Multivariate forecasting: Multiple regression

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Overview

1. The linear regression model
2. The (adjusted) determination coefficient
3. The quality of the regression function
4. Population and Sample
5. The t- and F-statistic
6. Betas
7. Multicollinearity
8. The autocorrelation in the residuals

Multiple linear regression analysis

- In general: **Most important** and **most frequently** used method of multivariate data analysis.
- Idea: Our time series can be predicted by a **linear combination** of one or more different time series.
- Also: **description** and **explanation** of relationships.
- Assumption: level of measurement of all time series variables is **metric** or **nominal binary**.

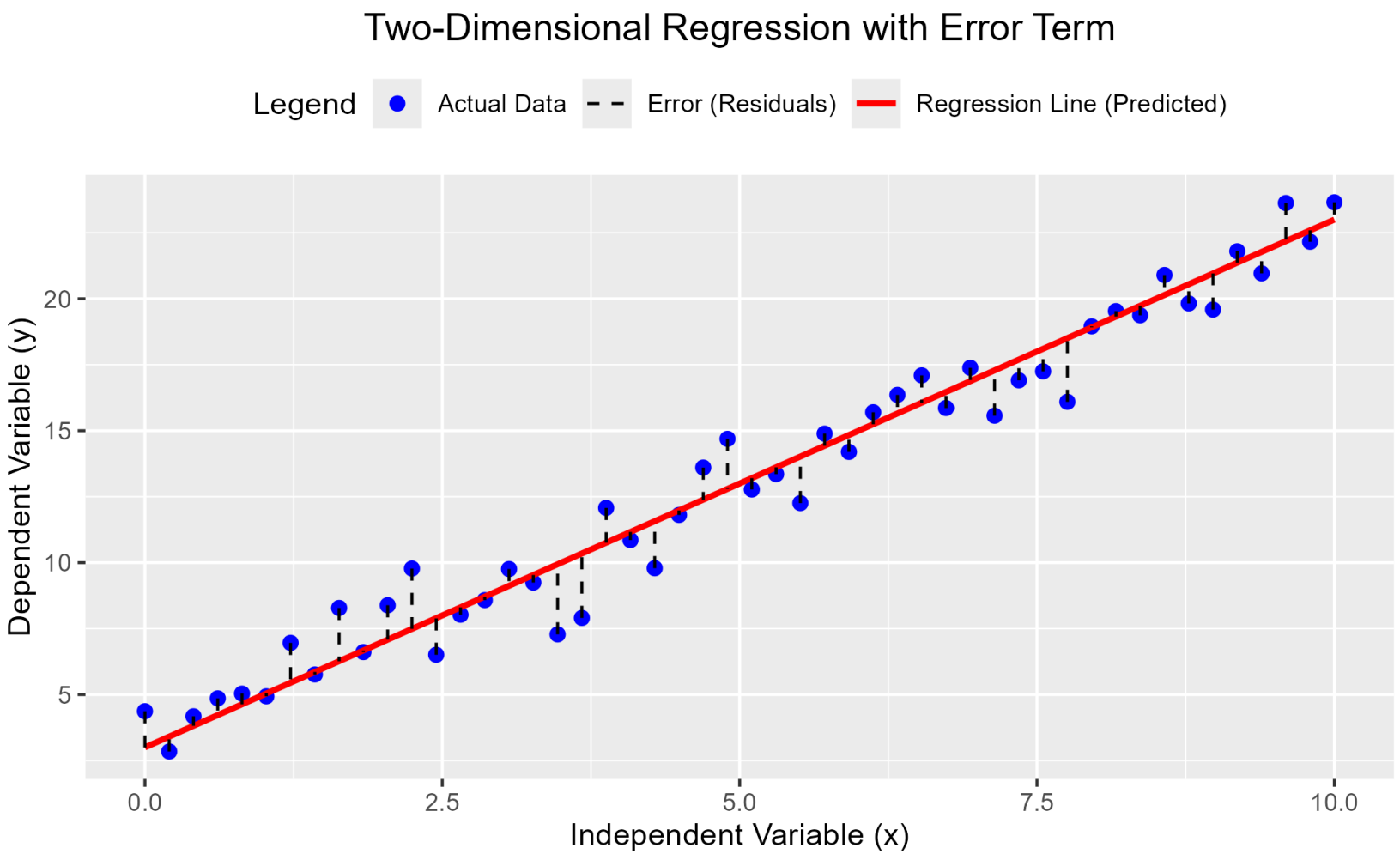
Linear regression model

$$y_{t+1} = b_0 + b_1 x_{1_t} + ... + b_J x_{J_t} + u_t$$

with y as depending time series to be forecasted, $x_1, x_2, ..., x_J$ as explaining time series with time lags and $u_t = y_{t+1} - \hat{y}_{t+1}$ as **random fluctuations** between observed and predicted time series value.
The linear coefficients $b_0, b_1, ..., b_J$ are determined by

$$\sum_{t=1}^{T} u_t^2 = \sum_{t=1}^{T} (y_{t+1} - \hat{y}_{t+1})^2 = \sum_{t=1}^{T} (y_{t+1} - b_0 - b_1 x_{1_t} - ... - b_J x_{J_t})^2 \rightarrow$$

