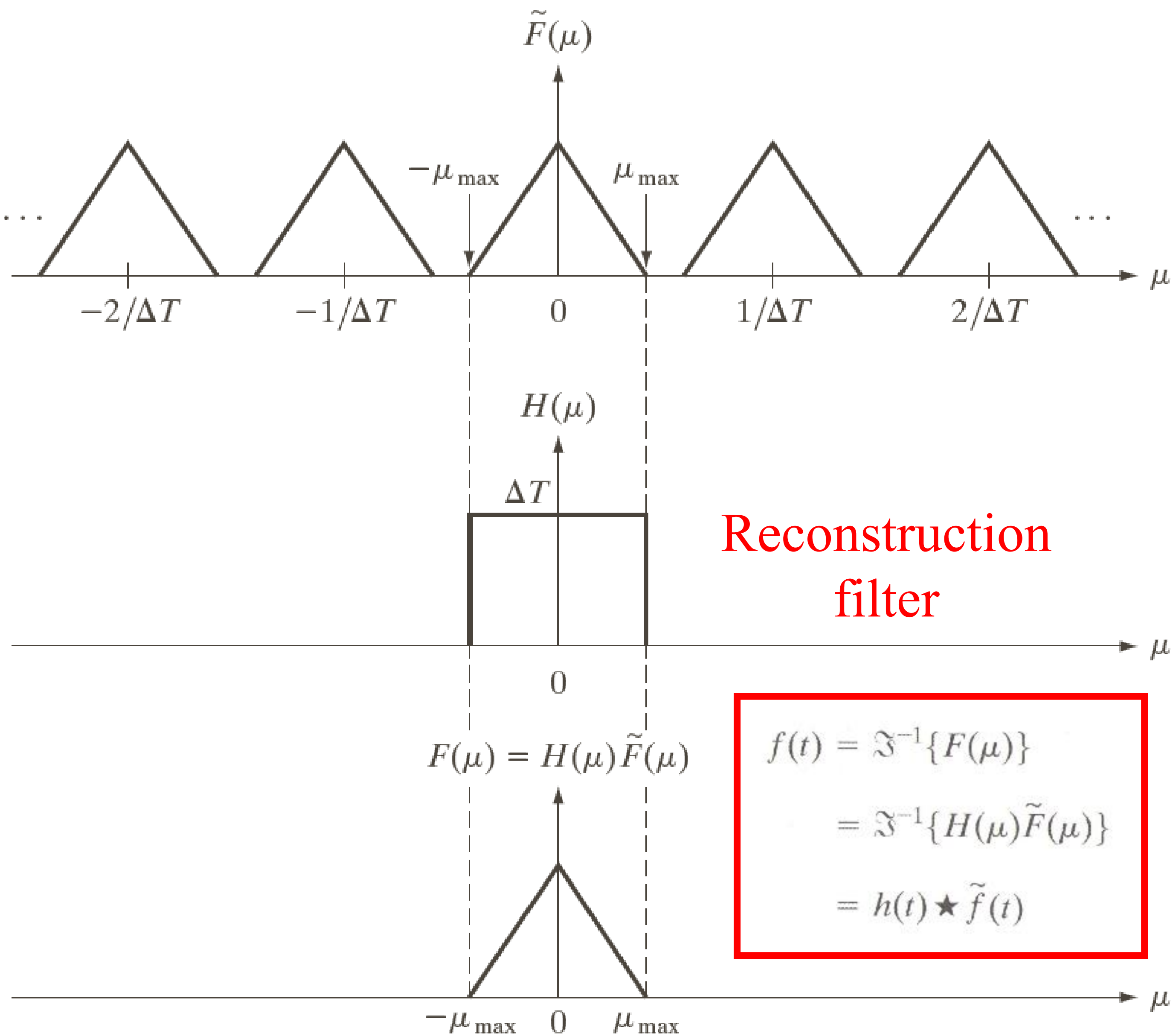
FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

Sampling Theorem

- Nyquist Rate: a sampling rate equal to exactly twice the highest frequency
- A continuous, band-limited function can be recovered completely from a set of its samples if the sampling rate exceeds the Nyquist rate.



a
b
FIGURE 4.7 (a) Transform of a band-limited function. (b) Transform resulting from critically sampling the same function.

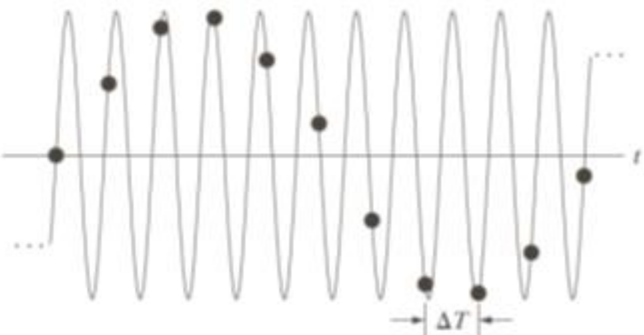


$$
\begin{aligned}
f(t) &= \Im^{-1}\{F(\mu)\} \\
&= \Im^{-1}\{H(\mu)\tilde{F}(\mu)\} \\
&= h(t) \star \tilde{f}(t)
\end{aligned}
$$

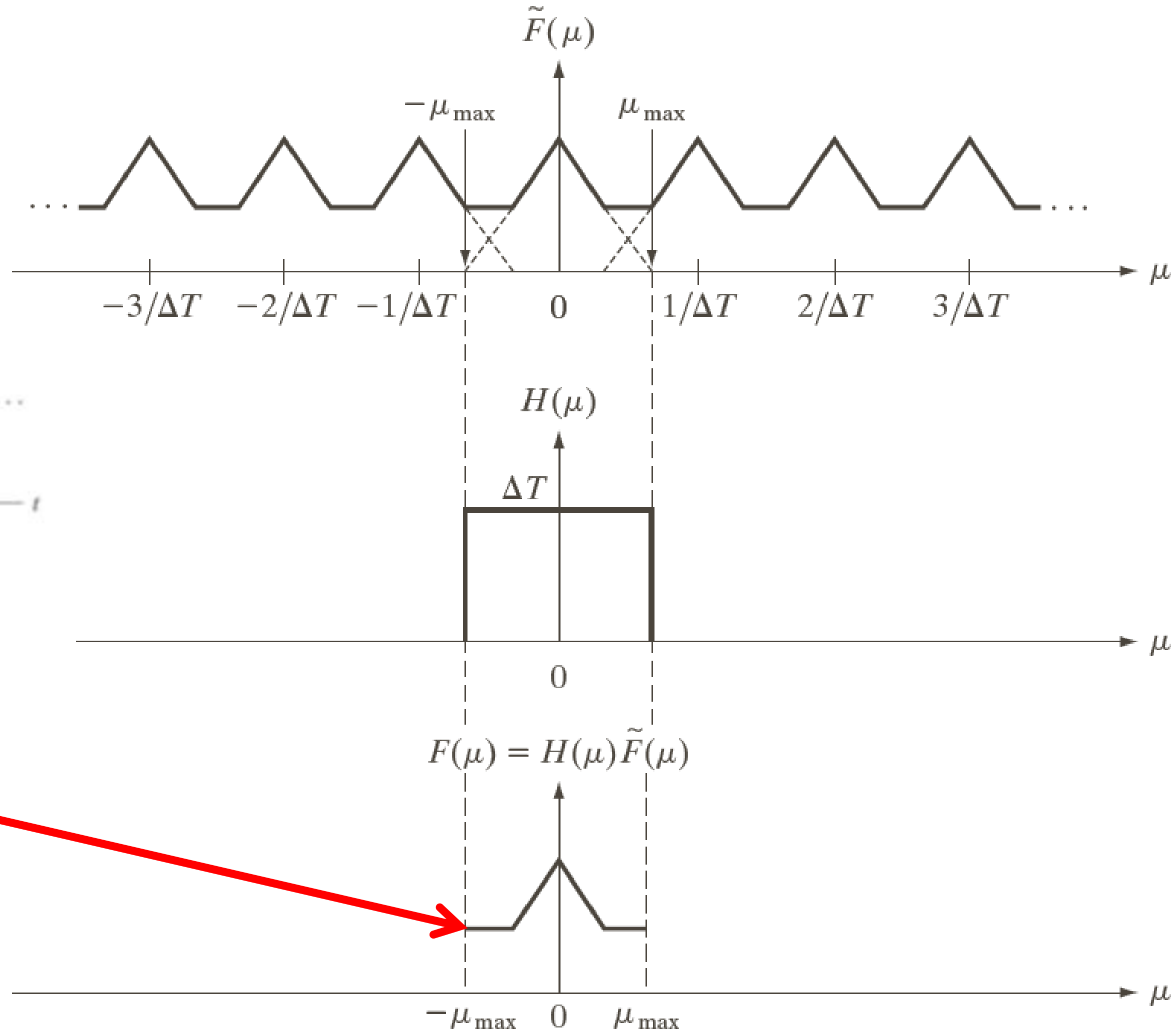
a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

$$
f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\,\mathrm{sinc}\left[(t - n\Delta T)/n\Delta T\right] \qquad (4.3\text{-}12)
$$



Aliasing

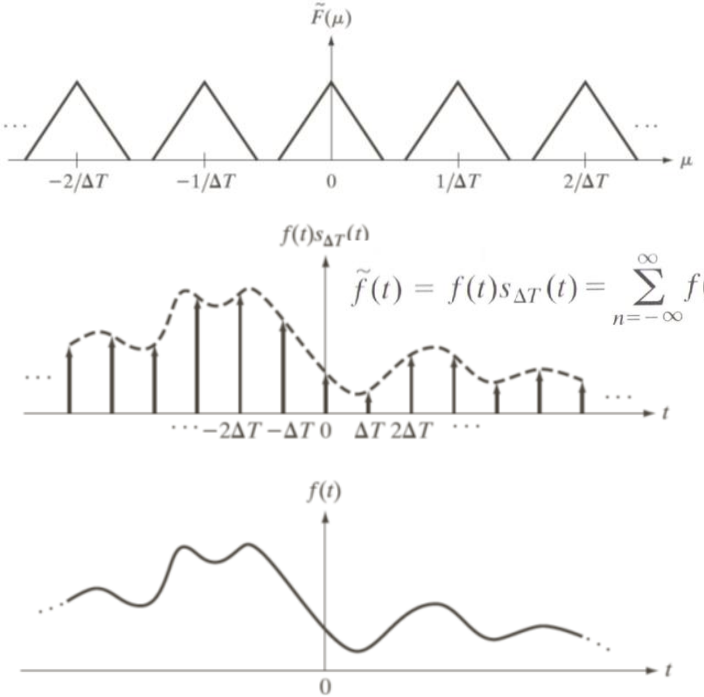


a
b
c

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Textbook
Section 4.4

Discrete Fourier Transform of One Variable



$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$\begin{aligned}
\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\,\Delta T) e^{-j2\pi\mu t}\, dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\,\delta(t - n\,\Delta T) e^{-j2\pi\mu t}\, dt \\
&= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}
\end{aligned}$$

