Constant additive and multiplicative seasonal figures

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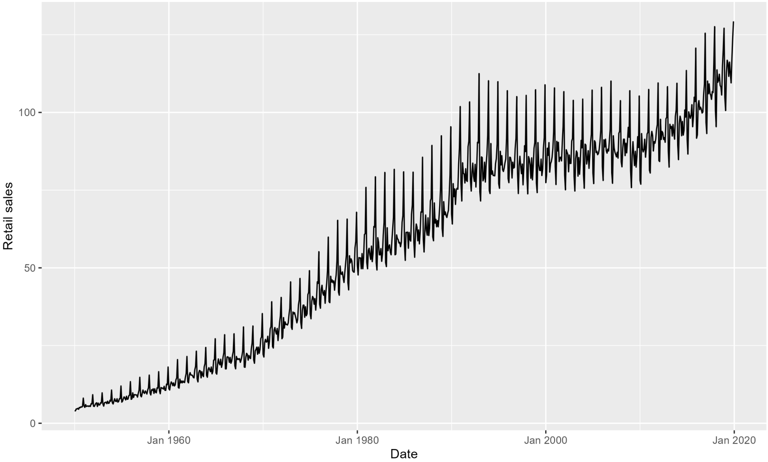
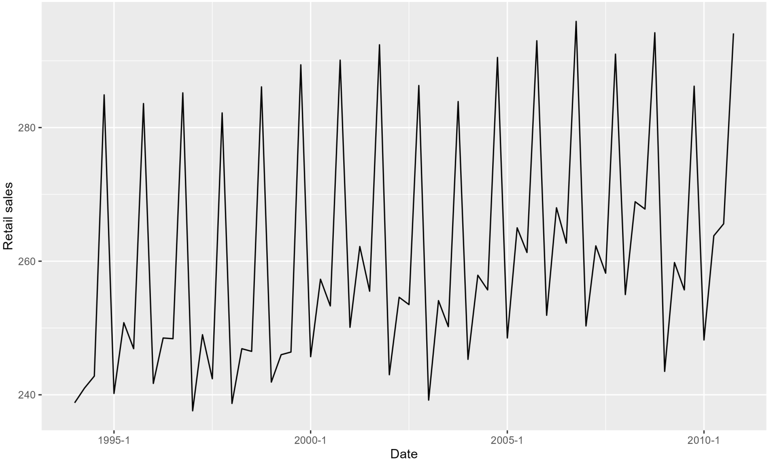
Overview

1. Phase average method
2. Seasonal dummies
3. Regression with dummy variables
4. Empirical example: Retail sales
5. Holt's Winter seasonal method
6. Forecasting error measures

Retail sales per Month (1950-2019)



Retail sales per quarter (1994-2010)



Constant additive seasonal figures

- **Assumption 1:** When there are stable periodic fluctuations for the whole observation period.
- **Assumption 2:** The time series can be represented by an additive model:

$$X = T + C + S + U$$

- **Assumption 3:** Seasonal influence increases or lowers the observation value by a constant factor.

$$X = T + C + S + U$$

- **Example:** In case of quarterly data we find four constant values

$$S_1, S_2, S_3, S_4!$$

Method I

In order to "free" the time series $X - G$ from its residual component U we introduce the **phase average method**:

1. Identification of the smooth component $G = T + C$ as
   - linear function of time,
   - polynomial of higher degree,
   - exponential function or
   - by use of moving average methods.
2. The difference $X - G = S + U$ just includes the seasonal component S and the residual component U.

Method II

3 For each phase (e.g. month, quarter) the mean deviation D_{ph} from the smooth component is computed: $\rightarrow$ the residual component U is "averaged to zero".

4 In case of **quarterly data** one gets four values D_1, D_2, D_3, D_4.

- If these four values do not add up to zero, they will be adjusted by its mean value $\bar{d} = \frac{1}{4} \sum D_{ph}$ which lead to seasonal index numbers

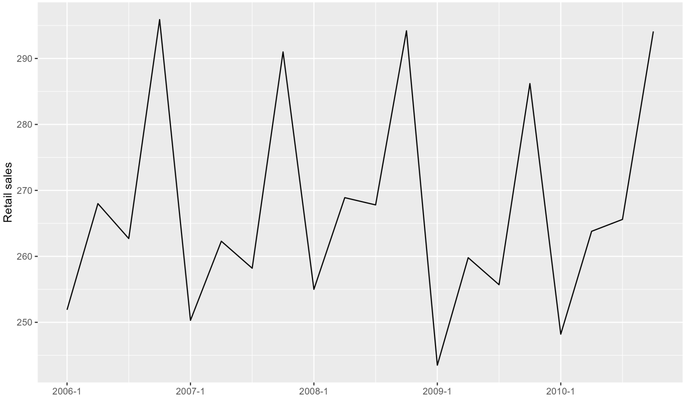
$$S_{ph:=} D_{ph} - \bar{d},$$

that add up to zero: $S_1 + S_2 + S_3 + S_4 = 0$

- A **seasonally adjusted time series** is obtained by subtracting the seasonal index number from the non adjusted time series!