$Min!$



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(A) Decomposition of total deviation

$$Total\ deviation = \quad explained\ deviation + \quad residual$$

$$y_t - \bar{y} \qquad = \qquad \hat{y}_t - \bar{y} \qquad + \quad y_t - \hat{y}$$

(B) Decomposition of total variation

$$Total\ variation = \quad explained\ variation \ + \ not\ explained\ variation$$

$$\sum_{t=1}^{T}(y_t - \bar{y})^2 \quad = \quad \sum_{t=1}^{T}(\hat{y}_t - \bar{y})^2 \quad + \quad \sum_{t=1}^{T}(y_t - \hat{y})^2$$

This non-trivial relationship applies only for **linear regression** with coefficients estimated by method of least squares!

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The determination coefficient r^2

$$R^2 = \frac{explained\ variation}{Total\ variation} = \frac{\sum_{t=1}^{T}(\hat{x}_t - \bar{x})^2}{\sum_{t=1}^{T}(x_t - \bar{x})^2}$$

The **determination coefficient** R^2 measures the goodness-of-fit of the trend line towards the time series. R^2 is normalized between zero and one. The higher the determination coefficient, the better the time series is described by the trend line.

The adjusted determination coefficient

The $\bar{R}^2$ is designed to provide a more accurate measure of goodness-of-fit in regression models, particularly when comparing models with different numbers of predictors or sample sizes.

$$\bar{R}^2 = R^2 - \frac{J(1-R^2)}{T-J-1}$$

- R^2: Ordinary determination coefficient (explains the proportion of variance in the dependent variable accounted for by the model).
- J: Number of explanatory variables (predictors).
- T: Total number of observations (sample size).
- $\bar{R}^2$ can be negative if the model performs poorly or if irrelevant variables are added.
- It is generally lower than R^2, reflecting the adjusted measure's stricter criteria.

Standard error of the estimation

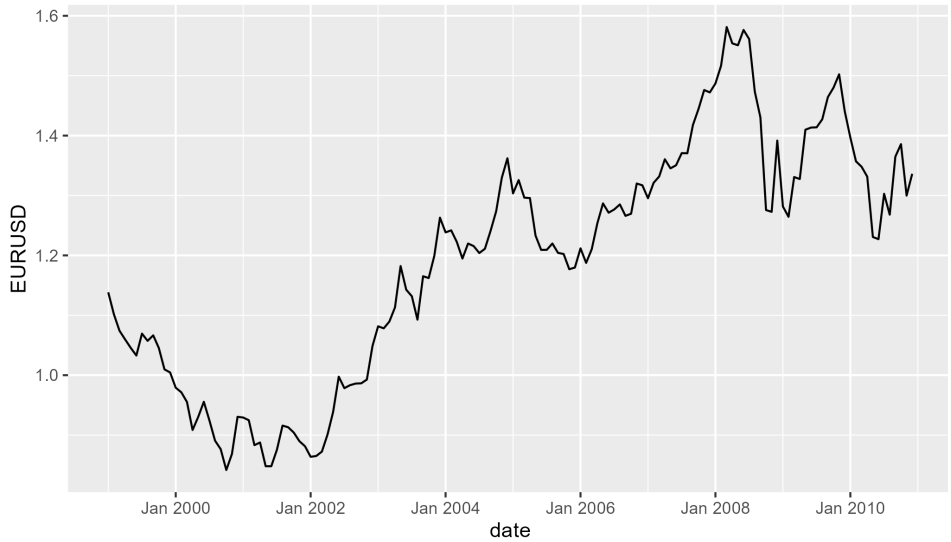
$$s = \sqrt{\frac{\sum_{t=1}^{T} u_t^2}{T - J - 1}} = \sqrt{\frac{not\ explained\ variation}{No.\ of\ observ. - No.\ of\ expl.\ - 1}}$$

