CARL VON OSSIETZKY universität OLDENBURG

Advanced Microeconomics

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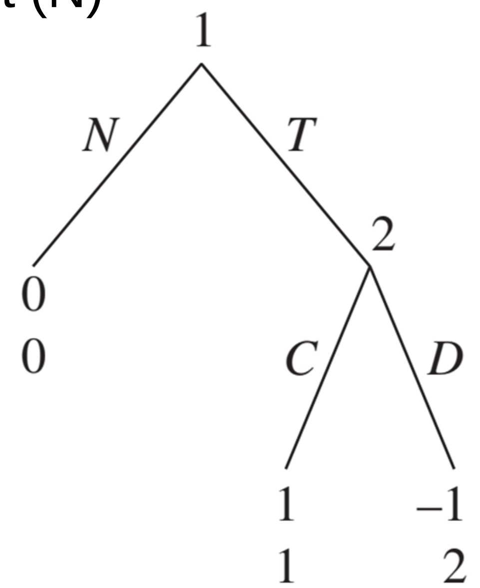
L 6: Dynamic games with complete and perfect information

- Gibbons, Chapter 2.1
- Tadelis, Section 7
- Osborne, Chapters 5 and 6
- McDonald and Solow (1981): "Wage bargaining and employment", American Economic Review, 71(5), 896-908.

- Until now, all players decided *simultaneously* on their actions
  – represented by a **normal form game**
- But in many situations players can decide *sequentially* and also more than once
- Example „Trust game“
  – Player 1 decides whether to „trust“ (T) player 2 or not (N)
    - example for trust: paying up front at Ebay and hoping that the seller will deliver
  – Player 2 decides whether to cooperate (C) or to defect (D)

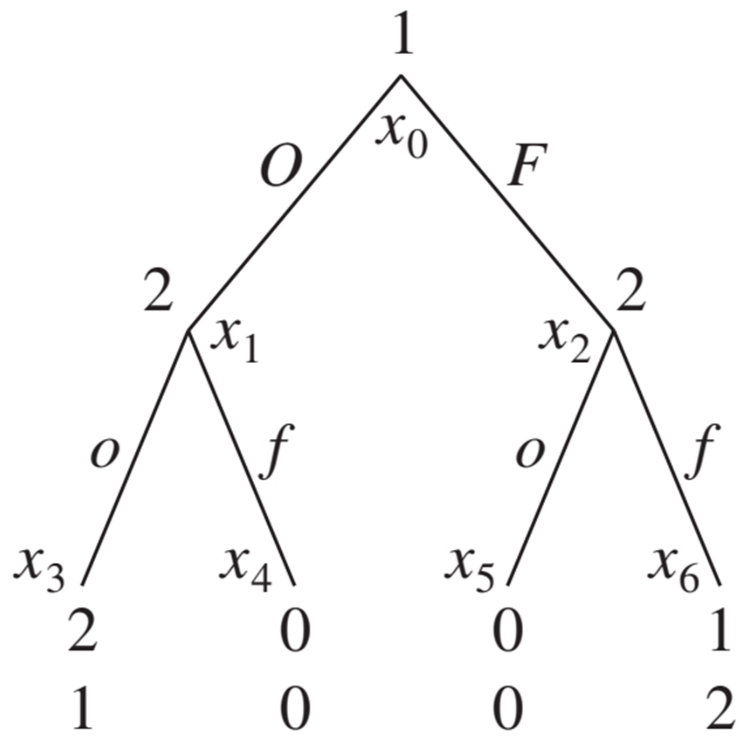


The Extensive-Form Game

- **Definition**. A **game in extensive-form** consists of:
  1) Set of players, N.
  2a) Order of moves.
  2b) Actions of players when they can move.
  2c) The knowledge that players have when they can move.
  2d) Probability distributions over exogenous events ("moves of nature").
  3) Players' payoffs as a function of outcomes, $\{v_i(.)\}_{i \in N}$.
- The structure of the extensive-form game (represented by the above points) is common knowledge among all the players.
- definitions of games in normal and extensive form have similar structure
  - But in normal form game, point (2) was simply a collection of sets of pure strategies $\{S_1, \ldots, S_n\}$ of the players
  - This aspect is much more complex in extensive form games

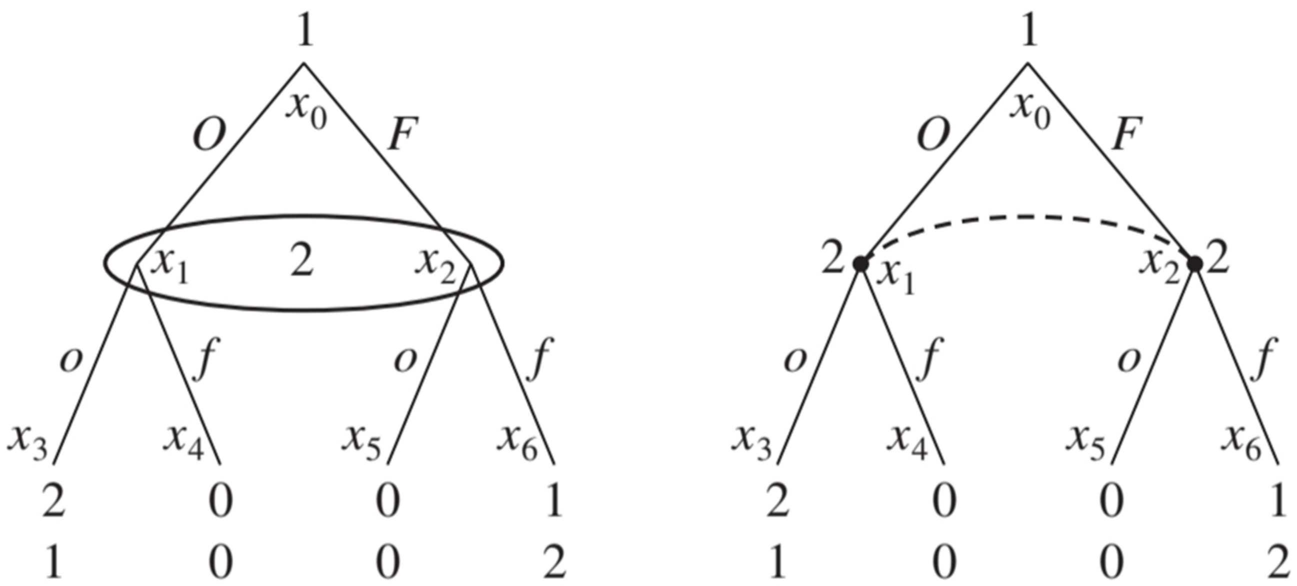
Extensive-form games are often represented by game trees

