CARL VON OSSIETZKY universität OLDENBURG

Advanced Microeconomics

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Lecture 11: Static games with incomplete information

Essential reading:

- Gibbons, Chapter 3
- Tadelis, Chapter 12
- Carpenter & Robbett, Chapter 11

Static games with incomplete information

- Static games with incomplete information are also called static Bayesian games
  - *incomplete information:* at least one player is unsure about the payoff function of another player
- Examples:
  - Uncertainty about valuation of seller in auction
  - Cournot duopoly with asymmetric information about costs
  - Poker player may not know the cards of other players
  - Entry game with asymmetric information on capacity expansion costs
- We model this by
  - saying that players maybe one of multiple possible types
  - forming beliefs about what each type of player would do
  - Maximizing expected payoffs based on these beliefs

Introductory example: market entry game

- Player 1 (the current monopolist) decides whether to build a new factory to increase capacity (I) or not (nI).
- Player 2 decides whether to enter the market (E) or not (nE)
- Player 1 knows the cost of his capacity expansion
- Player 2 only knows that the cost is 3 or 0
  - He believes that the probability of high costs is p
    - this belief is common knowledge

High costs 3

Belief p

Player 1	Player 2: E	Player 2: nE
I	0, -1	2, 0
nI	2, 1	3, 0

Low costs 0

Belief $1-p$

Player 1	Player 2: E	Player 2: nE
I	, -1	, 0
nI	2, 1	3, 0

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High costs 3

Belief p

		Player 2	
		E	nE
Player 1	I	0, -1	2, 0
	nI	2, 1	3, 0

Low costs 0

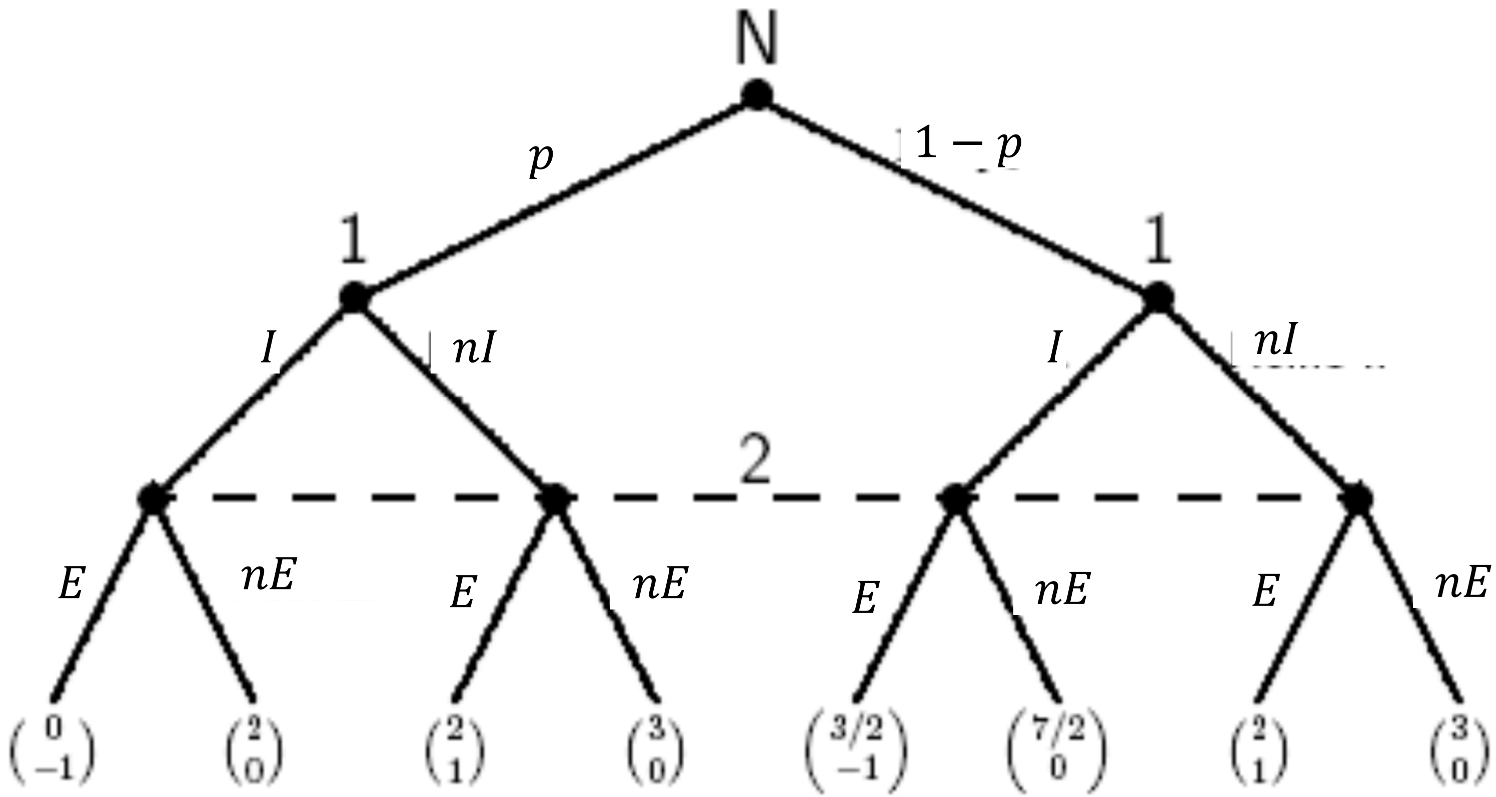
Belief $1-p$

		Player 2	
		E	nE