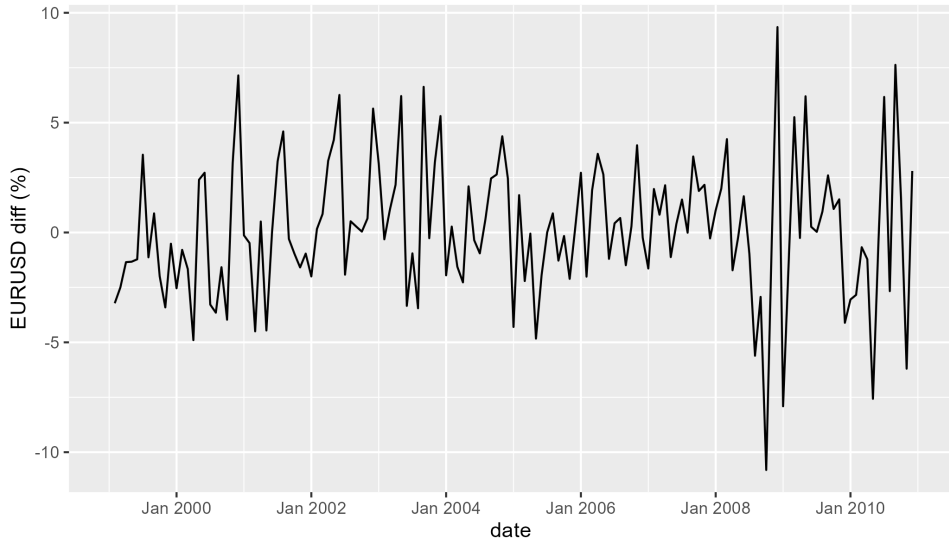
indicates the **mean error** of applying the regression function in order to estimate the values of the forecasting time series y.

Forecast of monthly percentage rate of change of EURO/USD exchange rate

DATE	EURUSD	EURUSD t+1 diff (%)	EURUSD diff (%)	Eurostoxx diff (%)	DJ diff (%)	Retail sales diff12 (%)	Unemployment diff (%)
JAN 2010	1.40	-2.84	-3.05	-6.35	-3.46	-3.61	10.46
FEB 2010	1.36	-0.67	-2.84	-1.74	2.56	1.04	0.70
MAR 2010	1.35	-1.22	-0.67	7.43	5.15	8.71	-2.06
APR 2010	1.33	-7.57	-1.22	-3.90	1.40	-2.86	-4.50
MAY 2010	1.23	-0.29	-7.57	-7.33	-7.92	1.07	-4.82
JUN 2010	1.23	6.17	-0.29	-1.42	-3.58	5.88	-2.71
JUL 2010	1.30	-2.67	6.17	6.56	7.08	4.47	1.22
AUG 2010	1.27	7.63	-2.67	-4.35	-4.31	3.63	-0.12
SEP 2010	1.36	1.53	7.63	4.76	7.72	3.06	-4.91
OCT 2010	1.39	-6.20	1.53	3.53	3.06	1.05	-2.83
NOV 2010	1.30	2.80	-6.20	-6.82	-1.01	5.53	-0.48
DEC 2010	1.34	NA	2.80	5.35	5.19	2.26	2.89