Definition. A **game tree** is a set of nodes $x \in X$ with a precedence relation $x > x'$, which means "x precedes x'." Every node in a game tree has only one predecessor. The precedence relation is *transitive* ($x > x', x' > x'' \Rightarrow x > x''$), *asymmetric* ($x > x' \Rightarrow$ not $x' > x$), and *incomplete* (not every pair of nodes x, y can be ordered). The ***root*** of the tree, denoted by x_0 precedes any other $x \in X$. Nodes that do not precede other nodes are called ***terminal nodes***, denoted by the set $Z \subset X$. Terminal nodes denote the final outcomes of the game with which payoffs are associated. Every node x that is not a terminal node is assigned either to a player, $i(x)$, with the action set $A_i(x)$, or to Nature.



Information Sets

- We can represent simultaneous-move games as game trees
  - Example: simultaneous-move Battle of the Sexes game.

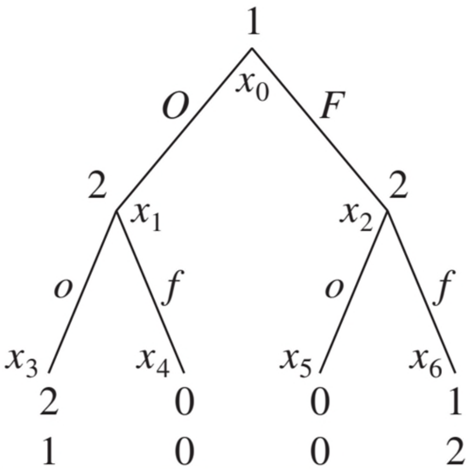


  - The fact that 2 has no information whether 1 has chosen O or F is represented alternatively by (i) an ellipse, or (ii) a dashed line.

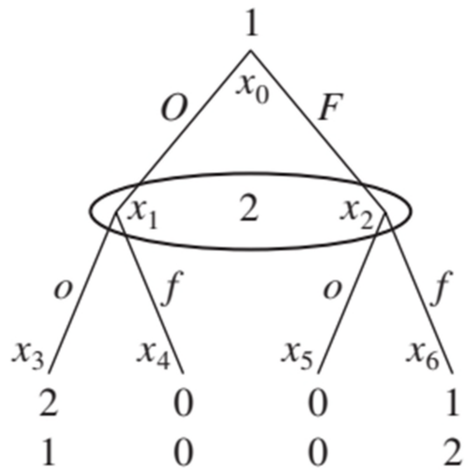
Information Sets

Definition. Every player i has a collection of **information sets** $h_i \in H_i$ that partition the nodes of the game at which player i moves with the following properties:

1. If the information set h_i is a singleton that includes only x, then player i who moves at x knows that he is at x.
2. If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$, then player i who moves at x does not know whether he is at x or x'.
3. If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$, then $A_i(x') = A_i(x)$.



Example Battle of Sexes game: sequential- and simultaneous move



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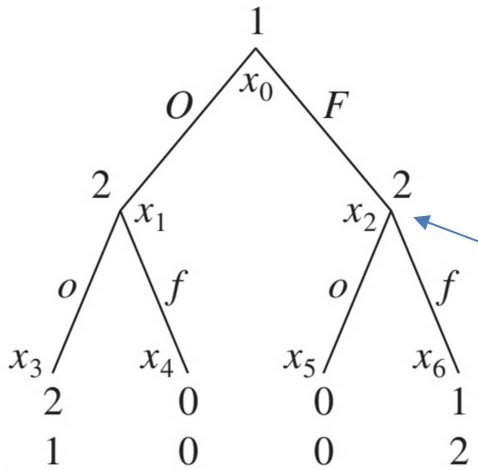
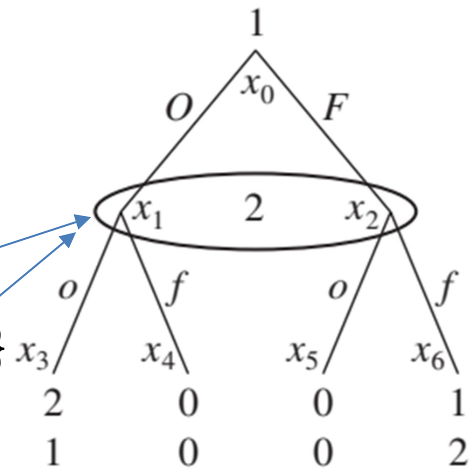


