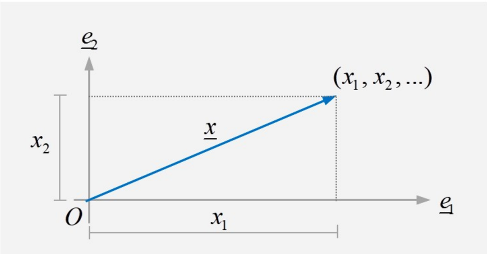
Vector Analysis

Vector Analysis

Let $\underline{x} = \sum_{i=1}^{n} x_i \underline{e}_i$, then

- the magnitude of a vector is $\left|\underline{x}\right| = \sqrt{\sum_{i=1}^{n} x_i^2}$;
- the unit vector of a vector is $\hat{\underline{x}} = \frac{\underline{x}}{\left|\underline{x}\right|}$.

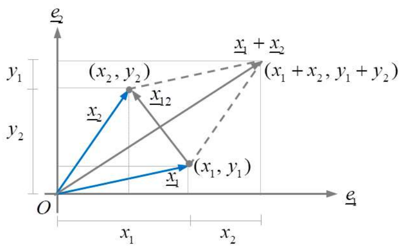


Addition & Subtraction of Vectors

Let $\underline{a} = \sum_{i=1}^{n} a_i \underline{e}_i$ and $\underline{b} = \sum_{i=1}^{n} b_i \underline{e}_i$, then $\underline{a} \pm \underline{b} = \sum_{i=1}^{n} (a_i \pm b_i) \underline{e}_i$.

Addition & Subtraction of Vectors

Example. [Addition of Vectors in 2D]



Let $\underline{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underline{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then $\underline{x}_1 + \underline{x}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$.

Since $\underline{x}_1 + \underline{x}_{12} = \underline{x}_2$, we have $\underline{x}_{12} = \underline{x}_2 - \underline{x}_1$.

Note that
- When the vectors represent positions, the vector difference is called *relative position* (相對位置);
- When the vectors represent velocities, the vector difference *relative velocity* (相對速度).

Dot Product of Two Vectors

Given two bases $\underline{e}_i$ and $\underline{e}_j$, the dot product $\cdot$ is defined as follows

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

where $\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$ is known as the *Kronecker delta*.

Dot Product of Two Vectors

Let $\underline{a} = \sum_{i=1}^{n} a_i \underline{e}_i = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\underline{b} = \sum_{i=1}^{n} b_i \underline{e}_i = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, then

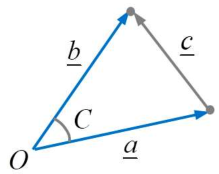
$$\underline{a} \cdot \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i b_i = \underline{a}^T \underline{b}.$$

Dot Product of Two Vectors

Consider $\underline{c} = \underline{b} - \underline{a}$, we have

$$\underline{c}^2 = (\underline{b} - \underline{a}) \cdot (\underline{b} - \underline{a})$$
$$= \underline{a}^2 + \underline{b}^2 - 2\underline{a} \cdot \underline{b}$$

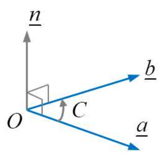
$$\therefore \quad \underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos C \qquad \text{(by law of cosine)}$$

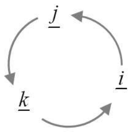


Cross Product of Two Vectors

The cross product $\times$ is defined in 3D.

Let $\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$, then





$$\underline{a} \times \underline{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{pmatrix} \begin{vmatrix} a_y & b_y \\ a_z & b_z \end{vmatrix} \\ -\begin{vmatrix} a_x & b_x \\ a_z & b_z \end{vmatrix} \\ \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix} \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

where $\underline{i} \times \underline{j} = \underline{k} = -\underline{j} \times \underline{i}$, $\underline{j} \times \underline{k} = \underline{i} = -\underline{k} \times \underline{j}$, $\underline{k} \times \underline{i} = \underline{j} = -\underline{i} \times \underline{k}$, $\underline{i} \times \underline{j} = \underline{k}$, and $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}$.

Cross Product of Two Vectors

$\because \quad |\underline{a}\times\underline{b}| = \sqrt{(a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_x b_y - a_y b_x)^2}$

$\Leftrightarrow \quad |\underline{a}\times\underline{b}| = \sqrt{(a_x^{\ 2} + a_y^{\ 2} + a_z^{\ 2})(b_x^{\ 2} + b_y^{\ 2} + b_z^{\ 2}) - (a_x b_x + a_y b_y + a_z b_z)^2}$

$\Leftrightarrow \quad |\underline{a}\times\underline{b}| = \sqrt{|\underline{a}|^2 |\underline{b}|^2 - (\underline{a}\cdot\underline{b})^2}$

$\Leftrightarrow \quad |\underline{a}\times\underline{b}| = \sqrt{|\underline{a}|^2 |\underline{b}|^2 (1 - \cos^2 C)}$

$\Leftrightarrow \quad |\underline{a}\times\underline{b}| = |\underline{a}||\underline{b}|\sin C$

Cross Product of Two Vectors

Given $\underline{c} = \underline{b} - \underline{a}$, we have

$$\begin{cases} \underline{c}\times\underline{a} = \underline{b}\times\underline{a} \\ \underline{c}\times\underline{b} = -\underline{a}\times\underline{b} \\ 0 = \underline{b}\times\underline{c} - \underline{a}\times\underline{c} \end{cases}$$

$\Rightarrow \quad \underline{a}\times\underline{b} = \underline{a}\times\underline{c} = \underline{b}\times\underline{c}$

$\Rightarrow \quad |\underline{a}\times\underline{b}| = |\underline{a}\times\underline{c}| = |\underline{b}\times\underline{c}|$

$\Rightarrow \quad |\underline{a}||\underline{b}|\sin C = |\underline{a}||\underline{c}|\sin B = |\underline{b}||\underline{c}|\sin A$

$\Rightarrow \quad \dfrac{\sin A}{|\underline{a}|} = \dfrac{\sin B}{|\underline{b}|} = \dfrac{\sin C}{|\underline{c}|}$