Player 1	I	3, -1	5, 0
	nI	2, 1	3, 0

- How shall we solve such a game of incomplete information?
- Trick of Harsanyi
  – Conversion of a game with **incomplete information** into a game with **imperfect information** with an additional player ("nature")
- Timing of events in a static game with incomplete information
  1. „Nature" draws a type realization for each player according to a probability distribution over all possible type profiles (*common knowledge*),
     - i.e. it determines the type vector $t \in (t_1, \dots, t_n)$ with $t_i \in T_i$ for all i
  2. Player i learns his own type t_i, but not the types of the other players.
  3. Players simultaneously choose an action $a_i \in A_i$
  4. Players receive the payoffs $u_i(a_1, \dots, a_n; t_i)$

Presentation of the introductory market entry game

- our introductory market entry game can be represented using Harsanyi's trick as follows.



Introductory example: market entry game

- Player 1 has a dominant strategy
  - If he is a high-cost type: "Do not invest" (nI)
  - If he is a low-cost type: "Invest" (I)
- Player 2
  - Player 2 knows about the dominant strategies of player 1
  - $E_2[\pi_2|E] =$
  - $E_2[\pi_2|nE] =$
  - Player 2 will enter if:

High costs

Belief p

		Player 2	
		E	nE
Player 1	I	0, -1	2, 0
	nI	2, 1	3, 0

Low costs

Belief $1 - p$

		Player 2	
		E	nE
Player 1	I	3, -1	5, 0
	nI	2, 1	3, 0

- Player 1 has a dominant strategy
  - If he is a high-cost type: "Do not invest" (nI)
  - If he is a low-cost type: "Invest" (I)
- Player 2
  - Player 2 knows about the dominant strategies of player 1
  - $E_2[\pi_2|E] = p \times 1 + (1-p)(-1) = 2p - 1$
  - $E_2[\pi_2|nE] = 0$
  - Player 2 will enter if $p > \frac{1}{2}$ (at $p = \frac{1}{2}$ he is indifferent)

