Advanced Microeconomics

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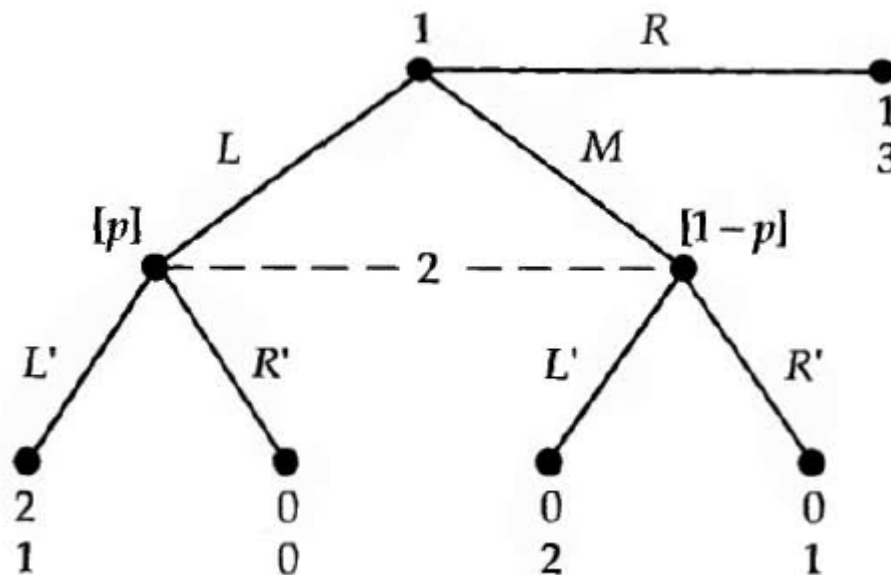# Lecture 11: Dynamic games with incomplete information

Essential reading:

- Gibbons, Chapter 4
- Tadelis, Chapter 15

- as in dynamic games with complete information, we need to eliminate equilibria that are based on implausible threats

- We will generalize the idea of subgame perfection by considering continuation games
  - these can also start in an information set that contains more than 1 decision nodes
    - in contrast to subgames

- the player with the move at such a nonsingleton information set must have a "Belief" – i.e. a probability distribution – about which node in the information set has been reached by the play of the game
  - Remember: an information set is a collection of all the possible decision nodes that a player cannot distinguish among when called on to act.
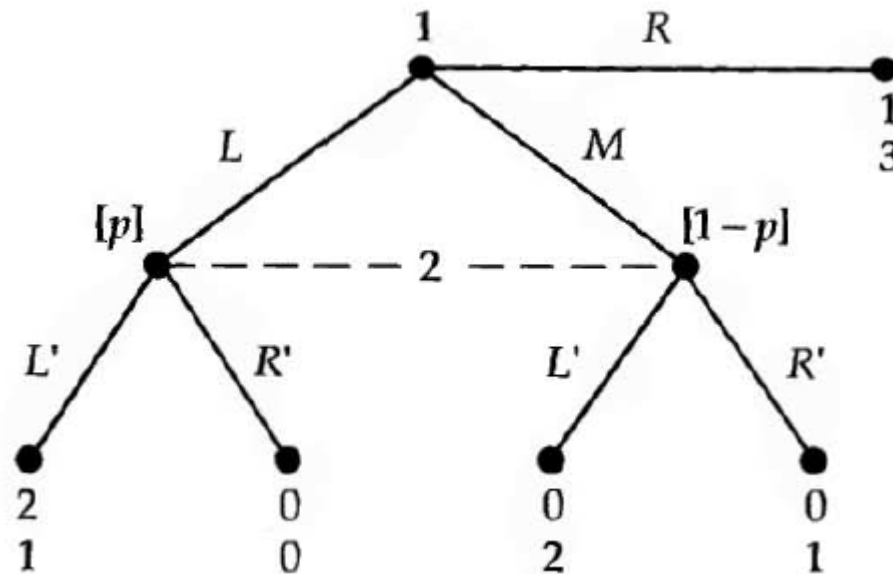
Comparison of equilibrium concepts

- This will lead us to the **perfect Bayesian equilibrium** (**PBE**)

- The PBE is consistent with

  - the SPE if it is a dynamic game with complete information,

  - the Bayesian Nash equilibrium if it is a static game with incomplete information, and

  - the Nash equilibrium if it is a static game with complete information.

