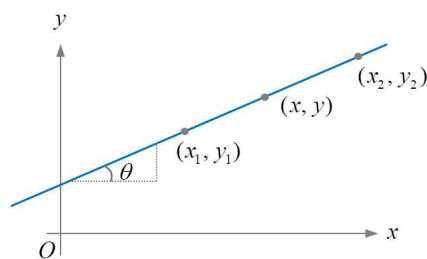


Calculus - Differentiation

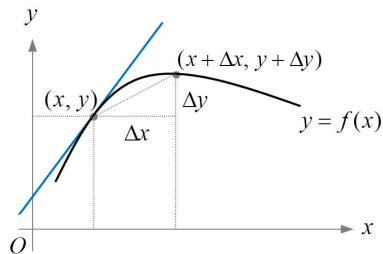
Equation of A Straight Line

For a straight line: we have

$$\text{the slope, } m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}.$$



Differentiation



The slope at point (x, y) is

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Differentiation

Example.

- $y = f(x) = x^2$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x$$

- $y = f(x) = x^3$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} = \lim_{\Delta x \rightarrow 0} [3x^2 + 3x\Delta x + (\Delta x)^2] = 3x^2$$

- $y = f(x) = x^n$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sum_{r=0}^n C_r^n x^{n-r} (\Delta x)^r - x^n}{\Delta x} = \lim_{\Delta x \rightarrow 0} \sum_{r=1}^n C_r^n x^{n-r} (\Delta x)^{r-1} = C_1^n x^{n-1} = nx^{n-1}$$

Differentiation

Example.

- $y = f(x) = \sin x$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} = \cos x$$

- $y = f(x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} = -\sin x$$

Chain Rule

Consider $y = f(x)$ where $x \equiv x(t)$, then

$$\begin{aligned} \frac{dy}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \frac{\Delta x}{\Delta t} \\ &= \frac{dy}{dx} \frac{dx}{dt} \end{aligned}$$

Product Rule

Consider $u \equiv u(x)$ and $v \equiv v(x)$, then

$$\begin{aligned}
 \frac{d}{dx}(uv) &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[u(x) + \Delta u(x)][v(x) + \Delta v(x)] - u(x)v(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left[\frac{\Delta u(x)}{\Delta x} v(x) + u(x) \frac{\Delta v(x)}{\Delta x} \right] \\
 &= \frac{du}{dx} v + u \frac{dv}{dx}
 \end{aligned}$$

Quotient Rule

$$\begin{aligned}
 \frac{d}{dx} \left(\frac{u}{v} \right) &= \frac{1}{v} \frac{du}{dx} + u \frac{d}{dx} \left(\frac{1}{v} \right) \\
 &= \frac{1}{v} \frac{du}{dx} - \frac{u}{v^2} \frac{dv}{dx} \\
 &= \frac{\frac{du}{dx} v - u \frac{dv}{dx}}{v^2}
 \end{aligned}$$