Let $\mu = \dfrac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1 \qquad (4.4\text{-}3)$

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\,\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} f_n\, e^{-j2\pi\mu n\Delta T} \quad \Leftarrow \quad \mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

$$\boxed{\begin{aligned} F_m &= \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1 \qquad (4.4\text{-}4) \\ f_n &= \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{\,j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1 \qquad (4.4\text{-}5) \end{aligned}}$$

Discrete Fourier Transform Pair

1D DFT pair

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1 \qquad (4.4\text{-}6)$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1 \qquad (4.4\text{-}7)$$

* For the above Fourier Transform pair, (1/M) can be put in eq. (4.4-6)

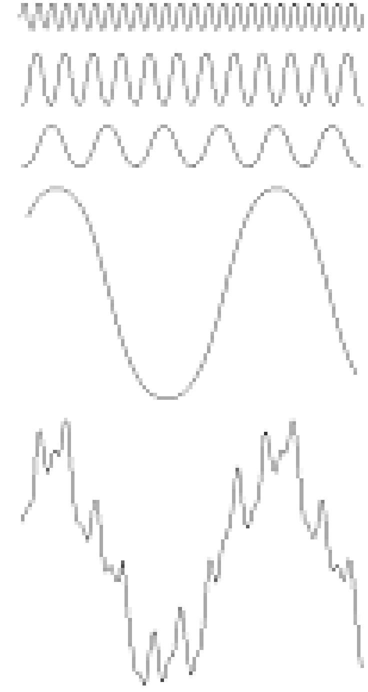
Discrete Fourier Transform (DFT)

Let $f(x) \triangleq f(x_0 + x\Delta x)$

$$F(u) \triangleq F(u\Delta u)$$

$$\left\{ \begin{array}{ll} F(u) = \dfrac{1}{M} \displaystyle\sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} & \text{for } u = 0, 1, 2, \ldots, M-1. \\ f(x) = \displaystyle\sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} & \text{for } x = 0, 1, 2, \ldots, M-1. \end{array} \right.$$

Mathematical prism



$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad \text{for } u = 0, 1, 2, \ldots, M-1.$$

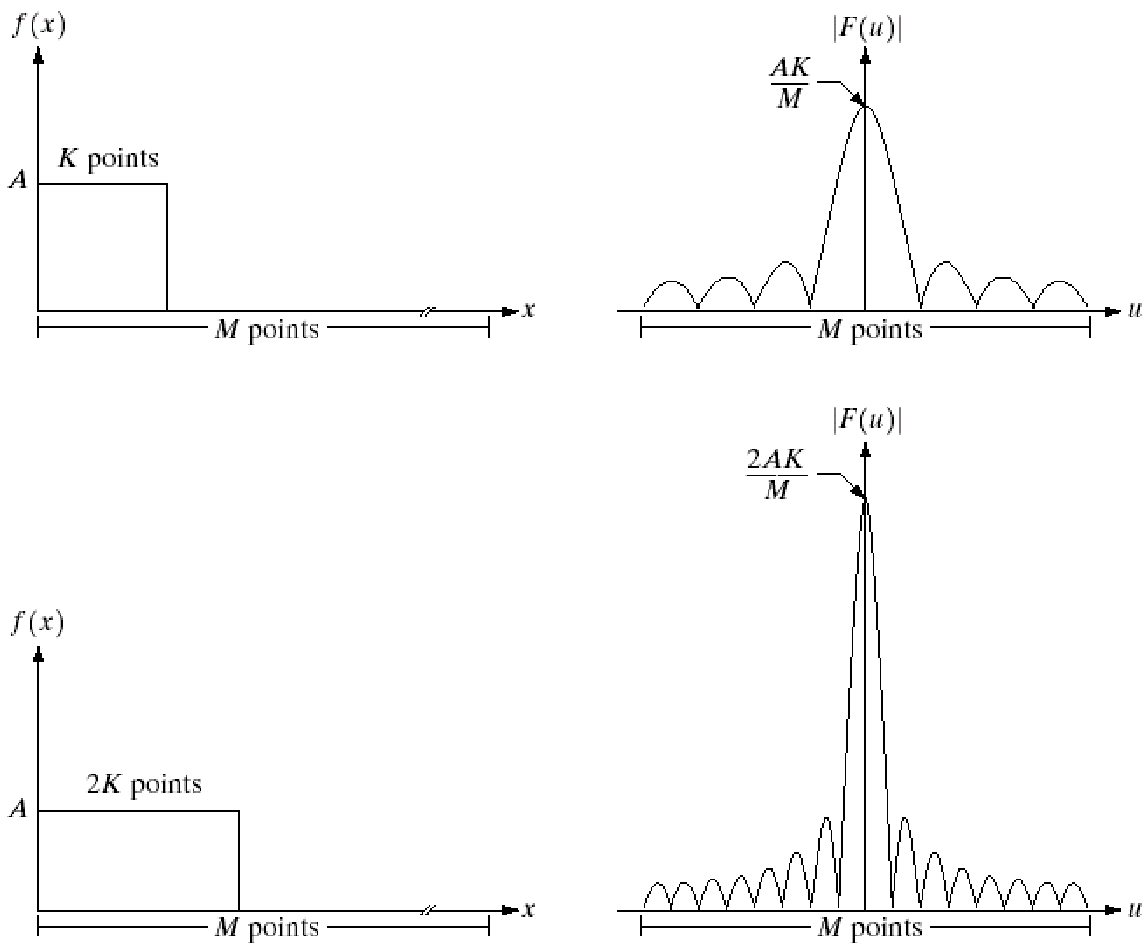
frequency

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos 2\pi ux/M - j\sin 2\pi ux/M]$$

frequency component

Fourier Spectrum | F(u) |

-- Inverse relationship between the frequency and spatial domains



a b
c d

FIGURE 4.2 (a) A discrete function of *M* points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Circular convolution

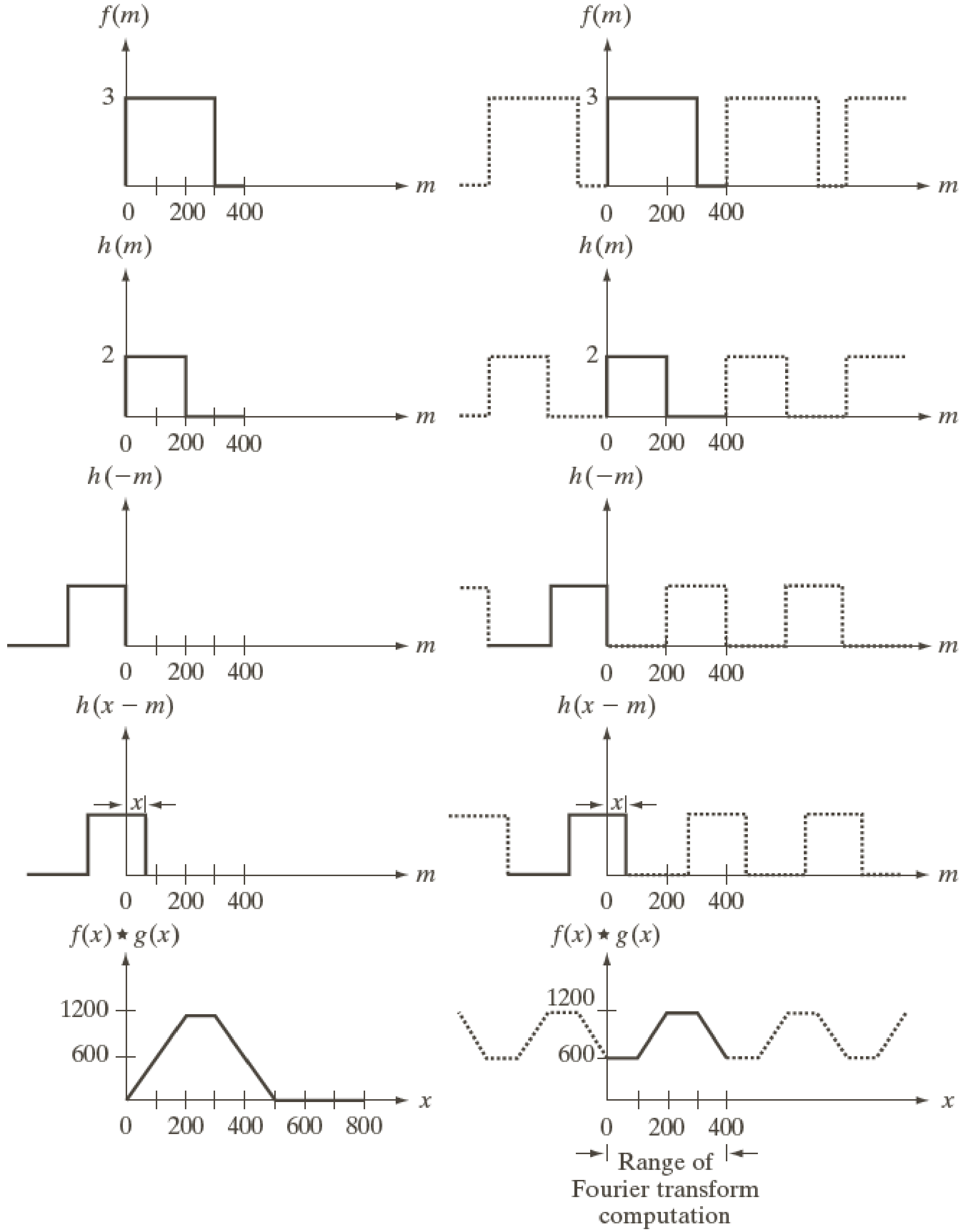
- DFT and IDFT are both infinitely periodic (not obvious)

$$F(u) = F(u + kM) \tag{4.4-8}$$

$$f(x) = f(x + kM) \tag{4.4-9}$$

- Circular Convolution

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m) \tag{4.4-10}$$



a f
b g
c h
d i
e j

FIGURE 4.28 Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

Solution: zero padding, Brigham [1988] (*textbook p. 273*)

