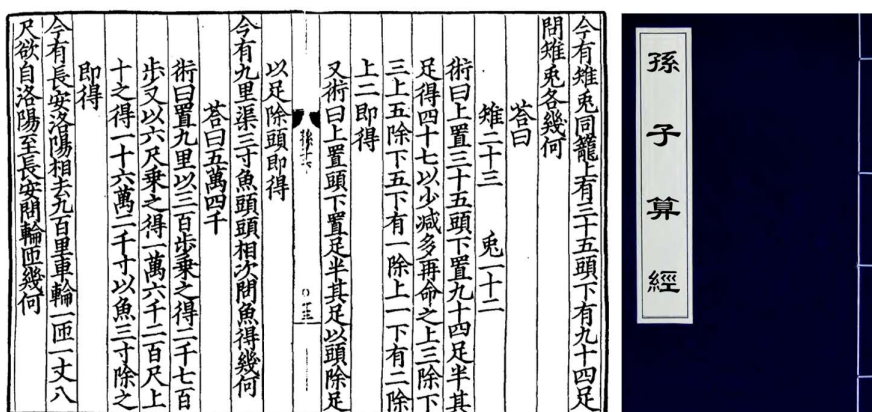


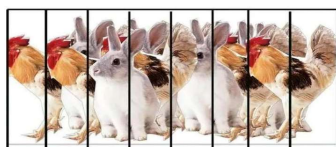
Linear Algebra

Example. 雞兔同籠



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「今有雉、兔同籠，上有三十五頭，下有九十四足。問：雉、兔各幾何？」



Let x_1 : number of chicken

x_2 : number of rabbits

Then we have the following system of linear equations:

$$\begin{cases} x_1 + x_2 = 35 & \text{(number of heads)} \\ 2x_1 + 4x_2 = 94 & \text{(number of legs)} \end{cases}$$

Hence $x_1 = 23$ and $x_2 = 12$.

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The system of linear equations can be written as

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 35 \\ 94 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 35 \\ 94 \end{pmatrix}$$

It can be checked that, given

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} = \underline{e}_1 \otimes \underline{e}_1 + \underline{e}_1 \otimes \underline{e}_2 + 2\underline{e}_2 \otimes \underline{e}_1 + 4\underline{e}_2 \otimes \underline{e}_2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \underline{e}_1 + x_2 \underline{e}_2, \text{ and } \begin{pmatrix} 35 \\ 94 \end{pmatrix} = 35 \underline{e}_1 + 94 \underline{e}_2 \text{ then}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 35 \\ 94 \end{pmatrix}$$

$$\Rightarrow (\underline{e}_1 \otimes \underline{e}_1 + \underline{e}_1 \otimes \underline{e}_2 + 2\underline{e}_2 \otimes \underline{e}_1 + 4\underline{e}_2 \otimes \underline{e}_2) \cdot (x_1 \underline{e}_1 + x_2 \underline{e}_2) = 35 \underline{e}_1 + 94 \underline{e}_2$$

$$\Rightarrow (x_1 + x_2) \underline{e}_1 + (2x_1 + 4x_2) \underline{e}_2 = 35 \underline{e}_1 + 94 \underline{e}_2$$

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Note that

- $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$,
- $\underline{e}_1 \otimes \underline{e}_2 \neq \underline{e}_2 \otimes \underline{e}_1$ (tensor product is not commutative).

The operation \cdot is called *dot product*, $\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$ is known as the *Kronecker delta*, and the operation \otimes is called *tensor product* or *Kronecker product*.

Tensors

Mathematically, we have the vector and matrix as follows:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n x_i \underline{e}_i \quad (\text{Vector})$$

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \sum_{j=1}^n \sum_{i=1}^m a_{ij} \underline{e}_i \otimes \underline{e}_j \quad (\text{Matrix})$$

Tensor is the extension of vector and matrix:

$$\underline{\underline{f}} = \sum_{i_p=1}^{n_p} \dots \sum_{i_2=1}^{n_2} \sum_{i_1=1}^{n_1} a_{i_1 i_2 \dots i_p} \underline{e}_{i_1} \otimes \underline{e}_{i_2} \otimes \dots \otimes \underline{e}_{i_p} \quad (\text{Tensor})$$

Tensors

The system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = f_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = f_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = f_n \end{cases}$$

can be written in the form of matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Symbolically we can write

$$\underline{\underline{A}} \underline{x} = \underline{f}$$

$$\text{where } \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } \underline{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Addition of Two Matrices