High costs

Belief p

		Player 2: E	Player 2: nE
Player 1	I	0, -1	2, 0
	nI	<u>2</u>, 1	<u>3</u>, 0

Low costs

Belief $1-p$

		Player 2: E	Player 2: nE
Player 1	I	<u>3</u>, -1	<u>5</u>, 0
	nI	2, 1	3, 0

Normal form representation of a static Bayesian game

- Remember: The normal form of a static game is given by $G = \{I; \{S_i\}; \{u_i\}\}$
  - Since the strategy in a static game contains only one action, you can also write $G = \{I; \{A_i\}; \{u_i\}\}$ with $u_i(a_1, \ldots, a_n)$

- Normal form representation of a static Bayesian game
  - Representation of the fact that players know their own payoff function, but may be **unsure about the payoff functions of other players**
  - Player i's private information is summarized in his "**type„** $t_i \in T_i$
    - T_i: set of all possible types (type space) of player i
      - In introductory example: high or low investment cost

- The type t_i of a player i is known only to him and determines his payoff function
- The possible payoff functions of player i are $u_i(a_1, \dots, a_n; t_i)$
  - i.e. each type t_i of player i has a different payoff function

- Example
  - High cost and low cost type of player 1
  - Only one type of player 2 → his payoffs are the same in both matrixes

High costs, Belief p

Player 1 \ Player 2	E	nE
I	0, -1	2, 0
nI	2, 1	3, 0

Low costs, Belief $1-p$

Player 1 \ Player 2	E	nE
I	3, -1	5, 0
nI	2, 1	3, 0

Normal form representation of a static Bayesian game

- The normal form representation of a static Bayesian game consists of
  1. the set of players,
  2. the set of action $A_1, \ldots, A_n$,
  3. the type spaces $T_1, \ldots, T_n$,
  4. the beliefs $p_1, \ldots, p_n$ and
  5. the payoff functions $u_1, \ldots, u_n$ of the players

  – briefly: $G = (I; \{A_i\}; \{T_i\}, \{p_i\}, \{u_i\})$

High costs, Belief p

Player 1 \ Player 2	E	nE
I	0, -1	2, 0
nI	$\underline{2}$, 1	$\underline{3}$, 0

Low costs, Belief $1-p$

Player 1 \ Player 2	E	nE
I	$\underline{3}$, -1	$\underline{5}$, 0
nI	2, 1	3, 0

A new concept of strategy

- Definition strategy
  - Let $G = (I; \{A_i\}; \{T_i\}, \{p_i\}, \{u_i\})$ be a static game with incomplete information. A strategy s_i of player i is a function $s_i(t_i)$ that prescribes an action $a_i \in A_i$ for each possible type $t_i \in T_i$ of player i.

- Action or strategy
  - Static game with complete information ... strategy and action are congruent
  - Dynamic game with complete information ... strategy is far more complex than action, because a strategy prescribes an action for each possible action of the other players
  - Static game with incomplete information ... strategy prescribes an action depending on a player‘s own type

Definition and existence

- Definition Bayesian Nash equilibrium
  - In the static game with incomplete information $G = (I; \{A_i\}; \{T_i\}, \{p_i\}, \{u_i\})$, the strategy vector $s^* = (s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium if for all players i and for all types $t_i \in T_i$ of the player i, the strategies $s_i^*(t_i)$ solve the maximization problem
    $$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i[s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), s_{i+1}^*(t_{i+1}), \ldots, s_n^*(t_n); t_i]$$
    - i.e., a player of type i chooses his action so as to maximize his expected payoff
      - hence none of the types of a player wants to change his action
- Theorem on existence
  - Any static game with incomplete information $G = (I; \{A_i\}; \{T_i\}, \{p_i\}, \{u_i\})$, in which the sets $T_1, \ldots, T_n$ and $A_1, \ldots, A_n$ have a finite number of elements, has a Bayesian Nash equilibrium, possibly in mixed strategies.