Time series plots / original vs. differences





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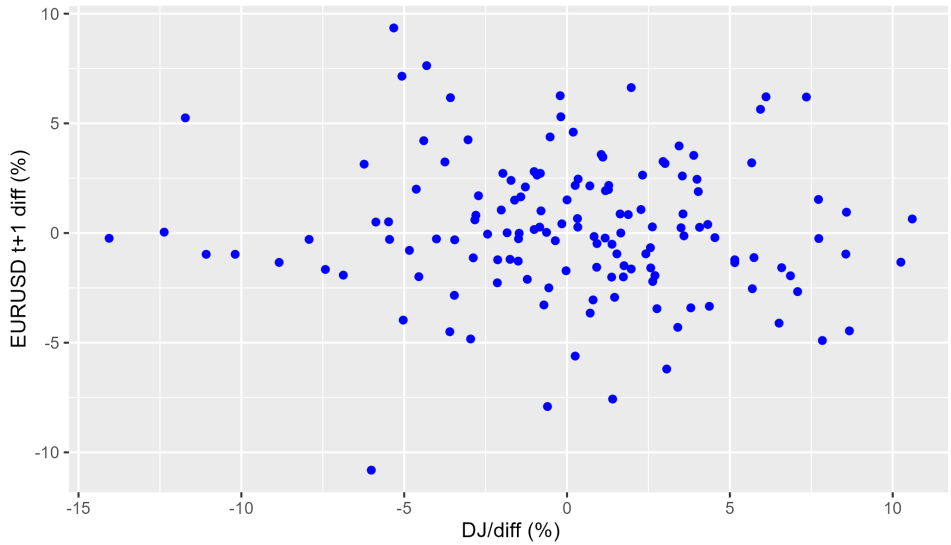
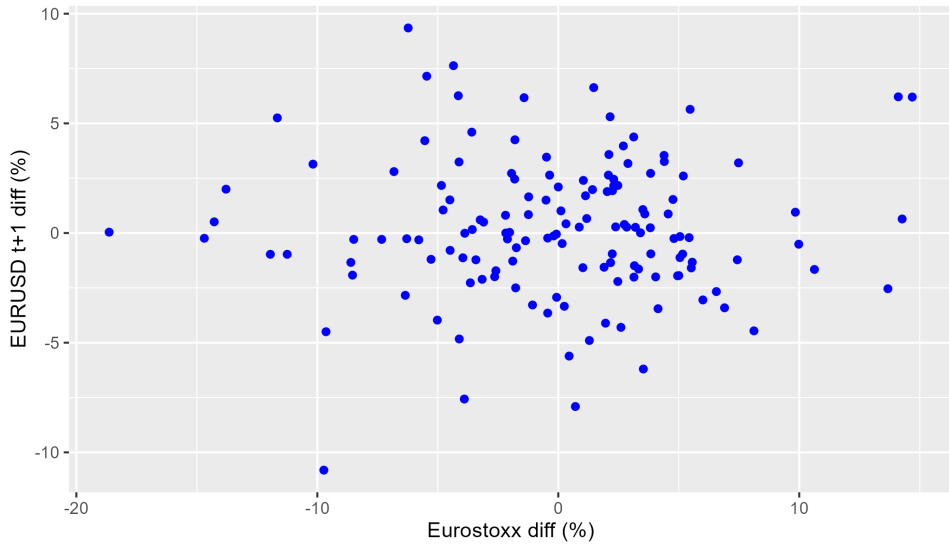
Scatter-plots (I)





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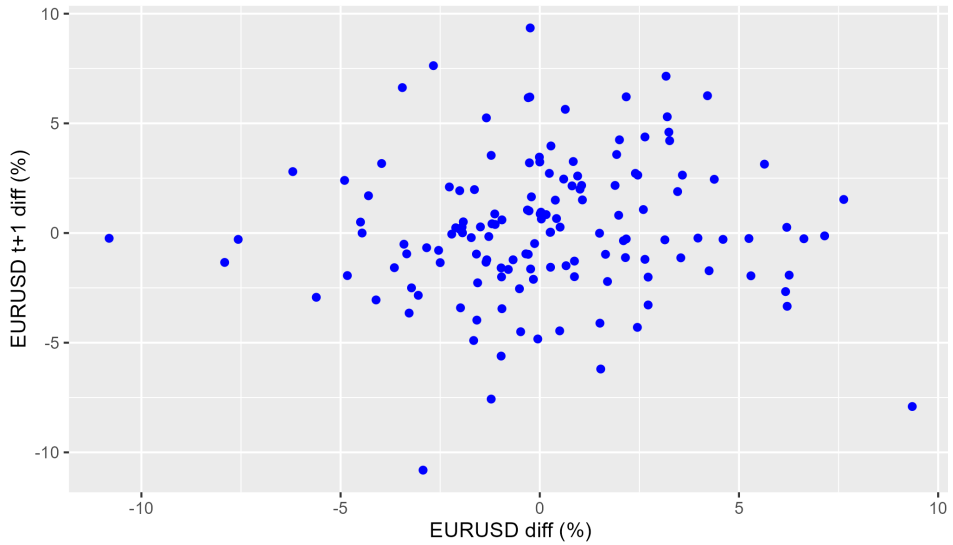
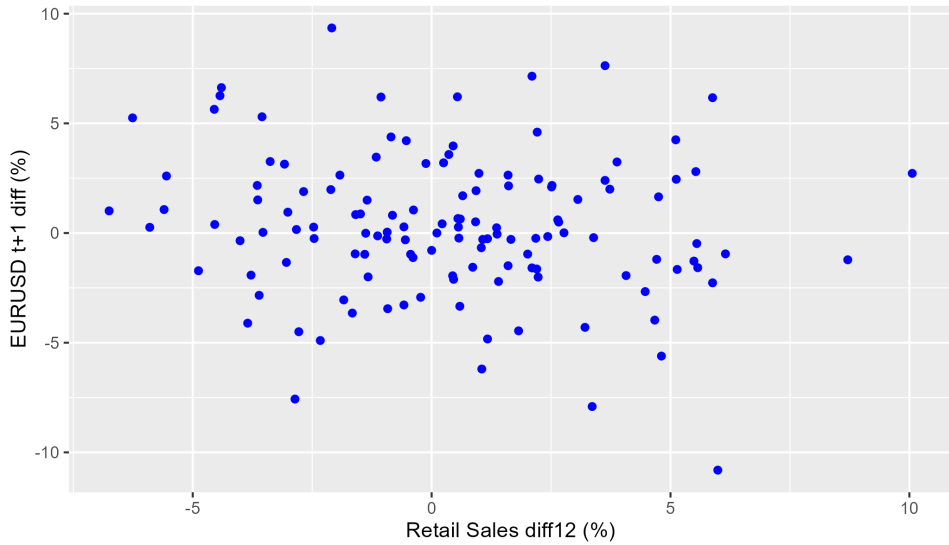
Scatter-plots (II)