- Definition: A system of **beliefs** p in an extensive form game is an assignment of probabilities $p(x) \in [0,1]$ for each decision node x in the game such that $\sum_{x \in H(x)} p(x) = 1$ for all information sets H.
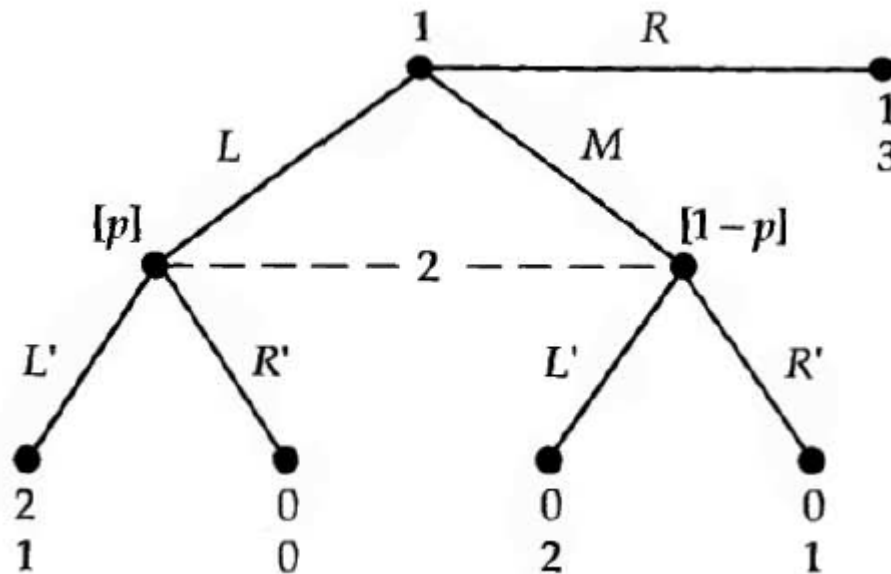
Player 2

	L′	R′
L		
Player 1 M		
R		

- Nash-equilibrium in pure strategies:

Player 2

Player 1		L'	R'
	L	2, 1	0, 0
	M	0, 2	0, 1
	R	1, 3	1, 3

- Nash-equilibrium in pure strategies: $(L, L'), (R, R')$

- In this game, there is no proper subgame

  - Since (R, R') is a Nash equilibrium of the total game, it is also subgame perfect. But it is not sequentially rational.

    - Player 2 threatens to play R', but for any Beliefs p, L' leads to a higher payoff.

- Additional equilibrium requirements (to exclude the SPE
  (R, R'))

  – **Requirement 1:** At each information set, the player with the move
  must have a belief about which node in the information set has been
  reached by the play of the game.

  – **Requirement 2:** Players' strategies must be sequentially rational
  (given their beliefs).

    - That is, given a player's beliefs, his strategy in any continuation
      game must be a best response against his opponents' strategies.

Application of requirements to example

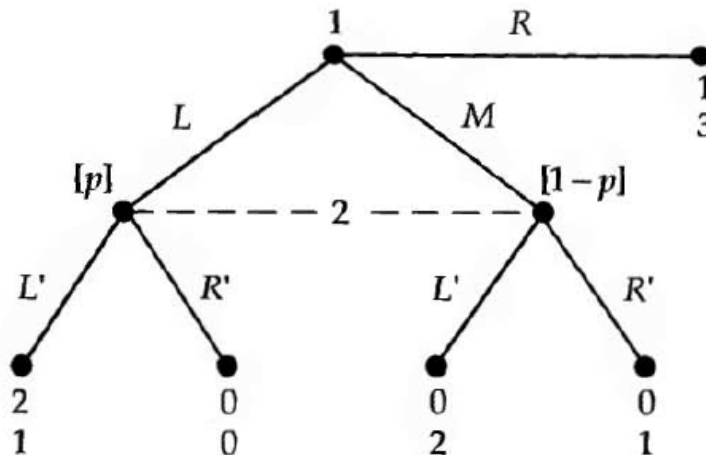# Application of requirements to example

- **Requirement 1:** At each information set, the player with the move must have a belief about which node in the information set has been reached by the play of the game.

- **Requirement 2:** Players' strategies must be sequentially rational (given their beliefs).