2D Discrete Fourier Transform

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

Fourier Spectrum: $|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$

Phase Angle: $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$

Power Spectrum:

$$\begin{aligned} P(u,v) &= |F(u,v)|^2 \\ &= R^2(u,v) + I^2(u,v) \end{aligned}$$

2D Unit Discrete Impulse

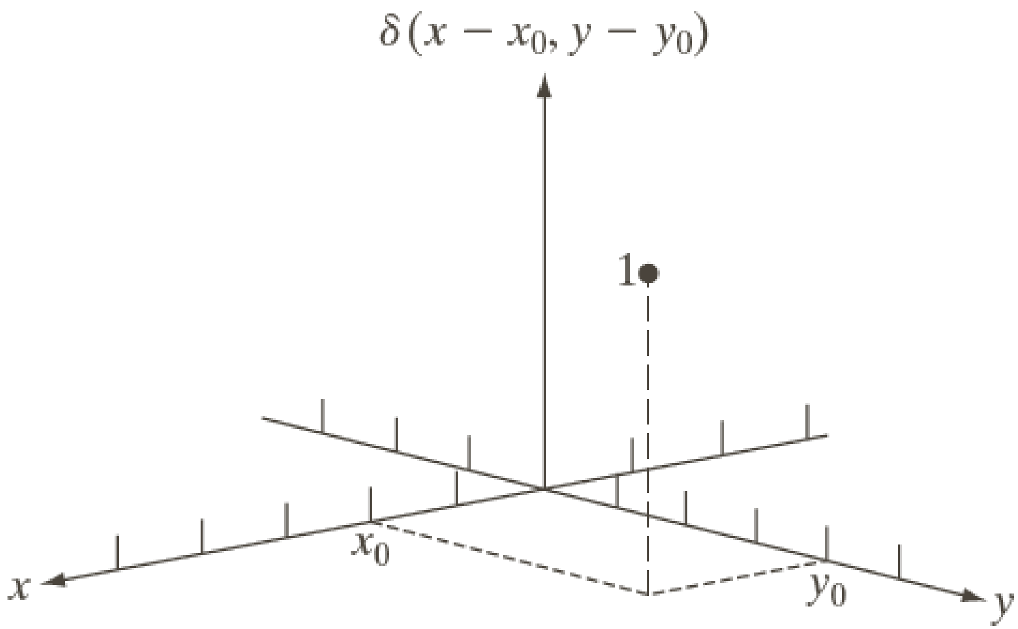


FIGURE 4.12 Two-dimensional unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0).

2D Sampling with 2D Impulse Train

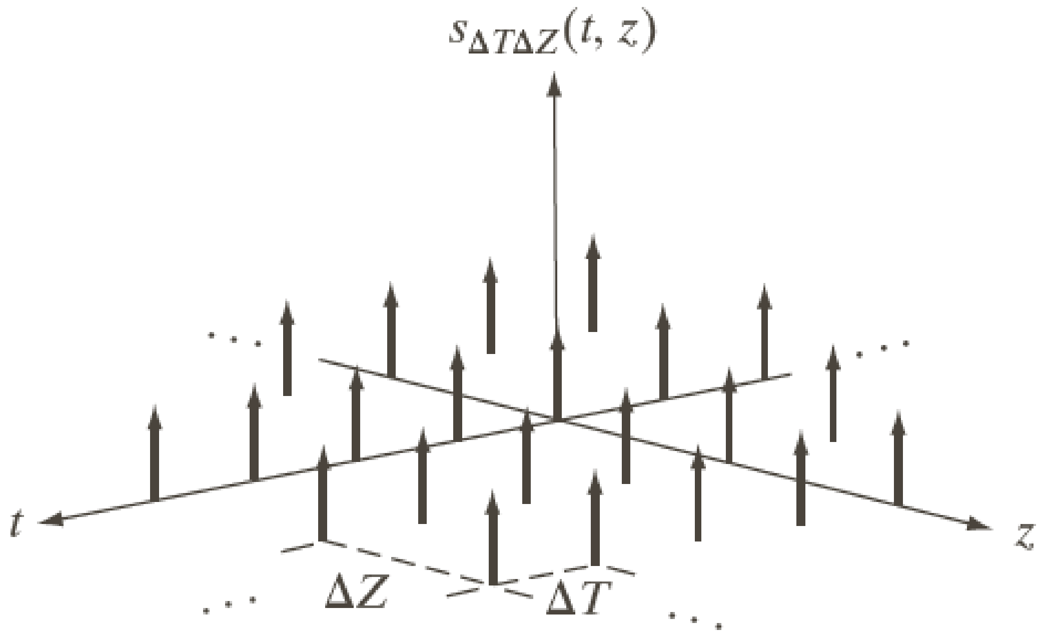
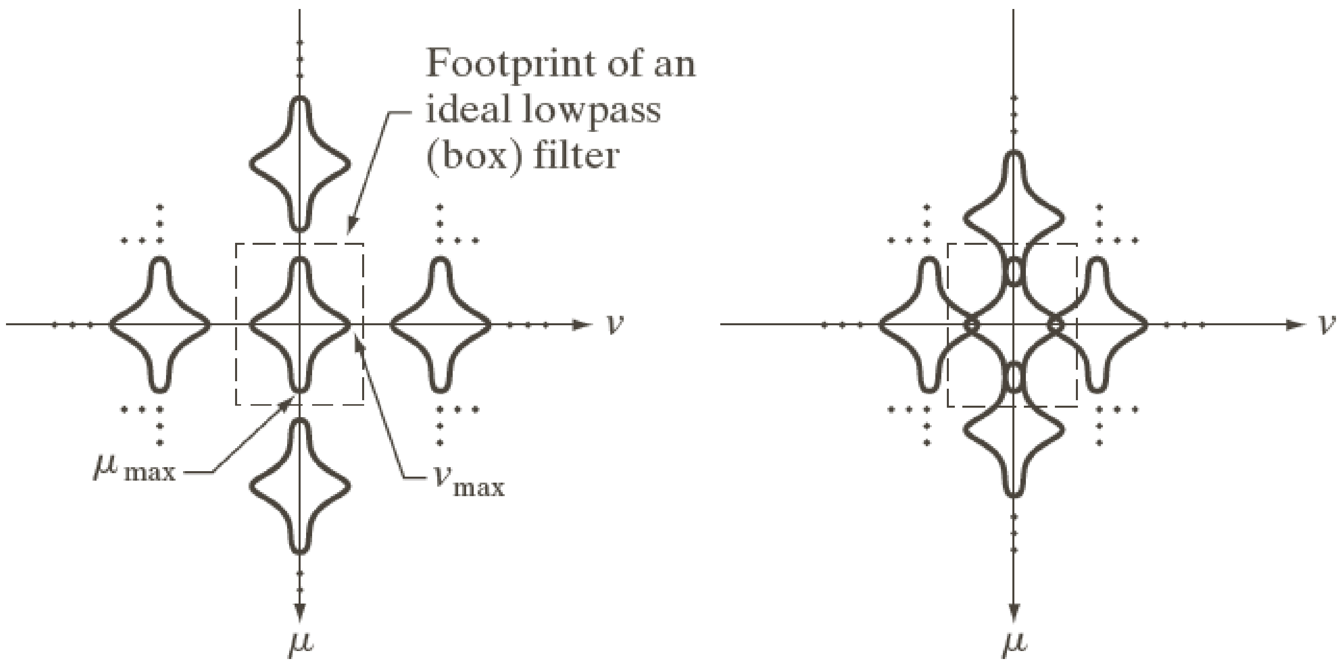


FIGURE 4.14
Two-dimensional impulse train.

Aliasing in Images



a b

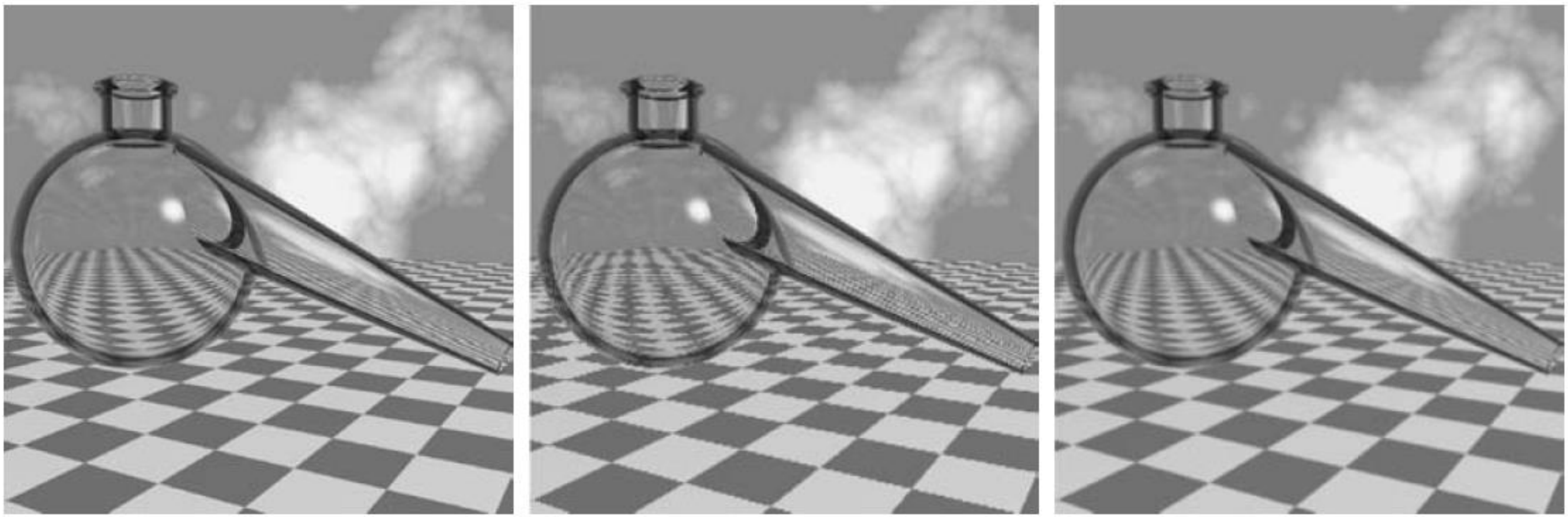
FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

Aliasing example

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Another aliasing example



a b c

FIGURE 4.18 Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)