Example Battle of Sexes game:
sequential- and simultaneous move

$h_2 = \{x_2\}$

$h_2 = \{x_1, x_2\}$

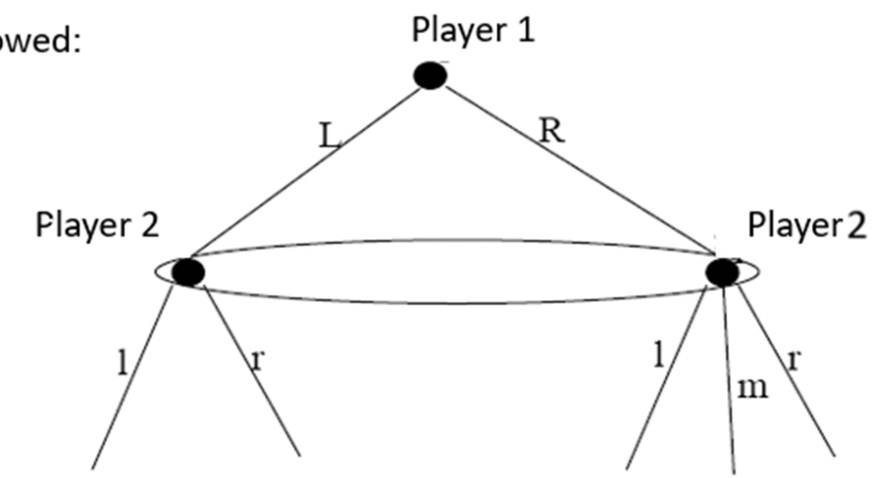
$A_2(x_1) = A_2(x_2) = \{o, f\}$



Information sets

- Properties 1 and 2 describe what knowledge players have when they can move.
- Property 3: „If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$, then $A_i(x') = A_i(x)$."
  - This excludes cases as the one depicted, where player 2 should be able to distinguish between the two nodes because he has different actions available.

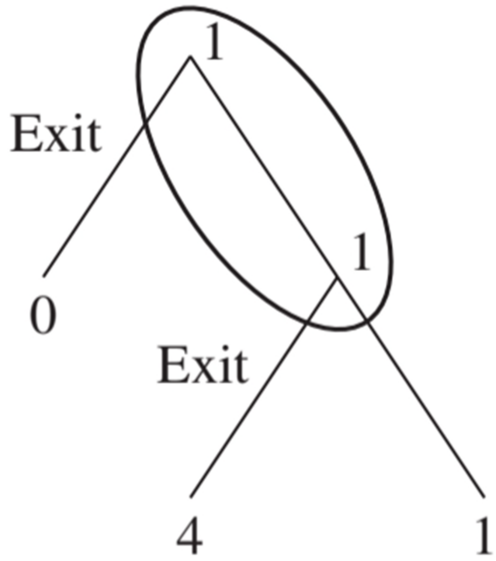
➔ Not allowed:



Information sets and perfect recall

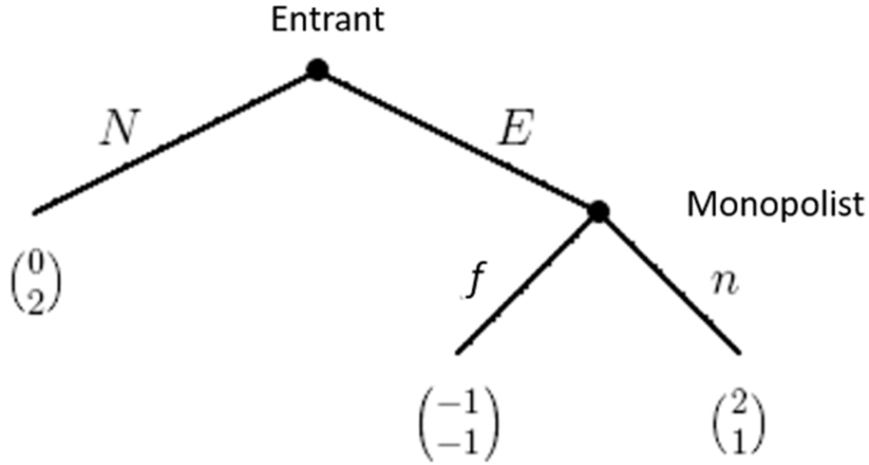
Definition. A game of **perfect recall** is one in which no player ever forgets information that he previously knew.

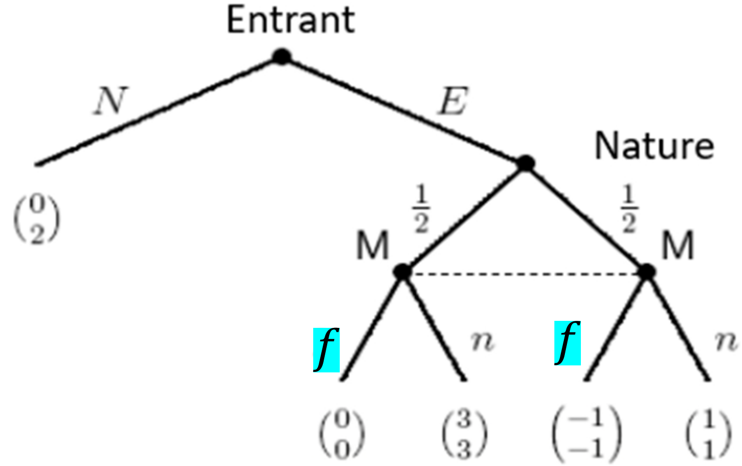
- In the whole lecture we focus on games with perfect recall.
- A game with *imperfect recall* is “The Absent-Minded Driver”
  - There are 2 exits from a motorway
    - The 1st leads to an unsafe neighbourhood
    - The 2nd is the safe way home
    - If he misses the 2nd exit, he has to take a diversion
  - Imperfect recall means that when the driver is at an exit, he can’t remember whether he has already passed one



