Stationarity of Time series

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Overview

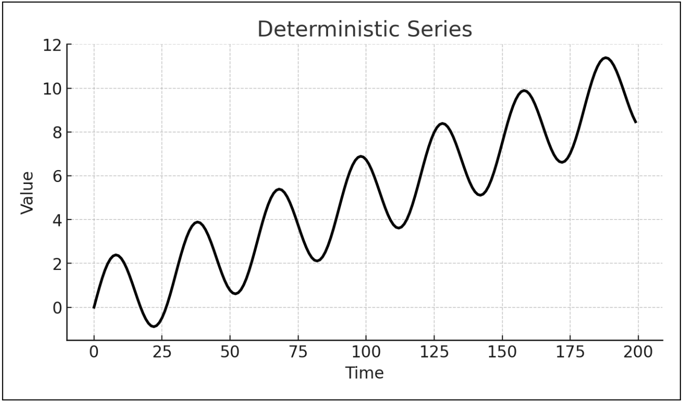
1. Stochastic processes
2. Stationarity
3. White Noise and Random Walk

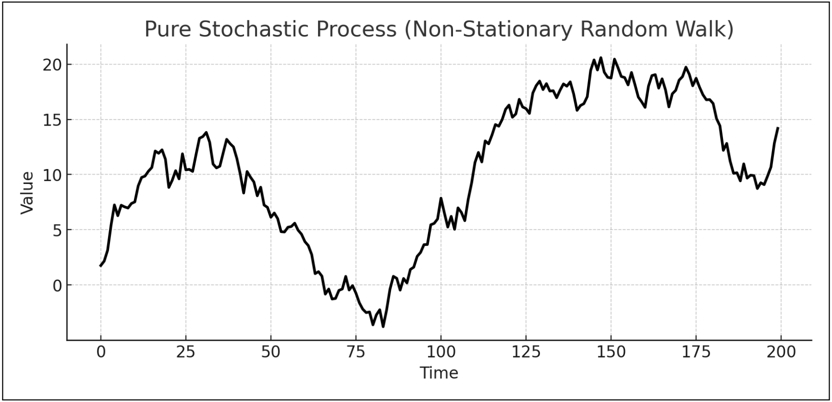
Stochastic Processes

- Time series model (additive components):

$$X_t = T_t + C_T + S_t + U_t$$

  - T_t: Trend component (long-term changes).
  - C_T: Cyclical component (periodic variations).
  - S_t: Seasonal component (repeating patterns).
  - U_t: Irregular/random component (noise).
- Random component (U_t):
  - Irregular and unpredictable.
  - Treated as a random variable.
  - Stochastically independent.


Deterministic Series
Value
Time
12
10
8
6
4
2
0
0
25
50
75
100
125
150
175
200


Pure Stochastic Process (Non-Stationary Random Walk)
Value
Time
20
15
10
5
0
0
25
50
75
100
125
150
175
200

Stochastic Processes

- **Shift in perspective:** Time series are now seen as realizations of stochastic processes.
- **Definition:** Stochastic processes are dynamic processes with randomness.
- **Key assumption change:** Independence is no longer assumed.
- **New focus:** Dependency structures in the data.

How does the past influence the present and future?

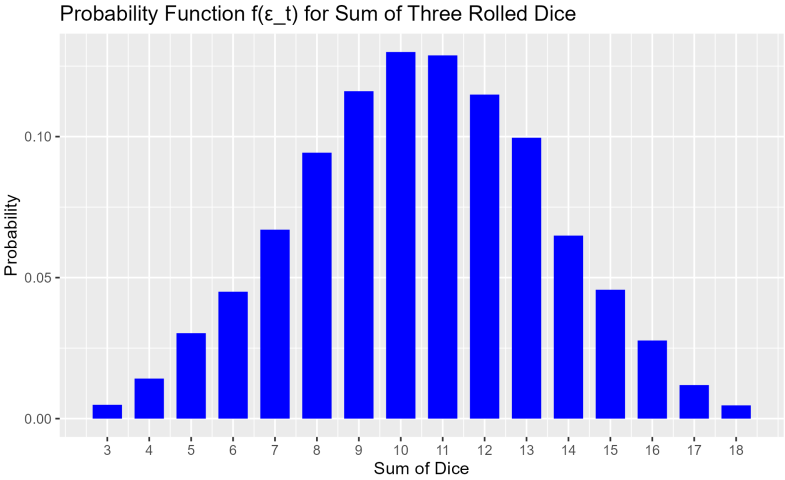
Stochastic Processes

- Definition: A stochastic process is a sequence (X_t) of random variables indexed by time.
- **A time series** is a sequence $x_1, ..., x_T$ of realizations of a section of (X_t) with t as an integer.
- Interpretations of a stochastic process:
  1. **Ensemble:** A collection of possible time series, with one chosen at random.
  2. **Sequence:** A series of random variables, for each time point t one random variable is assigned.

