Determinants

Wedge Products

Example [Wedge product of vectors in 2D]

$$\text{Let } \begin{cases} \underline{x}_1 = x_{11}\underline{e}_1 + x_{12}\underline{e}_2 \\ \underline{x}_2 = x_{21}\underline{e}_1 + x_{22}\underline{e}_2 \end{cases}$$

$$\begin{aligned} \text{then } \underline{x}_1 \wedge \underline{x}_2 &= (x_{11}\underline{e}_1 + x_{12}\underline{e}_2) \wedge (x_{21}\underline{e}_1 + x_{22}\underline{e}_2) \\ &= (x_{11}x_{22} - x_{12}x_{21})\underline{e}_1 \wedge \underline{e}_2 \\ &= \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \underline{e}_1 \wedge \underline{e}_2 . \end{aligned}$$

$$\text{Hence } \underline{x}_1 \wedge \underline{x}_2 = \det(\underline{x}_1, \underline{x}_2)\underline{e}_1 \wedge \underline{e}_2 \text{ where } \det(\underline{x}_1, \underline{x}_2) := \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} .$$

Wedge Products

Example [Wedge product of vectors in 3D]

$$\text{Let} \begin{cases} \underline{x}_1 = x_{11}\underline{e}_1 + x_{12}\underline{e}_2 + x_{13}\underline{e}_3 \\ \underline{x}_2 = x_{21}\underline{e}_1 + x_{22}\underline{e}_2 + x_{23}\underline{e}_3 \\ \underline{x}_3 = x_{31}\underline{e}_1 + x_{32}\underline{e}_2 + x_{33}\underline{e}_3 \end{cases}$$

then

$$\begin{aligned} \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 &= (x_{11}\underline{e}_1 + x_{12}\underline{e}_2 + x_{13}\underline{e}_3) \wedge (x_{21}\underline{e}_1 + x_{22}\underline{e}_2 + x_{23}\underline{e}_3) \wedge (x_{31}\underline{e}_1 + x_{32}\underline{e}_2 + x_{33}\underline{e}_3) \\ &= (x_{11}\underline{e}_1 + x_{12}\underline{e}_2 + x_{13}\underline{e}_3) \wedge [(x_{21}x_{32} - x_{22}x_{31})\underline{e}_1 \wedge \underline{e}_2 + (x_{22}x_{33} - x_{23}x_{32})\underline{e}_2 \wedge \underline{e}_3 + (x_{23}x_{31} - x_{21}x_{33})\underline{e}_3 \wedge \underline{e}_1] \\ &= [x_{11}(x_{22}x_{33} - x_{23}x_{32}) - x_{12}(x_{21}x_{33} - x_{23}x_{31}) + x_{13}(x_{21}x_{32} - x_{22}x_{31})]\underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \\ &= \left(x_{11}\begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} - x_{12}\begin{vmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{vmatrix} + x_{13}\begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \right) \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \\ &= \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \end{aligned}$$