Exogenous uncertainty: Modeling as random moves of nature

- **Random moves of nature** (exogenous uncertainty) can be represented by information sets.
- Example: Market entry game with/without move of nature
  - Left: Entrant decides whether to enter (E) or not (N). Then monopolist decides whether to fight (f) or not (n) to fight and share the market.
  - Right: Additional player - "**Nature**".
  - from the set of possible states of the world (high or low demand), nature selects one according to a given probability distribution $({}^{1}/_{2}, {}^{1}/_{2})$





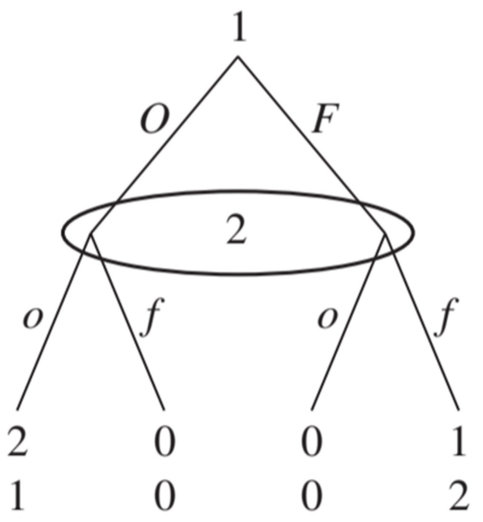
Perfect versus imperfect information

Definition. A game of complete information in which every information set is a singleton and there are no moves of Nature is called a **game of perfect information**. A game in which some information sets contain several nodes or in which there are moves of Nature is called a **game of imperfect information**.

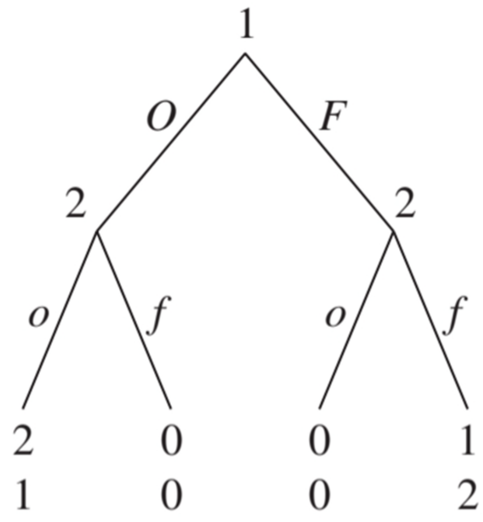
- Remember: complete information means that each player i knows the action set and the payoff function of each player $j \in N$, and this itself is common knowledge
- Definition implies that with **perfect information** all players play sequentially and each player observes all previous moves

A **pure strategy** for player i is a *complete plan of play* that describes which pure action player *i* will choose *at each of his information sets*.

- Pure strategies in simultaneous-move game:
  - $S_1 = \{O, F\}, S_2 = \{o, f\}$
- Pure strategies in sequential-move game:
  - $S_1 = \{O, F\}, S_2 = \{oo, of, fo, ff\}$, where "$of$" is shorthand for "play o if player 1 plays O and play f if he plays F."



Example Battle of Sexes game: simultaneous-move (left) and sequential-move (right)

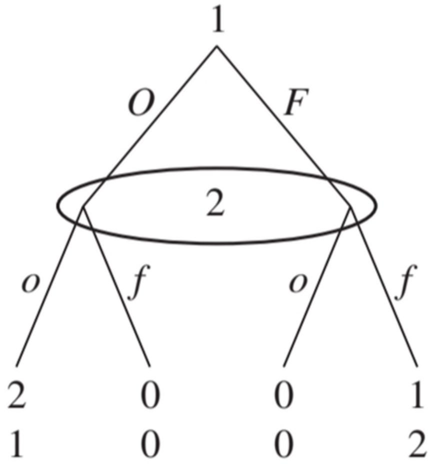


Pure Strategies: Formal definition

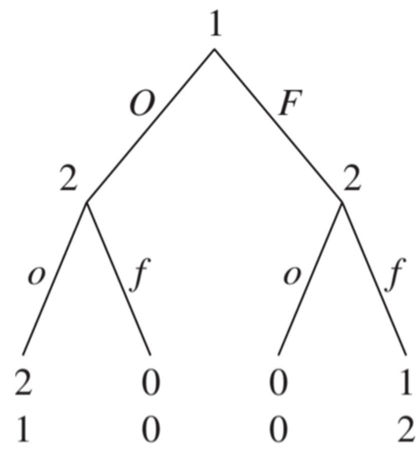
Some notation:

- $A_i(h_i)$: the actions that player i can take at information set h_i
- $A_i = \cup_{h_i \in H_i} A_i(h_i)$: set of all actions of player i, (i.e., the union of all the elements in all the sets $A_i(h_i)$

Definition. A **pure strategy** for player i is a mapping $s_i: H_i \rightarrow A_i$ that assigns an action $s_i(h_i) \in A_i(h_i)$ for every information set $h_i \in H_i$. We denote by S_i the set of all pure-strategy mappings $s_i \in S_i$.



