Smoothing Methods

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Overview

1. Local trends and linear filter
2. Moving averages
3. Simple exponential smoothing
4. Exponential smoothing by Holt-Winters

Local trends

- Methods of **regression** analysis are modeling trend or smooth component as a function of time index t.
- Requirement for regression analysis: Used function must be **stable** and its parameters are **constant** during the full observation period.
- Unrealistic (often)! $\rightarrow$ Many time series show a prevalent trend of development, but in **non constant** patterns.
- For this case: Methods in order to estimate **local trends** $\rightarrow$ Filter methods $\rightarrow$ **Smoothing** the time series.
- Also: Filtering out seasonal and residual component.

Smoothing Methods

- **Averaging values** over multiple periods in order to reduce the **noise** to uncover the **patterns**.
- Smoothing methods are **data driven**.
- **Useful** in series where the components **change overtime**.
- Different smoothing differ by the **number of values averaged**, how **many times averaging is performed** and how the **average is computed**.

Linear filter

- Definition: A linear transformation of a time series x into a new time series g:

$$g_{t+v} := \sum_{i=1}^{l} \alpha_i x_{t+i-1} \; with \;\; t = 1, ..., N - l + 1$$

is called **linear filter**.

- The parameters α_i are called **weights**, the number of windows l is called **length** of the filter. v shifts the time index, that is assigned to the specific value for g.

- The filtered time series is shorter than the original one:

$$N \rightarrow N - (l - 1)$$

Example 1

- A linear filter ϕ of length $l = 4$ with **weights**

$$\alpha_1 = 0.1, \alpha_2 = 0.6, \alpha_3 = 0.3, \alpha_4 = 0.2$$

- Filtering the time series

$$x = \{6, 12, 16, 13, 6, 16, 19, 17, 21, 8, 15, 21\}$$

leads to the following output for the **first** value (with $v = 0$)

$$\{g_1 = 0.1 \times 6 + 0.6 \times 12 + 0.3 \times 16 + 0.2 \times 13 = 15.2.\}$$

- The filtered time series then is:

$$g = 15.2, 15.9, 14.4, 13.5, 19.3, 22.3, 20.0, 19.7, 15.6$$

Differencing

- A simple and widely used filter method is the **difference filter** Δ, which is a linear filter of length 2 with weights

$$\alpha_1 = -1, \alpha_2 = 1$$

- Taking the difference between two consecutive values is the **first differences** of x_t

$$\Delta x_t := x_t - x_{t-1}$$

- The output is a differences series that measures the **changes** from one period to the next.

Moving Averages

- Definition: Averaging values across a window of consecutive values/times.

$$\sum_{i=1}^{l} \alpha_i = 1$$

- The mean values g are moving across the time series.
- If all i are of equal size, i.e. $\alpha_i = \frac{1}{l}$ it is called **simple moving average**, in case of varying weights it is called **weighted moving average**.
  - A simple average gives equal weight to all past observations.
  - When the process changes, old data become misleading, yet they still “pull” the smoother toward old values.
  - This causes slow adjustment
- The **number of data points** in each average **does not change** with time for **simple moving average**

The time index of moving averages

- Question: Which **window** l is to be chosen for moving averages?.
- Each value for g includes information from l consecutive values of x
- Number v specifies the temporal assignment, with $0 \leq v \leq l-1$
- Example: Moving average of five for x:

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}$$

- Time index of the **first** calculated value for g?

$$\rightarrow g_{1+v} \ with \ v = 0, 1, 2, 3, 4 \rightarrow g_1, g_2, g_3, g_4 \ or \ g_5$$

Centered, leading and lagging moving averages

- **Centered moving averages:** $v = (l-1)/2$

$$CMA_t = (x_{t-(l-1)/2} + ... + x_{t-1} + x_t + x_{t+1} + ... + x_{t+(l-1)/2})/l$$

