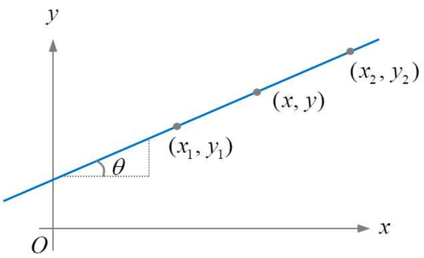
Calculus - Differentiation

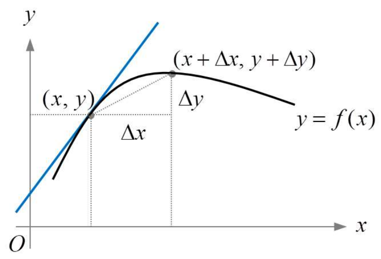
Equation of A Straight Line

For a straight line: we have

the slope, $m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y - y_1}{x - x_1}$.



Differentiation



The slope at point (x, y) is

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

Differentiation

Example.

- $y=f(x)=x^2$

$$\Rightarrow \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x$$

- $y=f(x)=x^3$

$$\Rightarrow \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}=\lim_{\Delta x\to 0}\frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{\Delta x}=\lim_{\Delta x\to 0}[3x^2+3x\Delta x+(\Delta x)^2]=3x^2$$

- $y=f(x)=x^n$

$$\Rightarrow \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sum_{r=0}^{n}C_r^n x^{n-r}(\Delta x)^r-x^n}{\Delta x}=\lim_{\Delta x\to 0}\sum_{r=1}^{n}C_r^n x^{n-r}(\Delta x)^{r-1}=C_1^n x^{n-1}=nx^{n-1}$$

Differentiation

Example.

- $y = f(x) = \sin x$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} = \cos x$$

- $y = f(x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} = -\sin x$$

L'Hôpital's rule

[L'Hôpital's rule] $\displaystyle \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ if $\displaystyle \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$.

Proof.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{\dfrac{f(x)-f(x_0)}{x-x_0}}{\dfrac{g(x)-g(x_0)}{x-x_0}} \qquad (\text{if } \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0)$$

$$= \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

□

Chain Rule

Consider $y = f(x)$ where $x \equiv x(t)$, then

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \frac{\Delta x}{\Delta t}$$
$$= \frac{dy}{dx} \frac{dx}{dt}$$

Product Rule

Consider $u \equiv u(x)$ and $v \equiv v(x)$, then

$$\frac{d}{dx}(uv) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{[u(x)+\Delta u(x)][v(x)+\Delta v(x)] - u(x)v(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[\frac{\Delta u(x)}{\Delta x} v(x) + u(x) \frac{\Delta v(x)}{\Delta x} \right]$$
$$= \frac{du}{dx} v + u \frac{dv}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{1}{v}\frac{du}{dx}+u\frac{d}{dx}\left(\frac{1}{v}\right)$$
$$=\frac{1}{v}\frac{du}{dx}-\frac{u}{v^2}\frac{dv}{dx}$$
$$=\frac{\frac{du}{dx}v-u\frac{dv}{dx}}{v^2}$$

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