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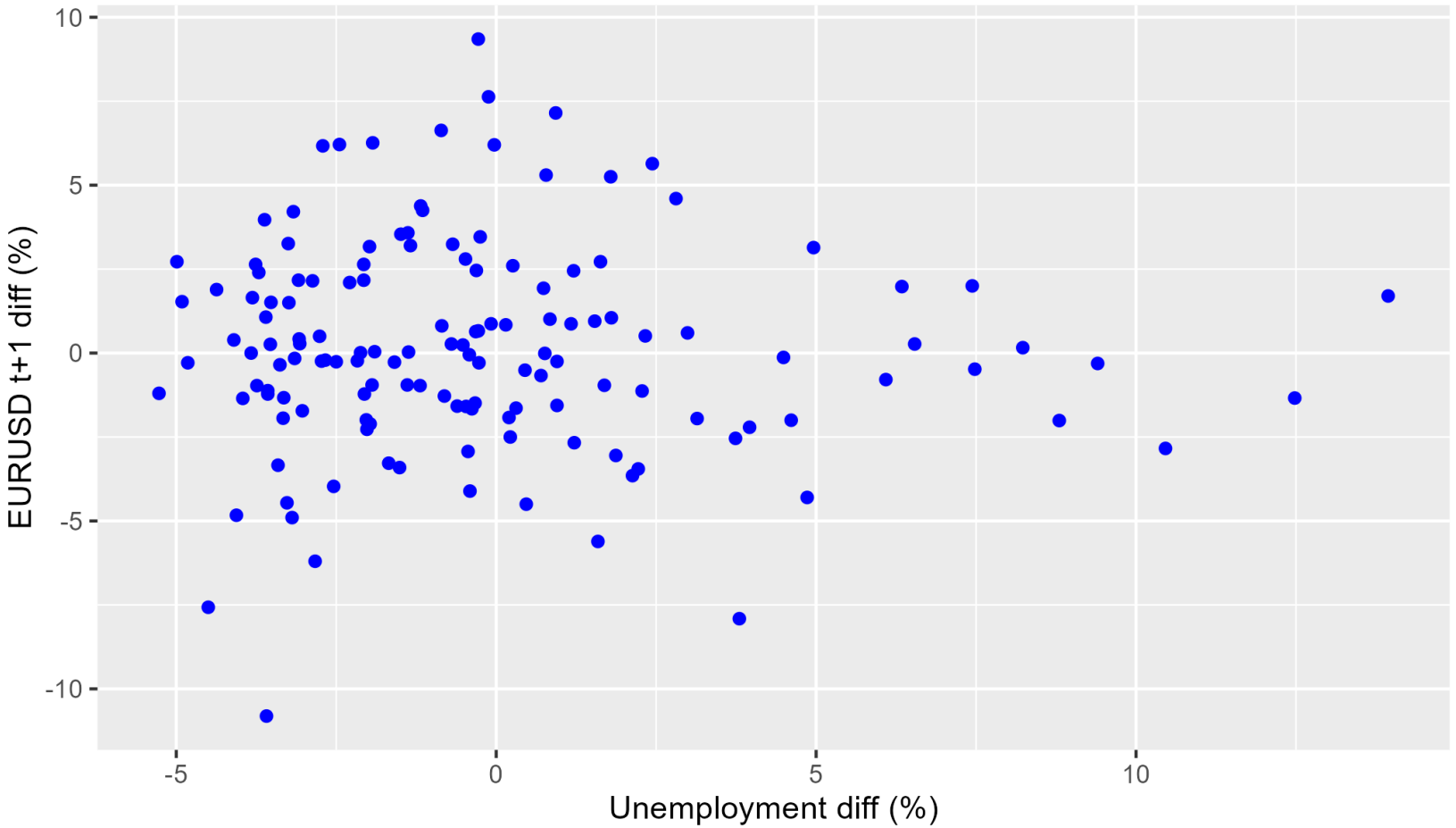
Scatter-plots (III)