- No matter what belief Player 2 has in his information set, it is always better to play L' than R'
  - i.e., L' is a dominant strategy

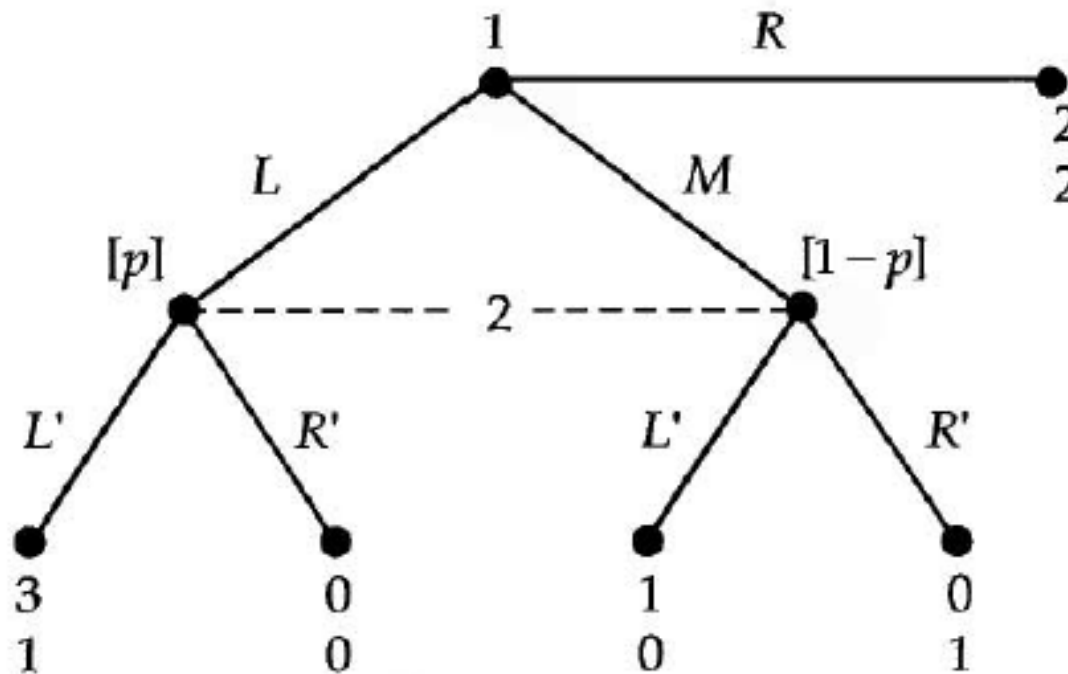- these claims eliminate the Nash equilibrium (R, R')

- Extension: players must not only have Beliefs (requirement 1), but beliefs must also be "reasonable".

  – Optimal action of a player depends on his beliefs

    - the beliefs must be consistent with the strategies of the players and the history of the game

- *Definition:* For a given equilibrium in a given extensive-form game, an information set is

  – **on the equilibrium path** if it will be reached with positive probability if the game is played according to the equilibrium strategies, and is

  – **off the equilibrium path** if it is certain not to be reached if the game is played according to the equilibrium strategies

- **Requirement 3**: In information sets on the equilibrium path, beliefs are determined by players' equilibrium strategies and by Bayes' rule.

- Consider the SPE (L, L') in the example. Player 2's belief must be $p = 1$

  – given the equilibrium strategy L of player 1, player 2 can anticipate which node is reached

- Suppose $p > \frac{1}{2}$. Player 2's sequentially rational strategy is L'

  – Player 1's optimal strategy in this case is L

  – Requirement 3 stipulates that beliefs are determined by players' equilibrium strategies. Thus: $p = 1$
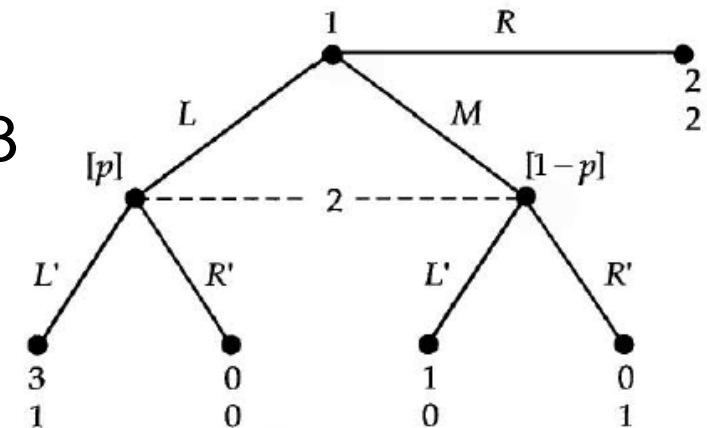
Example with new payoffs

- Suppose $p > \frac{1}{2}$. Player 2's sequentially rational strategy is L'