Example Battle of Sexes game: simultaneous- and sequential-move



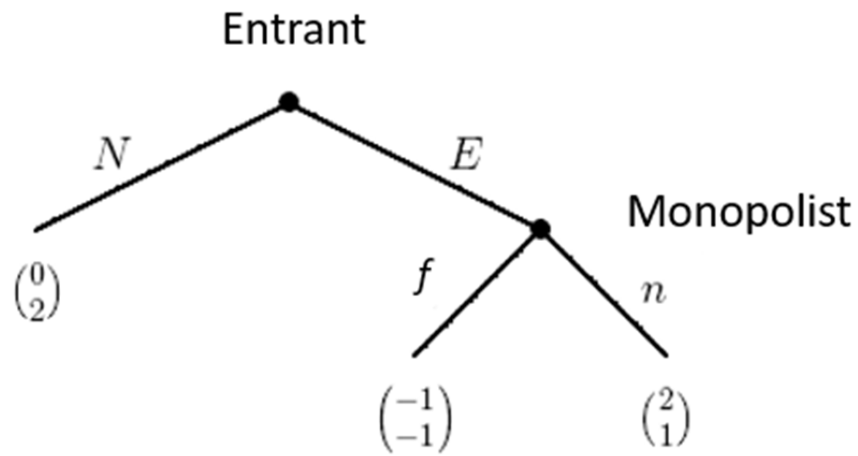
Mixed strategies in dynamic games

Definition. A mixed strategy for player i is a probability distribution over his pure strategies $s_i \in S_i$.

- Similar to definition for the case of normal form games
- But strategies usually more complex
  - Not just an action, but a complete plan of play

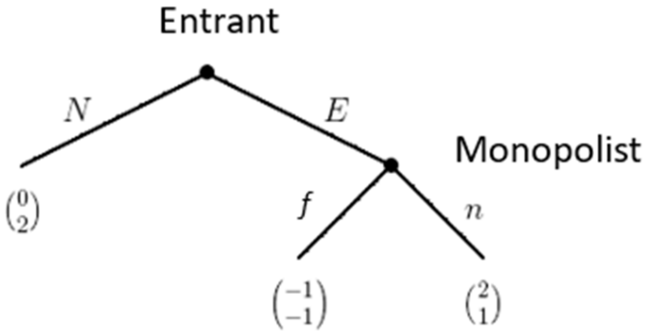
- In this lecture we don‘t analyze mixed strategies in sequential games

- A sequential game with a finite number of stages and perfect information is solved from behind by „**backward induction**"



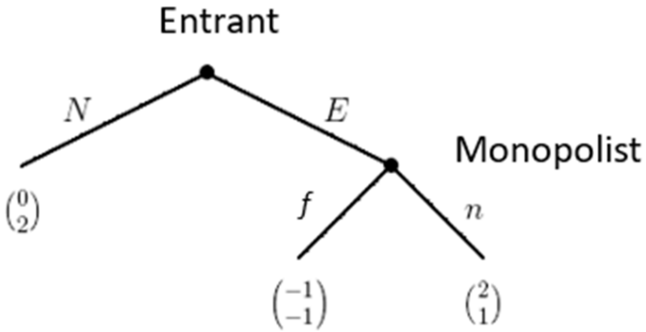
- Solution
  - Monopolist: Given that the entrant has entered, it is optimal for it to share the market
  - Entrant: If he enters, the monopolist will share the market; so he should enter
  - The result of the backward induction is therefore (E, n)

- Game in matrix representation



Sequential rationality and backwards induction: Market entry game

- Game in matrix representation



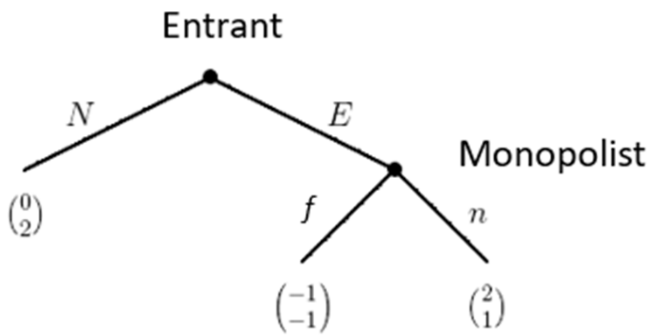
$\Rightarrow$

E \ M	f	n
N	0, 2	0, 2
E	-1, -1	2, 1

- The outcome of backward induction is a Nash equilibrium
  - Given that the entrants plays E, it is optimal for the monopolist to play n
  - Given that monopolist plays n, it is optimal for the entrant to play E

Sequential rationality and backwards induction: Market entry game

- Game in matrix representation



E \ M	f	n
N	0, 2	0, 2
E	-1, -1	2, 1

- Analysis of the normal form shows that there is a second Nash-equilibrium (N, f)
  - The monopolist "threatens" to fight if the entrant enters
- However, the Nash equilibrium (N, f) is not convincing
  - To fight is a **non-credible threat**, because it is not in the interest of the monopolist to actually carry out the threat

Sequential rationality

- The players' equilibrium strategies must satisfy the **principle of sequential rationality**

Definition. Given strategies $\sigma_{-i} \in \Delta S_{-i}$ of i's opponents, we say that σ_i is **sequentially rational** if and only if i is playing a best response to σ_{-i} in each of his information sets.

- i.e., a player's strategy should dictate an optimal action at each information set in the game tree
  - Including those that are not reached in equilibrium

Proposition (Zermelo's Theorem). Any **finite game of perfect information** has a **backward induction solution** that is **sequentially rational**. Furthermore if no two terminal nodes prescribe the same payoffs to any player then the backward induction solution is unique.