Inefficient trade and adverse selection

(Tadelis, pp. 258)

- competitive market analysis: markets allocate goods to the people who value them the most
  - assumes *perfect information* about the value of the good
  - often departs from reality
- with incomplete information, market outcome often inefficient
- Model
  - Player 1 owns an orange grove
  - yield of fruit depends on quality of soil and other local conditions
    - this is private information
  - Common knowledge: quality of land low, mediocre, or high with $p = \frac{1}{3}$
  - For each type of quality, player 2‘s valuation exceeds that of player 1
  - Static pricing model: simultaneous submission of strategies:
    - Player 2: what price he is willing to pay
    - Player 1: what prices he is willing to accept (A) or reject (R)

Inefficient trade and adverse selection
(Tadelis, pp. 258)

- This is a Bayesian game of incomplete information
  1. set of players: $\{1{,}2\}$
  2. set of actions: $A_2 = \{\text{price } p \geq 0\}$, $A_1 = \{A, R\}$
  3. type spaces: $\Theta_1 = \{L, M, H\}$
  4. beliefs $p_2(\theta_1 = L) = p_2(\theta_1 = M) = p_2(\theta_1 = H) = \frac{1}{3}$
  5. payoff functions:

$$v_1(\theta_1) = \begin{cases} 10 & if \quad \theta_1 = L \\ 20 & if \quad \theta_1 = M \\ 30 & if \quad \theta_1 = H \end{cases} \quad \text{and} \quad v_2(\theta_1) = \begin{cases} 14 & if \quad \theta_1 = L \\ 24 & if \quad \theta_1 = M \\ 34 & if \quad \theta_1 = H \end{cases}$$

- Strategy of player 2: a price offer $p \geq 0$
- Strategy of player 1: $s_1: [0, \infty] \times \Theta_1 \to [A, R]$
  - A mapping from the offered price and the type space to a response

Inefficient trade and adverse selection

(Tadelis, pp. 258)

$$v_1(\theta_1) = \begin{cases} 10 & if \quad \theta_1 = L \\ 20 & if \quad \theta_1 = M \\ 30 & if \quad \theta_1 = H \end{cases} \quad \text{and} \quad v_2(\theta_1) = \begin{cases} 14 & if \quad \theta_1 = L \\ 24 & if \quad \theta_1 = M \\ 34 & if \quad \theta_1 = H \end{cases}$$

- Efficient outcome: land is always traded
  - with complete information, this would be the outcome of the game
- Outcome with incomplete information
  - player 2 should not offer more than his expected value of land: 24
    - But player 1 of type $\theta_1 = H$ would not sell at this price
  - Player 2 anticipates this and offers no more than $\frac{1}{2} \cdot 14 + \frac{1}{2} \cdot 24 = 19$
    - But player 1 types $\theta_1 = M, H$ would not sell at this price
  - Player 2 anticipates this and offers no more than 14

Result. In a Bayesian Nash equilibrium (BNE) only the lowest type of player 1 is trading. Furthermore, any price $p^* \in [10, 14]$ can be supported as a Bayesian Nash equilibrium.

- Proof of last sentence:
  - consider the following equilibrium strategies
  - Strategy of player 2: offer a price $p^* \in [10, 14]$
  - Strategy of player 1:

$$s_1(\theta_1) = \begin{cases} A \text{ if and only if } p \geq p^* \text{ and } R \text{ otherwise} & \text{when } \theta_1 = L \\ A \text{ if and only if } p \geq 20 \text{ and } R \text{ otherwise} & \text{when } \theta_1 = M \\ A \text{ if and only if } p \geq 30 \text{ and } R \text{ otherwise} & \text{when } \theta_1 = H. \end{cases}$$

  - these strategies are mutual best responses, and therefore a BNE
  - We have **adverse selection**: only the lowest quality is traded

Inefficient trade and adverse selection
(Tadelis, pp. 258)

Remark. This is a game with **common values:**

- the type of one player affects the payoffs of another player
  - it is this feature that causes adverse selection

Inefficient trade and adverse selection
(Tadelis, pp. 258)