Centering Property and Conjugate Symmetry

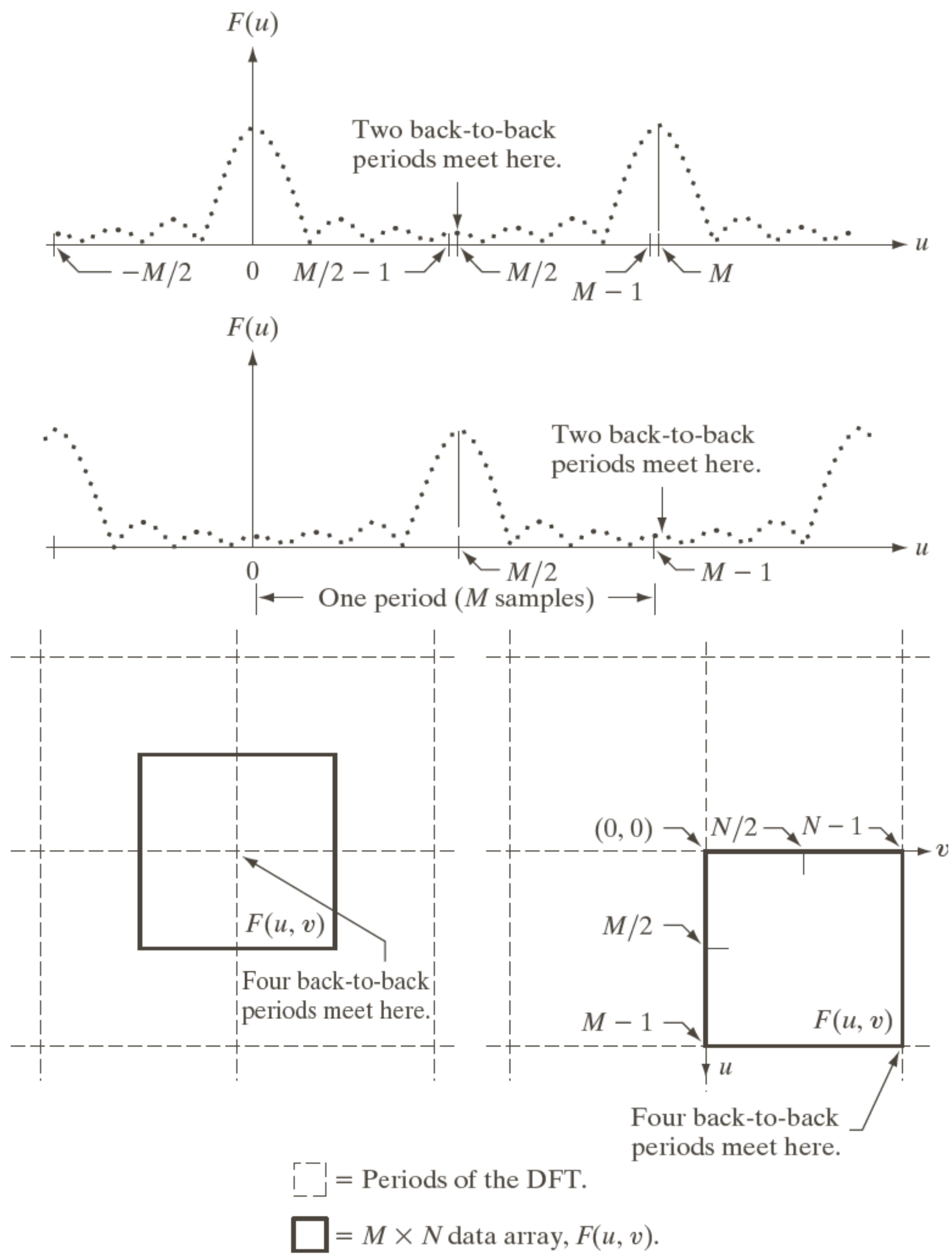
Centering Property (p.195, eq. 4.6-3)

$$\Im\left[f(x,y)(-1)^{x+y}\right] = F(u - M/2, v - N/2)$$

Conjugate Symmetry

If *f(x,y)* is real $\Rightarrow$ $F(u,v) = F^*(-u,-v)$

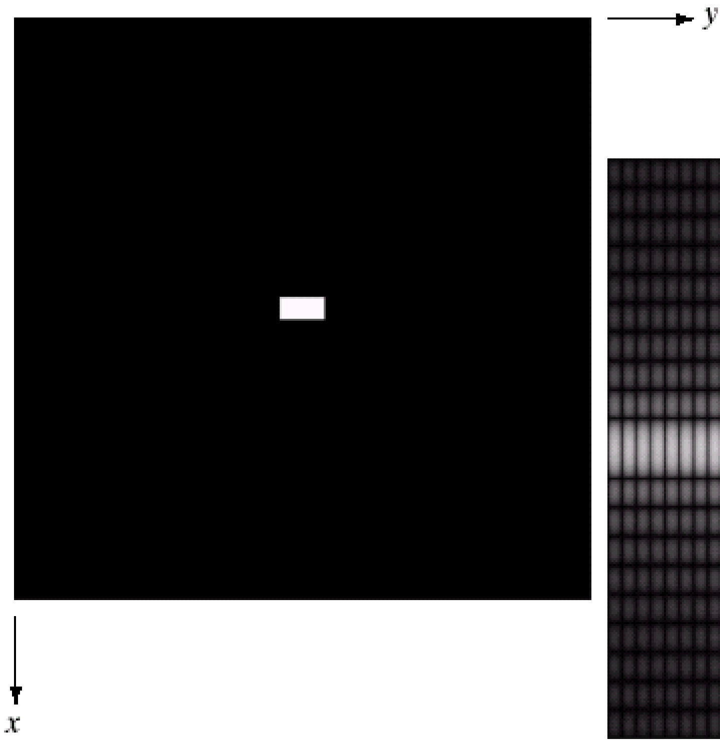
$\Rightarrow$ $|F(u,v)| = |F(-u,-v)|$

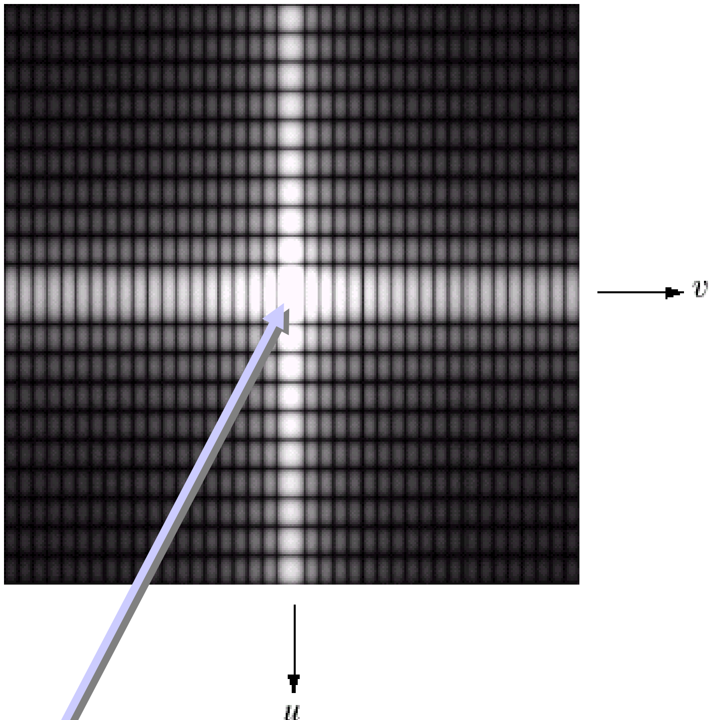


a
b
c d

FIGURE 4.23 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The solid area is the $M \times N$ data array, $F(u, v)$, obtained with Eq. (4.5-15). This array consists of four quarter periods. (d) A Shifted DFT obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ before computing $F(u, v)$. The data now contains one complete, centered period, as in (b).