Correlation table

	EURUSD t+1 diff (%)	EURUSD diff (%)	Eurostoxx diff (%)	DJ diff (%)	Retail Sales diff12 (%)	Unemployment diff (%)
EURUSD t+1 diff (%)	1.00	0.05	0.04	-0.06	-0.14	-0.04
EURUSD diff (%)	0.05	1.00	0.11	0.24	-0.11	-0.07
Eurostoxx diff (%)	0.04	0.11	1.00	0.81	0.05	-0.08
DJ diff (%)	-0.06	0.24	0.81	1.00	-0.05	-0.10
Retail Sales diff12 (%)	-0.14	-0.11	0.05	-0.05	1.00	-0.09
Unemployment diff (%)	-0.04	-0.07	-0.08	-0.10	-0.09	1.00

Table: Pearson Correlation Table

Multiple regression analysis

	(1)
Constant	0.401
	(0.275)
EURUSD diff (%)	−0.253*
	(0.138)
Eurostoxx diff (%)	0.182
	(0.114)
DJ diff (%)	0.073
	(0.107)
Retail Sales diff12 (%)	−0.167*
	(0.098)
Unemployment diff (%)	−0.048
	(0.067)
Num.Obs.	131
R2	0.064
R2 Adj.	0.026
AIC	682.5
BIC	702.6
RMSE	3.10

The Quality of the Regression Function

- The determination coefficient R^2 is a measure of the **goodness of fit** of the regression function towards the empirical data.
- However, R^2 increases if the number of explanatory variables J is high or if the number of observation points T is low.
- The adjusted determination coefficient R^2_{adj} accounts for the influence of J and T, providing a more accurate measure.
- Question: What is the critical value of R^2_{adj}, i.e., at what value can we consider the **quality of the regression function** to be sufficient?

Population and Sample

Assumption: The regression coefficients $b_0, b_1, \ldots, b_J$ are random samples from a distribution, serving as **estimators** for the unknown population coefficients $\beta_0, \beta_1, \ldots, \beta_J$.

- **Null Hypothesis (H_0):**

$$\beta_1 = \beta_2 = \cdots = \beta_J = 0$$

This implies no relationship/dependency between forecasting and explaining time series.

- If H_0 is true:
  - The "true" determination coefficient ρ^2 equals zero.
  - However, the empirical coefficients $b_1, b_2, \ldots, b_J$ and R^2 may not be zero due to:
    - Sampling variability.
    - Random noise in the data.

Population and Sample

Key Question: How large must $b_0, b_1, \ldots, b_J$ or R^2 be to reject H_0 and conclude that $\beta_0, \beta_1, \ldots, \beta_J$ or $\rho^2 > 0$? This depends on:

- Sample size T.
- Number of predictors J.
- Chosen significance level α.

F Statistic

- The F statistic tests whether there is a linear relationship between the dependent variable y and the independent variables $x_1, x_2, \ldots, x_J$.
- **Key Question:** Does at least one of the independent variables contribute to explaining the variation in y?
- Equivalent: Is at least one regression coefficient β_j significantly different from zero?

Hypotheses in the F Test

- **Null Hypothesis (H_0):**

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_J = 0$$

This implies:

  - There is no relationship between y and any of the independent variables.
  - The regression model does not improve upon a simple prediction.

- **Alternative Hypothesis (H_1):** At least one regression coefficient β_j is not zero, suggesting a linear relationship exists.

Step 1: Computing the Empirical F statistic

- The F statistic is calculated as a ratio:

$$F_{emp} = \frac{\frac{\sum_{t=1}^{T}(\hat{y}_t - \bar{y})^2}{J}}{\frac{\sum_{t=1}^{T}(y_t - \hat{y})^2}{T-J-1}} = \frac{explained\ variation/J}{not\ explained\ variation/T-J-1}$$

- Components:
  - **Explained variation:** Variation due to the regression model, adjusted for the number of predictors J.
  - **Unexplained variation:** Residual variation, adjusted for degrees of freedom $(T - J - 1)$.

