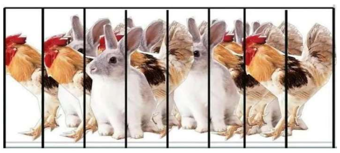
Linear Algebra

Example. 雞兔同籠

今有雉兎同籠上有三十五頭下有九十四足
問雉兎各幾何
荅曰
雉二十三　兎一十二
術曰上置三十五頭下置九十四足半其
足得四十七以少減多再命之上三除下
三上五除下五下有一除上一下有二除
上二即得
又術曰上置頭下置足半其足以頭除足
以足除頭即得
今有九里渠三寸魚頭頭相次問魚得幾何
荅曰五萬四千
術曰置九里以三百步乘之得二千七百
步又以六尺乘之得一萬六千二百尺上
十之得一十六萬二千寸以魚三寸除之
即得
今有長安洛陽相去九百里車輪一匝一丈八
尺欲自洛陽至長安問輪匝幾何

Example. 雞兔同籠

「今有雉、兔同籠，上有三十五頭，下有九十四足。問：雉、兔各幾何？」



Let x_1 : number of chicken

x_2 : number of rabbits

Then we have the following system of linear equations:

$$\begin{cases} x_1 + x_2 = 35 & \text{(number of heads)} \\ 2x_1 + 4x_2 = 94 & \text{(number of legs)} \end{cases}$$

Hence $x_1 = 23$ and $x_2 = 12$.

Example. 雞兔同籠

Let x_1 : number of chicken

x_2 : number of rabbits

Then we have the following system of linear equations:

$$\begin{cases} x_1 + x_2 = 35 & \text{(number of heads)} \\ 2x_1 + 4x_2 = 94 & \text{(number of legs)} \end{cases}$$

Hence $x_1 = 23$ and $x_2 = 12$.

The system of linear equations can be written as

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 35 \\ 94 \end{pmatrix}$$

Example. 雞兔同籠

$$\begin{pmatrix}1 & 1\\ 2 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}35\\ 94\end{pmatrix}$$

It can be checked that, given

$\begin{pmatrix}1 & 1\\ 2 & 4\end{pmatrix} = \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + 2\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_1 + 4\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2$, $\begin{pmatrix}x_1\\ x_2\end{pmatrix} = x_1\underset{\sim}{e}_1 + x_2\underset{\sim}{e}_2$, and $\begin{pmatrix}35\\ 94\end{pmatrix} = 35\underset{\sim}{e}_1 + 94\underset{\sim}{e}_2$ then

$$\begin{pmatrix}1 & 1\\ 2 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}35\\ 94\end{pmatrix}$$

$$\Rightarrow \quad (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + 2\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_1 + 4\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2)\cdot(x_1\underset{\sim}{e}_1 + x_2\underset{\sim}{e}_2) = 35\underset{\sim}{e}_1 + 94\underset{\sim}{e}_2$$

$$\Rightarrow \quad (x_1 + x_2)\underset{\sim}{e}_1 + (2x_1 + 4x_2)\underset{\sim}{e}_2 = 35\underset{\sim}{e}_1 + 94\underset{\sim}{e}_2$$

$$\Rightarrow \quad \begin{cases} x_1 + x_2 = 35 \\ 2x_1 + 4x_2 = 94 \end{cases}$$

Example. 雞兔同籠

Note that

- $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$,
- $\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 \neq \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_1$ (tensor product is not commutative).

The operation $\cdot$ is called *dot product*, $\delta_{ij} = \begin{cases}1 & (i=j)\\ 0 & (i\neq j)\end{cases}$ is known as the *Kronecker delta*, and the operation $\otimes$ is called *tensor product* or *Kronecker product*.

Tensors

Mathematically, we have the vector and matrix as follows:

- $$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i \underline{e}_i \qquad \text{(Vector)}$$
- $$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}\underline{e}_i \otimes \underline{e}_j \qquad \text{(Matrix)}$$

Tensor is the extension of vector and matrix:

- $$\underset{\sim}{f} = \sum_{i_p=1}^{n_p} \cdots \sum_{i_2=1}^{n_2}\sum_{i_1=1}^{n_1} a_{i_1 i_2 \ldots i_p}\underline{e}_{i_1} \otimes \underline{e}_{i_2} \otimes \ldots \otimes \underline{e}_{i_p} \qquad \text{(Tensor)}$$

Tensors

The system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = f_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = f_2 \\ \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = f_n \end{cases}$$

can be written in the form of matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Symbolically we can write

$$\underline{\underline{A}}\underline{x} = \underline{f}$$

where $\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $\underline{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$.

