

Probability - Gaussian Distribution

Probability

	Discrete Random Variable	Continuous Random Variable
Probability	$P(X = x_i) = P(x_i)$	$P(a \leq x \leq b) = \int_a^b f(x)dx$
Expectation	$E(X) = \sum_{i=1}^m x_i P(x_i)$	$E(X) = \int_{-\infty}^{\infty} xf(x)dx$
Variance	$Var(X) = \sum_{i=1}^m (x_i - \mu)^2 P(x_i)$	$Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$

where $f(x)$ is the probability density function.

Probability

$$\begin{aligned}
 \text{Var}(X) &= \sum_{i=1}^m (x_i - \mu)^2 P(x_i) \\
 &= \sum_{i=1}^m (x_i^2 - 2\mu x_i + \mu^2) P(x_i) \\
 &= \sum_{i=1}^m x_i^2 P(x_i) - 2\mu \sum_{i=1}^m x_i P(x_i) + \mu^2 \sum_{i=1}^m P(x_i) \\
 &= \sum_{i=1}^m x_i^2 P(x_i) - 2\mu^2 + \mu^2 \\
 &= \sum_{i=1}^m x_i^2 P(x_i) - \mu^2 \\
 &= E(X^2) - [E(X)]^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(X) &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\
 &= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x) dx \\
 &= \int_{-\infty}^{\infty} x^2 f(x) dx - 2\mu \int_{-\infty}^{\infty} x f(x) dx + \mu^2 \int_{-\infty}^{\infty} f(x) dx \\
 &= E(X^2) - 2\mu^2 + \mu^2 \\
 &= E(X^2) - \mu^2
 \end{aligned}$$

Gaussian Integral

The *Gaussian integral* is



Gaussian Distribution

Let $z = \frac{x-\mu}{\sigma}$ s.t. $\int_{-\infty}^{\infty} f(z)dx = 1$ where $f(z) = ae^{-bz^2}$ is known as *Gaussian function*.

$$\Rightarrow \int_{-\infty}^{\infty} ae^{-b\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1 \Rightarrow a \int_{-\infty}^{\infty} e^{-\left(\sqrt{b}\frac{x-\mu}{\sigma}\right)^2} d\left(\sqrt{b}\frac{x-\mu}{\sigma}\right) = \frac{\sqrt{b}}{\sigma} \Rightarrow a = \frac{\sqrt{b}}{\sigma\sqrt{\pi}}$$

Take $b = \frac{1}{2}$, we have the probability density function, $f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ as the *Gaussian distribution* (a.k.a. *normal distribution*).

Standard Normal Distribution

(1) Normal Distribution: $X \sim N(\mu, \sigma^2)$

A random variable X follows a normal distribution with mean μ and variance σ^2 is represented by $X \sim N(\mu, \sigma^2)$

(2) Standard Normal Distribution: $Z \sim N(0, 1)$

$$Z = \frac{X - \mu}{\sigma}$$

$$E(Z) = E\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma} E(X - \mu) = \frac{1}{\sigma} (\mu - \mu) = 0$$

$$Var(Z) = Var\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X) = \frac{\sigma^2}{\sigma^2} = 1$$

Sum of Normally Distributed Random Variables

If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, then $(X_1 + X_2) \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Normal Distribution

Example.

- (1) $X \sim N(10, 4)$. Find $P(X \geq 12)$. (Ans. 0.1587)
- (2) $X \sim N(10, 4)$. Find $P(11 \geq X \geq 9.5)$. (Ans. 0.2902)
- (3) $X \sim N(50, 100)$. If $P(X > a) = 0.04$, find the values of a . (Ans. $a = 67.5$)
- (4) $X \sim N(50, 100)$. If $P(X < b) = 0.209$, find the values of b . (Ans. $b = 41.9$)