Example

Retail sales per quarter (2006-2010)



Calculation of additive seasonal index numbers – Retail Sales

Year	Quarter	x	G	x-G
2006	1	251.9	NA	NA
2006	2	268.0	NA	NA
2006	3	262.7	269.4250	-6.7250
2006	4	295.9	268.5125	27.3875
2007	1	250.3	267.2375	-16.9375
2007	2	262.3	266.0625	-3.7625
2007	3	258.2	266.0375	-7.8375
2007	4	291.0	267.4500	23.5500
2008	1	255.0	269.4750	-14.4750
2008	2	268.9	271.0750	-2.1750
2008	3	267.8	270.0375	-2.2375
2008	4	294.2	267.4625	26.7375
2009	1	243.5	264.8125	-21.3125
2009	2	259.8	262.3000	-2.5000
2009	3	255.7	261.8875	-6.1875
2009	4	286.2	262.9750	23.2250
2010	1	248.2	264.7125	-16.5125
2010	2	263.8	266.9375	-3.1375
2010	3	265.6	NA	NA
2010	4	294.1	NA	NA

Quarter	Mean	adjusted	N
1	-17.309375	-17.128125	4
2	-2.893750	-2.712500	4
3	-5.746875	-5.565625	4
4	25.225000	25.406250	4
Total	-0.725	0	16

Regression analysis

- Calculation of seasonal index numbers by means of regression analysis.

  Regression with **dummy variables**.
- Dummy variables are auxiliary variables with two outcomes (e.g., 0 and 1) for measuring certain qualitative impacts $\rightarrow$ **binary** or **dichotomous** variables.
- **Four** quarters of a year can be represented by four seasonal dummies.

Calculation of additive seasonal index numbers

t	1	2	3	4	...	68
x_t	x_1	x_2	x_3	x_4	...	x_{68}
constant	1	1	1	1	...	1
$Dummy_1$	1	0	0	0	...	0
$Dummy_2$	0	1	0	0	...	0
$Dummy_3$	0	0	1	0	...	0
$Dummy_4$	0	0	0	1	...	1

Linear regression approach II

- By using the general approach with constant b_0 just **three** seasonal dummies are adequate:

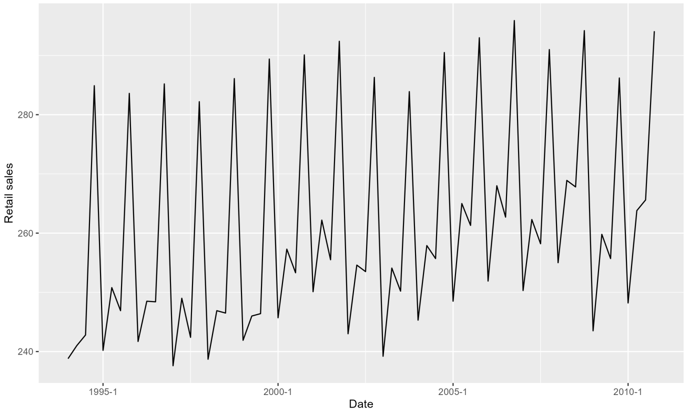
  $$x_t - g_t = b_0 + b_1 \cdot Dummy_1 + b_2 \cdot Dummy_2 + b_3 \cdot Dummy_3 + u_t$$

  with

  $$\begin{aligned} D_1 &= b_0 \\ D_2 &= b_0 + b_1 \\ D_3 &= b_0 + b_2 \\ D_4 &= b_0 + b_3 \end{aligned}$$

- The seasonal index numbers are obtained by adjusting the phase averages D_{ph} for their means.
- The residuals of the regression equation are representing the residual component U. The time series $x_t^{SA} = g_t + u_t$ is referred to as **seasonally adjusted**.