  – Player 1's optimal strategy in this case is L

  – Requirement 3 stipulates that beliefs are determined by players' equilibrium strategies. Thus: $p = 1$

- Suppose $p < \frac{1}{2}$. Player 2's sequentially rational strategy is R'

  – Player 1's optimal strategy in this case is R

- The information set of the game is not reached at all, **therefore requirement 3 has no effect** (it only refers to information sets on the GG path)

- Two equilibria meet requirements 1 to 3

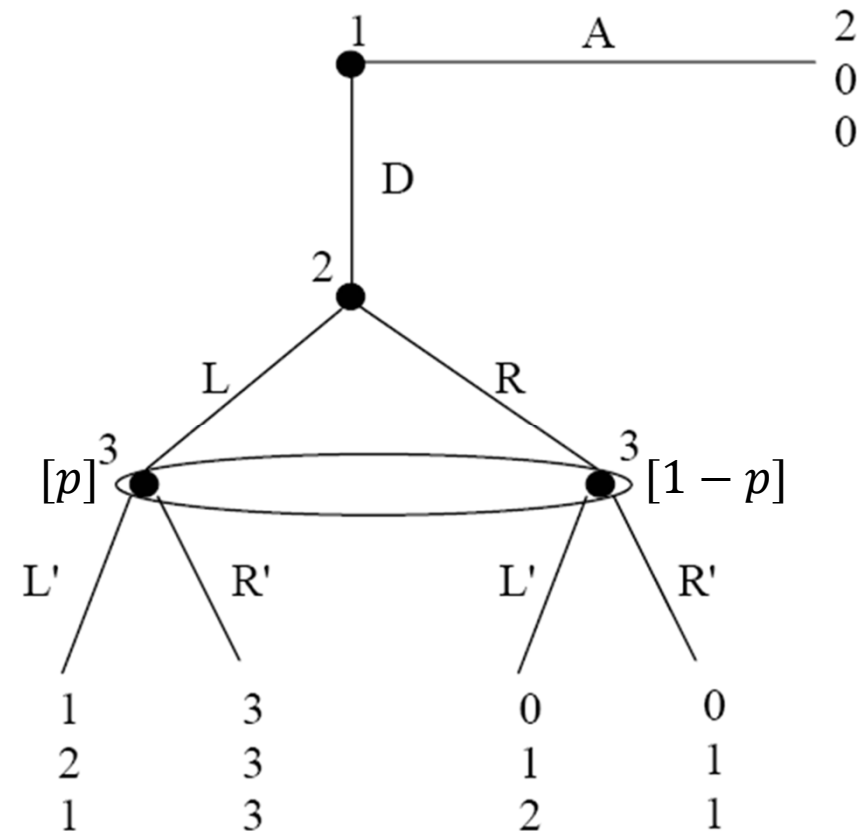 – $s^* = \left((L, L'), p = 1\right), \left((R, R'), p < \frac{1}{2}\right)$

- *Definition:* A **weakly perfect Bayesian equilibrium** (weak PBE) consists of a strategy vector and a system of beliefs satisfying requirements 1 to 3.

- When strategies are not completely mixed, certain information sets are reached with probability 0.

  – In such information sets, the concept of weak PBE allows for **arbitrary beliefs**, so that often very many results can be supported as weak PBE.

    - such as $\left((R, R'), p < \frac{1}{2} \right)$ in the example above

Perfect Bayesian Equilibrium

- refinement of the equilibrium concept if some information sets are reached with probability 0
  - **Requirement 4**: At information sets *off the equilibrium path*, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.

