Advanced Microeconomics

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Lecture 9: Nash bargaining solution

Essential reading:

- see slides for references

The Nash bargaining solution

- In lecture 8 we have analyzed bargaining solutions by explicitly modelling a situation with offers and counter-offers
- In this model, bargaining power resulted from
  - being the first to make an offer, and
  - being more patient than the opponent
    - which was captured by the discount factor
- 2 drawbacks of this approach
  - there may be other sources of bargaining power
    - being more experienced, being in a higher position, …
  - the modelling was a bit tricky
- Hence we now investigate an alternative approach to bargaining that is not based on game theory
  - The Nash bargaining solution

The Nash bargaining solution

Nash (1950), The Bargaining Problem, Econometrica, 18(2), 155–162.

- If reaching an agreement, 2 players attain a joint value of v^*
- they bargain about how to distribute v^*
  - x_1, x_2: shares of players 1 and 2
  - d_1, d_2: disagreement points of players 1 and 2
    - i.e., what they get if bargaining fails
- The **Nash bargaining solution** (x_1^*, x_2^*) to this problem is defined as follows:

$$(x_1^*, x_2^*) = \arg\max_{x_1, x_2} (x_1 - d_1)(x_2 - d_2) \qquad \text{s.t.} \quad x_1 + x_2 \leq v^*$$

The Nash bargaining solution

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- The constraint must hold with equality. Why?
- Substitution of the constraint into the objective function yields
- FOC:
- … and by symmetry (or upon substitution): $x_2 =$
- In words: each player gets
  - his fall-back option d_i
  - plus half of the surplus from negotiations

The Nash bargaining solution

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- The constraint must hold with equality. Why?
- Substitution of the constraint into the objective function yields

$$x_1^* = \arg\max_{x_1} (x_1 - d_1)(v^* - x_1 - d_2)$$

- FOC:

$$(v^* - x_1 - d_2) - (x_1 - d_1) = 0$$

$$2x_1 = v^* - d_2 + d_1$$

$$x_1 = d_1 + 0.5(v^* - d_1 - d_2)$$

- … and by symmetry (or upon substitution): $x_2 = d_2 + 0.5(v^* - d_1 - d_2)$
- In words: each player gets
  – his fall-back option d_i
  – plus half of the surplus from negotiations

An example

(see Watson (2013): Strategy – an introduction to game theory, chapter 18)

- Rosemary chairs the English department at a high school
- Jerry, a professional actor, is interested in working there
  - Jerry gets personal value of $10,000 when working as a drama teacher (because he loves teaching)
  - The school values Jerry's work as drama teacher at $40,000
  - Joint value if bargaining succeeds: $v^* = 50.000$
- they bargain about salary t
- Payoffs from employment:
  - $x_R = 40{,}000 - t$
  - $x_J = 10{,}000 + t$
- Payoffs from non-employment, ie. the disagreement point d_i:
  - Jerry: $d_J =$ $15,000; from working on his own
  - Rosemary: $d_R =$ $10,000; her value of hiring a less qualified applicant

An example

- Assumption: both have equal bargaining power
- Nash bargaining solution: $x_i = d_i + 0.5(v^* - d_J - d_R)$
  - In our example:
    - $v^* = 50.000$, $x_J = 10{,}000 + t$, $x_R = 40{,}000 - t, d_J = 15{,}000$, $d_R = 10{,}000$
- Substitution for Jerry yields
- Substitution for Rosemary yields

An example

- Assumption: both have equal bargaining power
- Nash bargaining solution: $x_i = d_i + 0.5(v^* - d_J - d_R)$
  - In our example:
    - $v^* = 50.000$, $x_J = 10{,}000 + t$, $x_R = 40{,}000 - t, d_J = 15{,}000,$
      $d_R = 10{,}000$
- Substitution for Jerry yields

$$10{,}000 + t = 15{,}000 + 0.5(50.000 - 15{,}000 - 10{,}000)$$
$$t = 5{,}000 + 12{,}500 = 17{,}500$$

- Substitution for Rosemary yields

$$40{,}000 - t = 10{,}000 + 0.5(50.000 - 15{,}000 - 10{,}000)$$
$$t = 30{,}000 - 12{,}500 = 17{,}500$$

