Integration

Revision - Differentiations

$$f^{(1)}(x) = \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$$

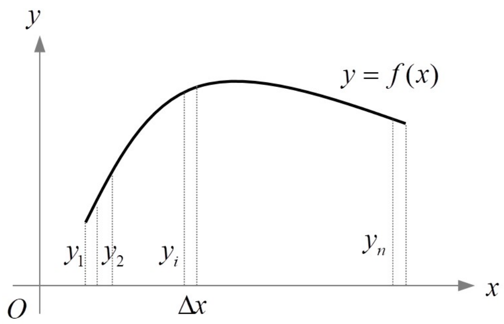
$$f^{(2)}(x) = \frac{d}{dx} f^{(1)}(x) = \frac{d^2}{dx^2} f(x)$$

$$f^{(3)}(x) = \frac{d}{dx} f^{(2)}(x) = \frac{d^2}{dx^2} f^{(1)}(x) = \frac{d^3}{dx^3} f(x)$$

$$\vdots$$

$$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x) = \frac{d^2}{dx^2} f^{(n-2)}(x) = \frac{d^3}{dx^3} f^{(n-3)}(x) = \ldots = \frac{d^{n-1}}{dx^{n-1}} f^{(1)}(x) = \frac{d^n}{dx^n} f(x)$$

Integration - Area under a Curve



The area under the curve is

$$\boxed{\int f(x)dx \doteq \lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x}$$

Integration as an Inverse Process of Differentiation

$$\begin{aligned}
\because \frac{d}{dx}\int f(x)dx &= \lim_{\Delta x \to 0} \frac{\int f(x+\Delta x)dx - \int f(x)dx}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\int [f(x+\Delta x) - f(x)]dx}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\int \left[\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\Delta x)^n - f(x) \right] dx}{\Delta x} \\
&= \int f'(x)dx
\end{aligned}$$

$$\Rightarrow \boxed{\frac{d}{dx}\int f(x)dx = \int f'(x)dx = f(x)}$$

$$\begin{aligned}
&\because \frac{df(x)}{dx} \doteq f'(x) \\
&\Rightarrow df(x) = f'(x)dx \\
&\Rightarrow \int df(x) = \int f'(x)dx \\
&\Rightarrow f(x) = \int f'(x)dx
\end{aligned}$$

Hence we have

$$\int f(x)dx \underset{\int(\cdot)dx}{\overset{\frac{d}{dx}(\cdot)}{\rightleftarrows}} f(x) \underset{\int(\cdot)dx}{\overset{\frac{d}{dx}(\cdot)}{\rightleftarrows}} f'(x)$$

Integration as an Inverse Process of Differentiation

Differentiations	**Integrations**
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln x + C$

Fundamental Theorem of Calculus (微積分基本定理)

The *fundamental theorem of calculus* states that

$$\boxed{\int_{x_1}^{x_2} f'(x)dx = \int_{x_1}^{x_2} d\left[f(x)\right] = f(x_2) - f(x_1)} .$$

Integration by Substitution

$$\int f(g(x))\cdot g'(x)dx = \int f(g(x))d(g(x)) = \int f(u)du \text{ where } u = g(x)$$

Integration by Parts

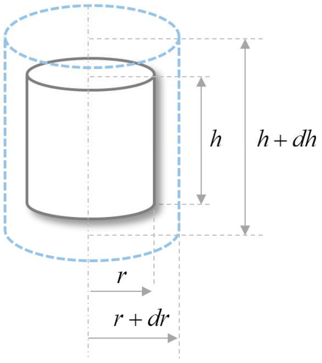
$\because \; d(uv) = udv + vdu$ (Product rule in differentiation)

$\Rightarrow \; \boxed{\int udv = \int d(uv) - \int vdu}$.

Example. $\int \ln x dx = x\ln x - \int xd(\ln x) = x\ln x - \int dx = x\ln x - x + C = x(\ln x - 1) + C$

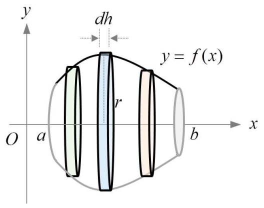
Volumes of Solids of Revolution

The volume of a cylinder is $V = \pi r^2 h$ where r is the radius and h is the height.



Volumes of Solids of Revolution

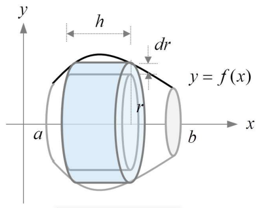
Disk method



$$V = \int_a^b \pi r^2 dh$$

where
$r = y$ and $h = x$

Shell method

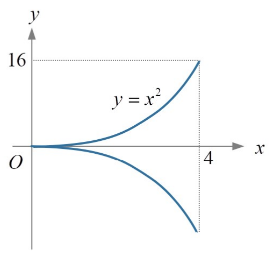


$$V = \int_a^b 2\pi rh dr$$

where
$r = y$ and
$h = \max(f^{-1}(y)) - \min(f^{-1}(y))$

Volumes of Solids of Revolution

Example.



Disk method:

$$V = \int_0^4 \pi y^2 dx$$
$$= \int_0^4 \pi x^4 dx$$
$$= \left[\pi \frac{x^5}{5}\right]_{x=0}^{x=4}$$
$$= \frac{1024\pi}{5}$$

Shell method:

$$V = \int_0^{16} 2\pi y(4-x)dy$$
$$= \int_0^{16} 2\pi y(4-\sqrt{y})dy$$
$$= 2\pi\left[\frac{4y^2}{2} - \frac{y^{\frac{5}{2}}}{\frac{5}{2}}\right]_{x=0}^{y=16}$$
$$= 2\pi\left(2\cdot 16^2 - 2\cdot\frac{4^5}{5}\right)$$
$$= \frac{1024\pi}{5}$$