Advanced Microeconomics

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Lecture 4 - Mixed strategies

- Gibbons, Chapter 1
- Tadelis, Chapter 6
- (Osborne, Chapters 2-4)

Mixed strategies

- Matching Pennies

Player 2

		Heads	Tails
Player 1	Heads		
	Tails		

Mixed strategies

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		Player 2	
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Player 1	Heads	-1, 1	1, -1
	Tails	1, -1	-1, 1

- Is there a Nash equilibrium in "pure" strategies?
- Alternative: Players choose their pure strategies only with a certain probability
- Similarly, a penalty taker does not always aim for the same corner

• Instead of expanding the set of possible actions, we expand the choices by introducing mixed strategies.

Mixed strategies

- Definition
  - Let $S_i = \{s_{i1}, \dots, s_{iK}\}$ be the finite set of pure strategies of player i in the normal form game $G = \{I; \{S_i\}; \{u_i\}\}$. A **mixed strategy of** player i is a probability distribution $p_i = (p_{i1}, \dots, p_{iK})$ over his pure strategies with $0 \leq p_{ik} \leq 1$ for $k = 1, \dots, K$ and $p_{i1} + \cdots + p_{iK} = 1$.
    - p_{i1}: propability that player i chooses strategy s_{i1}
- Definition can also be adjusted for **continuous strategy sets**
    - E.g. when a dupolist chooses a quantity
  - In this case the strategy set S_i is an interval and a mixed strategy is a cumulative distribution function $F_i: S_i \to [0,1]$ where $F_i(x) = \Pr[s_i \leq x]$.

- Every pure strategy is also a (trivial) mixed strategy

Mixed strategies

- Expected payoffs for mixed strategies
  1. Weighting the payoff of each pure strategy with the probability of playing that strategy
  2. Adding up the weighted payoffs
  - 2 players, the 1st index represents the player, the 2nd the pure strategy
    - $p_2 = (p_{21}, \dots, p_{2K})$, a mixed strategy of player 2
  - Expected payoff for player 1 when playing the **pure strategy** s_{1j} :
    - $v_1(s_{1j}, p_2) =$

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    - $p_2 = (p_{21}, \dots, p_{2K})$, a mixed strategy of player 2
  - Expected payoff for player 1 when playing the **pure strategy** s_{1j} :
    - $v_1(s_{1j}, p_2) = p_{21}u_1(s_{1j}, s_{21}) + \cdots + p_{2K}u_1(s_{1j}, s_{2K})$

$$= \sum_{k=1}^{K} p_{2k}u_1(s_{1j}, s_{2k})$$

  - Expected payoff for player 1 when playing the **mixed strategy** $p_1 = (p_{11}, \dots, p_{1J})$:

$v_1(p_1, p_2)$

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$$v_1(p_1, p_2) = p_{11}v_1(s_{11}, p_2) + \cdots + p_{1J}v_1(s_{1J}, p_2)$$
$$= \sum_{j=1}^{J} p_{1j}v_1(s_{1j}, p_2) = \sum_{j=1}^{J}\sum_{k=1}^{K} p_{1j}p_{2k}u_1(s_{1j}, s_{2k})$$

Nash equilibrium in mixed strategies

– Definition

In the normal form game $G = \{I; \{S_i\}; \{u_i\}\}$, the **mixed-strategy** vector $p^* = (p_1^*, \ldots, p_n^*)$ is a **Nash equilibrium (NE)** if for each player his mixed strategy p_i^* is a best response to the mixed-strategy vector p_{-i}^* of the other players. That is, if for all $i = 1, \ldots, n$,

$$v_i(p_i^*, p_{-i}^*) \geq v_i(p_i, p_{-i}^*)$$

for all $p_i \in \Delta S_i$.

- ΔS_i is the simplex of S_i
  - i.e. the set of all probability distributions over the strategy set S_i

– Note on interpretation of NE: We can think of p_{-i}^* as the belief of player i about his opponents behavior

- rationality requires that a player plays a best response given his belief
- A Nash equilibrium requires that these beliefs be correct

Matching Pennies: Nash-equilibrium in mixed strategies

- Matching Pennies

Player 2

		Heads	Tails
Player 1	Heads	-1, 1	1, -1
	Tails	1, -1	-1, 1

  - r, probability with which player 1 plays Heads
    - $(r, 1 - r)$: mixed strategy of player 1
  - q, probability with which player plays 2 Heads
    - $(q, 1 - q)$: mixed strategy of player 2
  - Expected payoff of player 1
    - $v_1(r, q) =$

Matching Pennies: Nash-equilibrium in mixed strategies

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  - $(r, 1\text{–}r)$: mixed strategy of player 1
- q, probability with which player plays 2 Heads
  - $(q, 1\text{–}q)$: mixed strategy of player 2
- Expected payoff of player 1
  - $$v_1(r,q) = r[-q + (1-q)] + (1-r)[q-(1-q)] \\ = r(2-4q) + (2q-1)$$
- Expected payoff of player 2: $v_2(r,q) = q[r-(1-r)] + (1-q)[-r+(1-r)]$
  $= q(4r-2) + (1-2r)$

Matching Pennies: Nash equilibrium in mixed strategies

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– Is $v_1(r,q) = r(2-4q) + (2q-1)$ rising or falling in r?