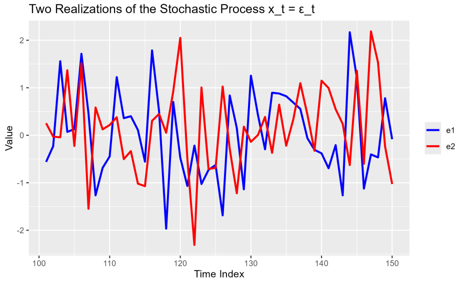
Examples of Stochastic Processes

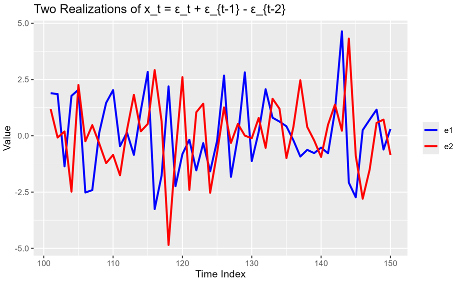
- Example: Dice game.
- Random variable ε_t: Sum of three dice rolls.
  - Possible realizations: 3 to 18.
  - Long-term mean: $E(\varepsilon_t) = 10.5$.
  - Different trajectories: Each player experiences unique paths due to randomness.

Probability function $f(\varepsilon_t)$ for sum of numbers of three rolled dices with $T = 10000$



Two realizations $x_{101}, ..., x_{150}$ of the stochastic process $x_t = \varepsilon_t$ and $x_t = \varepsilon_t + \varepsilon_{t-1} - \varepsilon_{t-2}$





Autocorrelation

- Definition: Measures dependency between observations at different time lags (τ).

$$r_\tau = \frac{\sum_{t=1}^{N-|\tau|}(x_t - \bar{x})(x_{t+|\tau|} - \bar{x})}{\sum_{t=1}^{N}(x_t - \bar{x})^2} = \frac{c_\tau}{c_0}$$

- Properties:
  - $r_\tau \in [-1, 1]$:
    - $+1$: Perfect positive correlation.
    - 0: No correlation.
    - -1: Perfect negative correlation.
- Use: Analyze dependency patterns in time series.

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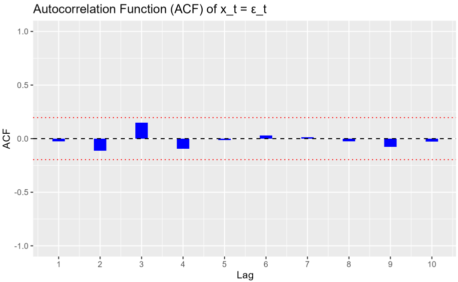
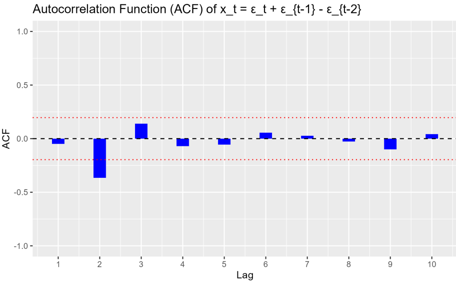
Autocorrelation functions (ACF) for $x_t = \varepsilon_t$ and $x_t = \varepsilon_t + \varepsilon_{t-1} - \varepsilon_{t-2}$





Examples of stochastic processes

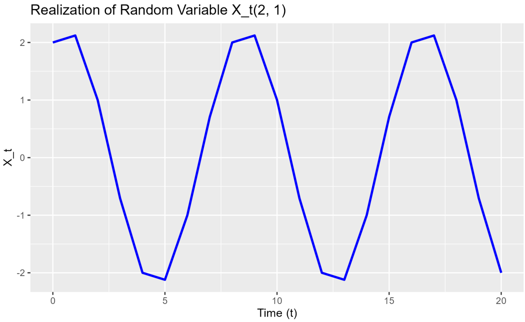
- Be random variable X_t defined by

$$X_t(\omega) = A(\omega) \cdot cos 2\pi \frac{1}{8} t + B(\omega) \cdot sin 2\pi \frac{1}{8} t$$

with $\omega = (a, b)$, where a is the sum of numbers of a **first** and b the sum of numbers of a **second** independently rolled dice.

