Linear Systems

Linear Systems - Linearity

Suppose $f: X \to f[X]$, a function f is said to be *linear* if

$$f(c_1 x_1 + c_2 x_2) = c_1 f(x_1) + c_2 f(x_2)$$

where $x_1, x_2 \in X$, and c_1 and c_2 are constants.

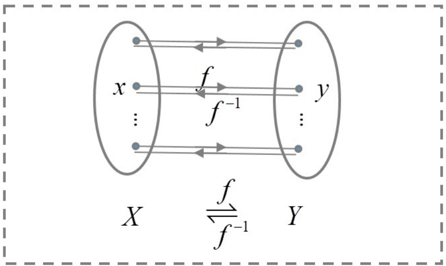
Linear Systems

Suppose $\begin{array}{c} f: X \to Y \\ \quad \Psi \qquad \Psi \\ x \mapsto y \end{array}$

and $\begin{array}{c} f^{-1}: Y \to X \\ \quad \Psi \qquad \Psi \\ y \mapsto x \end{array}$

then f^{-1} is called the *inverse function*,

s.t. $f^{-1}\left(f(x)\right) = x$ and $f\left(f^{-1}(y)\right) = y$.



Matrices

Let $\underline{\underline{A}}$ be a matrix.

- If there exists $\underline{\underline{I}}$ such that $\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{I}}\,\underline{\underline{A}} = \underline{\underline{A}}$, then $\underline{\underline{I}}$ is called the *identity matrix* (單位矩陣).
- If there exists $\underline{\underline{A}}^{-1}$ such that $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}$, then $\underline{\underline{A}}^{-1}$ is called the *inverse matrix* (逆矩陣).

Note that

- $\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$;
- $\underline{\underline{A}}^{-1} = \dfrac{adj(\underline{\underline{A}})}{\det(\underline{\underline{A}})}$ where $adj(\underline{\underline{A}}) = [cof(\underline{\underline{A}})]^T$.

Linear Systems

$\underline{\underline{A}}\underline{x} = \underline{f}$ is a linear system.

This linear system is said to be

- *homogeneous* if $\underline{f} = \underline{0}$;
- *non-homogeneous* if $\underline{f} \neq \underline{0}$;

System of Linear Equations

Theorem [System of linear equations] The system of linear equations $\underline{\underline{A}}\underline{x} = \underline{0}$ has a non-trivial solution if $\left|\underline{\underline{A}}\right| = 0$.

Proof.

$\because \underline{\underline{A}}^{-1} = \dfrac{adj\underline{\underline{A}}}{\left|\underline{\underline{A}}\right|}$, hence $\underline{\underline{A}}\underline{x} = \underline{0}$ has a trivial solution (i.e. $\underline{x} = \underline{0}$) if $\underline{\underline{A}}^{-1}$ exists.

$\therefore \underline{\underline{A}}\underline{x} = \underline{0}$ has a non-trivial solution only if $\underline{\underline{A}}^{-1}$ does not exist, i.e. $\left|\underline{\underline{A}}\right| = 0$. □