Centered Fourier Spectrum

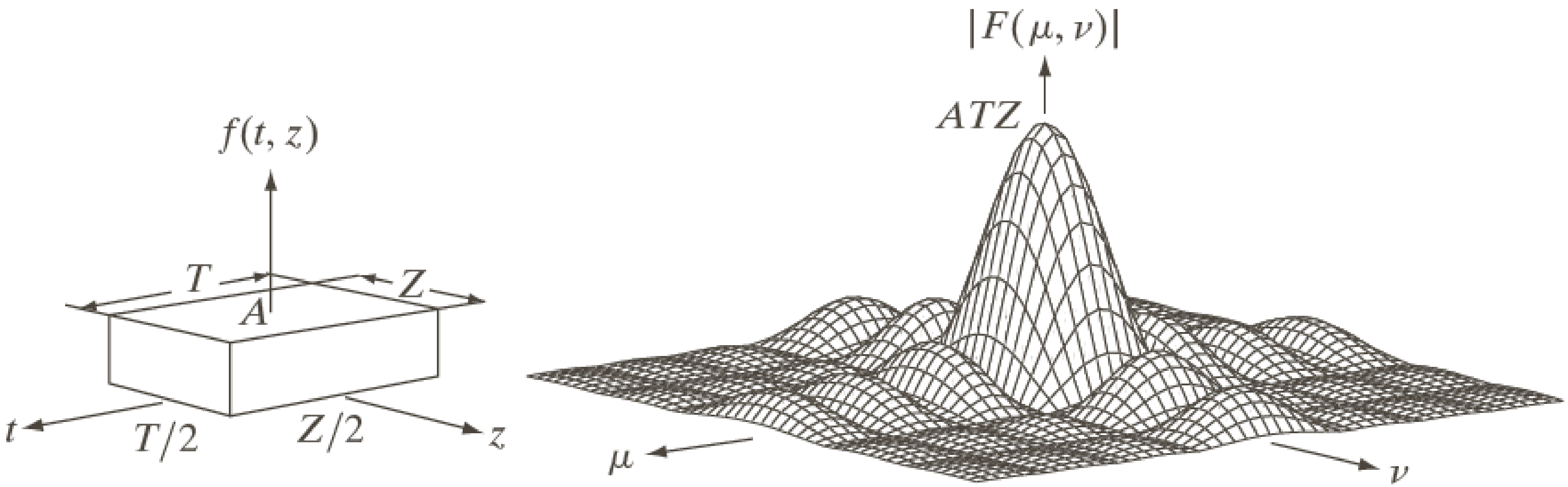




a b

FIGURE 4.3 (a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$



a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

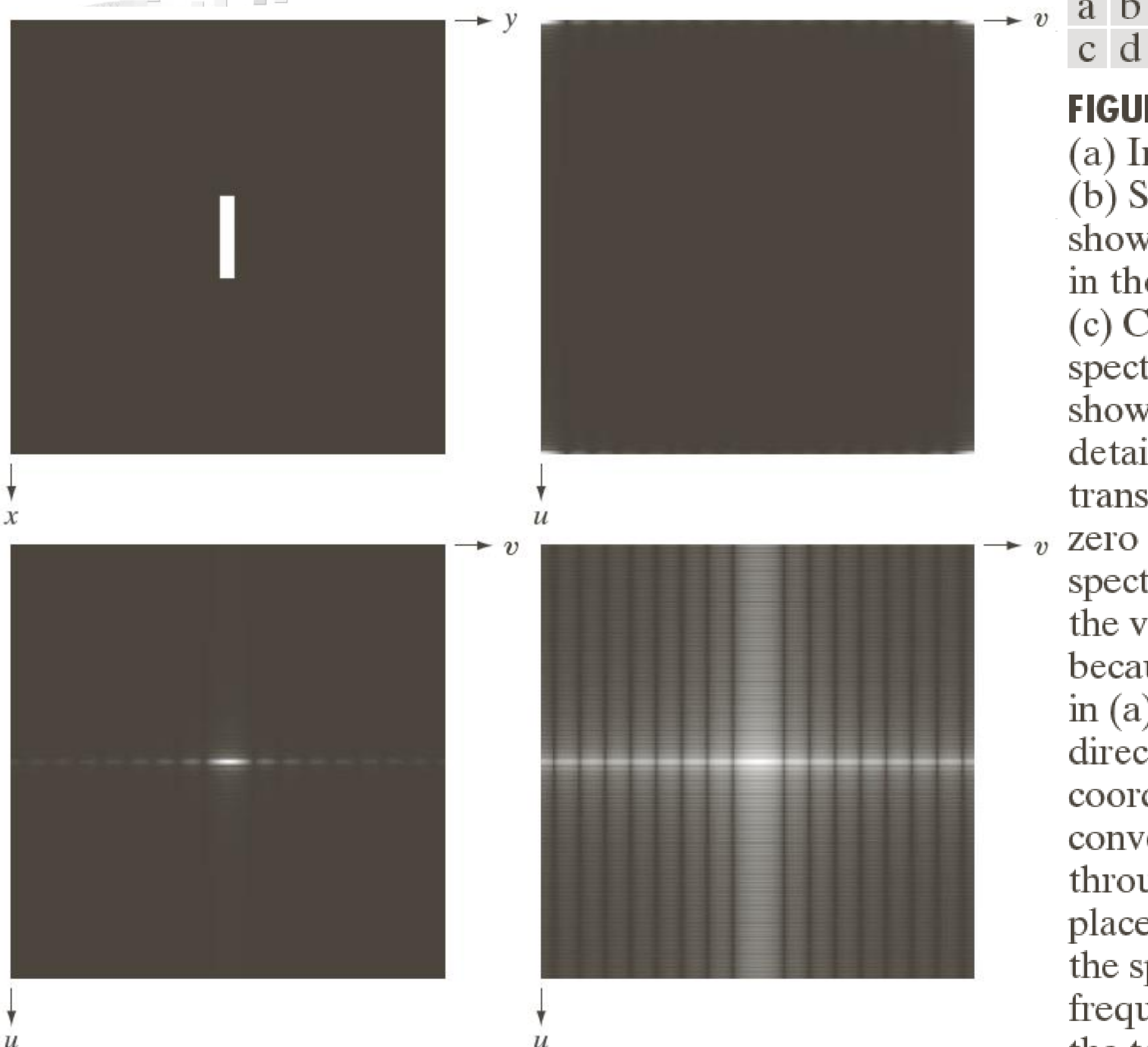
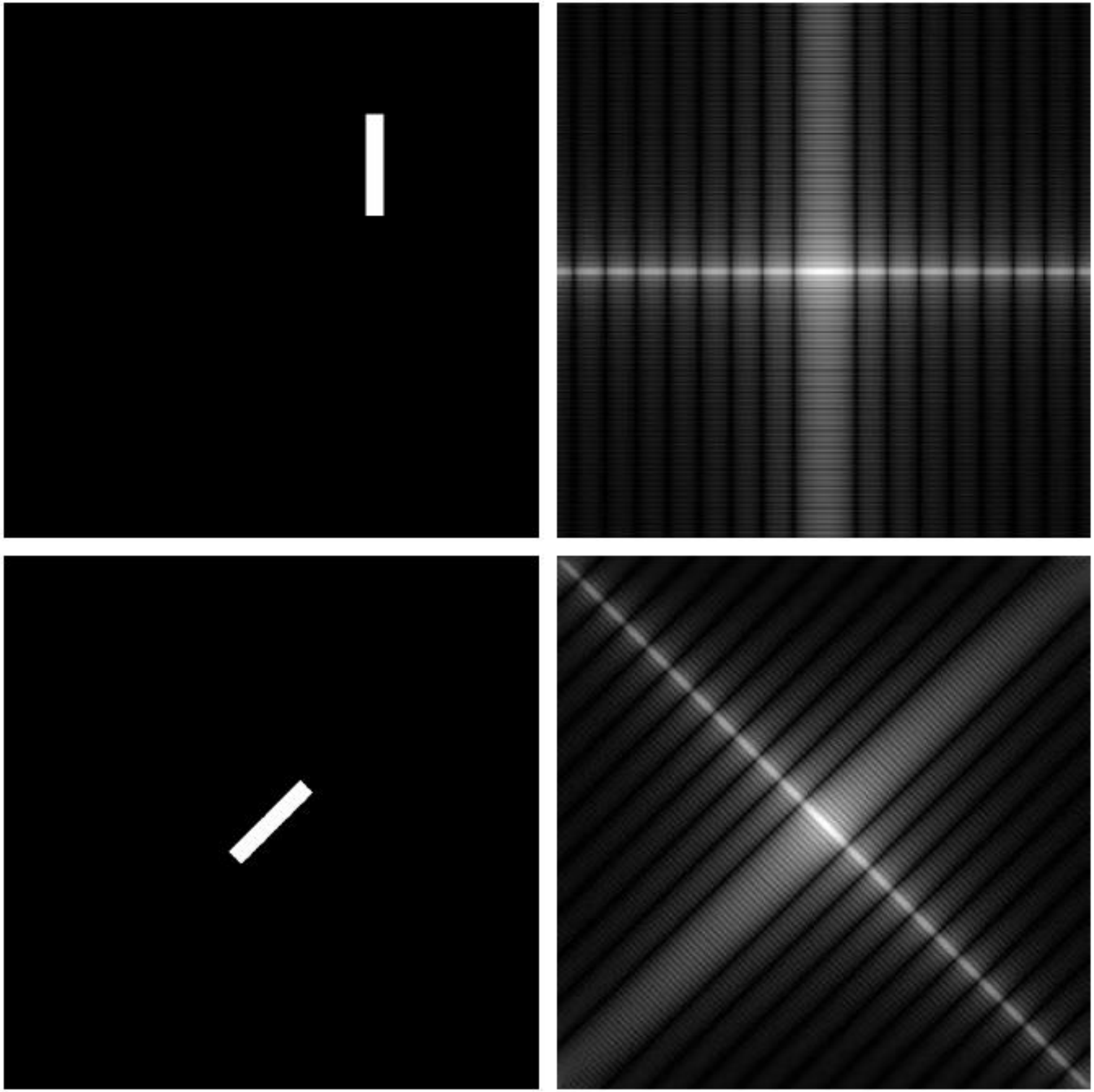
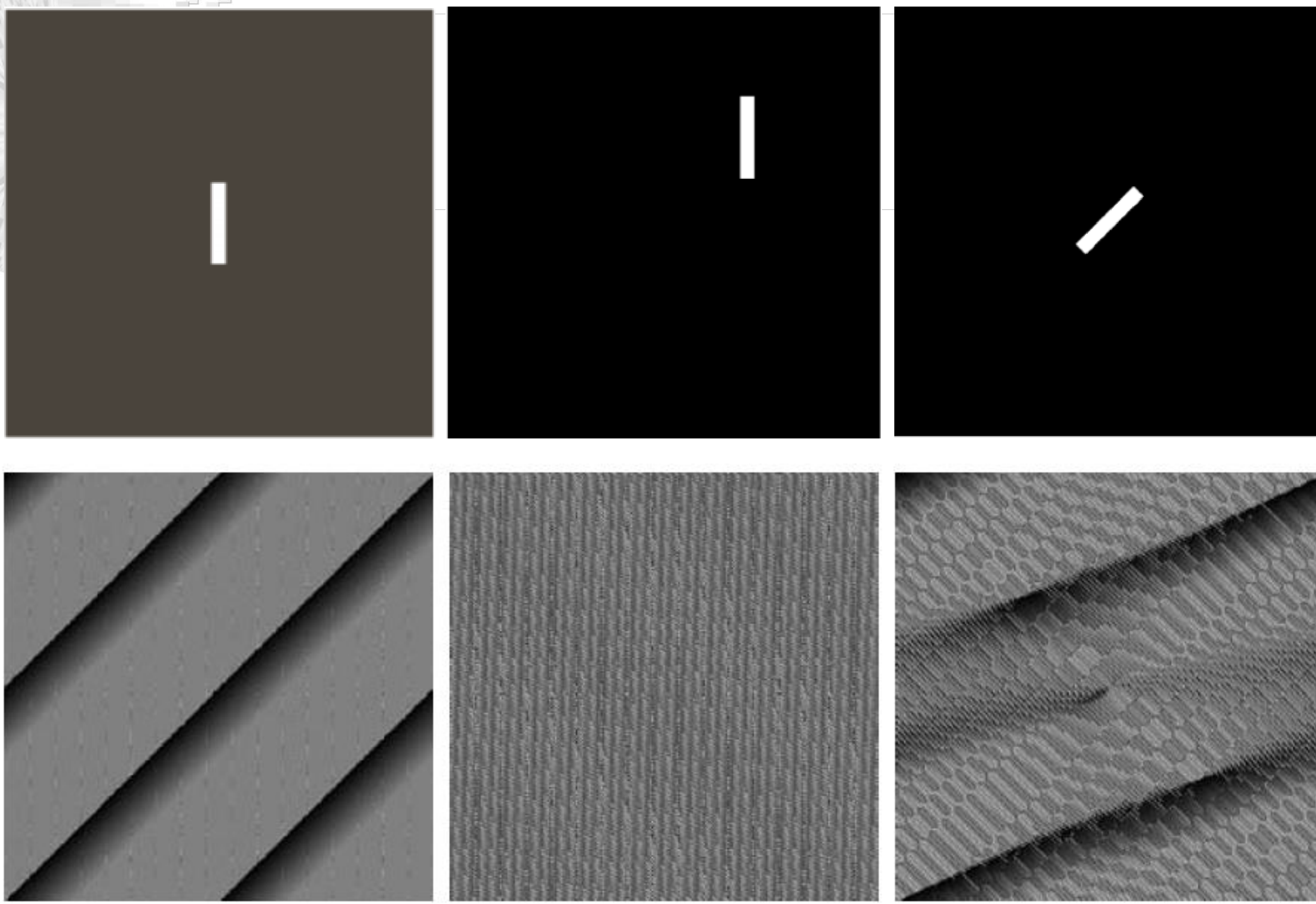


FIGURE 4.24
(a) Image.
(b) Spectrum showing bright spots in the four corners.
(c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.