- **Leading moving averages:** $v = 0$

$$LMA_{t+k} = (x_t + x_{t+1} + ... + x_{t+l-1} +)/l$$

- **Lagging moving averages:** $v = (l-1)$

$$TMA_{t+k} = (x_t + x_{t-1} + ... + x_{t-l+1} +)/l$$

with $k = 1, 2, 3, 4...$

Properties of moving averages

1. The smaller length l, more jittery and faster reaction to changes.
2. The larger length l, smoother line and slower reaction to changes.
3. The **residual component** U of a time series will be regulated, because the values of the smoothing period add up to about zero, i.e. $\sum_{i=1}^{l} u_{t+i-1} \cong 0$
4. The **seasonal component** S is largely regulated, if l equals the number of phases (or a multiple) in a smoothing period, i.e. $\sum_{i=1}^{l} s_{t+i-1} \cong 0$
5. Therefore the filtered series g reads

$$g = \phi x = \phi[T+C+S+U] = \phi T+\phi C+\phi S+\phi U \cong \phi T+\phi C+0+0$$

and eventually $g \cong \phi[T + C]$, i.e. the moving average is just a mean value from **trend and cyclical component**.

Centered moving averages for even length l

- Example: Time series with **quarterly data**.
- In order to adjust **seasonal variations** in time series with quarterly data an even filter length of $l = 4$ is required.
- To this end $v = (l-1)/2 = 1.5$ is used.
- However, **inappropriate time indexes** are the result:

$$g_{1+v} = g_{2.5} = \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$$

$$g_{2+v} = g_{3.5} = \frac{1}{4}(x_2 + x_3 + x_4 + x_5)$$

$$g_{3+v} = g_{4.5} = \frac{1}{4}(x_3 + x_4 + x_5 + x_6)$$

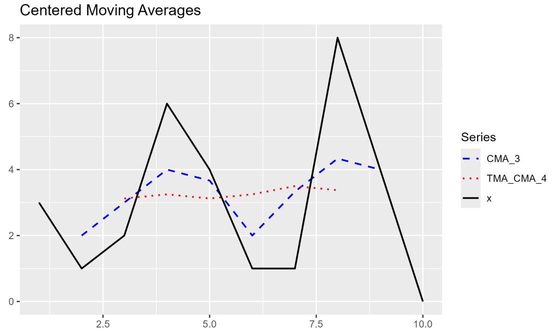
Centered moving averages for even length l

- **Indexes in whole numbers** are obtained by calculating moving **averages of two**:

$$g_3^* = \frac{1}{2}(g_{2.5} + g_{3.5}) = \frac{1}{8}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4 + \frac{1}{8}x_5$$

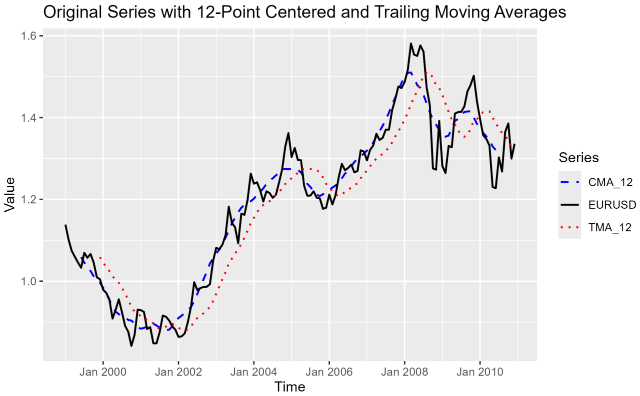
- These **second order moving averages** incorporate the first and the last observation value just half-weighted.
- Instead of simple centered moving averages of length 4 we use **weighted moving averages** of length 5.