Addition of Two Matrices

Suppose we have two systems of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = f_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = f_2 \\ \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = f_n \end{cases} \quad \text{and} \quad \begin{cases} b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n = g_1 \\ b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n = g_2 \\ \qquad\qquad \vdots \\ b_{n1}x_1 + b_{n2}x_2 + \ldots + b_{nn}x_n = g_n \end{cases}$$

then addition of these two systems are

$$\begin{cases} (a_{11}+b_{11})x_1 + (a_{12}+b_{12})x_2 + \ldots + (a_{1n}+b_{1n})x_n = f_1 + g_1 \\ (a_{21}+b_{11})x_1 + (a_{22}+b_{12})x_2 + \ldots + (a_{2n}+b_{1n})x_n = f_2 + g_2 \\ \qquad\qquad \vdots \\ (a_{n1}+b_{n1})x_1 + (a_{n2}+b_{n2})x_2 + \ldots + (a_{nn}+b_{nn})x_n = f_n + g_n \end{cases}.$$

Addition of Two Matrices

Suppose we have two systems of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = f_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = f_2 \\ \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = f_n \end{cases} \quad \text{and} \quad \begin{cases} b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n = g_1 \\ b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n = g_2 \\ \qquad\qquad \vdots \\ b_{n1}x_1 + b_{n2}x_2 + \ldots + b_{nn}x_n = g_n \end{cases}$$

then addition of these two systems are

$$\begin{cases} (a_{11}+b_{11})x_1 + (a_{12}+b_{12})x_2 + \ldots + (a_{1n}+b_{1n})x_n = f_1 + g_1 \\ (a_{21}+b_{11})x_1 + (a_{22}+b_{12})x_2 + \ldots + (a_{2n}+b_{1n})x_n = f_2 + g_2 \\ \qquad\qquad \vdots \\ (a_{n1}+b_{n1})x_1 + (a_{n2}+b_{n2})x_2 + \ldots + (a_{nn}+b_{nn})x_n = f_n + g_n \end{cases}.$$

In other words, we have

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{2n} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1+g_1 \\ f_2+g_2 \\ \vdots \\ f_n+g_n \end{pmatrix}.$$

Hence we have

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{2n} \end{pmatrix}.$$

Addition of Two Matrices

Let $\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\underline{e}_i \otimes \underline{e}_j$, and $\underline{\underline{B}} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \sum_{k=1}^{n}\sum_{r=1}^{n} b_{rk}\underline{e}_r \otimes \underline{e}_k$,

then
$$\underline{\underline{A}} + \underline{\underline{B}} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\underline{e}_i \otimes \underline{e}_j + \sum_{k=1}^{n}\sum_{r=1}^{n} b_{rk}\underline{e}_r \otimes \underline{e}_k$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{n} (a_{ij} + b_{ij})\underline{e}_i \otimes \underline{e}_j$$

Matrix Multiplication

Let $\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\underline{e}_i \otimes \underline{e}_j$, and $\underline{\underline{B}} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \sum_{k=1}^{n}\sum_{r=1}^{n} b_{rk}\underline{e}_r \otimes \underline{e}_k$,

then the multiplication of the two matrices is
$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\underline{e}_i \otimes \underline{e}_j\right) \cdot \left(\sum_{k=1}^{n}\sum_{r=1}^{n} b_{rk}\underline{e}_r \otimes \underline{e}_k\right)$$
$$= \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\underline{e}_i \otimes \underline{e}_j\right) \cdot \left(\sum_{k=1}^{n}\sum_{j=1}^{n} b_{jk}\underline{e}_j \otimes \underline{e}_k\right)$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{jk}\underline{e}_i \otimes \underline{e}_k \ .$$

Note that $\underline{\underline{A}} \cdot \underline{\underline{B}} \neq \underline{\underline{B}} \cdot \underline{\underline{A}}$ in general (matrix multiplication is not commutative in general).

Matrix Multiplication

Example 1.1. [Multiplication of 2×2 matrices] Let $\underline{\underline{A}}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and $\underline{\underline{B}}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then

$$\begin{aligned}\underline{\underline{A}}\cdot\underline{\underline{B}} &= (a_{11}\underline{e_1}\otimes\underline{e_1}+a_{12}\underline{e_1}\otimes\underline{e_2}+a_{21}\underline{e_2}\otimes\underline{e_1}+a_{22}\underline{e_2}\otimes\underline{e_2})\cdot(b_{11}\underline{e_1}\otimes\underline{e_1}+b_{12}\underline{e_1}\otimes\underline{e_2}+b_{21}\underline{e_2}\otimes\underline{e_1}+b_{22}\underline{e_2}\otimes\underline{e_2}) \\ &= (a_{11}b_{11}+a_{12}b_{21})\underline{e_1}\otimes\underline{e_1}+(a_{11}b_{12}+a_{12}b_{22})\underline{e_1}\otimes\underline{e_2}+ \\ &\quad (a_{21}b_{11}+a_{22}b_{21})\underline{e_2}\otimes\underline{e_1}+(a_{21}b_{12}+a_{22}b_{22})\underline{e_2}\otimes\underline{e_2} \\ &= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}\end{aligned}$$

Hence $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$.