Retail sales per quarter (1994-2010)



Linear regression

	WITHOUT constant term	WITH constant term
Constant		244.682***
		(1.271)
quarter1	244.682***	
	(1.271)	
quarter2	256.241***	11.559***
	(2.006)	(2.375)
quarter3	253.700***	9.018***
	(1.868)	(2.259)
quarter4	288.765***	44.082***
	(1.035)	(1.639)
Num.Obs.	68	68
R2	0.999	0.872
R2 Adj.	0.999	0.866
AIC	455.2	455.2
BIC	466.3	466.3
RMSE	6.39	6.39

Empirical example: seasonal index numbers

- The regression equation reads:

$$x_t - g_t = b_0 + b_1 \cdot Dummy_1 + b_2 \cdot Dummy_2 + b_3 \cdot Dummy_3 + u_t$$

with

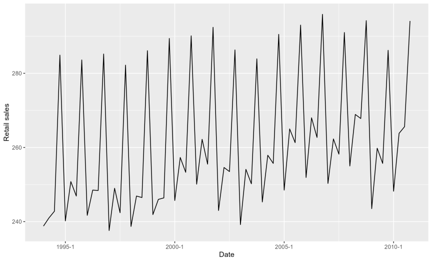
$$\begin{aligned} D_1 &= b_0 = 244.682 \\ D_2 &= b_0 + b_1 = 244.682 + 11.559 = 256.241 \\ D_3 &= b_0 + b_2 = 244.682 + 9.018 = 253.7 \\ D_4 &= b_0 + b_3 = 244.682 + 44.082 = 288.764 \end{aligned}$$

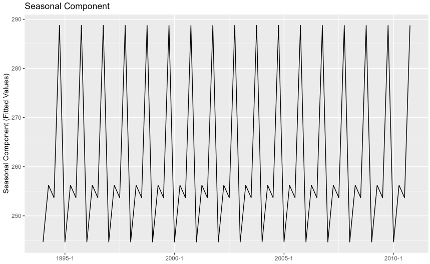
- The seasonal index numbers are obtained by adjusting the phase averages D_{ph} for their means, i.e.:

$$\bar{d} = \frac{1}{4} \sum D_{ph} = \frac{1}{4}(244.682 + ... + 253.7) = 260.8468$$

and therefore $S_1 = D_1 - \bar{d} = -16.1648; S_2 = -4.6068; S_3 = -7.1468; S_4 = 27.9172$

Retail sales / seasonal component





Regression with trend component

- The real advantage of the regression method is, that it can be used to determine seasonal **and** trend component simultaneously.
- Therefore seasonal dummies and trend function are combined in the regression equation.
- The approach for the **linear** trend now reads:

$$x_t = b_0 + b_1 \cdot Dummy_1 + b_2 \cdot Dummy_2 + b_3 \cdot Dummy_3 + b_4 \cdot t + u_t$$

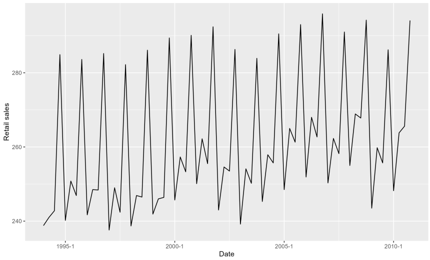
- for the **linear** and **quadratic** trend:

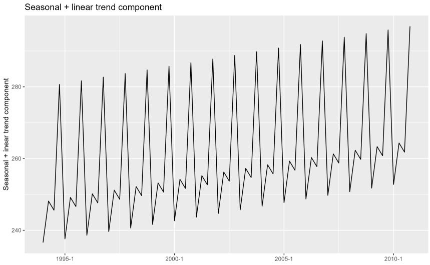
$$x_t = b_0 + b_1 \cdot Dummy_1 + b_2 \cdot Dummy_2 + b_3 \cdot Dummy_3 + b_4 \cdot t + b_5 \cdot t^2 + u_t$$

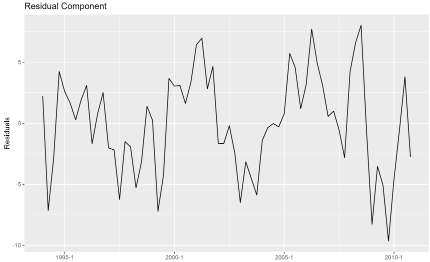
Regression with linear and quadratic trend component

	linear trend	quadratic trend
Constant	236.332***	235.045***
	(1.156)	(1.653)
trend	0.253***	0.364***
	(0.027)	(0.104)
$trend^2$		-0.002
		(0.002)
quarter2	11.306***	11.303***
	(1.510)	(1.500)
quarter3	8.512***	8.508***
	(1.374)	(1.392)
quarter4	43.323***	43.323***
	(1.352)	(1.351)
Num.Obs.	68	68
R2	0.949	0.950
R2 Adj.	0.946	0.946
AIC	394.4	395.1
BIC	407.7	410.7
RMSE	4.03	3.99