Next lecture: extended solution concept that covers imperfect information

Example: Wages and employment

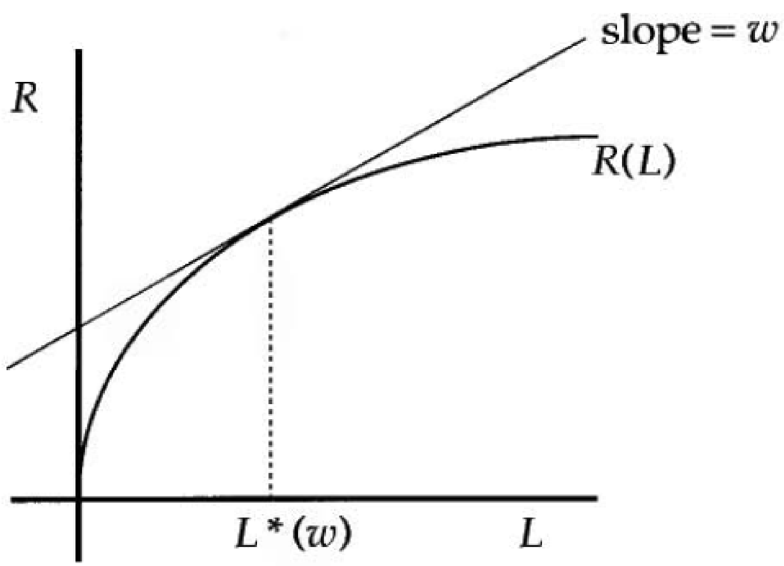
- Leontief (1946), later in numerous variations (e.g. McDonald and Solow 1981, see reference).

- Two-stage game:
  1. Union (player 1) determines the wage rate w
  2. Company (player 2) determines the employment level L

- Solution by backward induction
  – i.e., we first determine the firm‘s best response: $L(w)$

Wages and employment

- Stage 2 - Decision by the firm
  - $R(L)$: Revenues as a function of employment level L
  - $R(L)$ is increasing and strictly concave, i.e. $R'(L) > 0, R''(L) < 0$
  - Profit maximization:
  - The optimal employment level is a decreasing function of w,
    - i.e. $L'(w) < 0$ (why?)

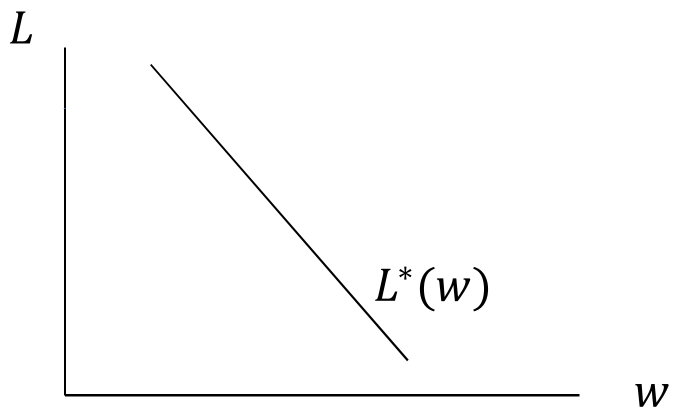


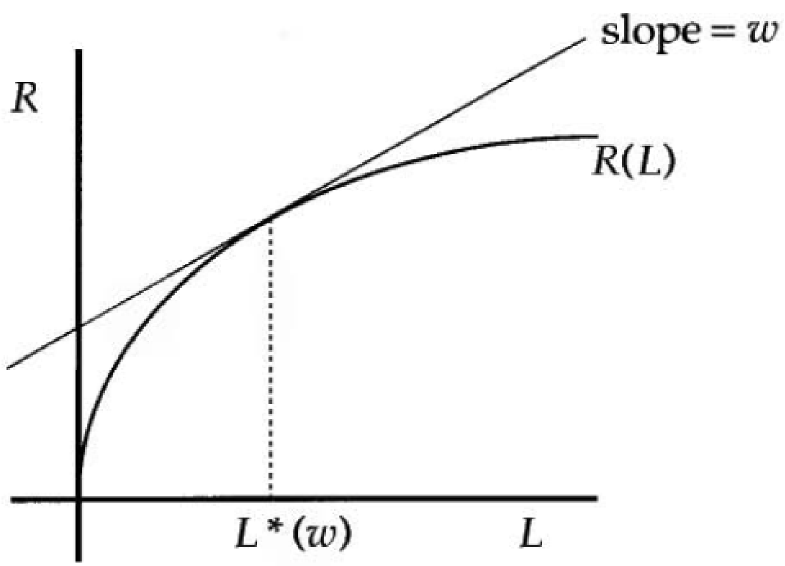
Wages and employment

- Stage 2 - Decision by the firm
  - $R(L)$: Revenues as a function of employment level L
  - $R(L)$ is increasing and strictly concave, i.e. $R'(L) > 0, R''(L) < 0$
  - Profit maximization:

$$\max_L \pi(L) = R(L) - wL \implies R'(L) = w$$

  - The optimal employment level is a decreasing function of w,
    - i.e. $L'(w) < 0$ (why?)