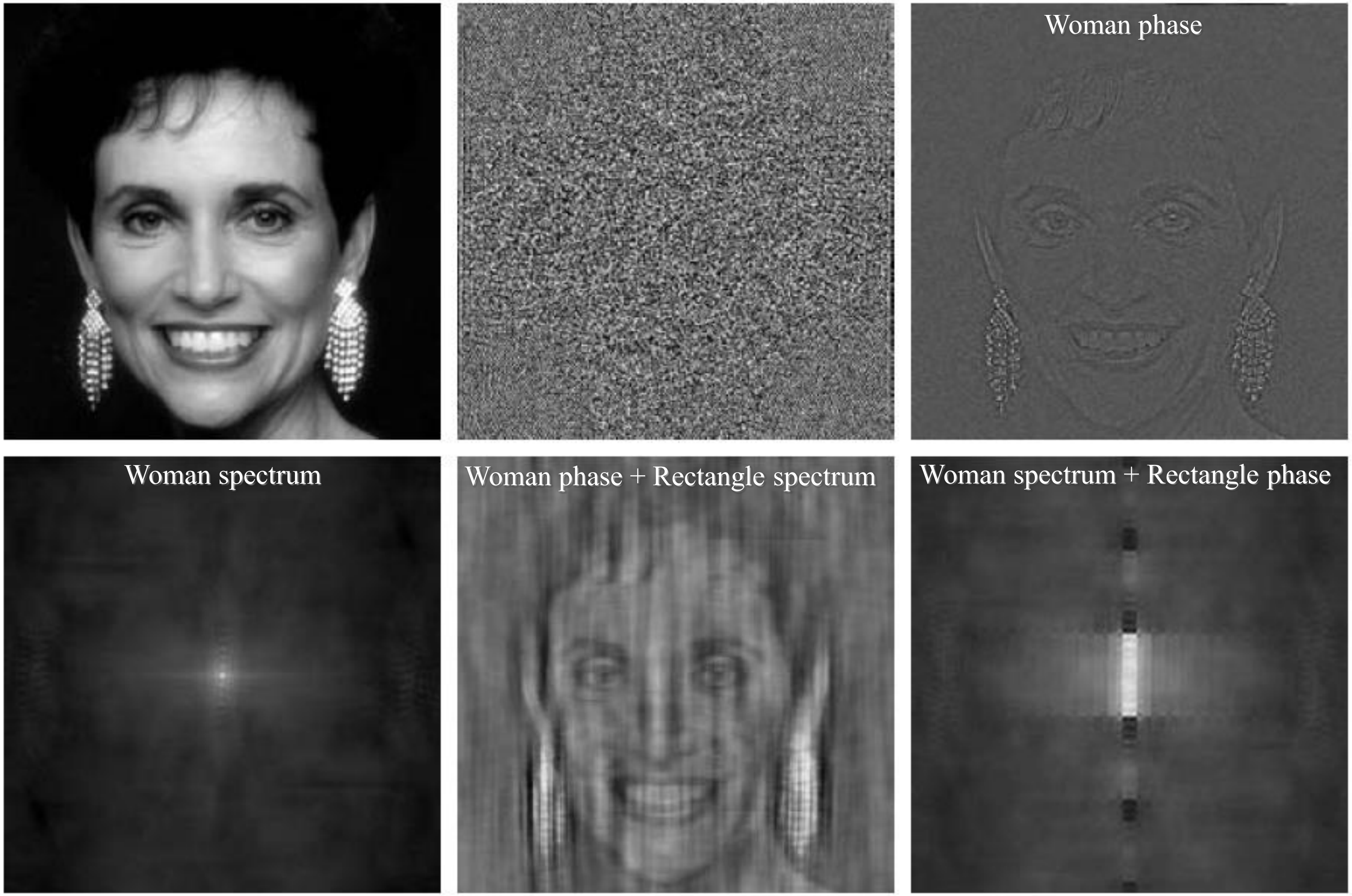
a	b
c	d

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).



a b c

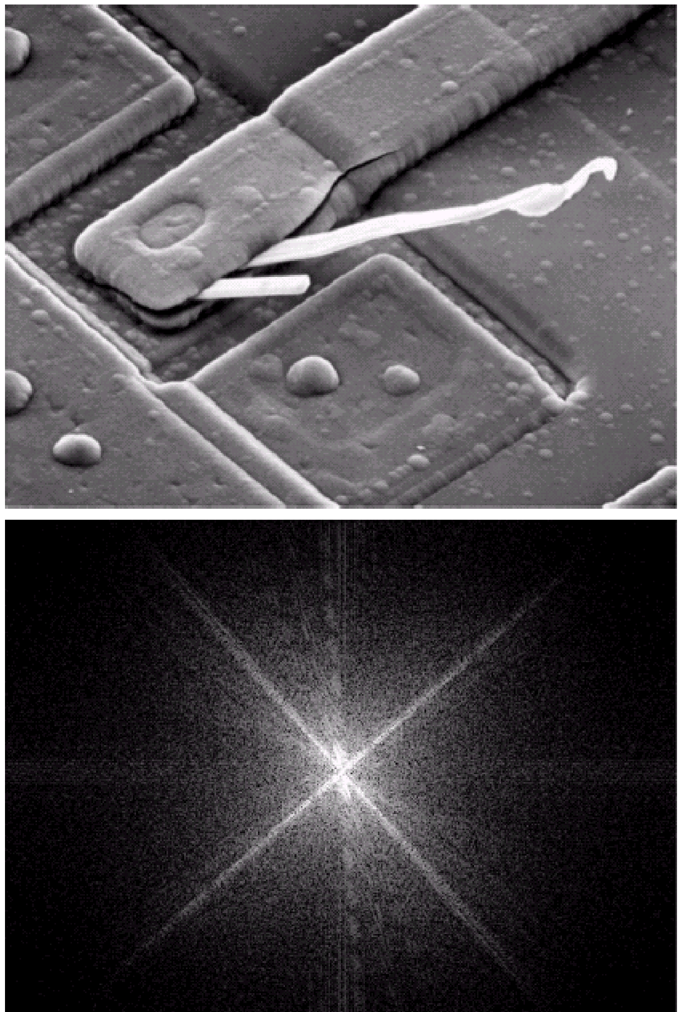
FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).



a	b	c
d	e	f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

Image Features and Fourier Spectrum

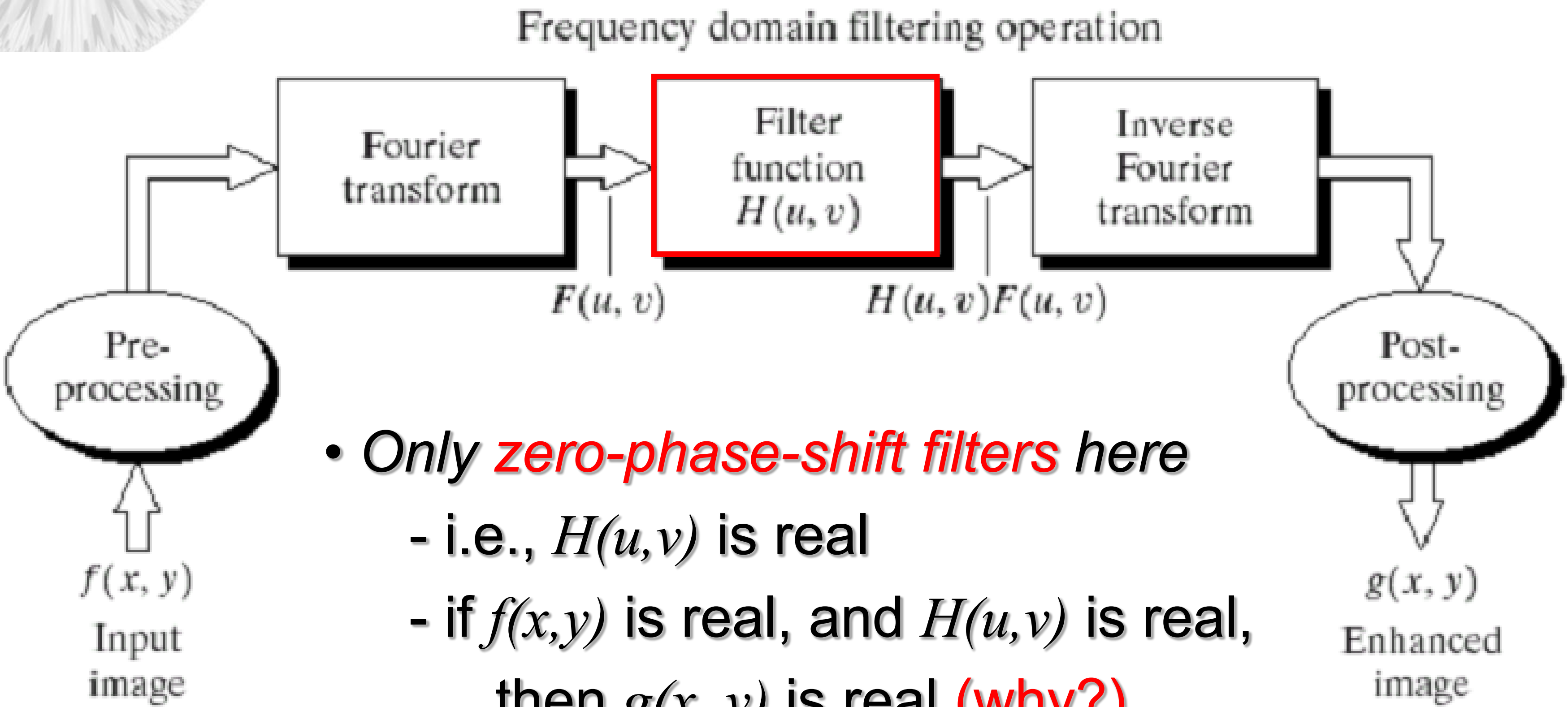


a
b

FIGURE 4.4 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Filtering in the Frequency Domain