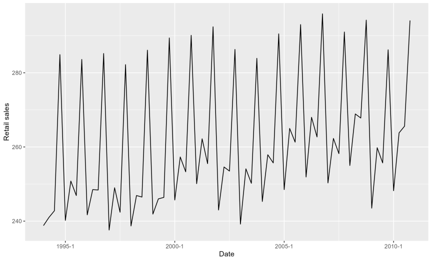
seasonal + linear trend component

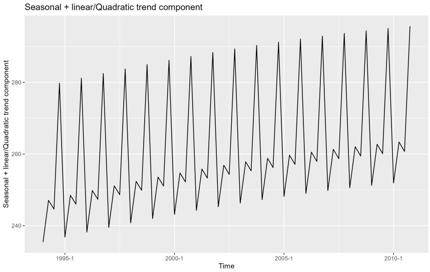


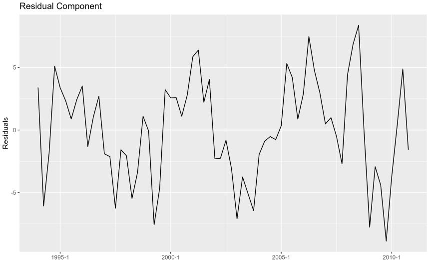




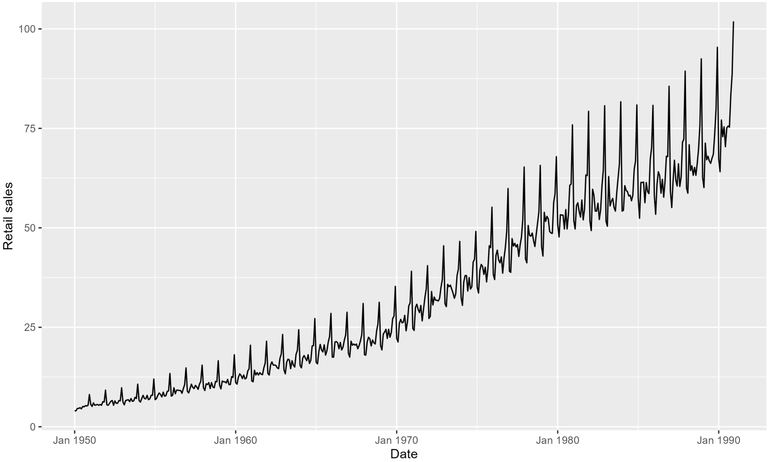
seasonal + linear and quadratic trend component



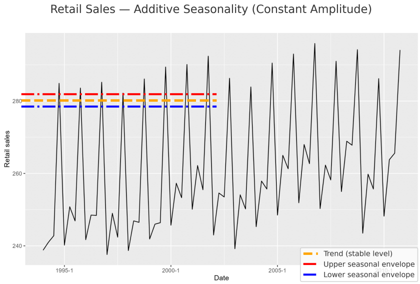


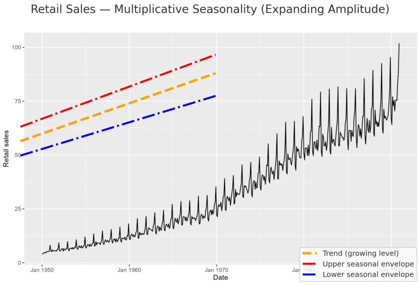


Retail sales per Month (1950-1990)



How to visually inspect





Constant multiplicative seasonal figures

- **Assumption 1:** When there is a proportional or percentage increase in periodic fluctuations for the whole observation period.
- **Assumption 2:** The time series can be represented by a multiplicative model: model:

$$X = T \cdot C \cdot S \cdot U$$

- **Assumption 3:** Seasonal influence increases or lowers the observation value by a proportional or percentage factor.

$$X = T + C + S + U$$

- **Example** In case of quarterly data we find four constant values

$$S_1, S_2, S_3, S_4!$$

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Phase average method I

In order to "free" the time series $\frac{X}{G}$ from its residual component U we introduce the **phase average method**:

1. After identification of the smooth component G as

$$\frac{X}{G} = S \cdot U$$

   is used, including just the seasonal and the residual component.

2. The quotients

$$\frac{x_1}{g_1}, \frac{x_2}{g_2}, ..., \frac{x_T}{g_T}$$

   show the observation values in relation to their smooth component. A quotient of 1.08 means, that the current value exceeds the smooth component by 8%.

Phase average method II

3 For each phase the average quotient Q_{ph} is computed as geometrical mean of the quotients $\frac{X}{G}$ from the same phases. In practice: often arithmetic mean or median is used.

