3.2. **Beyond the Space of Planar Unilateral Pentagons.** The space of unilateral planar pentagons sampled in `mystery data_3` is the quotient

$$M_5(1,1,1,1,1) = \left\{(\mathbf{z}_1,\ldots,\mathbf{z}_5)\in\mathbb{C}^5 \;:\; |\mathbf{z}_k - \mathbf{z}_{k+1}| = 1,\; k=1,\ldots,5\right\} / \sim$$

where $\mathbf{z}_6 = \mathbf{z}_1$, and two pentagons $\mathbf{p} = (\mathbf{z}_1,\ldots,\mathbf{z}_5)$ and $\mathbf{p}' = (\mathbf{z}'_1,\ldots,\mathbf{z}'_5)$ are deemed equivalent, $\mathbf{p}\sim\mathbf{p}'$, if and only if there are $a,b\in\mathbb{C}$ with $|a|=1$ and so that $\mathbf{z}'_k = a\mathbf{z}_k + b$ for all $k=1,\ldots,5$.

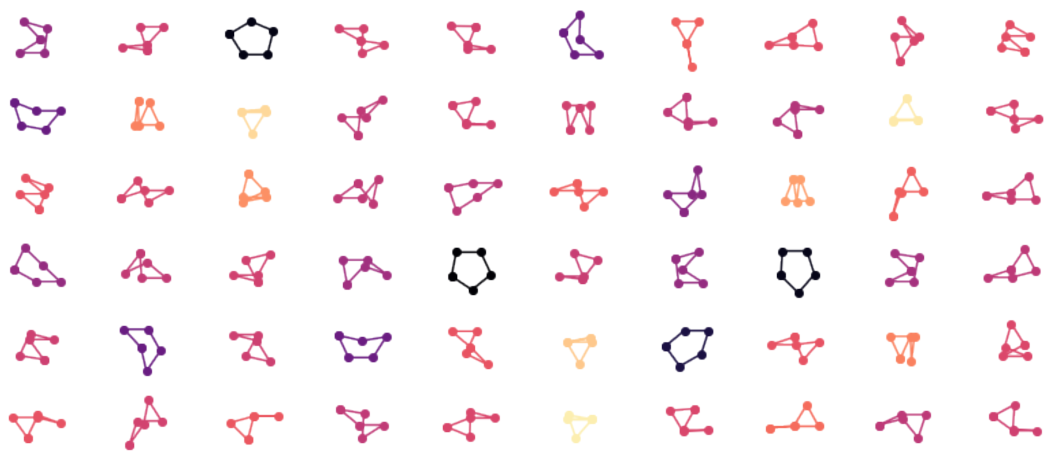

FIGURE 2. Some of the pentagons sampled in `mystery data_3`

Note that if $a = \frac{1}{\mathbf{z}_1-\mathbf{z}_5}$ and $b = \frac{-\mathbf{z}_5}{\mathbf{z}_1-\mathbf{z}_5}$, then $0 = a\mathbf{z}_1 + b$ and $1 = a\mathbf{z}_5 + b$; in other words, each pentagon is equivalent to one of the form $(1,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,0)$, for unique $\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\in\mathbb{C}$ so that $|1-\mathbf{w}_1| = |\mathbf{w}_1-\mathbf{w}_2| = |\mathbf{w}_2-\mathbf{w}_3| = |\mathbf{w}_3| = 1$. The pentagon $(1,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,0)$ is said to be in standard form. Note that the points in `mystery data_3` are in standard form; they have only 6 coordinates (as opposed to 10).

One can define similar spaces $M_5(d_1,\ldots,d_5)$, given prescribed edge lengths $d_1,\ldots,d_5>0$. The goal of this project is to sample and provide coordinates, using DREiMac, for them. In particular, the paper by Klaus and Kojima mentions that for appropriate choices of $d_1,\ldots,d_5$ one can get the following spaces: the sphere, the torus, the sphere with two, three, or four handles, and the disjoint union of two torii. The challenge here is to choose appropriate edge lengths realizing these spaces, generate data sets that sample them fully, and to use DREiMac to generate faithful dimensionality reductions as in our reconstruction of `mystery data 3` inside of $\mathbb{T}^3$.

A place to start:

(1) Describe what the equivalence relation above means geometrically, and check that every equivalence class does indeed have a unique representative in standard form.

References.

(1) On the moduli space of equilateral plane pentagons S. Klaus and S. Kojima, https://link.springer.com/article/10.1007/s13366-018-0429-z
(2) DREiMac, https://github.com/scikit-tda/DREiMac/tree/experimental
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(4) Multiscale Projective Coordinates via Persistent Cohomology of Sparse Filtrations, J. A. Perea, https://arxiv.org/pdf/1612.02861.pdf