Exercises

Exercise

A gamble (賭局) is said to be

- a *fair gamble* (公平賭局) if the expected value = 0;
- a *favorable gamble* (有利賭局) if the expected value > 0;
- an *unfavorable gamble* (不利賭局) if the expected value <0.

Example.

(1) The probability of gaining x_1 dolloars is p, and the probability of losing x_2 dollars is $1-p$. The expected value is $E(X) = px_1 + (1-p)(-x_2)$.

(2) x_0 dollars are required to be paid to start a game. The probability of gaining x_1 dolloars is p, and the probability of losing x_2 dollars is $1-p$. The expected value is $E(X) = 1 \cdot (-x_0) + px_1 + (1-p)(-x_2)$.

Exercise

- A company plans to invest in a particular construction project. There is a 35% chance that it will lose $30,000, a 40% chance that it will break even, and a 25% chance that it will make a profit of $55,000. How much can the company expect to make or lose on this project? (Ans. $3,250)

- The organizers at a balloon festival know that if it's windy in town, only 9,000 people will show up for the festival. But if it's not windy, 30,000 people will show up. The probability of wind is 45%. What is the expectation for the number of people who attend? (Ans. 20,550)

Ref. :

[1]https://math.libretexts.org/Courses/Prince_Georges_Community_College/MAT_1130_Mathematical_Ideas_Mirtova_Jones_(PGCC%3A_Fall_2022)/03%3A_Probability/3.03%3A_Expected_Value

Exercise

3. On two different occasions during a high school soccer game, the ball was kicked simultaneously by players on opposing teams. In which case (Case 1 or Case 2) does the ball undergo the greatest acceleration? Explain your answer.

Ref. :

[1] https://www.physicsclassroom.com/class/vectors/Lesson-3/Addition-of-Forces

Exercise

EXERCISE 2

1. *If* $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, *find*

(*i*) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ *and*

(*ii*) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. *Hence, show that* $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}).\mathbf{c}$.

Sol. (*i*)
$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = (4-1)\mathbf{i} - (-2-3)\mathbf{j} + (-1-6)\mathbf{k}$$
$$= 3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$$

$$\therefore \quad \mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (2\mathbf{i} + \mathbf{j} + 3\mathbf{k}).(3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k})$$
$$= 2(3) + 1(5) + 3(-7) = -10 \quad \text{...(1)}$$

$$(ii)\ \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = (1-6)\mathbf{i} - (2+3)\mathbf{j} + (4+1)\mathbf{k}$$
$$= -5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$$

$$\therefore \quad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (-5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}) \cdot (3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$
$$= (-5)(3) + (-5)(1) + 5(2) = -10 \quad \text{...(2)}$$

From (1) and (2), $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}).\mathbf{c}$.

Ref. :

[1] https://books.google.com.hk/books?id=0VLMn_OHpcgC&newbks=1&newbks_redir=0&printsec=frontcover&dq=vector+analysis+exercise&hl=en&redir_esc=y#v=onepage&q&f=false

Exercise

Example 2.4.1

Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution

The matrices are inverses if the product AB and BA both equal the identity matrix of dimension 2×2: I_2,

$$\mathrm{AB} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

and

$$\mathrm{BA} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

Clearly that is the case; therefore, the matrices A and B are inverses of each other.

Ref. :

[1]https://math.libretexts.org/Bookshelves/Applied_Mathematics/Applied_Finite_Mathematics_(Sekhon_and_Bloom)/02%3A_Matrices/2.04%3A_Inverse_Matrices

Exercise

Example 2.4.2

Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

Solution

Suppose A has an inverse, and it is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $AB = I_2$: $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

After multiplying the two matrices on the left side, we get

$$\begin{bmatrix} 3a+c & 3b+d \\ 5a+2c & 5b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries, we get four equations with four unknowns:

$$\begin{array}{ll} 3a+c=1 & 3b+d=0 \\ 5a+2c=0 & 5b+2d=1 \end{array}$$

Solving this system, we get: $a=2 \quad b=-1 \quad c=-5 \quad d=3$

Therefore, the inverse of the matrix A is $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Ref. :