Suppose we have two systems of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = f_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = f_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = f_n \end{cases} \quad \text{and} \quad \begin{cases} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n = g_1 \\ b_{21}x_1 + b_{22}x_2 + \dots + b_{2n}x_n = g_2 \\ \vdots \\ b_{n1}x_1 + b_{n2}x_2 + \dots + b_{nn}x_n = g_n \end{cases}$$

then addition of these two systems are

$$\begin{cases} (a_{11} + b_{11})x_1 + (a_{12} + b_{12})x_2 + \dots + (a_{1n} + b_{1n})x_n = f_1 + g_1 \\ (a_{21} + b_{21})x_1 + (a_{22} + b_{22})x_2 + \dots + (a_{2n} + b_{2n})x_n = f_2 + g_2 \\ \vdots \\ (a_{n1} + b_{n1})x_1 + (a_{n2} + b_{n2})x_2 + \dots + (a_{nn} + b_{nn})x_n = f_n + g_n \end{cases}$$

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then addition of these two systems are

$$\begin{cases} (a_{11} + b_{11})x_1 + (a_{12} + b_{12})x_2 + \dots + (a_{1n} + b_{1n})x_n = f_1 + g_1 \\ (a_{21} + b_{21})x_1 + (a_{22} + b_{22})x_2 + \dots + (a_{2n} + b_{2n})x_n = f_2 + g_2 \\ \vdots \\ (a_{n1} + b_{n1})x_1 + (a_{n2} + b_{n2})x_2 + \dots + (a_{nn} + b_{nn})x_n = f_n + g_n \end{cases}$$

In other words, we have

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nn} + b_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 + g_1 \\ f_2 + g_2 \\ \vdots \\ f_n + g_n \end{pmatrix}$$

Hence we have

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nn} + b_{nn} \end{pmatrix}$$

Addition of Two Matrices

$$\text{Let } \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^n \sum_{i=1}^n a_{ij} \underline{e}_i \otimes \underline{e}_j, \text{ and } \underline{\underline{B}} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \sum_{k=1}^n \sum_{r=1}^n b_{rk} \underline{e}_r \otimes \underline{e}_k,$$

$$\text{then } \underline{\underline{A}} + \underline{\underline{B}} = \sum_{j=1}^n \sum_{i=1}^n a_{ij} \underline{e}_i \otimes \underline{e}_j + \sum_{k=1}^n \sum_{r=1}^n b_{rk} \underline{e}_r \otimes \underline{e}_k$$

$$= \sum_{j=1}^n \sum_{i=1}^n (a_{ij} + b_{ij}) \underline{e}_i \otimes \underline{e}_j$$

Matrix Multiplication

$$\text{Let } \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^n \sum_{i=1}^n a_{ij} \underline{e}_i \otimes \underline{e}_j, \text{ and } \underline{\underline{B}} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \sum_{k=1}^n \sum_{r=1}^n b_{rk} \underline{e}_r \otimes \underline{e}_k,$$

then the multiplication of the two matrices is

$$\begin{aligned} \underline{\underline{A}} \cdot \underline{\underline{B}} &= \left(\sum_{j=1}^n \sum_{i=1}^n a_{ij} \underline{e}_i \otimes \underline{e}_j \right) \cdot \left(\sum_{k=1}^n \sum_{r=1}^n b_{rk} \underline{e}_r \otimes \underline{e}_k \right) \\ &= \left(\sum_{j=1}^n \sum_{i=1}^n a_{ij} \underline{e}_i \otimes \underline{e}_j \right) \cdot \left(\sum_{k=1}^n \sum_{j=1}^n b_{jk} \underline{e}_j \otimes \underline{e}_k \right) \\ &= \sum_{k=1}^n \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{jk} \underline{e}_i \otimes \underline{e}_k. \end{aligned}$$

Note that $\underline{\underline{A}} \cdot \underline{\underline{B}} \neq \underline{\underline{B}} \cdot \underline{\underline{A}}$ in general (matrix multiplication is not commutative in general).

Matrix Multiplication

Example 1.1. [Multiplication of 2×2 matrices] Let $\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and $\underline{\underline{B}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then

$$\begin{aligned} \underline{\underline{A}} \cdot \underline{\underline{B}} &= (a_{11}\underline{e}_1 \otimes \underline{e}_1 + a_{12}\underline{e}_1 \otimes \underline{e}_2 + a_{21}\underline{e}_2 \otimes \underline{e}_1 + a_{22}\underline{e}_2 \otimes \underline{e}_2) \cdot (b_{11}\underline{e}_1 \otimes \underline{e}_1 + b_{12}\underline{e}_1 \otimes \underline{e}_2 + b_{21}\underline{e}_2 \otimes \underline{e}_1 + b_{22}\underline{e}_2 \otimes \underline{e}_2) \\ &= (a_{11}b_{11} + a_{12}b_{21})\underline{e}_1 \otimes \underline{e}_1 + (a_{11}b_{12} + a_{12}b_{22})\underline{e}_1 \otimes \underline{e}_2 + \\ &\quad (a_{21}b_{11} + a_{22}b_{21})\underline{e}_2 \otimes \underline{e}_1 + (a_{21}b_{12} + a_{22}b_{22})\underline{e}_2 \otimes \underline{e}_2 \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \end{aligned}$$

$$\text{Hence } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$