- $\frac{\partial v_1(r,q)}{\partial r} = 2 - 4q$

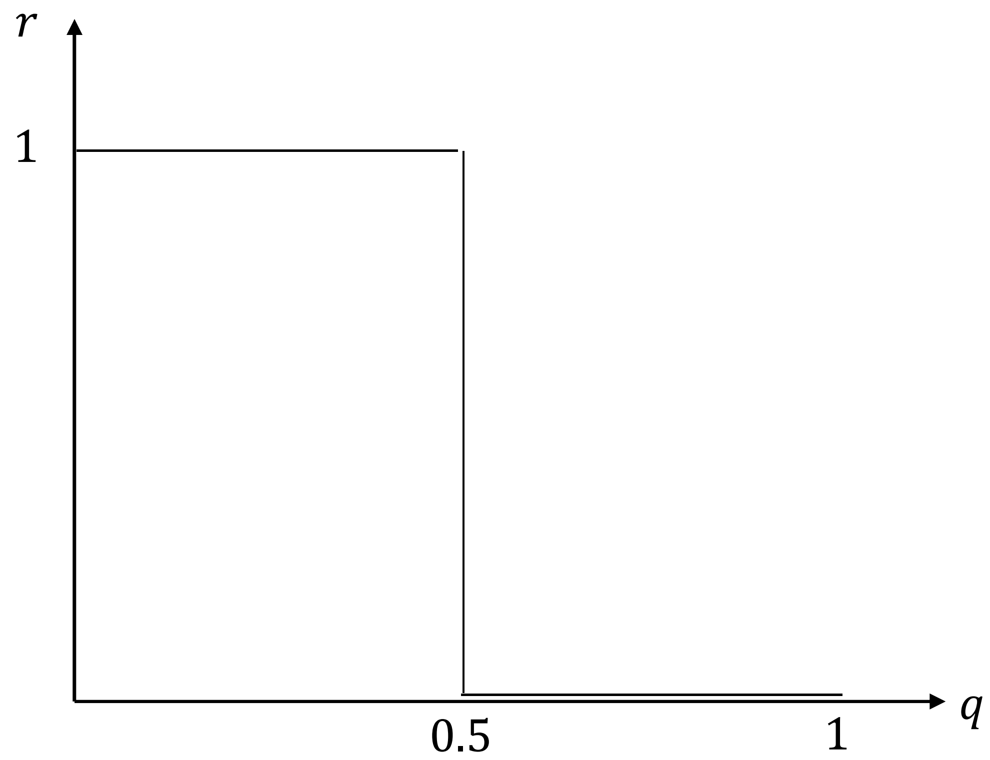
- $v_1(r,q)$ *rises* in r if $q < 0.5$
- $v_1(r,q)$ falls in r if $q > 0.5$

Matching Pennies: Nash equilibrium in mixed strategies

- r: Probability that player 1 plays Heads
- q: Probability that player plays 2 Heads
- Expected payoff of player 1, $v_1(r,q) = r(2-4q) + (2q-1)$
  - rises in r if $q < 0.5$,
  - falls in r if $q > 0.5$

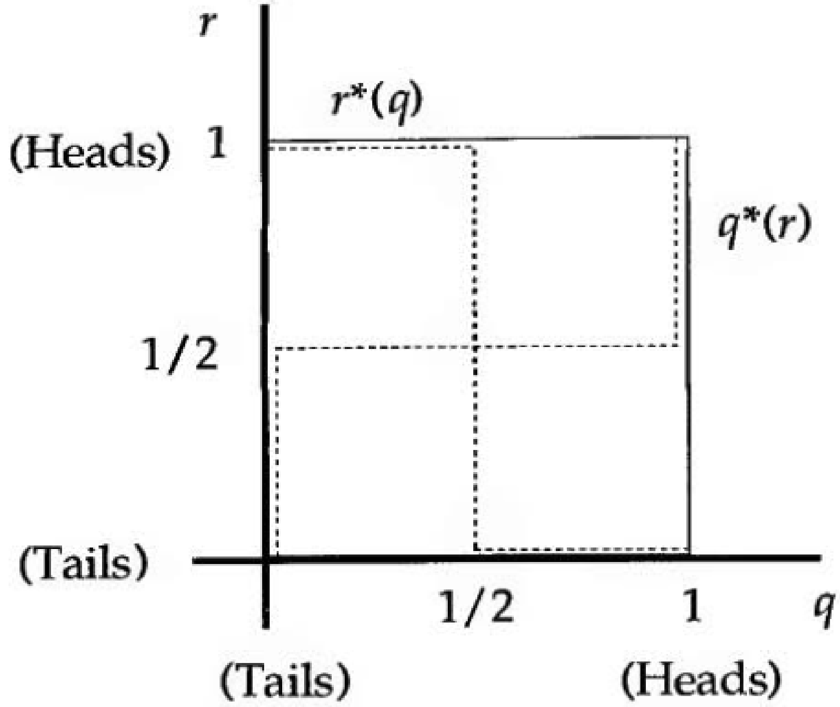
Matching Pennies: Nash equilibrium in mixed strategies