Example: Centered moving averages calculation

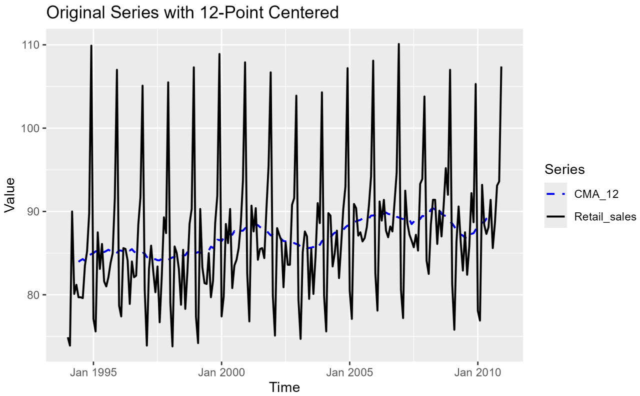


t	1	2	3	4	5	6	7	8	9	10
x_t	3	1	2	6	4	1	1	8	4	0
$l = 3$	-	2.0								
$l = 4$	-	-	3.1							

Empirical example: EUR/USD exchange rate 1999-2010



Empirical example: Retail sales 1994-2010



Simple Exponential smoothing (SES)

- Similar to moving average **except** instead of using **fixed number** of weights, we take a **weighted average** of all past values.
- Often of particular interest: is to give **new** values **more weights**.
- Goal: **Only** for forecasting series that do **not** show apparent **trends or seasonality**.
- Output g of the first N values $x_1, ..., x_N$ is combined with the new observation $x_{N+1} \rightarrow$ **recursive filter**.

SES

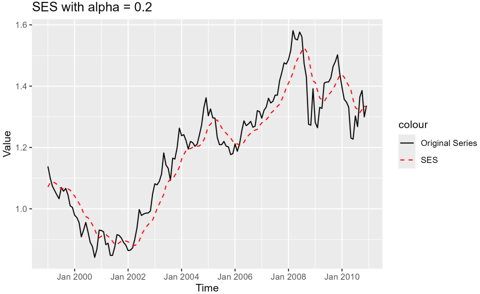
- The Filtering

$$g_{t+1} := \alpha x_t + \alpha(1-\alpha)x_{t-1} + \alpha(1-\alpha)^2 x_{t-2} + ...$$

with $t = 2, 3, ..., N$ is called **simple exponential smoothing**.

- The real number $0 < \alpha < 1$ is called **smoothing parameter**. Its initial value is set to $g_1 := x_1$ as a general rule.
- This filter is **recursive:** the smoothed value is a weighted mean of the smoothed value of the **previous period** and the **current value**.
- Two equations
  - Smoothing/Updating equation: $\ell_t = \alpha x_t + (1-\alpha)\ell_{t-1}$
  - Forecast equation: $F_{t+h} = \ell_t$

Example: EUR/USD 1999-2010



Month (t)	EUR/USD	g_{t+1}
1999-01	1.1384	1.071084
1999-02	1.1018	1.084547
1999-03	1.0742	1.087998
1999-04	1.0597	1.085238
1999-05	1.0456	1.08013
1999-06	1.0328	1.073224
1999-07	1.0694	1.065139
1999-08	1.0573	1.065992
1999-09	1.0665	1.064253
1999-10	1.0453	1.064703
1999-11	1.0097	1.060822
1999-12	1.0046	1.050598

Exponential Smoothing by Holt-Winters

- Simple exponential smoothing is not very suitable if there is a **trend/seasonality** in the original time series.

- For time series with trend: we can use *double exponential smoothing* also called **Holt's linear trend model** (Holt-Winters)
- The Holt's assumes a **local trend** by **updating** parameter b_t.
- Adjustment parameter b_t is calculated **recursive** and changes overtime.

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Exponential Smoothing by Holt-Winters

- The two updating equations equals:

$$\ell_t = \alpha x_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$$
$$b_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta)b_{t-1}$$
$$F_{t+h} = \ell_t + b_t h$$

with $0 < \alpha < 1$ and $0 < \beta < 1$. The initial values for g_t and b_t are usually set to $g_1 = x_1$ and $b_1 = 0$

- Parameter α is responsible for the speed of level adjustment. Large values for β will change the slope parameter b_t slowly. $\beta = 1$: Simple exponential smoothing!
- Choice of parameters α and β depends on the practical application.
- h is the number of periods ahead to be forecast.

SES vs. Holt-Winters