Regression analysis for trend and smooth component

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Overview

1. Linear trends and linear regression
2. Coefficient of Determination
3. Higher-order polynomials
4. Exponential trend

Basic Steps in Forecasting

- Plot the series to determine it components
- Remove any trend or seasonal pattern
- Develop a forecasting model
- Validate the forecasting model
- Bonus tips:
  - Scale of original time series
  - Prediction intervals
  - Monitor forecast

Data Transformations

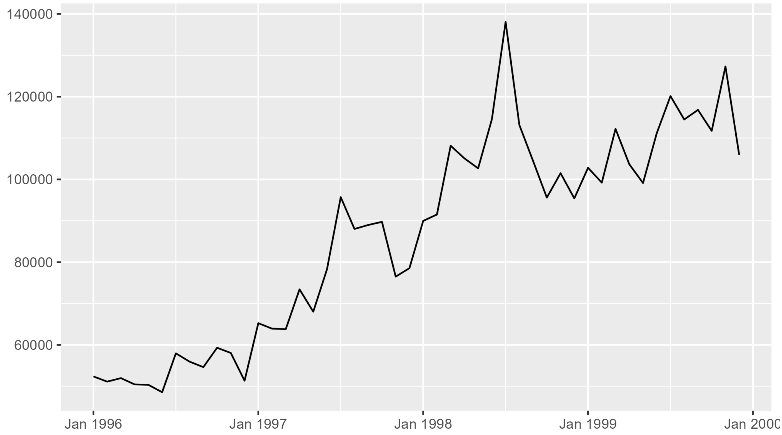
Data transformations are normally done to stabilize the variance of the data.

- Transformations: e.g. taking logarithms, square roots, or Box-Cox transformations to stabilize variance.
- Trend and Seasonal Adjustments
  - Differencing: subtracting the previous value to remove trend or seasonality.
  - Regression model

Linear Trend

- For many economical time series an obvious steady increase or decrease over time (**linear orientation**) can be observed.
- Assumption: the time series can be modeled by a straight line (**trend line**).
  - **Goal 0:** Describe the time series.
  - **Goal 1:** Predict the long term development of the time series.
  - **Goal 2:** Trend adjustment, i.e. elimination of the trend component $\rightarrow$ stationarity!
- Identification of the trend line: **using linear regression**.

Empirical example: DAX 1996-2000



Trend line I

- Assumption: We use linear regression model to capture a time series with a linear trend. The variable is described by a deterministic function m_t of time.
- Observation value x_t deviates from m_t just by random influence or measurement error u_t.
- x_t is composed of m_t and u_t additively, i.e.

$$x_t = m_t + u_t \quad t = 1, 2, ..., N$$

- The random influences u_t are realizations of independent random variables,
  - expectation of the mean **equals zero** and
  - the variance is **constant** through time.

Trend line II

- The equation of the trend line is a linear function:

$$m_t = b_0 + b_1 t$$

- The coefficients b_0 and b_1 are determined by **method of least squares**.
- Based on the differences

$$u_t = x_t - m_t \quad = x_t - b_0 - b_1 t \quad t = 1, 2, ..., N$$

a straight line can be chosen, where the sum of squared errors (S) is minimum, i.e.

$$S = \sum_{t=1}^{N} (x_t - b_0 - b_1 t)^2 \rightarrow MIN!$$

Minimizing the objective function I

- The **objective function** reads:

$$S = \sum_{t=1}^{N} (x_t - b_0 - b_1 t)^2 \rightarrow MIN!$$

- The first **partial derivatives** of S by coefficients b_0 and b_1 equal

$$\frac{\partial S}{\partial b_0} = 2 \sum_{t=1}^{N} (x_t - b_0 - b_1 t)(-1) \quad and$$

$$\frac{\partial S}{\partial b_1} = 2 \sum_{t=1}^{N} (x_t - b_0 - b_1 t)(-t)$$

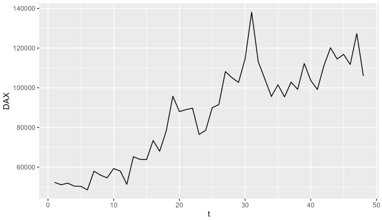
Minimizing the objective function II

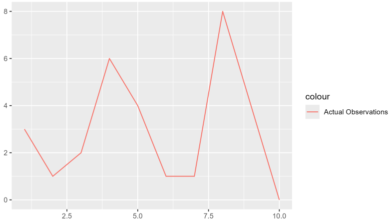
- The **method of least squares** used for estimation of the regression function minimizes S for both regression coefficients b_0 and b_1.
- To this end the first derivatives $\frac{\partial S}{\partial b_0}$ and $\frac{\partial S}{\partial b_1}$ are **set to zero** and solved for b_0 and b_1.
- Finally, the **derived regression coefficients** read:

$$b_1 = \frac{N \sum t x_t - (\sum t)(\sum x_t)}{N \sum t^2 - (\sum t)^2} = \frac{\overline{t x_t} - \bar{t} \cdot \bar{x}}{\overline{t^2} \cdot (\bar{t})^2}$$

$$b_0 = \bar{x} - b_1 \bar{t}$$

DAX 1996-2000 with trend line





Linear Regression – Dax data

	(1)
Intercept	46 975.060***
	(1983.780)
trend	1617.418***
	(86.509)
Num.Obs.	48
R2	0.826
R2 Adj.	0.822
AIC	1029.1
BIC	1034.7
RMSE	10 286.44

Decomposition of the total variation

For linear models estimated by method of least squares it applies:

Total variation = explained variation + not explained variation resp.

$$\sum_{t=1}^{N}(x_t-\bar{x})^2=\sum_{t=1}^{N}(\hat{x}_t-\bar{x})^2+\sum_{t=1}^{N}(x_t-\hat{x})^2$$

with $\hat{x}_t = x_t - u_t = m_t = b_0 + b_1 t$ as **estimated value** for x_t

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Goodness-of-fit: coefficient of determination

$$r^2 = \frac{explained\ variation}{Total\ variation} = \frac{\sum_{t=1}^{N}(\hat{x}_t - \bar{x})^2}{\sum_{t=1}^{N}(x_t - \bar{x})^2}$$

- The determination coefficient r^2 measures the goodness-of-fit of the trend line towards the time series. R^2 is normalized between zero and one.
- Higher R^2 values indicate a better fit of the model to the data.
- The **adjusted** R^2 is R^2 corrected for degrees of freedom so that the total number of explanatory variables used is accounted for.

Higher-order polynomials for trend and smooth component

- In many cases a **linear** function is not very appropriate to describe the average movement of a time series.
- Example: Unemployment data 1990-2010
  - curved process
  - trend reversal
- Problem: The smooth component (trend + cyclical component) must **not** follow every single movement of the time series. It has to be smooth: the typical process of the curve shall be visible.

Higher-order polynomial I

- The general linear regression model includes k+1 well-known functions of time $m_0(t), m_1(t), m_2(t), ..., m_k(t)$ and assumes a linear combination in order to model the trend process:

$$m(t) = b_0 m_0(t) + b_1 m_1(t) + b_2 m_2(t) + ... + b_k m_k(t)$$

  with $b_0, b_1, ..., b_k$ chosen with the highest goodness-of-fit for the observed time series $\rightarrow$ method of **least squares!**

- The model of the linear trend is a special case, while $m_0(t) = 1$ and $m_1(t) = t$.

- If no ordinary linear trend can be assumed $\rightarrow$ **polynomials!**

Higher-order polynomial II

- A **polynomial** trend of order k

$$m_(t) = b_0 t + b_1 t + b_2 t^2 + ... + b_k t^k$$

arises from the linear regression model by choosing

$$m_0(t) = 1, m_1(t) = t, m_2(t) = t^2, ..., m_k(t) = t^k$$

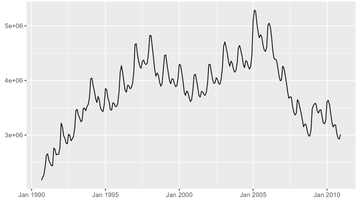
- Again the coefficients are estimated by the method of least squares, i.e. $b_0, b_1, ..., b_k$ so that

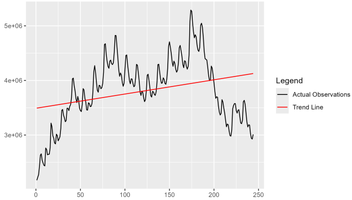
$$S = \sum_{t=1}^{N} (x_t - m(t))^2 = \sum_{t=1}^{N} (x_t - b_0 - b_1 t - b_2 t^2 - ... - b_k t^k)^2$$

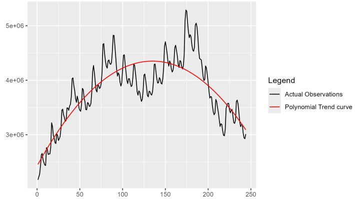
becomes minimal!