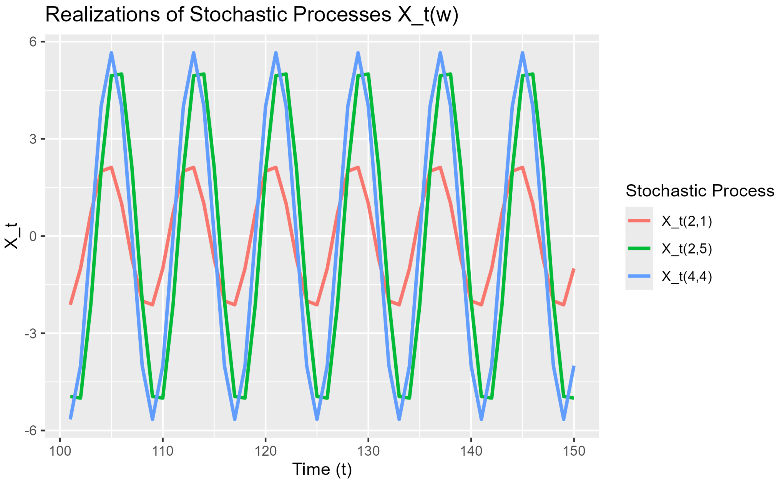
- A and B are the corresponding random variables, i.e. $A(\omega) = a$ and $B(\omega) = b$.
- All 36 results $(1, 1), (1, 2), ..., (6, 6)$ own the same probability $\frac{1}{36}$
- The stochastic process (X_t) includes 36 different realizations (or time series) $(X_t(\omega))$, **one** for each **fixed** value of ω.

Realizations of random variable $X_t(2, 1)$

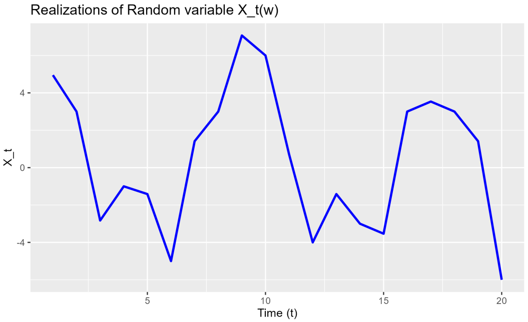


t	$X_t(2,1)$
1	2.12
2	1.00
3	-.71
4	-2.00
5	-2.12
6	-1.00
7	.71
8	2.00
9	2.12
10	1.00
11	-.71
12	-2.00
13	-2.12
14	-1.00
15	.71
16	2.00
17	2.12
18	1.00
19	-.71
20	-2.00

Three realizations of the stochastic process $X_t(\omega)$



Realizations of random variable $X_t(w)$



t	a	b	$X_t(w)$
1	6	1	4.95
2	3	3	3.00
3	5	1	-2.83
4	1	4	-1.00
5	1	1	-1.41
6	6	5	-5.00
7	6	4	1.41
8	3	6	3.00
9	6	4	7.07
10	1	6	6.00
11	2	3	.71
12	4	3	-4.00
13	1	1	-1.41
14	6	3	-3.00
15	1	6	-3.54
16	3	3	3.00
17	2	3	3.54
18	2	3	3.00
19	3	5	1.41
20	6	5	-6.00