4 In case of **quarterly data** one gets four values Q_1, Q_2, Q_3, Q_4.

- The mean of the seasonal factors must be equal to one (the neutral element of multiplication). Adjusting by the mean value $\bar{q} = \sqrt[4]{\Pi Q_{ph}}$ leads to seasonal factors

$$S_{ph} := \frac{Q_{ph}}{\bar{q}}$$

with a geometrical mean of one.

- A **seasonally adjusted time series** is obtained by dividing the non adjusted time series by the seasonal index number!

Calculation of multiplicative seasonal index numbers

Year	Quarter	x	G	x/G
2006	1	251.9	NA	NA
2006	2	268.0	NA	NA
2006	3	262.7	269.4250	0.9750
2006	4	295.9	268.5125	1.1020
2007	1	250.3	267.2375	0.9366
2007	2	262.3	266.0625	0.9859
2007	3	258.2	266.0375	0.9705
2007	4	291.0	267.4500	1.0881
2008	1	255.0	269.4750	0.9463
2008	2	268.9	271.0750	0.9920
2008	3	267.8	270.0375	0.9917
2008	4	294.2	267.4625	1.1000
2009	1	243.5	264.8125	0.9195
2009	2	259.8	262.3000	0.9905
2009	3	255.7	261.8875	0.9764
2009	4	286.2	262.9750	1.0883
2010	1	248.2	264.7125	0.9376
2010	2	263.8	266.9375	0.9882
2010	3	265.6	NA	NA
2010	4	294.1	NA	NA

Mean quotient X/G per quarter

Quarter	Geometric Mean	Adjusted	N
1	0.935	0.936	4
2	0.989	0.990	4
3	0.978	0.979	4
4	1.09	1.10	4
Total	0.9992872	1	16

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Holt-Winters with Additive Seasonal Fluctuations

- It is suitable for time series with trend but with additive seasonality.

$$F_{t+h} = \ell_t + b_t h + s_{t+h-m}$$

- three updating equations:

$$\ell_t = \alpha(x_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta)b_{t-1}$$

$$s_t = \gamma(x_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$$

- m which indicates seasonal indices used for forecasting is from the final year of the sample.

Holt-Winters with Additive Seasonal Fluctuations

- m is the length of the seasonality.
- We assume seasonality with m to be the number of the seasons in the first half of the sample.
- α is the smoothing constant, β is the slope parameter and γ is the parameter for adjusting seasonal fluctuations.
- We need m start values for the seasonal factors.
- We also need start values for g_t and b_t.

Holt-Winters with Multiplicative Seasonal Fluctuations

- It is suitable for time series with trend but with multiplicative seasonality.

$$F_{t+h} = (g_t + b_t h)s_{t+h-m}$$

- Updating equations:

$$\ell_t = \alpha \frac{x_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta)b_{t-1}$$

$$s_t = \gamma \frac{x_t}{(\ell_{t-1} + b_{t-1})} + (1-\gamma)s_{t-m}$$

- In multiplicative seasonality, values in different seasons differ by percentage amount

Evaluating Forecasting Model Performance

Forecasters are usually concerned about the accuracy of their forecast.

- Measuring the forecasting error serves several purposes:
  - as a **target criterion** for model and parameter estimation,
  - in order to assess **goodness of fit** and **adequacy** of the model,
  - in order to **compare** different forecasting approaches.

Forecasting Error

1. Forecasting error:

$$e_t = X_t - \hat{X}_{t|t-1}$$

2. Absolute error:

$$AE_t = |e_t|$$

3. Percentage error:

$$PE_t = \frac{e_t}{X_t} \cdot 100$$

4. Absolute percentage error:

$$APE_t = |\frac{e_t}{X_t}| \cdot 100$$

Accuracy Measure I

1. Mean error:

$$ME = \frac{1}{T}\sum_{t=1}^{T} e_t$$

2. Mean absolute error:

$$MAE/MAD = \frac{1}{T}\sum_{t=1}^{T} |e_t|$$

3. Mean square error:

$$MSE = \frac{1}{T}\sum_{t=1}^{T} e_t^2$$

Accuracy Measure II

1. Mean error in percent:

$$MPE = \frac{1}{T}\sum_{t=1}^{T}\frac{e_t}{X_t}\cdot 100$$

2. Mean absolute error in percent:

$$MAPE = \frac{1}{T}\sum_{t=1}^{T}|\frac{e_t}{X_t}|\cdot 100$$

3. Root mean square error::

$$RMSE = \sqrt{MSE} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}e_t^2}$$