Step 2: Specification of the Error Probability

- The error probability α defines the likelihood of rejecting the null hypothesis H_0 when it is actually true (**Type I error**).
- **Common Values of α:**
  - $\alpha = 0.05$ (5%) is the most commonly used level.
  - Other common values: $\alpha = 0.01$ (1%) or $\alpha = 0.10$ (10%).
- **Confidence Level:** Equal to $1 - \alpha$. For example:

$$\text{If } \alpha = 0.05, \text{ then confidence level } = 95\%.$$

Step 3: Finding the Theoretical F statistic

- The theoretical F value F_{theor} is obtained from the F-distribution table.
- **Required Inputs:**
  - **Numerator Degrees of Freedom:** $df_1 = J$, the number of predictors.
  - **Denominator Degrees of Freedom:** $df_2 = T - J - 1$, based on the number of observations T and predictors J.
- The choice of significance level α (e.g., 5%) determines the threshold for rejecting H_0.
- Use the F-distribution table to find F_{theor} for the given df_1, df_2, and α.

Step 4: Comparing Empirical and Theoretical F statistic

- **Decision Rule:**
  - If $F_{emp} > F_{theor}$:
    - Reject the null hypothesis H_0.
    - Conclude that the regression model is **statistically significant**.
  - If $F_{emp} \leq F_{theor}$:
    - Fail to reject the null hypothesis H_0.
    - Conclude that there is **no significant relationship** between the variables.
- This comparison determines whether the relationship described by the regression function is meaningful or likely due to chance.

The Connection Between F and R^2

- The coefficient of determination R^2 is defined as:

$$R^2 = \frac{\text{explained variation (EV)}}{\text{total variation (TV)}} = 1 - \frac{\text{not explained variation (NEV)}}{\text{total variation (TV)}}$$

- The F statistic can be expressed in terms of R^2:

$$F = \frac{\frac{\text{EV}}{\text{TV}}/J}{\frac{\text{NEV}}{\text{TV}}/(T-J-1)} = \frac{R^2/J}{(1-R^2)/(T-J-1)}$$

- This equation links F and R^2, allowing determination of model significance using R^2.

Beta Values

- **Definition:** Beta values (β_j) are standardized regression coefficients:

$$\beta_j = b_j \cdot \frac{s_{x_j}}{s_y}$$

Where:

- b_j: Non-standardized regression coefficient for predictor x_j.
- s_{x_j}: Standard deviation of predictor x_j.
- s_y: Standard deviation of the dependent variable y.

- **Reasons:**
  - Standardizes the influence of predictors, making them comparable across variables with different units or scales.
  - Helps identify which predictor has the largest influence on the dependent variable y.
- **Insight:**
  - The predictor x_j with the largest absolute Beta value ($|\beta_j|$) has the strongest effect on y.
  - The sign of β_j indicates the direction of the relationship.

Example

Scenario: Study the impact of Income (x_1), Education (x_2), and Age (x_3) on SWB (y).

- Non-standardized regression coefficients:

$$b_1 = 0.03, \quad b_2 = 0.8, \quad b_3 = -0.1$$

- Standard deviations:

$$s_{x_1} = 15,000, \quad s_{x_2} = 2, \quad s_{x_3} = 10, \quad s_y = 5$$

Empirical example–EURO/USD

- Forecast of monthly percentage rate of change of EURO/USD exchange rate
- Observation period JAN/1999 – DEC/2019

Correlation table

	EURUSD t+1 diff (%)	EURUSD diff (%)	Eurostoxx diff (%)	DJ diff (%)	Retail Sales diff12 (%)	Unemployment diff (%)
EURUSD t+1 diff (%)	1.00	0.01	0.01	-0.10	-0.10	-0.02
EURUSD diff (%)	0.01	1.00	0.12	0.26	-0.06	-0.01
Eurostoxx diff (%)	0.01	0.12	1.00	0.78	0.08	-0.01
DJ diff (%)	-0.10	0.26	0.78	1.00	0.01	-0.04
Retail Sales diff12 (%)	-0.10	-0.06	0.08	0.01	1.00	-0.04
Unemployment diff (%)	-0.02	-0.01	-0.01	-0.04	-0.04	1.00

Table: Pearson Correlation Table

Regression with Standardized Coefficients

	Coefficients	Standardized coefficients
Constant	0.346*	0.000*
	(0.208)	(0.000)
EURUSD diff (%)	0.061	0.061
	(0.079)	(0.079)
Eurostoxx diff (%)	0.152**	0.275**
	(0.071)	(0.128)
DJ diff (%)	−0.234***	−0.334***
	(0.085)	(0.121)
Retail Sales diff12 (%)	−0.100	−0.114
	(0.063)	(0.071)
Unemployment diff (%)	−0.026	−0.032
	(0.042)	(0.053)
Num.Obs.	239	239
R2	0.051	0.051
R2 Adj.	0.031	0.031
F-statistic	2.327	2.327

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t-Test for the Regression Coefficient

- After using the F-test to check if at least one regression coefficient (β_j) is not zero, we test each coefficient individually to identify which ones are significant.
- The empirical t-value formula:

$$t_{\text{emp}} = \frac{b_j - \beta_j}{s_{b_j}}$$

Where:

- b_j: Estimated coefficient for the j-th variable (from the data).
- β_j: True (unknown) coefficient.
- s_{b_j}: Standard error of b_j, measuring the variability of b_j.