Stationarity

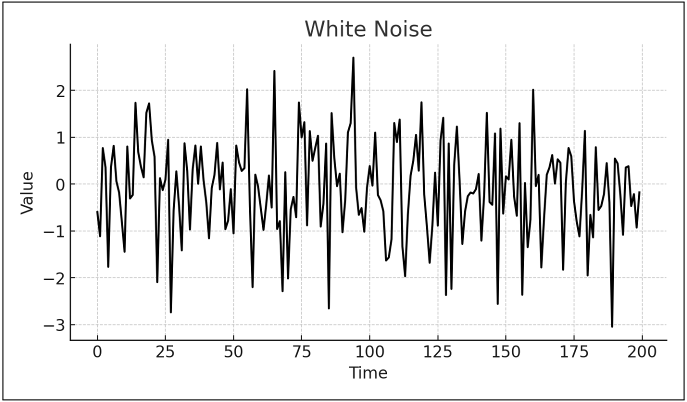
- Definition: A process (X_t) is stationary if:
  1. Mean is constant: $E(X_t) = \mu$.
  2. Variance is constant: $VAR(X_t) = \sigma^2$.
  3. Covariance depends only on lag: $COV(X_t, X_{t+\tau}) = \gamma_\tau$.
- Importance: Stationarity ensures consistent statistical properties over time.
- Therefore the system of n random variables $(X_{t1}, X_{t2}, ..., X_{tn})$ owns the same structure of expectation value, variance and co-variance as the (for s units) shifted system $(X_{t1+s}, X_{t2+s}, ..., X_{tn+s})$.
- Non-stationary series (e.g., trends) require transformation before analysis.

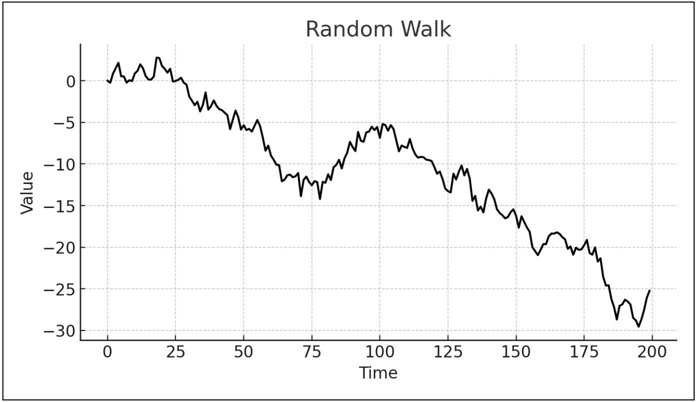
White Noise and Random Walk

- White Noise:
  - Sequence of i.i.d. random variables (ε_t).
  - Stationary but unforecastable.
- Random Walk:

$$X_t = \begin{cases} \varepsilon_1 & t = 1 \\ X_{t-1} + \varepsilon_t & t > 1 \end{cases}$$

  - Non-stationary due to dependence on time t.
  - Variance increases over time.


White Noise
Value
Time
2
1
0
−1
−2
−3
0
25
50
75
100
125
150
175
200


Random Walk
Value
Time
0
−5
−10
−15
−20
−25
−30
0
25
50
75
100
125
150
175
200

Random walk examination

- Testing the linear regression function:

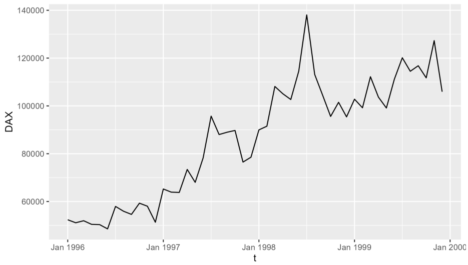
$$x_t = b_0 + b_1 x_{t-1} + u_t$$

- <u>Rule of thumb:</u> In case of $b_1 \in [0.9; 1.1]$ it applies approximately:

$$x_t = x_{t-1} + \varepsilon_t$$

with $\varepsilon_t = b_0 + u_t$ as white-noise-process
and x_t as random-walk-process.

Is DAX a random walk?



	(1)
Constant	8883.539*
	(5139.215)
Dax_{t-1}	0.910***
	(0.067)
Num.Obs.	47
R2	0.853
R2 Adj.	0.849
AIC	999.0
BIC	1004.6
RMSE	9370.41

Summary

- Time series are realizations of stochastic processes.
- Stationarity ensures consistent statistical properties over time.
- White noise is stationary but unpredictable.
- Nevertheless time series show certain structures $\rightarrow \mu, \sigma^2, \gamma_\tau$.
- **Important:** These structures should remain constant over time $\rightarrow$ stationarity!
- Random walks are non-stationary due to time dependence.
- For non-stationary series:
  - Detrending or differencing may be required for analysis.
- In general: Time series with trend are non stationary $\rightarrow$ trend adjustment!