- Hence Jerry will be employed at a wage of €17,500

The generalized Nash bargaining solution

- A generalization of this is the **Generalized Nash bargaining solution** (x_1^*, x_2^*), defined as follows:

$$(x_1^*, x_2^*) = \arg\max_{x_1, x_2} \ (x_1 - d_1)^{\alpha}(x_2 - d_2)^{1-\alpha}$$

$$\text{s.t.} \quad x_1 + x_2 \leq v^*, \quad \alpha \in (0,1)$$

- It yields the solution

$$x_1^* = d_1 + \alpha(v^* - d_1 - d_2)$$

$$x_2^* = d_2 + (1 - \alpha)(v^* - d_1 - d_2)$$

  - α is a measure of the bargaining power of player 1

- An appendix at the end of these lecture slides contains a formal derivation

Two ways to motivate the Nash bargaining solution

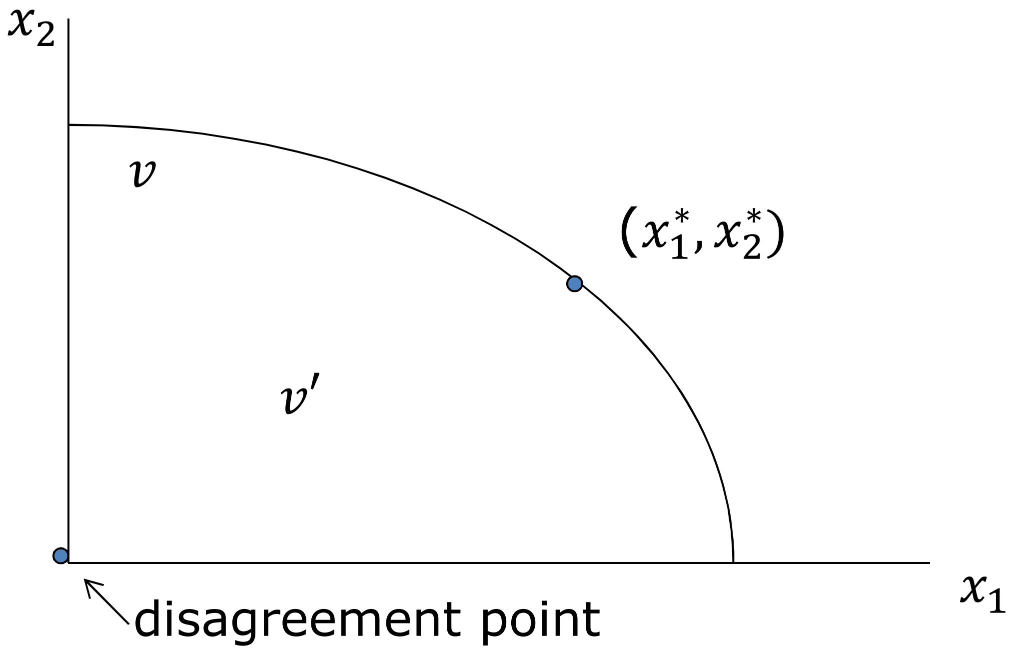
1) Equivalence of generalized Nash bargaining solution and solution of alternating offer game (lecture 7)
   - Suppose that 2 agents bargain over v^* and the Nash bargaining solution leads to an allocation (x_1^*, x_2^*)
   - Then there exists a profile of discount factors (δ_A, δ_B) such that the alternating offer game leads to the same allocation
     - Binmore/Rubinstein/Wolinsky. 1982. “The Nash Bargaining Solution in Economic Modelling”, RAND Journal of Economics 17(2), 176-188.

Axiomatic approach to motivate the Nash bargaining solution

2) Appealing to normative axioms that a bargaining solution should satisfy (this was the original approach of Nash).
   - The generalized Nash solution is the only bargaining solution that satisfies the following axioms:
     - Pareto efficiency
     - individual rationality (i.e. every agent receives at least his reservation utility)
     - invariance to independent changes of units in which utility is measured
       - i.e. although the bargaining solution uses cardinal information on preferences, it does not in any way involve interpersonal comparisons of utilities
     - independence of irrelevant alternatives (see next slide)

Independence of irrelevant alternatives

- if (x_1^*, x_2^*) is a "reasonable" outcome in the bargaining set v and we consider a v' that is smaller than v but retains the feasibility of (x_1^*, x_2^*) – that is, we eliminate from v only "irrelevant alternatives" – then (x_1^*, x_2^*) remains the reasonable outcome.