[1]https://math.libretexts.org/Bookshelves/Applied_Mathematics/Applied_Finite_Mathematics_(Sekhon_and_Bloom)/02%3A_Matrices/2.04%3A_Inverse_Matrices

Exercise

Composite Functions

3. Find dy/dx when $y = (x^2 - 5x + 7)^4$.

Answer. Define $u = (x^2 - 5x + 7)$. Then $y = u^4$, and hence

$$\frac{dy}{du} = 4u^3 \qquad \text{and} \qquad \frac{du}{dx} = 2x - 5.$$

By using the chain rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4(x^2 - 5x + 7)^3(2x - 5).$$

Ref. :

[1] https://www.le.ac.uk/users/dsgp1/EXERCISE/MATHSEX/FSERIES/f2ans.pdf

Exercise

Products and Quotients

7. Differentiate $y = (2x+1)^3(x-8)^7$ with respect to x.

Answer. Define $u = (2x+1)^3$ and $v = (x-8)^7$. Then

$$\frac{du}{dx} = 6(2x+1)^2 \qquad \text{and} \qquad \frac{dv}{dx} = 7(x-8)^6;$$

whence the derivative of $y = uv$ is found via the product rule:

$$\begin{aligned}\frac{d}{dx}(uv) &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= 7(2x+1)^3(x-8)^6 + 6(x-8)^7(2x+1)^2 \\ &= 5(2x+1)^2(x-8)^6(4x-11).\end{aligned}$$

Ref. :

[1] https://www.le.ac.uk/users/dsgp1/EXERCISE/MATHSEX/FSERIES/f2ans.pdf

Exercise

9. Differentiate $3x^2/(x-1)^4$ with respect to x.

Answer. Let $y = u/v$ with $u = 3x^2$ and $v = (x-1)^4$. Then

$$\frac{du}{dx} = 6x \qquad \text{and} \qquad \frac{dv}{dx} = 4(x-1)^3,$$

and so the quotient rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{v(du/dx) - u(dv/dx)}{v^2} \\ &= \frac{6x(x-1)^4 - 12x^2(x-1)^3}{(x-1)^8} \\ &= -\frac{6x(x-1)^3(x+1)}{(x-1)^8} = -\frac{6x(x+1)}{(x-1)^5}.\end{aligned}$$

Ref. :

[1] https://www.le.ac.uk/users/dsgp1/EXERCISE/MATHSEX/FSERIES/f2ans.pdf

Exercise

Integration by Substitution: Indefinite Integrals: Solutions

Use a suitable subsitution to find the following integrals

Exercise 1

$$\int x\cos(x^2)dx$$

Let $u = x^2$

then $\frac{du}{dx} = 2x$

so $dx = \frac{1}{2x} du$

Making the substitution:

$$\int x.\cos(u)\frac{1}{2x}\,du$$

$$= \frac{1}{2}\int \cos(u)du$$

$$= \frac{1}{2}\sin(u) + C$$

$$= \frac{1}{2}\sin(x^2) + C$$

Exercise 2

$$\int \frac{e^x}{1+e^x}\,dx$$

Let $u = 1 + e^x$

then $\frac{du}{dx} = e^x$

so $dx = \frac{1}{e^x} du$

Making the substitution:

$$\int \frac{e^x}{u}\frac{1}{e^x}\,du$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln(1 + e^x) + C$$

(note that the absolute value signs can be dropped, as $1+e^x > 0 \ \forall x$)

Exercise 3

$$\int\sqrt{2x+1}\,dx$$

Let $u = 2x + 1$

then $\frac{du}{dx} = 2$

so $dx = \frac{1}{2} du$

Making the substitution:

$$\int\sqrt{u}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\,\frac{2}{3}\,u^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

Ref. :

[1] https://www.math.canterbury.ac.nz/php/resources/math100/integration/integration-by-substitution-indefinite-integrals-sol.gif

Exercise

Integration by Parts: Exercise 1: Solutions

Use the formula $\int u \frac{dv}{dx}\,dx = u.v - \int v \frac{du}{dx}\,dx$ to find the following integrals

Exercise 1

$\int x^2 \ln(x)dx$

Let $u = \ln(x)$ (as $\ln(x)$ is hard to integrate)

Then $\frac{dv}{dx} = x^2$.