- *Definition:* A **perfect Bayesian equilibrium** (PBE) consists of a strategy vector and a system of beliefs satisfying requirements 1 to 4.
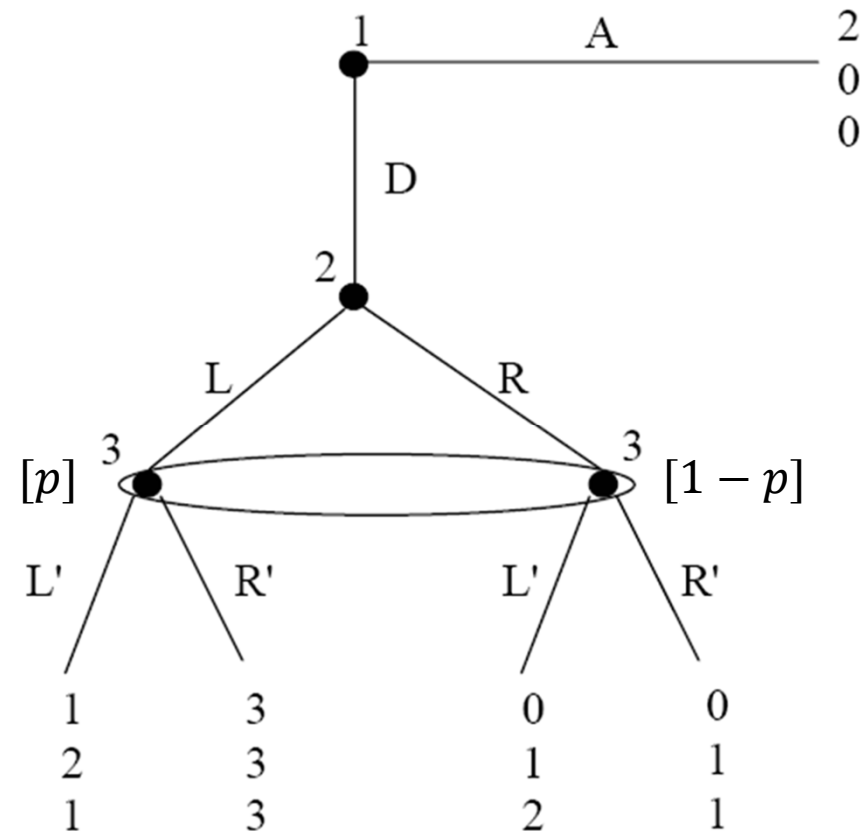
- one proper subgame with a unique Nash equilibrium (L, R'):

  - since (3,3) highest payoff for P2 & P3

- unique SPE: (D, L, R')

- Unique PBE: $((D, L, R'), p = 1)$: Requirements 1 to 3 are fulfilled

- Requirement 4 is trivially satisfied, since there is no information set off the equilibrium path
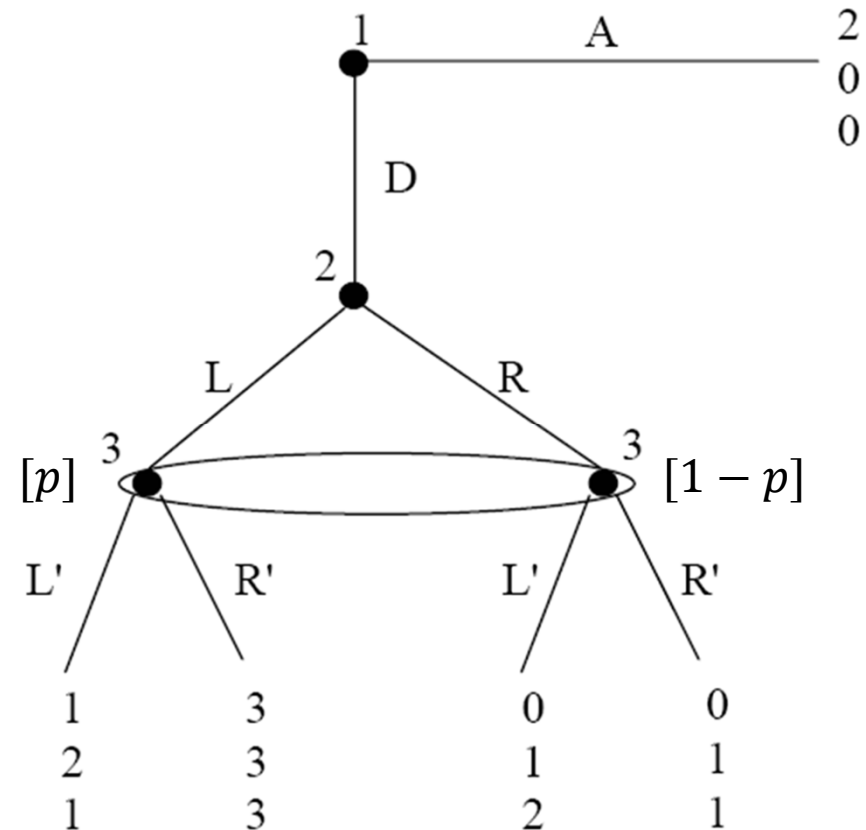
- **Further Nash GG:** (A, L, L')

  - no one gets higher payoff from deviating unilaterally

- $((A, L, L'), p = 0)$ is a *weak* PBE as it meets requirements 1 to 3

  - Problem: belief $p = 0$ of player 3 is inconsistent with the strategy of player 2