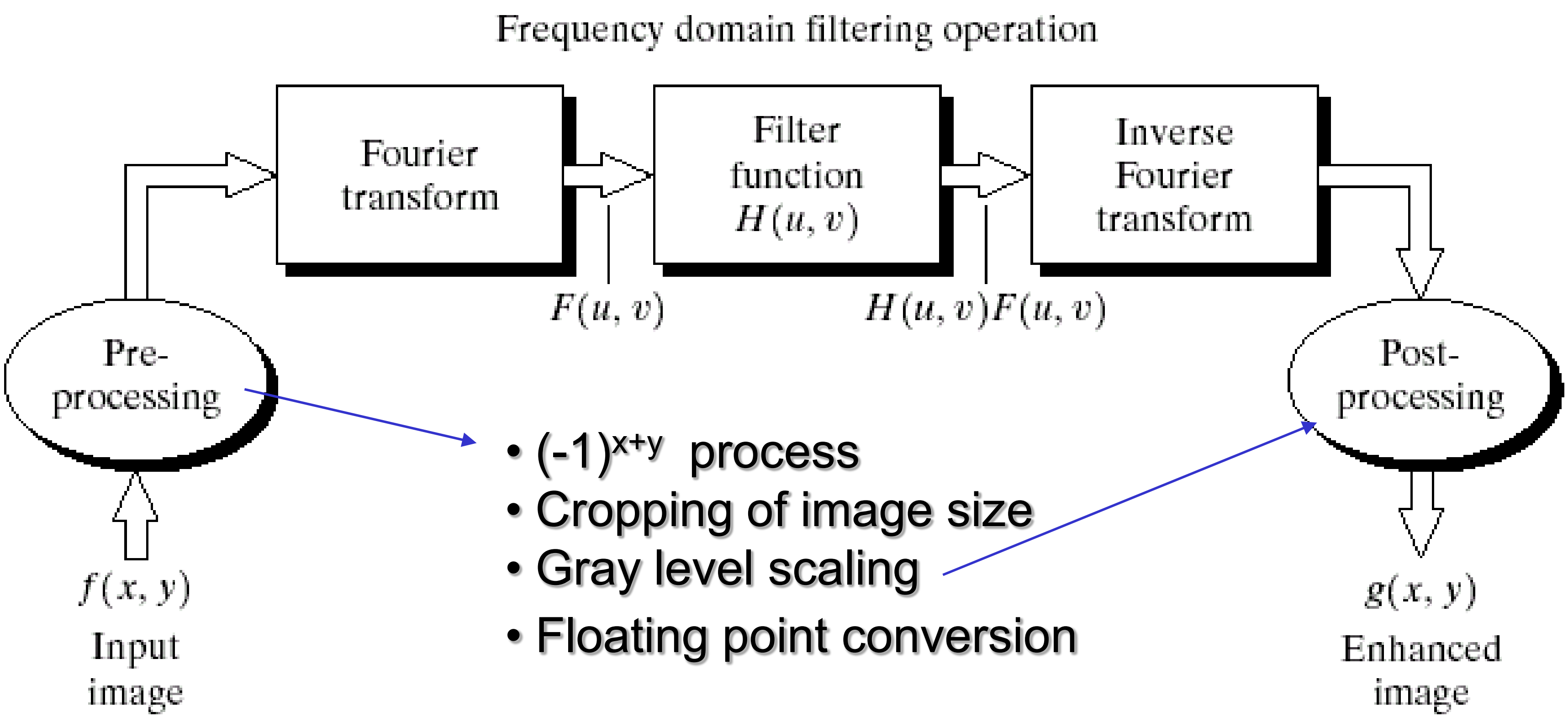
- Notch Filters: chap 5
- Lowpass Filters: chap 4
- Highpass Filters: chap 4



- *Only* *zero-phase-shift filters* *here*
  - i.e., $H(u,v)$ is real
  - if $f(x,y)$ is real, and $H(u,v)$ is real, then $g(x, y)$ is real (why?)

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

Preprocessing before frequency domain filtering



- $(-1)^{x+y}$ process
- Cropping of image size
- Gray level scaling
- Floating point conversion

FIGURE 4.5 Basic steps for filtering in the frequency domain.

Lowpass and Highpass

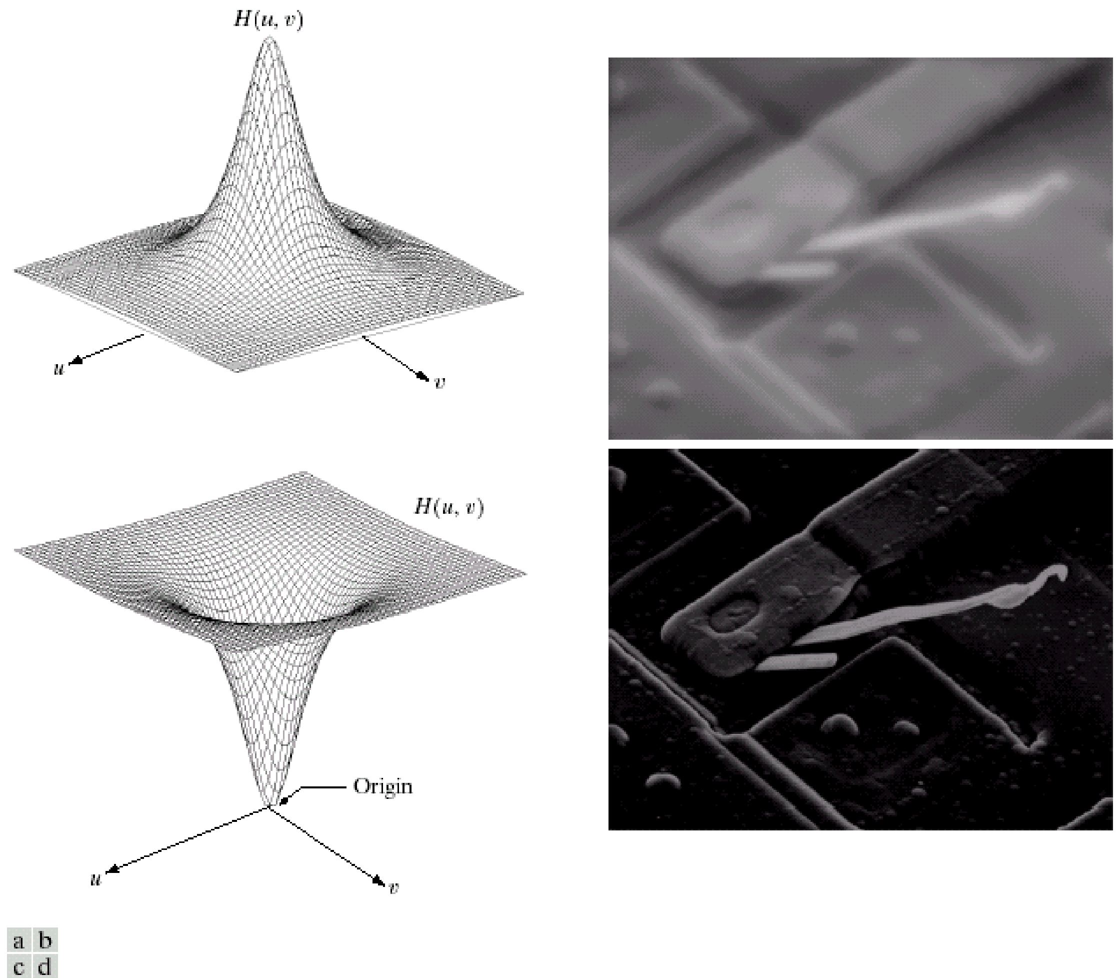
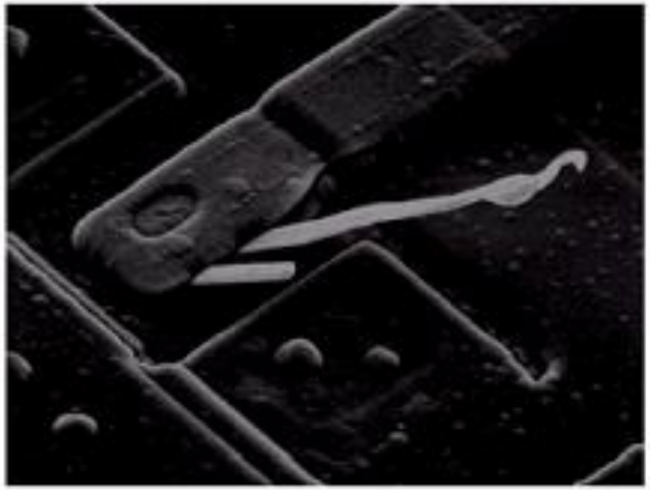


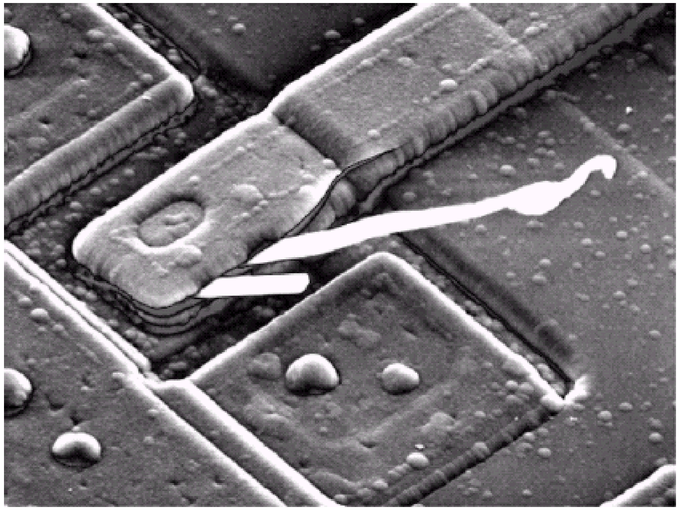
FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Modified Highpass

Highpass



Modified Highpass



-- add a constant of one-half the filter height to the filter to avoid zero $F(0,0)$