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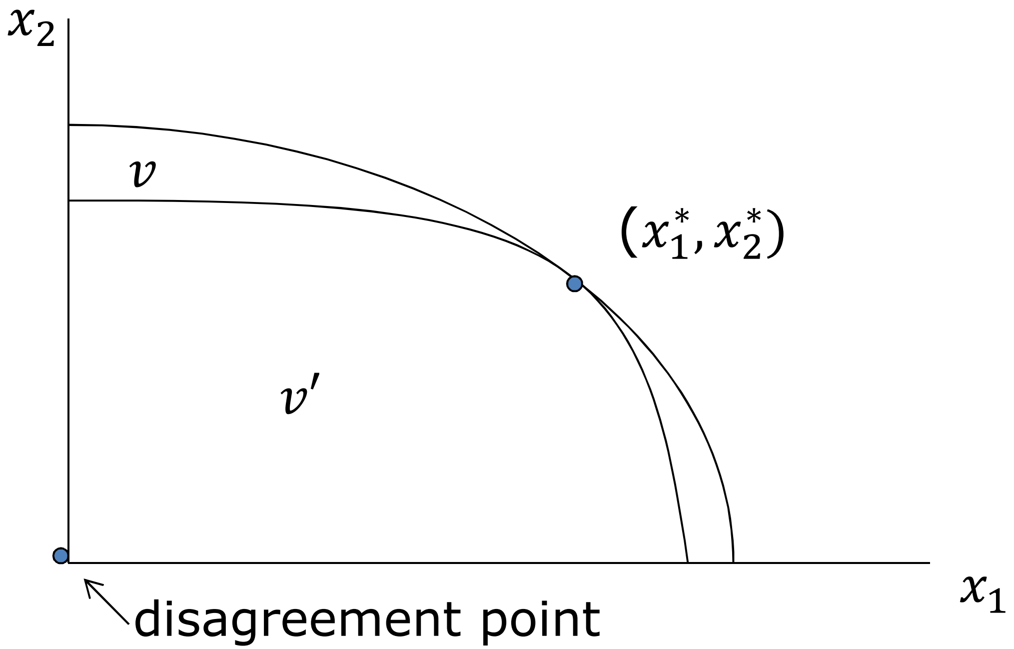

Appendix (not relevant for exams): Derivation of generalized Nash bargaining solution

$$(x_1^*, x_2^*) = \arg\max_{x_1,x_2} \ (x_1 - d_1)^\alpha (x_2 - d_2)^{1-\alpha} \quad \text{s.t.} \quad x_1 + x_2 \leq v^*, \qquad \alpha \in (0,1)$$

- The constraint binds so that substitution yields:

$$x_1^* = \arg\max_{x_1} \ (x_1 - d_1)^\alpha (v^* - x_1 - d_2)^{1-\alpha}$$

- FOC: $\alpha(x_1^* - d_1)^{\alpha-1}(v^* - x_1^* - d_2)^{1-\alpha} - (1-\alpha)(x_1^* - d_1)^\alpha (v^* - x_1^* - d_2)^{-\alpha} = 0$

- Step 1: rearranging terms so that they have the same basis

$$\alpha \frac{(v^* - x_1^* - d_2)^{1-\alpha}}{(v^* - x_1^* - d_2)^{-\alpha}} - (1-\alpha)\frac{(x_1^* - d_1)^\alpha}{(x_1^* - d_1)^{\alpha-1}} = 0$$

- Step 2: applying rules of exponentials

$$\alpha(v^* - x_1^* - d_2)^{1-\alpha+\alpha} - (1-\alpha)(x_1^* - d_1)^{\alpha-(\alpha-1)} = 0$$

- Step 3: isolating x_1^*

$$-\alpha x_1^* - (1-\alpha)x_1^* + \alpha(v^* - d_2) + (1-\alpha)d_1 = 0$$
$$x_1^* = \alpha(v^* - d_2) + (1-\alpha)d_1$$
$$x_1^* = d_1 + \alpha(v^* - d_1 - d_2)$$