- Important application: insurance markets
  - Insurance provider does not know the type/risk of insurance holder
  - Adverse selection: no insurance offered for low risk types
- Government can (partly) correct this market failure by making insurance obligatory
  - E.g., health insurance, pension insurance, unemployment insurance

Appendix (not relevant for exam): Mixed Strategies revisited: Harsanyi‘s Interpretation (Tadelis, pp. 264)

		Player 2	
		Heads	Tails
Player 1	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

- In lecture 3 we have shown that the matching pennies game has a unique mixed-strategy Nash equilibrium:
  - each player chooses Heads with probability 0.5
- critique: players are indifferent between H and T, yet they are prescribed to randomize in a precise way

Appendix (not relevant for exam): Mixed Strategies revisited: Harsanyi‘s Interpretation (Tadelis, pp. 264)

Player 2

		Heads	Tails
Player 1	Heads	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
	Tails	$-1, 1+\varepsilon_2$	$1, -1$

- each player has a slight preference for H over T, or vice versa
  - Leads to above “perturbed” Matching Pennies game
- Value of ε_i is private information
- Distribution is common knowledge: ε_1 and ε_2 are independent and uniformly distributed on the interval $[-\varepsilon, \varepsilon]$ for some small ε
  - thus: if player i believes that the other player chooses H with probability 0.5, then i has a strict preference for
    - for H if $\varepsilon_i > 0$, and
    - for T if $\varepsilon_i < 0$

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Player 2

Player 1		Heads	Tails
	Heads	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
	Tails	$-1, 1+\varepsilon_2$	$1, -1$

- This is a Bayesian game of incomplete information
  1. set of players: $\{1,2\}$
  2. set of actions: $A_1 = A_2 =$ {Heads, Tails}
  3. type spaces: $T_1 = T_2 = [-\varepsilon, \varepsilon]$
     - ε_i is the type of player
  4. beliefs $p_1(H)$, $p_2(H)$: belief that the other player has preference for H
  5. the payoff functions: payoffs in matrix
- A pure strategy is a mapping $s_i: [-\varepsilon, \varepsilon] \rightarrow [H, T]$
  – Strategy assigns a choice to every type of player i

Appendix (not relevant for exam): Mixed Strategies revisited: Harsanyi‘s Interpretation (Tadelis, pp. 264)

Player 1 \ Player 2	Heads	Tails
Heads	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
Tails	$-1, 1+\varepsilon_2$	$1, -1$

Claim. In the Bayesian perturbed Matching Pennies game, there is a unique pure-strategy Bayesian Nash equilibrium (BNE) in which $s_i(\varepsilon_i) = H$ if and only if $\varepsilon_i \geq 0$, and $s_i(\varepsilon_i) = L$ if and only if $\varepsilon_i < 0$. This equilibrium converges in outcomes and payoffs to the Matching Pennies game when $\varepsilon \to 0$.

- BNE: strategies must be a mutual best response,
  - i.e. for each player her strategy must maximize her expected payoff, given the other player‘s equilibrium strategy
- Based on i‘s belief that ε_j is uniformly distributed on the interval $[-\varepsilon, \varepsilon]$, player j types play H with probability 0.5
  - thus: above strategy maximizes i‘s expected payoff

Appendix (not relevant for exam): Mixed Strategies revisited: Harsanyi‘s Interpretation (Tadelis, pp. 264)

- Main message:
  - each type of player chooses a *pure* strategy
  - but: the distribution of types makes a player have beliefs as if he were facing a player who is playing a mixed strategy
  - Hence the outcomes and payoffs of the mixed strategy equilibrium and the above game with a little bit of private information are the same
- Harsany’s “purification” argument: using mixed-strategy equilibria in simple games of complete information can be thought of as a solution of more complex games of incomplete information
- Remember: we informally motivated mixed strategy equilibria with the idea that a game is played repeatedly
  - the previous game is played once, but by different, randomly chosen types of players