Taylor Sereis

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Let $f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+a_3(x-x_0)^3+\ldots$, then

- $f(x)\big|_{x\to x_0}=a_0 \quad\Rightarrow\quad a_0=f(x)\big|_{x\to x_0}$;
- $f'(x)=\sum_{n=1}^{\infty}a_n n(x-x_0)^{n-1}=a_1+2a_2(x-x_0)+3a_3(x-x_0)^2+\ldots \quad\Rightarrow\quad a_1=f'(x)\big|_{x\to x_0}$;
- $f''(x)=\sum_{n=2}^{\infty}a_n n(n-1)(x-x_0)^{n-2}=2\cdot 1a_2+3\cdot 2a_3(x-x_0)+4\cdot 3a_4(x-x_0)^2+\ldots \quad\Rightarrow\quad a_2=\frac{1}{2!}f''(x)\big|_{x\to x_0}$;
- …
- $f^{(r)}(x)=\sum_{n=r}^{\infty}a_n n(n-1)\ldots(n-r+1)(x-x_0)^{n-r} \quad\Rightarrow\quad a_r=\frac{1}{r!}f^{(r)}(x)\big|_{x\to x_0}$;
- …

Hence we have

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(x)\big|_{x\to x_0}(x-x_0)^n .$$

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Example.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

Local Extrema

$$\because\ f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \ldots.$$

Assume the series is convergent.

When $f'(x_0) = 0$, then $f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \ldots.$.

- If $f''(x_0) < 0$, then $f(x) < f(x_0)$ $\Rightarrow$ $f(x)$ has a local maximum at $x = x_0$;
- If $f''(x_0) > 0$, then $f(x) > f(x_0)$ $\Rightarrow$ $f(x)$ has a local minimum at $x = x_0$;
- If $f''(x_0) = 0$, then $f(x)$ has a point of inflection at $x = x_0$.

Local Extrema

Alternatively, we can determine the local extremum in the following:

- $$f'(x)\begin{cases} >0 & \text{when } x = x_0^- \\ =0 & \text{when } x = x_0 \\ <0 & \text{when } x = x_0^+ \end{cases} \Rightarrow f(x) \text{ has a local maximum at } x = x_0\,;$$

- $$f'(x)\begin{cases} <0 & \text{when } x = x_0^- \\ =0 & \text{when } x = x_0 \\ >0 & \text{when } x = x_0^+ \end{cases} \Rightarrow f(x) \text{ has a local minimum at } x = x_0\,.$$