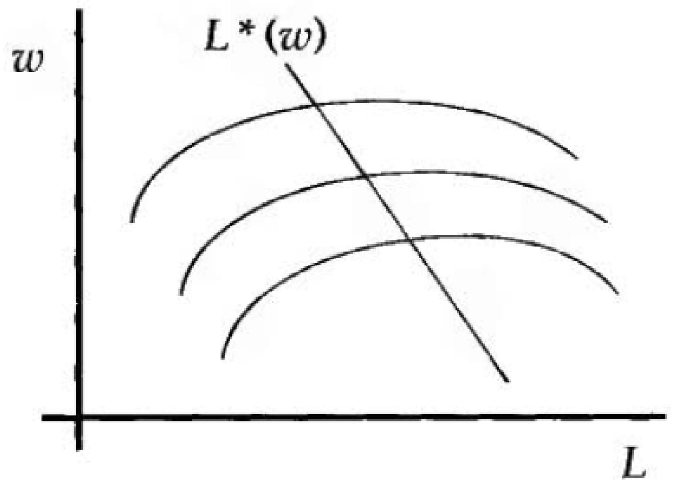
- Best response functions $L^*(w)$ and iso-profit lines
  - Iso-profit lines (= wage at which firm makes constant profits $\bar{\pi}$)
  - $\lim_{L \to 0} R'(L) = \infty$ and $\lim_{L \to \infty} R'(L) = 0$. Therefore:
  - If L is small, then $\frac{dw}{dL} > 0$, i.e. a higher employment level leads to a higher wage so as to keep profits constant
  - If L is large, then $\frac{dw}{dL} < 0$, i.e. a higher employment level leads to a lower wage so as to keep profits constant

Wages and employment

- Best response functions $L^*(w)$ and iso-profit lines
  - Iso-profit lines (= wage at which firm makes constant profits $\bar{\pi}$)

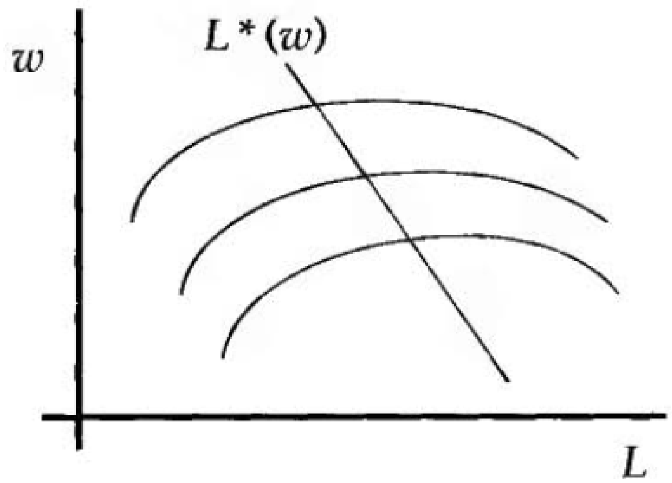
$$R(L) - wL = \bar{\pi} \iff w = \frac{R(L)-\bar{\pi}}{L} \text{ with } \frac{dw}{dL} = \frac{R'(L)L-[R(L)-\bar{\pi}]}{L^2}$$

  - Assumption: $\lim_{L\to 0} R'(L) = \infty$ and $\lim_{L\to\infty} R'(L) = 0$. Therefore:
  - If L is small, then $\frac{dw}{dL} > 0$, i.e. to keep profits constant, a higher employment level must go along with higher wage
  - If L is large, then $\frac{dw}{dL} < 0$, i.e. to keep profits constant a higher employment level must be compensated by lower wage



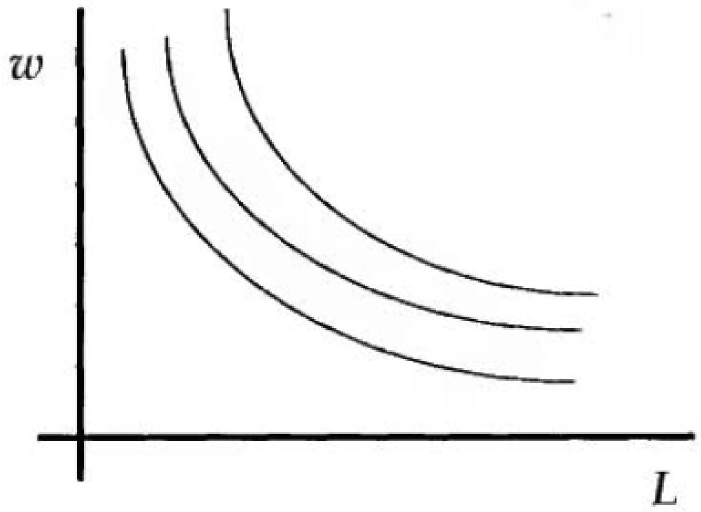
Wages and employment

- Best response functions $L^*(w)$ and iso-profit lines
  - $L^*(w)$ is a decreasing function (see above)
  - $L^*(w)$ intersects each of the iso profit lines at its maximum
    - Because for each w, firms choose the L, where they achieve the iso-profit line with the highest profits
      - lower iso-profit lines mean higher profits

