## 1. Topological Time Series Analysis (TTSA)

1.1. **Persistent Cup Products for QuasiPeriodicity Detection.** A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be quasiperiodic if it can be written as $f(t) = F(t\omega_1, \ldots, t\omega_N)$ for a continuous function $F : \mathbb{T}^N \longrightarrow \mathbb{C}$ on the $N$-torus $\mathbb{T}^N = S^1 \times \cdots \times S^1$, and $\mathbb{Q}$-linearly independent real numbers $\omega_1, \ldots, \omega_N$. For these types of functions and appropriate parameters $d \in \mathbb{N}$ and $\tau > 0$, the sliding window point cloud

$$\mathbb{SW}_{d,\tau} f = \left\{ SW_{d,\tau} f(t) \ : \ t \in \mathbb{R} \right\} \quad , \quad SW_{d,\tau} f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ \vdots \\ f(t+d\tau) \end{bmatrix}$$

can be shown to be dense in an $N$-torus embedded in $\mathbb{C}^{d+1}$.

Let $\mathsf{dgm}_j^{\mathcal{R}}(\mathbb{SW}_{d,\tau} f)$ denote the persistence diagram for the $j$-th persistent cohomology over a field $\mathbb{F}$, $j \geq 0$, of the Rips filtration on the Sliding window point cloud $\mathbb{SW}_{d,\tau} f$. In TTSA the diagrams $\mathsf{dgm}_j^{\mathcal{R}}(\mathbb{SW}_{d,\tau} f)$ are used to certify that $f$ is quasiperiodic (i.e., that the closure of $\mathbb{SW}_{d,\tau} f$ is an $N$-torus) – by checking that there are $\binom{N}{j}$ highly persistent features. The main issue with this approach is that even if the dimension of the cohomology vector spaces looks right, one will not have an $N$-torus unless the cup product structure is also correct. The goal of this project is to bridge this gap; the challenge is to define and implement a quasiperiodicity score involving persistent cup products.

*A place to start:*

(1) Confirm the advertised cup product structure on $\mathbb{T}^2$. Use a small triangulation.

*References.*


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