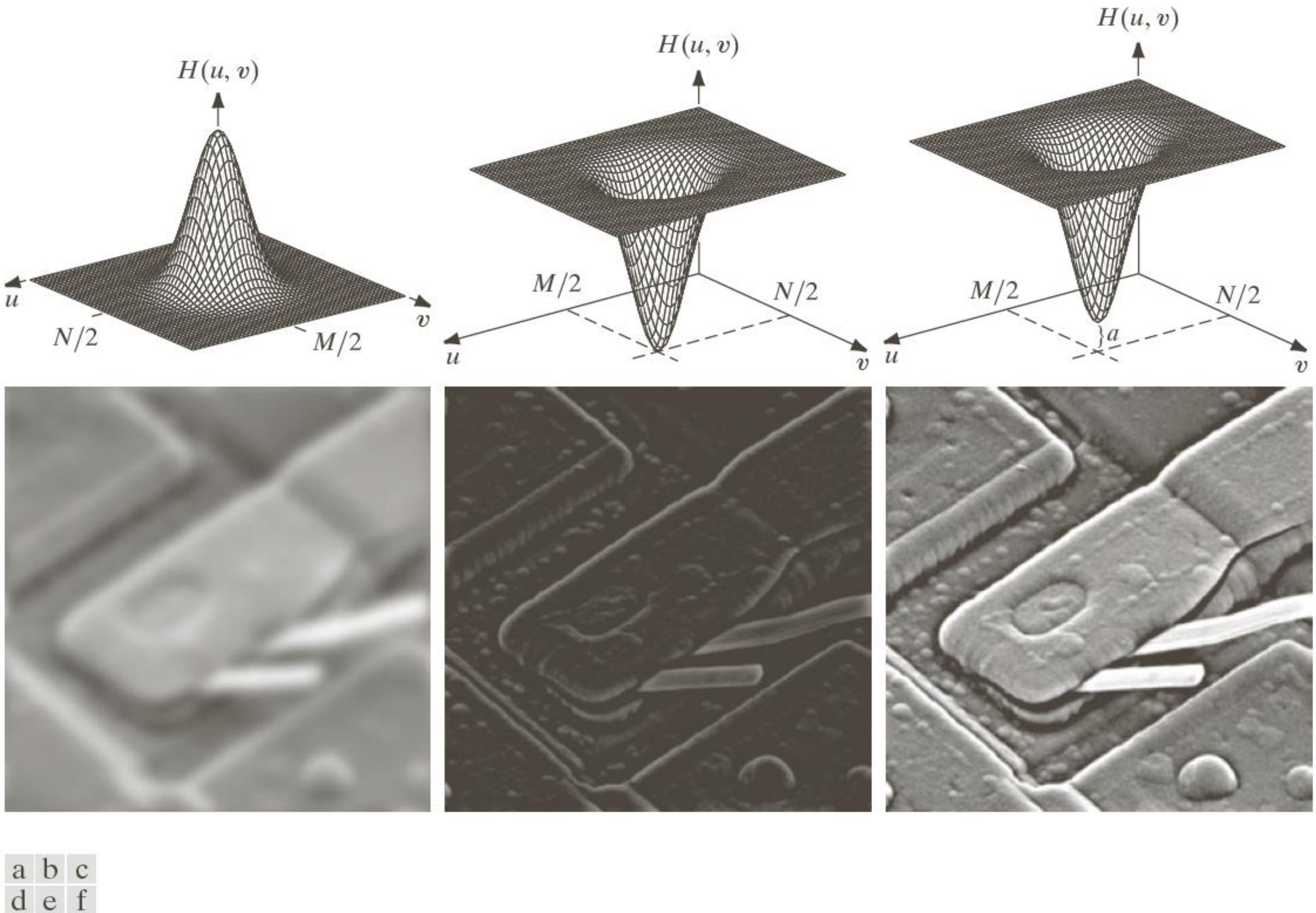


FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

Discrete Convolution

Remind -- Spatial Filtering (chapter 3):

$$g(x, y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s, t) f(x + s, y + t)$$

Discrete Convolution:

$$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$$

-- *Flipping, Shifting, Sum of product*

Review: Linear Systems

two-dimensional impulse function

$$\delta(m-p, n-q) = \begin{cases} 1, & m = p \text{ and } n = q \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Any discrete-space image f may be expressed in terms of the impulse function in Eq. (1):

$$f(m, n) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} f(p, q)\delta(m-p, n-q) \quad (2)$$

A two-dimensional system **L** is a process of image transformation, as shown in Fig. 1.

We can write

$$g(m, n) = \mathbf{L}[f(m, n)]. \tag{3}$$

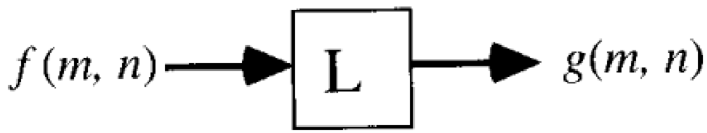


FIGURE 1 Two-dimensional input–output system.

The system **L** is *linear* if and only if for any $f_1(m, n)$, $f_2(m, n)$ such that

$$g_1(m, n) = \mathbf{L}[f_1(m, n)], \qquad g_2(m, n) = \mathbf{L}[f_2(m, n)] \tag{4}$$

and any two constants a, b, then

$$ag_1(m, n) + bg_2(m, n) = \mathbf{L}[af_1(m, n) + bf_2(m, n)] \tag{5}$$

for every (m, n). This is often called the *superposition property* of linear systems.

The system **L** is *shift invariant* if for every $f(m, n)$ such that Eq. (3) holds, then also

$$g(m-p, n-q) = \mathbf{L}[f(m-p, n-q)] \qquad (6)$$

for any (p, q). Thus, a spatial shift in the input to **L** produces nc change in the output, except for an identical shift.

The *unit impulse response* of a two-dimensional input–output system **L** is

$$\mathbf{L}[\delta(m-p, n-q)] = h(m, n; p, q). \tag{7}$$

This is the response of system **L**, at spatial position (m, n), to an impulse located at spatial position (p, q). Generally, the impulse response is a function of these four spatial variables. However, if the system **L** is space invariant, then if

$$\mathbf{L}[\delta(m, n)] = h(m, n) \tag{8}$$

is the response to an impulse applied at the spatial origin, then also

$$\boxed{\mathbf{L}[\delta(m - p, n - q)] = h(m - p, n - q),} \tag{9}$$

which means that the response to an impulse applied at any spatial position can be found from the impulse response in Eq. (8).

Consider the generic system **L** shown in Fig. 1, with input $f(m, n)$ and output $g(m, n)$. Assume that the response is due to the input f only (the system would be at rest without the input). Then, from Eq. (2):

$$\begin{aligned} g(m, n) &= \mathbf{L}[f(m, n)] \\ &= \mathbf{L}\left[\sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} f(p, q)\delta(m-p, n-q)\right] \end{aligned} \tag{10}$$

If the system is known to be linear, then

$$g(m, n) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} f(p, q)\mathbf{L}[\delta(m-p, n-q)] \tag{11}$$

$$= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} f(p, q)h(m, n; p, q), \tag{12}$$

which is all that generally can be said without further knowledge of the system and the input. If it is known that the system is space invariant (hence LSI), then Eq. (12) becomes

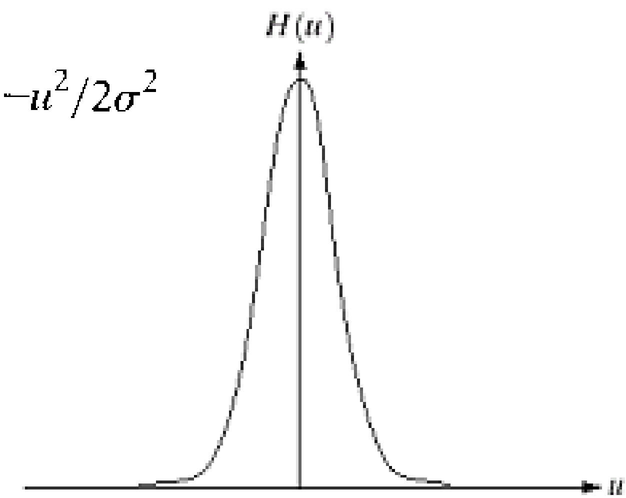
$$g(m, n) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} f(p, q)h(m-p, n-q) \tag{13}$$

$$= f(m, n) * h(m, n), \tag{14}$$

which is the two-dimensional discrete space linear convolution of input f with impulse response h.

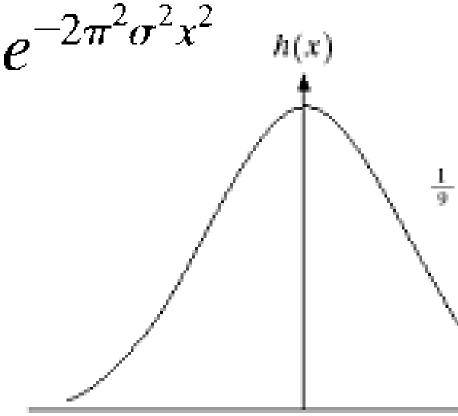
Gaussian Filter Functions

$$H(u) = Ae^{-u^2/2\sigma^2}$$



Notice that all the values are positive in both domain

$$h(x) = \sqrt{2\pi}\sigma Ae^{-2\pi^2\sigma^2x^2}$$



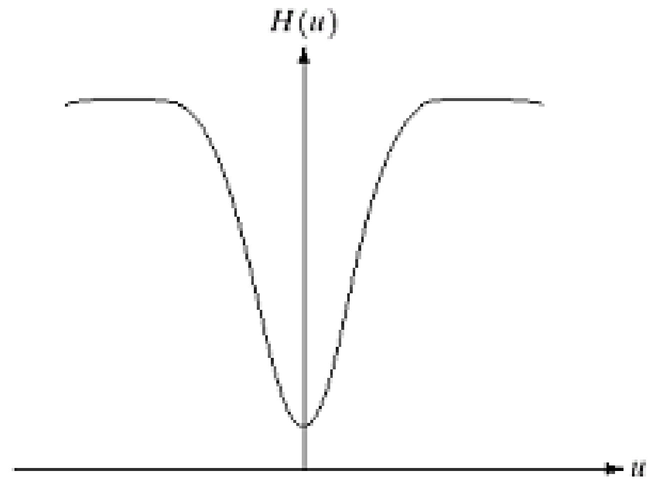
$$\frac{1}{9} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \frac{1}{16} \times \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

→ smoothing masks used in section 3.4.1 are reasonable

→ reciprocal relationship

Highpass using a difference of Gaussians

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$



$$h(x) = \sqrt{2\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2 x^2}$$