$\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^3}{3}$.

Substituting into the integration by parts formula:

$$\int x^2 \ln(x)dx$$

$$= \frac{x^3}{3}\ln(x) - \int \frac{x^3}{3}\frac{1}{x}\,dx$$

(always simplify at this step before integrating!)

$$= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx$$

$$= \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C$$

Exercise 2

$\int x\cos(x)dx$

Let $\frac{dv}{dx} = \cos(x)$ (as trig functions are easy to integrate)

Then $u = x$.

$\frac{du}{dx} = 1$ and $v = \sin(x)$.

Substituting into the integration by parts formula:

$$\int x\cos(x)dx$$

$$= x\sin(x) - \int \sin(x).1dx$$

$$= x\sin(x) + \cos(x) + C$$

Exercise 3

$\int \sin(\ln(x))dx = I_1$ (for future reference)

Let $u = \sin(\ln(x))$ and $\frac{dv}{dx} = 1$.

$\frac{du}{dx} = \cos(\ln(x))\frac{1}{x}$ (chain rule)

and $v = x$.

Substituting into the integration by parts formula:

$$I_1 = x.\sin(\ln(x)) - \int \cos(\ln(x))\frac{1}{x}x\,dx$$

$$= x\sin(\ln(x)) - \int \cos(\ln(x))\,dx$$

Applying integration by parts to $\int \cos(\ln(x))\,dx = I_2$:

Let $u = \cos(\ln(x))$ and $\frac{dv}{dx} = 1$.

$\frac{du}{dx} = -\sin(\ln(x))\frac{1}{x}$ (chain rule)

and $v = x$.

$$I_2 = x.\cos(\ln(x)) - \int -\sin(\ln(x))\frac{1}{x}x\,dx$$

$$= x\cos(\ln(x)) + \boxed{\int \sin(\ln(x))dx}$$

I_1 again!

Therefore,

$$I_1 = x\sin(\ln(x)) - I_2$$

$$= x\sin(\ln(x)) - (x\cos(\ln(x)) + I_1)$$

$$2I_1 = x\sin(\ln(x)) - x\cos(\ln(x))$$

$$I_1 = \frac{1}{2}x\{\sin(\ln(x)) - \cos(\ln(x))\} + C$$

Ref. :

[1] https://www.math.canterbury.ac.nz/php/resources/math100/integration/integration-by-substitution-and-parts-sol.gif

Exercise

Partial Fractions: Linear Factors: Solutions

Find the following integrals by first writing each integral as a series of partial fractions

Exercise 1

$$\int \frac{2-x}{x(x+1)}\,dx$$

$$\frac{2-x}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)}$$

Equating numerators:

$2-x = A(x+1) + Bx$

When $x = -1$:
$3 = -B$, so $B = -3$

When $x = 0$:
$2 = A$

(Alternative method:
equate coefficients of x)

Thus

$$\frac{2-x}{x(x+1)} = \frac{2}{x} + \frac{-3}{x+1}$$

and

$$\int \frac{2-x}{x(x+1)}\,dx = \int \frac{2}{x} + \frac{-3}{x+1}\,dx = 2\ln|x| - 3\ln|x+1| + C$$

Exercise 2

$$\int \frac{1}{x^2+5x+6}\,dx$$

$x^2 + 5x + 6 = (x+2)(x+3)$
and

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$

Equating numerators:

$1 = A(x+3) + B(x+2)$

When $x = -3$:
$1 = -B$, so $B = -1$

When $x = -2$:
$1 = A$

(Alternative method:
equate coefficients of x)

