MSRI UP 2023 - HW 1 - DAY 4

Due: Saturday 6/17 by 7:00am. Submit your assignment to your personal dropbox.

Important: You don't have to complete the entire assignment, but do as much of it as possible. Make sure to present your (partial) solutions in a neat and organized manner. Discuss the problems with at least two other people, but write your solutions on your own.

(1) Let $L \in \mathbb{N}$ and let $f(t) = \sin(Lt)$ for $t \in \mathbb{R}$. Recall that the sliding window of f at t, with parameters $d \in \mathbb{N}$ and $\tau > 0$, is given by the vector

$$SW_{d,\tau}f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ \vdots \\ f(t+d\tau) \end{bmatrix} \in \mathbb{R}^{d+1}$$

The goal of this problem is to determine optimal choices for d and τ so that the sliding window point cloud

$$\mathbb{SW}_{d,\tau}f = \left\{ SW_{d,\tau}f(t) \ : \ t \in [0, 2\pi) \right\}$$

has maximal persistence in dimension 1. That is, we seek to maximize the quantity

$$mp_1 = \max\{b - a \ : \ (a,b) \in \mathsf{dgm}_1(\mathcal{R}(\mathbb{SW}_{d,\tau}f))\} \tag{1}$$

(a) If $f(t) = \sin(Lt)$, show that there exist vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{d+1}$, which do not depend on t, and so that

$$SW_{d,\tau}f(t) = \sin(Lt)\mathbf{u} + \cos(Lt)\mathbf{v} \tag{2}$$

(b) Note that if the vectors from part (a) are linearly independent, then the curve given by equation (2) traces a planar ellipse. Determine the minimum value of d needed for $\mathbf{u}$ and $\mathbf{v}$ to be linearly independent.

(c) Please provide an argument for the following fact: Of all the planar ellipses E with fixed perimeter ℓ, the circle (i.e., the ellipse with equal major and minor axes) has maximal 1-persistence (i.e., the largest $mp_1 = \max\{b - a \ : \ (a,b) \in \mathsf{dgm}_1(\mathcal{R}(E))\}$).

(d) Determine values of τ, as a function of L and d, which maximize mp_1 in equation (1).

(2) Let (X, d_X) be a metric space, let $\mathcal{C}_X$ be the set of Cauchy sequences in X, and let $\sim$ be the equivalence relation in $\mathcal{C}_X$ given by $\underline{a} = \{a_n\}_{n\in\mathbb{N}} \sim \{b_n\}_{n\in\mathbb{N}} = \underline{b}$ if and only if $\lim_{n\to\infty} d_X(a_n, b_n) = 0$.

(a) Show that
$$\tilde{d}_{\mathcal{C}_X}([\underline{a}],[\underline{b}]) = \lim_{n\to\infty} d_X(a_n,b_n)$$
defines a metric in the space of equivalence classes $\mathcal{C}_X/\sim = \{[\underline{a}] \ : \ \underline{a}\in\mathcal{C}_X\}$.

(b) Show that $(\mathcal{C}_X/\sim, \tilde{d}_{\mathcal{C}_X})$ is a complete metric space.

(c) Show that there is an isometric embedding $\varphi : X \longrightarrow \mathcal{C}_X/\sim$, so that the closure of its image — i.e., $\overline{\varphi(X)}$ — is equal to $\mathcal{C}_X/\sim$.