Advanced Microeconomics

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VL 10 - Repeated games

- Gibbons, Chapter 2
- Osborne, Chapters 14 and 15

Repeated games

- Games in which *the same stage-game* is being played at every stage
- All players are informed about the result of the last round (stage) before the next round (stage) starts

- Repeated games are actually a special case of dynamic games
- However, they have some interesting properties that do not apply to general dynamic games
  - In particular, threats or promises about behavior in the future can influence present behavior

Finitely often repeated games

- Definition
  - Let $G = (I, \{S_i\}, \{u_i\})$ be the stage-game in normal form. Then $G(T, \delta)$ denotes the **finitely repeated game** in which the stage-game G is played T consecutive times and δ is the **common discount factor**.

- Example: Prisoner's dilemma is the stage game G
  - (O, L) is the unique Nash equilibrium of the stage-game with payoff (1,1)

Player 2

Player 1		L	R
	O	1, 1	5, 0
	U	0, 5	4, 4

Finitely often repeated games

- Prisoner's dilemma played 2 times (no discounting)
  - Backward induction: unique Nash equilibrium of the lowest subgame (stage-game) is (O_2, L_2) with payoff (1,1)

Player 2

Player 1		L_2	R_2
	O_2	1, 1	5, 0
	U_2	0, 5	4, 4

  - then the „reduced" game at the first stage is (index stands for stage)

Player 2

Player 1		L_1	R_1
	O_1	2, 2	6, 1
	U_1	1, 6	5, 5

Finitely often repeated games

- Prisoner's dilemma played 2 times
  - The unique Nash equilibrium of the reduced game is (O_1, L_1)
  - the unique SPE of the once repeated prisoner's dilemma is $\big((O_1, O_2), (L_1, L_2)\big)$

- If the stage-game G has a unique Nash equilibrium, then the finitely often repeated game $G(T, \delta)$ has a unique SPE
  - The Nash equilibrium of the stage-game is played at every stage
  - In this class of games, SPE are independent of the "history" of the game.

Infinitely repeated games

- Definition
  - Let $G = (I, \{S_i\}, \{u_i\})$ be the stage-game in normal form. Then the payoffs of the **infinitely repeated game are** defined as follows: Let $\pi_1, \pi_2, \pi_3, \ldots$ be the payoffs at each stage. Then the payoff for the total game (i.e., the present value of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, \ldots$) is:

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 \ldots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t,$$

    where $\delta \in (0,1)$ is the discount factor.

  - Sometimes such a game is written as $G(\infty, \delta)$, or simply $G(\delta)$

Infinitely repeated games

- In the above Prisoner's Dilemma, can cooperation (U, R) occur at every stage of the infinitely repeated game?
  - Even though the only Nash equilibrium of the stage-game is (O, L)?
- Claim: For a sufficiently large discount factor δ, the reciprocal play of the following **(grim) trigger strategy** is a SPE of the infinitely repeated prisoner's dilemma:
  - Player 2: "Play R at the first stage ($t = 1$). Play R at the stage $t \geq 2$ if the result of all previous $t-1$ stages was (U, R), otherwise play L."
  - Player 1 with equivalent strategy for actions U and O

Player 2

Player 1		L	R
	O	1, 1	5, 0
	U	0, 5	4, 4

- Proof - Step 1
  - Show that there is a δ (sufficiently large) such that reciprocal play of this strategy is a Nash equilibrium of the infinitely repeated prisoner's dilemma
    - i.e., of the total game

a) Suppose both players stick to the **(grim) trigger strategy**, then (U, R) is played at each stage.
   - Each player receives the sequence of payoffs 4, 4, 4, ...
   - The present value of this payoff stream is

Infinitely repeated games

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a) Suppose both players stick to the **(grim) trigger strategy**, then (U, R) is played at each stage.
   - Each player receives the sequence of payoffs 4, 4, 4, ...
   - The present value of this payoff stream is

$$4 + 4\delta + 4\delta^2 + \cdots = 4\sum_{t=1}^{\infty} \delta^{t-1} = \frac{4}{1-\delta}$$

- Proof - Step 1 continued
  b) Assuming player 1 sticks to the (grim) trigger strategy, but player 2 deviates

    - In the event of deviation, her payoff sequence would be $5, 1, 1, \ldots$
    - The present value of its payoff sequence is

  – Deviation is not worthwhile if

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta}$$

Infinitely repeated games

- Proof - Step 1 continued

  b) Assuming player 1 sticks to the (grim) trigger strategy, but player 2 deviates

    - In the event of deviation, her payoff sequence would be $5, 1, 1, \ldots$
    - The present value of its payoff sequence is

$$5 + \delta + \delta^2 + \cdots = 5 + \sum_{t=2}^{\infty} \delta^{t-1} = 5 + \frac{\delta}{1-\delta}$$

  – Deviation is not beneficial if

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta} \implies 4 \geq 5 - 5\delta + \delta \implies \delta \geq \frac{1}{4}$$

Proof - Step 2: Show that the (grim) trigger strategies induce a Nash equilibrium in each subgame.

- Each subgame of the infinitely repeated game is identical to the whole game.
- these subgames can be split into two groups:
  1) Subgames in which the results of all previous periods were (U, R):
     - If $\delta \geq 1/4$, then the (grim) trigger strategy is a Nash equilibrium of this subgame.
       - ➢ this is exactly what we have shown above
  2) subgames in which the result of at least one previous stage was not (U, R):
     - in these subgames, the (grim) trigger strategies of the players dictate playing (O, L) at each stage.
       - – Deviating unilaterally isn't worth it, so the (grim) trigger strategy is also a Nash equilibrium of this subgame

Folk Theorem

- Alternative popular strategy: "**Tit-for-tat**"
  - Play 'cooperation' in period 1 and then always replicate the behavior of the other player in the previous period, i.e.
    - if the other player is cooperative, you behave cooperatively in the following round
    - if the other player is non-cooperative, you also behave non-cooperatively in the following round
  - For δ sufficiently large, these punishment strategies also lead to mutual cooperation in the SPE

Folk Theorem

- What payoffs can you get in an equilibrium of any infinitely repeated game?