- This type of trend models is well suited in order to describe a time series.
- Warning: Be careful with forecasts by trend extrapolation. Out of its adjustment range polynomials tend towards $\pm\infty$ rapidly.

Empirical example: Unemployment data 1990-2010







Regressions – Unemployment Data

	Linear	Polynomial
Intercept	3 490 640.303***	2 427 880.803***
	(83 986.186)	(44 379.343)
trend	2605.813***	28 526.776***
	(655.111)	(1057.007)
$trend^2$		−105.800***
		(4.155)
Num.Obs.	244	244
R2	0.089	0.675
R2 Adj.	0.086	0.672
AIC	7179.2	6929.8
BIC	7189.7	6943.8
RMSE	585 462.64	349 802.82

Exponential trend I

- **Exponential trend** is the main approach in order to model **non linear** trend or smooth components.
- The trend function is represented by an **exponential function**:

$$m(t) = a \cdot e^{bt}$$

- The slope of the trend function

$$\frac{\partial m}{\partial t} = b \cdot a \cdot e^{bt} = b \cdot m(t)$$

  is not constant, but proportional to the actual level.
- Coefficient b is a **constant rate** for the growth of the trend function.

Exponential trend II

- In the **multiplicative model** (excluding the seasonal S and cyclical component C) time series x_t can be represented by:

$$x_t = a \cdot e^{bt} \cdot u_t$$

- By taking the **logarithm** of:

$$ln\ x_t = ln\ a \cdot bt \cdot ln\ u_t$$

- **Substitution** by $ln\ a = b_0; b = b_1; ln\ u_t = u_t^*$:

$$ln\ x_t = b_0 + b_1 t + u_t^* \;\; resp. \;\; m(t)^* = b_0 + b_1 t$$

with $m(t)^* = in\ x_t - u_t^*$

- The **calculation of coefficients** again can be carried out by simple linear regression.

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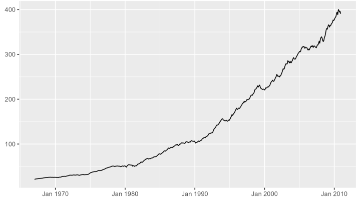
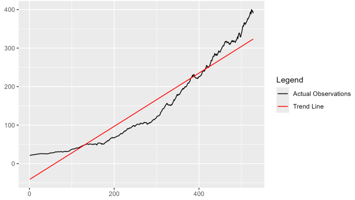
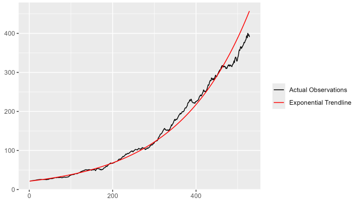
REX 1967-2010







Regression – Rex Data

	Linear	Exponential
Intercept	-41.391^{***}	3.063^{***}
	(3.154)	(0.005)
trend	0.691^{***}	0.006^{***}
	(0.011)	(0.000)
Num.Obs.	528	528
R2	0.915	0.994
R2 Adj.	0.915	0.994
AIC	5164.3	6958.8
BIC	5177.1	6971.6
RMSE	32.00	175.06

Important Lessons

- **Identification stage:** the choice of model
- **Estimation stage:** estimate the parameters of the model
- **Diagnostic checking stage:** criterion to evaluate the relative goodness of fit of the models

- **Linear regression** is powerful for capturing linear trends in data.

- **Linear regression** can be used to fit global trend that applies to the entire series.
- R^2 helps assess the goodness-of-fit of the regression model..
- **Higher-order polynomials** can model more complex trends but require careful use to avoid overfitting.
- **Exponential trends** are useful in modeling growth patterns where the dependent variable grows at a constant percentage rate, fitting well with the multiplicative model.
- **Avoid** selecting overly **complicated trend patterns**.
- Always **evaluate** how the model performs on the **test data**.