1 Exponential Function

The *exponential function of a scalar* is

$$\boxed{e^{\theta} = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}} \quad . \tag{1.1}$$

Similarly, the *exponential function of a matrix* is

$$\boxed{e^{\underline{\underline{A}}} = \lim_{m\to\infty}\left(\underline{\underline{I}}+\frac{\underline{\underline{A}}}{m}\right)^{m}} \quad . \tag{1.2}$$

1.1 Euler's Number

The *Euler's number* is
$e = 2.718281828459045235360287471352662497757247093699959574966967627724076630353547594571...$.

1.2 Euler's Formula

The *Euler's formula* is

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad . \tag{1.3}$$

<u>Proof</u>. $$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + ... = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...\right)$$
$$= \cos\theta + i\sin\theta$$

□

When $\theta = \pi$, we have

$$\boxed{e^{i\pi} + 1 = 0} \tag{1.4}$$

which is known as the *Euler's identity*.

1.3 Derivative of an Exponential Function

The derivative of a logarithmic function is

$$\boxed{\frac{d}{dx}e^{x} = e^{x}} \quad . \tag{1.5}$$

<u>Proof</u>. $$\frac{d}{dx}e^{x} = \frac{\lim\limits_{m\to\infty}\left(1+\frac{x+\Delta x}{m}\right)^{m} - \lim\limits_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}}{\Delta x} = \lim_{\Delta x\to 0}\frac{\lim\limits_{m\to\infty}\left(1+\frac{x}{m}+\frac{\Delta x}{m}\right)^{m} - \lim\limits_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{\lim\limits_{m\to\infty}\sum\limits_{r=0}^{m}C_r^m\left(1+\frac{x}{m}\right)^{m-r}\left(\frac{\Delta x}{m}\right)^{r} - \lim\limits_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}}{\Delta x} = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m-1} = \lim_{m\to\infty}\frac{\left(1+\frac{x}{m}\right)^{m}}{\left(1+\frac{x}{m}\right)} = \frac{e^{x}}{1} = e^{x}$$

□

1.4 Derivative of a Logarithmic Function

The derivative of a logarithmic function is

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}} \quad . \tag{1.6}$$

<u>Proof</u>. $$\frac{d}{dx}\ln x = \lim_{\Delta x\to 0}\frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \lim_{\Delta x\to 0}\ln\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} = \lim_{\Delta x\to 0}\ln\left(1+\frac{1/x}{1/\Delta x}\right)^{\frac{1}{\Delta x}} = \ln e^{\frac{1}{x}} = \frac{1}{x}$$ □