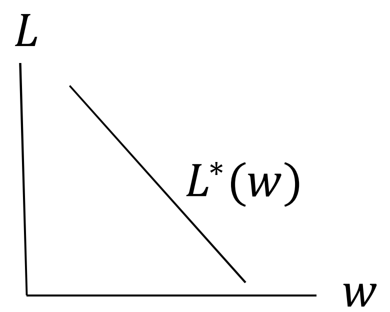


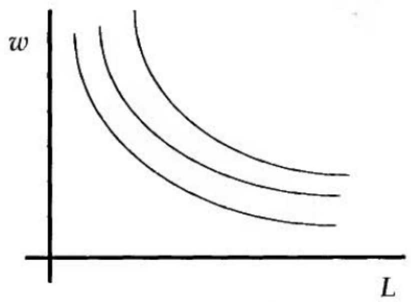
Wages and employment

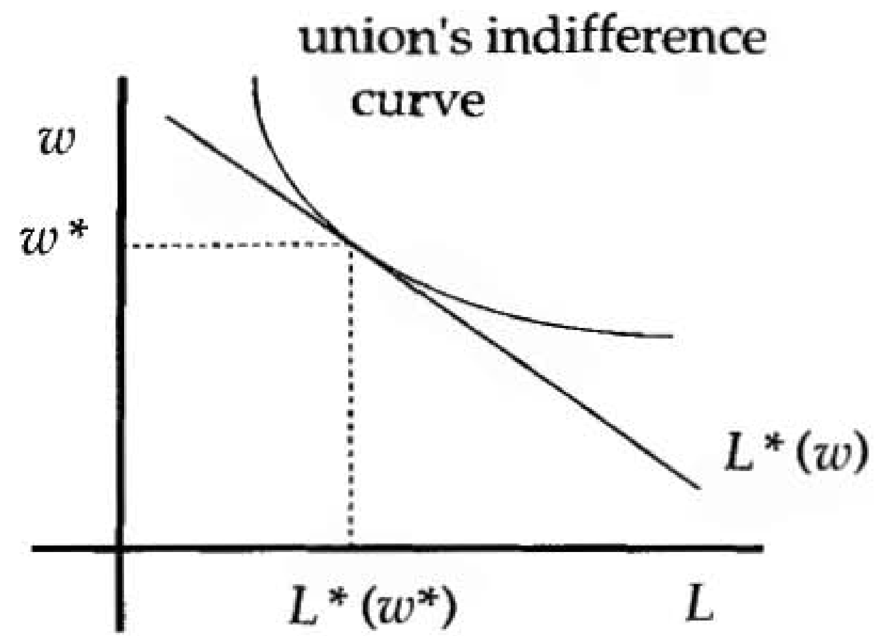
- Stage 1 - Decision of the trade union
  - Payoff functions are based on the utility of (potential) workers, $U(w, L)$
  - $U(w, L)$ is strictly monotonically increasing in w and L (a higher wage w is better, a higher employment level L is better) and quasiconcave
    - leads to convex indifference curves in figure



- Stage 1 - Decision of the trade union
  - Union anticipates the reaction function $L^*(w)$ of the firm and chooses the highest indifference curve compatible with it

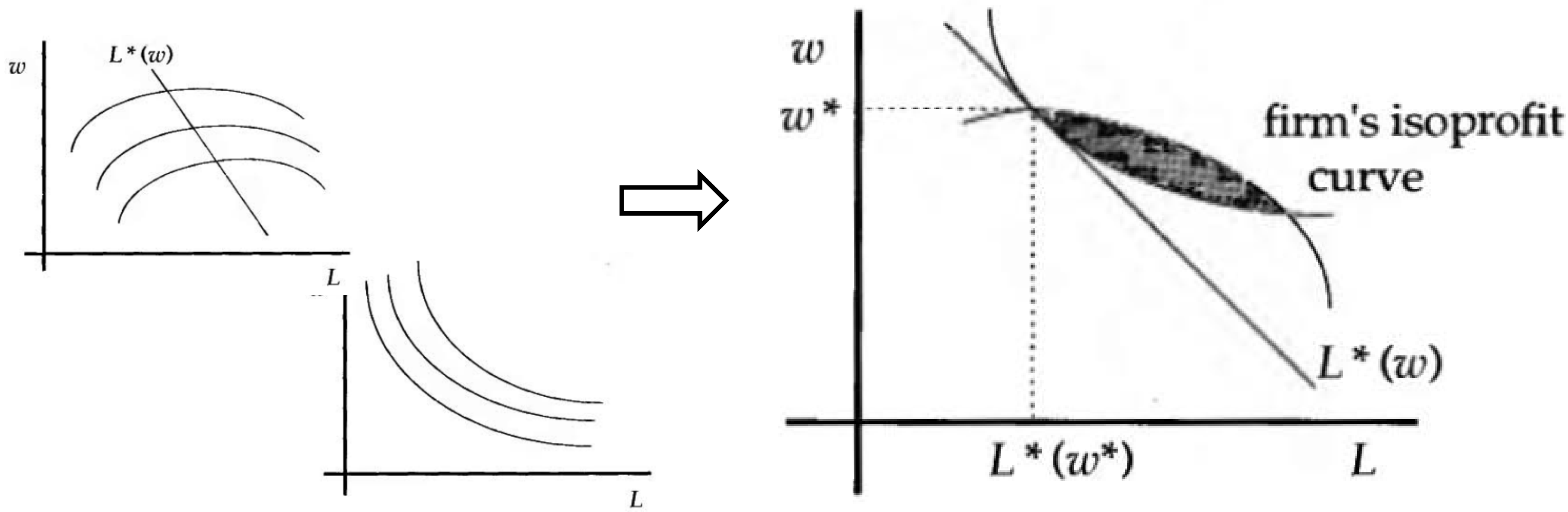






Wages and employment

- Stage 1 - Decision of the trade union
  - (w^*, L^*) is **inefficient** because the union and the company do not negotiate employment and wage levels at the same time
  - Unions and companies could both do better by choosing a point within the shaded ellipse (reduce wages a little and increase employment a little)



Wages and employment: Policy Implications

- The inefficiency arises because the union and the company do not negotiate employment and wage levels at the same time
- Model thus provides an argument why employment and wage levels should be negotiated together
  - Which is often the case