Thus

$$\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} + \frac{-1}{x+3}$$

and

$$\int \frac{1}{x^2+5x+6}\,dx = \int \frac{1}{x+2} + \frac{-1}{x+3}\,dx = \ln|x+2| - \ln|x+3| + C$$

Exercise 3

$$\int \frac{9x+32}{x^2+6x+8}\,dx$$

$x^2 + 6x + 8 = (x+4)(x+2)$
and

$$\frac{9x+32}{(x+4)(x+2)} = \frac{A}{x+4} + \frac{B}{x+2} = \frac{A(x+2)+B(x+4)}{(x+4)(x+2)}$$

Equating numerators:

$9x+32 = A(x+2) + B(x+4)$

When $x = -2$:
$14 = 2B$, so $B = 7$

When $x = -4$:
$-4 = -2A$, so $A = 2$

(Alternative method:
equate coefficients of x)

Thus

$$\frac{9x+32}{(x+4)(x+2)} = \frac{2}{x+4} + \frac{7}{x+2}$$

and

$$\int \frac{9x+32}{x^2+6x+8}\,dx = \int \frac{2}{x+4} + \frac{7}{x+2}\,dx = 2\ln|x+4| + 7\ln|x+2| + C$$

Ref. :

[1] https://www.math.canterbury.ac.nz/php/resources/math100/integration/integration-using-partial-fractions-1-sol.gif

Exercise

Integration by Substitution: Definite Integrals: Solutions

The Fundamental Theorem of Calculus enables us to find indefinite integrals by finding antiderivatives.
If F(t) is an antiderivative of f(t) then

$$\underbrace{\int_a^b f(t)dt}_{\text{definite integral (signed area)}} = \underbrace{F(t)}_{\text{indefinite integral (reverse of differentiation)}}\Big]_a^b = F(b) - F(a)$$

Use a suitable substitution to find the following definite integrals
(Note: if you change the x limits of integration to u limits first, you won't need to substitute back in terms of x after the integration step!)

Exercise 1

$$\int_0^2 \frac{1}{3+5x}\,dx$$

Let $u = 3+5x$

then $\frac{du}{dx} = 5$

and $dx = \frac{1}{5}\,du$

At x=0, u = 3+5*0 = 3
At x=2, u = 3+5*2 = 13

Making the substitution:

$$\int_3^{13} \frac{1}{u}\,\frac{1}{5}\,du$$

$$= \frac{1}{5}\ln|u|\Big]_3^{13}$$

$$= \frac{1}{5}(\ln(13) - \ln(3))$$

= 1/5 ln(13/3)

(= 0.29 (2dp))

Exercise 2

$$\int_0^2 \frac{1}{(3+5x)^2}\,dx$$

Let $u = 3+5x$

then $\frac{du}{dx} = 5$

and $dx = \frac{1}{5}\,du$

At x=0, u = 3+5*0 = 3
At x=2, u = 3+5*2 = 13

Making the substitution:

$$\int_3^{13} \frac{1}{u^2}\,\frac{1}{5}\,du$$

$$= \frac{1}{5}\int_3^{13} u^{-2}\,du$$

$$= \frac{1}{5}\left(-u^{-1}\right]_3^{13}$$

$$= -\frac{1}{5}\left(\frac{1}{13} - \frac{1}{3}\right)$$

$$= \frac{2}{39}$$

Exercise 3

$$\int_1^e \frac{\ln(x)}{x}\,dx$$

Let $u = \ln(x)$

then $\frac{du}{dx} = \frac{1}{x}$

and $dx = x\,du$

At x=1, u = ln(1) = 0
At x=e, u = ln(e) = 1

Making the substitution:

$$\int_0^1 \frac{u}{x}.x\,du$$

$$= \int_0^1 u\,du$$

$$= \frac{u^2}{2}\Big]_0^1$$

$$= \frac{1^2}{2} - \frac{0^2}{2}$$

$$= \frac{1}{2}$$

Ref. :

[1] https://www.math.canterbury.ac.nz/php/resources/math100/integration/integration-by-substitution-definite-integrals-sol.gif