- r: Probability that player 1 plays Heads
- q: Probability that player plays 2 Heads
- Expected payoff of player 2, $v_2(r, q) = q(4r - 2) + (1 - 2r)$
  - falls in q if $r < 0.5$,
  - rises in q if $r > 0.5$



Matching Pennies: Nash equilibrium in mixed strategies

- Symmetric game (player 2 has the same payoff function)
- i.e. strategy of player 2 is analogous to that of player 1



- The Nash equilibrium is $r^* = 0.5$, $q^* = 0.5$

Determination of the Nash equilibrium in mixed strategies

- There is a more common approach to determine the Nash equilibrium in mixed strategies with 2 players.
  - *Idea*: if a player is mixing several pure strategies, then he must be indifferent between them
    - Otherwise, the player would always choose the pure strategy with the highest expected value (i.e. with a probability of 1)
  - *Approach*: Each player chooses his mixed strategy in such a way that the other player is indifferent between his pure strategies
    - i.e., these pure strategies must have the same expected value, given the mixed strategy of the other player

Determination of the Nash equilibrium in mixed strategies

- r, probability with which player 1 plays Heads
- q, probability with which player plays 2 Heads

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- Example matching pennies
  - Expected payoff of player 1‘s pure strategies must be the same:

Determination of the Nash equilibrium in mixed strategies

- r, probability with which player 1 plays Heads
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- Example matching pennies
  - Expected payoff of player 1‘s pure strategies must be the same:

$$v_1(H, q) = v_1(T, q) \Leftrightarrow -q + (1 - q) = q - (1 - q)$$

    - can be solved for $q = 0.5$

Battle of the Systems

		Woman (2)	
		Soccer	Opera
Man (1)	Soccer	2, 1	0, 0
	Opera	0, 0	1, 2

- 3 Nash equilibria:
  - Two in pure strategies, $(s_1^*, s_2^*) =$
  - One in mixed strategies: $(p_1^*, p_2^*) =$
- Calculation
  - r, probability that man (1) chooses soccer
  - q, probability that woman (2) chooses soccer
  - Player 1 chooses r such that
  - Player 2 chooses q such that

Battle of the Sescape

			Woman (2)	
			Soccer	Opera
Man (1)	Soccer		2, 1	0, 0
	Opera		0, 0	1, 2

- 3 Nash equilibria:
  - Two in pure strategies, $(s_1^*, s_2^*) = (S, S), (O, O)$
  - One in mixed strategies: $(p_1^*, p_2^*) = \left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$
- Calculation
  - r, probability that man (1) chooses soccer
  - q, probability that woman (2) chooses soccer
  - Player 1 chooses r such that $v_2(S|r) = v_2(O|r) \Leftrightarrow r = 2(1-r)$
  - Player 2 chooses q such that $v_1(S|q) = v_1(O|q) \Leftrightarrow 2q = (1-q)$

Discussion of the concept of "Nash-equilibrium in mixed strategies".

- Scepticism about the utility of mixed strategies
  - Motivation through examples with multiple interaction (as in the example of the penalty taker)
  - De-facto, however, we consider only one-time interaction (otherwise we would have a dynamic game)

- "purification argument" by Harsanyi

  A Nash equilibrium in mixed strategies can (almost always) be viewed as an equilibrium in pure strategies of a "similar game" in which there is a "bit" of private information.

  - Idea: instead of one opponent with mixed strategies, there are several types of one player, each with pure strategies

Existence of a Nash equilibrium

- John Nash has shown that every finite normal form game $G = \{I; \{S_i\}; \{u_i\}\}$ has a Nash equilibrium in mixed strategies
  - Nash equilibrium can also be one in pure strategies, because any pure strategy can be conceived as a mixed strategy in which only one strategy is played with positive probability

- There can be more than one Nash equilibrium
  - In this case, one must examine which of them seems to be a more plausible prediction for the game
  - Topic of the next lecture