- Typically, the null hypothesis $H_0 : \beta_j = 0$ is tested:

$$t_{\text{emp}} = \frac{b_j}{s_{b_j}}$$

Four Steps of the t-Test

1. Compute empirical t-value:

$$t_{\text{emp}} = \frac{b_j}{s_{b_j}}$$

2. Set significance level (α), e.g., 5%.
3. Find theoretical t-value using degrees of freedom ($df = T - J - 1$).
   - The $t-test$ is a **two-sided test**. The theoretical t-value therefore must be found in the column $\frac{\alpha}{2}$ of the $t-table$!
4. Decision Rule:
   - Reject H_0 if $|t_{\text{emp}}| > t_{\text{theor}}$. It suggests that x_j significantly affects y.
   - Fail to reject H_0 if $|t_{\text{emp}}| \leq t_{\text{theor}}$.

Multicollinearity

- Multicollinearity occurs when predictors are highly correlated, leading to unreliable coefficient estimates.
- Why is it a problem?
  - Coefficients become unstable (small data changes result in large differences in estimates).
  - Inflated standard errors reduce the accuracy of t-tests.
- Detection Methods:
  1. **Correlation Matrix**: Check for correlations > 0.9 or < -0.9.
  2. **Tolerance (TOL)**: $TOL = 1 - R_j^2$, where R_j^2 is the determination coefficient of x_j against other predictors. Low $TOL (< 0.1)$ indicates multicollinearity.
  3. **Variance Inflation Factor (VIF)**: $VIF = \frac{1}{TOL}$. High $VIF (> 10)$ signals multicollinearity.

Strategies Against Multicollinearity

- **Extend Observation Period (T)**: Increasing the sample size can reduce correlations between predictors.
- **Combine or Transform Variables**: Highly correlated predictors can be merged into a composite index or transformed to reduce dependence.
- **Eliminate Predictors**: Remove variables that contribute most to multicollinearity without compromising the theoretical basis of the model.

Autocorrelation in Residuals

- Residuals ($u_t = y_t - \hat{y}_t$) should be **uncorrelated**.
- Problem: Autocorrelation inflates standard errors and produces unreliable t-values.
- **Durbin-Watson Statistic**:

$$d = \frac{\sum_{t=2}^{T}(u_t - u_{t-1})^2}{\sum_{t=1}^{T} u_t^2}$$

Measures the correlation of consecutive residuals.

- Interpretation:
  1. For positive autocorrelation ($u_t \approx u_{t-1}$) d tends to zero.
  2. For negative autocorrelation ($u_t \approx u_{t-1}$) d tends to $+4$.
  3. No autocorrelation ($u_t \approx u_{t-1}$) d tends to $+2$ for large values of T.
  - For large datasets (T is large), the expected value $E(d) = 2 + \frac{2 \cdot J}{T-J-1}$, d approaches 2, which is the ideal scenario of no autocorrelation.

Durbin-Watson Statistic in Practice

- Steps:
  1. Set significance level (α), e.g., 5%.
  2. Find critical values (d_{lo} and d_{up}) based on T and J.
  3. There is an **uncertainty area** for the Durbin-Watson statistic:
     a. $d_{lo} < d > d_{up}$.
     b. $(4 - d_{up}) < d > (4 - d_{lo})$.
  4. The **null hypothesis** H_0 reads $\rho = 0$, i.e.. the "true" **autocorrelation** coefficient between u_t and u_{t-1} is equal to zero.
  5. Compare d_{emp}:
     - $d_{\text{emp}} < d_{lo}$ or $d_{\text{emp}} > 4 - d_{lo}$: Reject H_0 (autocorrelation detected).
     - $d_{up} < d_{\text{emp}} < 4 - d_{up}$: Fail to reject H_0 (no autocorrelation detected).
- Example: For $T = 142$, $J = 5$, and $\alpha = 5\% : d_{lo} = 1.57, d_{up} = 1.78$:
  - Reject H_0 if $d_{\text{emp}} < 1.57$ or $d_{\text{emp}} > 2.43$.
  - Fail to reject H_0 if $1.78 < d_{\text{emp}} < 2.22$.