  - but the Belief $p = 0$ concerns an information set *off the equilibrium-path*

- $((A, L, L'), p = 0)$ is a weak PBE

- Requirement 4: also off the equilibrium path, Player 3 must derive his belief from player 2's strategy

- this implies $p = 1$

  – For $p = 1$, player 3 must play R' (requirement 2)

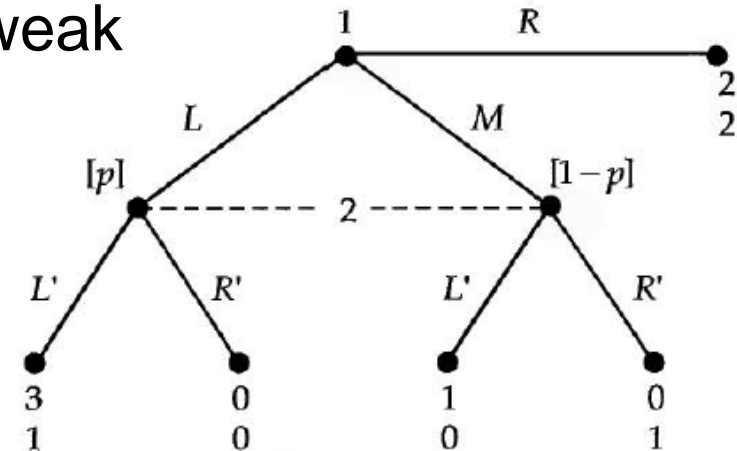 – $((A, L, L'), p = 0)$ violates requirement 4

- In preceding example, there were 2 weak

  PBE: $\left((L, L'); p = 1\right), \left((R, R'), p < \dfrac{1}{2}\right)$
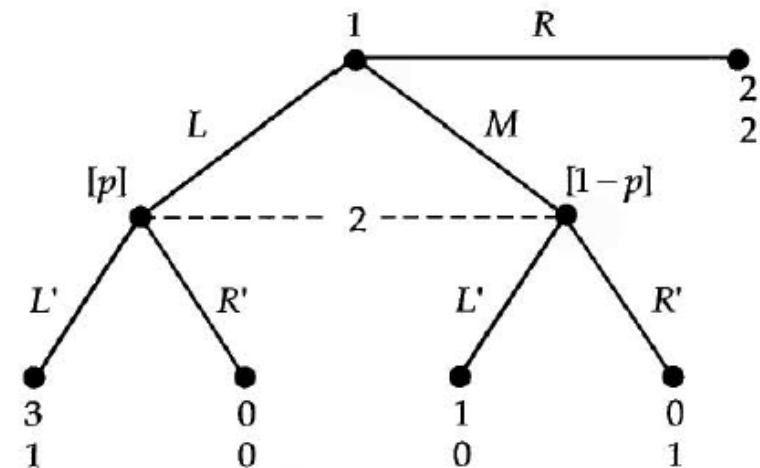

- requirement 4
  - also off the equilibrium path, beliefs must be consistent with players' strategies

  - the weak PBE $\left((R, R'), p < \dfrac{1}{2}\right)$ has an information set off the equilibrium path that concerns the beliefs of player 2

  - But: in this weak PBE $\left((R, R'), p < \dfrac{1}{2}\right)$ the strategy of player 1 has no implications for the beliefs of player 2

  - Therefore, also the PBE $\left((R, R'), p < \dfrac{1}{2}\right)$ satisfies **all** 4 requirements

- The literature has considered „equilibrium refinements" to exclude equilibrium candidates that appear less plausible

  – **Requirement** 5: If possible, each player's beliefs off the equilibrium path should place zero probability on nodes that are reached only if another player plays a strategy that is strictly dominated beginning at some information set.

- Consider again our PBE $\left((R, R'), p < \dfrac{1}{2} \right)$

Equilibrium refinements

- **Requirement** 5: If possible, each player's beliefs off the equilibrium path should place zero probability on nodes that are reached only if another player plays a strategy that is strictly dominated beginning at some information set.

- Consider again our PBE $\left((R, R'), p < \frac{1}{2} \right)$

- For player 1, M is strictly dominated by R

- Thus, requirement 5 implies that $1 - p = 0$

  – As that node is only reached if player 1 plays his strictly dominated strategy M

- The weak PBE $\left((R, R'), p < \frac{1}{2} \right)$ therefore violates requirement 5

- only one PBE that satisfies all 5 requirements: $s^* = \left((L, L'); p = 1 \right)$