- The Folk Theorem or Friedman Theorem (1971) gives information about this
  - Definition: feasible payoffs
  - Definition: Average payoffs
  - Folk Theorem
  - Proof

- Definition *feasible*
  - The payoffs $(x_1, \ldots, x_n)$ of players $i = 1, \ldots n$ are called feasible if they can be described as a **convex combination of** the payoff combinations of the stage-game
    - Convex combinations are the **weighted average**, where the weights must be non-negative and sum to 1
  - Fig. shows **convex combination of** the payoff combinations of the stage-game

		Player 2	
		L	R
Player 1	O	1, 1	5, 0
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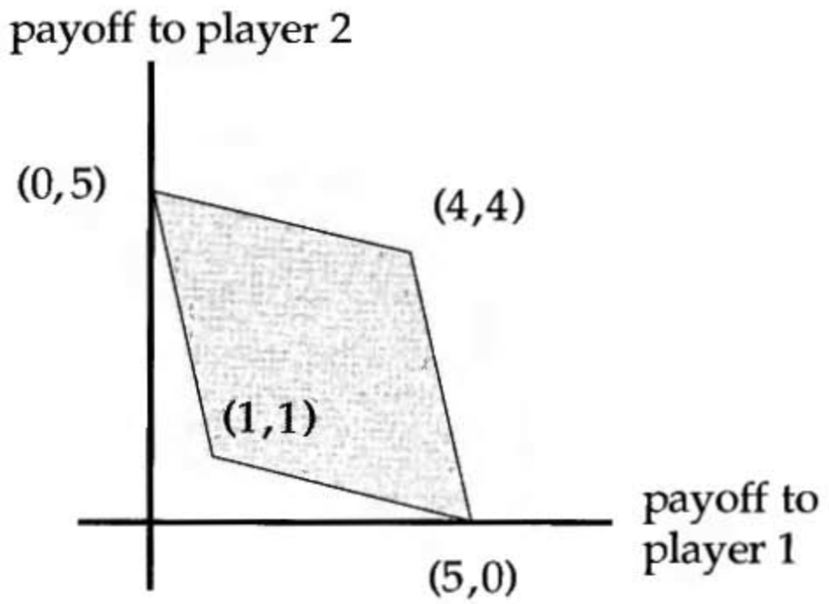

Folk Theorem

- Definition of **average payoff**
  - The **present value of** the payoff sequence $\pi_1, \pi_2, \ldots$ is $\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t$
    - For $\pi_t = \pi = constant$, we have….

      so that $(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\pi = \pi$
  - In general, $(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\pi_t$ is called the **average payoff** of the sequence of payoffs $\pi_1, \pi_2, \ldots$
  - The average payoff is therefore merely a rescaling of the present value $\sum_{t=1}^{\infty} \delta^{t-1}\pi_t$ with the factor $1-\delta$
    - Therefore, maximizing the average payoff is equivalent to maximizing the present value
  - Advantage of using the average payoff: it is directly comparable to the payoffs from the stage-game

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    - For $\pi_t = \pi = constant$, we have $\sum_{t=1}^{\infty} \delta^{t-1}\pi_t = \pi \sum_{t=1}^{\infty} \delta^{t-1} = \frac{\pi}{1-\delta}$

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  - In general, $(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\pi_t$ is called the **average payoff** of the sequence of payoffs $\pi_1, \pi_2, \ldots$
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- **Folk Theorem** (Friedman, 1971)
  - Let G be a finite, static game with complete information. Let $(e_1, \ldots, e_n)$ be the payoffs of a Nash equilibrium of G and let $(x_1, \ldots, x_n)$ be a feasible payoff of G. If $x_i > e_i$ for all $i = 1, \ldots, n$ and δ if is sufficiently close to 1, then there exists a SPE of the infinitely repeated game $G(\infty, \delta)$ that achieves an average payoff $(x_1, \ldots, x_n)$.

- Implication
  - With δ sufficiently large, we can achieve a SPE with better payoffs than in the Nash-equilibrium of the stage-game
    - E.g. mutual cooperation in the prisoner's dilemma

Folk Theorem

- An infinitely repeated game has an infinite number of SPE with extremely different outcomes
  - Therefore, the Folk theorem has little predictive power
  - "Almost all" payoffs can be achieved as a SPE if players are sufficiently patient

- Question: Is the game with alternating offers from the last lecture a repeated game?

- Nice link to play: https://ncase.me/trust/

Appendix: Geometric series

- The sum of a geometric sequence can also be represented as a fraction
- Consider the following infinite geometric sequence

$$\pi + \pi\delta + \pi\delta^2 + \pi\delta^3 + \cdots = \pi(1 + \delta + \delta^2 + \cdots) = \pi \sum_{t=0}^{\infty} \delta^t .$$

- Define x as the value of the geometric sequence

$$x \coloneqq 1 + \delta + \delta^2 + \cdots$$

- Multiply both sides by δ

$$\delta x = \delta + \delta^2 + \delta^3 + \cdots$$

- Subtraction of the two equations gives:

$$(1 - \delta)x = 1 \rightarrow x = \frac{1}{1-\delta}$$