# Assignment 1

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# Question 1

Prove theorem 3 of §1. [Hint: define a quasi-inverse $\mathbf{s} : \mathrm{Sh}(\mathcal{B}) \to \mathrm{Sh}(X)$ for $\mathbf{r}$ as follows. Given a sheaf $F$ on $\mathcal{B}$, and an open set $U \subset X$, consider the cover $\{B_i \mid i \in I\}$ of $U$ by all basic open sets $B_i \in \mathcal{B}$ which are contained in $U$. Define $\mathbf{s}(F)(U)$ by the equalizer $\mathbf{s}(F)(U) \to \Pi_{i\in I}F(B_i) \rightrightarrows \Pi_{i,j}F(B_i \cap B_j)$]

*Proof.* Claim 1: For any $B \in \mathcal{B}$, $\mathbf{s}(F)(B) = F(B)$.
The claim follows from the fact that $F$ is a sheaf on the base, and the fact that equalizer is unique.

Claim 2: For any functor $F : \mathcal{O}(X) \to \mathbf{Sets}$, to check it is a sheaf, for any open set $U$, it suffices to check the sheaf condition for the cover $U = \bigcup_{i\in I} B_i$ where $B_i \in \mathcal{B}$ for all $i \in I$.
Proof: Suppose the sheaf condition holds for the bases, we check the sheaf condition holds for any arbitary cover. Consider an open set $U$ covered by $U = \bigcup_{i\in I}$, we check $F(U) \to \Pi_{i\in I}F(U_i) \rightrightarrows \Pi_{i,j}F(U_i \cap U_j)$ is an equalizer. For each $U_i$, we have $U_i = \bigcup_k B_{ik}$ for $B_{ik} \in \mathcal{B}$. We have a diagram:

$$\begin{tikzcd}
F(U) \arrow[r, "e"] \arrow[dr, "e'"'] & \Pi_i F(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] \arrow[d, "t_1"] & \Pi_{i,j}F(U_i \cap U_j) \arrow[d, "t_2"] \\
& \Pi_{i,k}(B_{ik}) \arrow[r, shift left, "p'"] \arrow[r, shift right, "q'"'] & \Pi_{i,j,k,h}F(B_{ik} \cap B_{jh})
\end{tikzcd}$$

The map $t_1, t_2$ are injective, and the diagram commutes. Take $\alpha \in \Pi_{i\in I}F(U_i)$ such that $p(\alpha) = q(\alpha)$. Then for $t_1(\alpha) \in \Pi_{i,k}(B_{ik})$, we have $p'(t_1(\alpha)) = q'(t_1(\alpha))$. By the sheaf condition on the basis, there exists a unique $\alpha_0 \in F(U)$ such that $e'(\alpha_0) = t_1(\alpha)$. $e'$ is a mono and hence is injective, $e$ must be injective as well, $\alpha_0$ is the unique element which is mapped to $\alpha$.

There are three things to check:
(1) $\mathbf{s}(F)$ is indeed a sheaf:
By claim 2, it suffices to check the sheaf condition on basis, the sheaf condition on basis holds by claim 1 and definition of $\mathbf{s}$.
(2) $\mathbf{r} \circ \mathbf{s} = id$:
We need to check for any $B \in \mathcal{B}$, $\mathbf{r}\,\mathbf{s}(F)(B) = F(B)$. By the claim, as $B \in \mathcal{B}$, $\mathbf{s}(F)(B) = F(B)$, hence $\mathbf{r}\,\mathbf{s}(F)(B) = \mathbf{r}(F)(B) = F(B)$ since $\mathbf{r}$ is merely a restriction.

(3) $\mathbf{s} \circ \mathbf{r} = id$:
We need to check for a sheaf $F_0$ on $X$, $\mathbf{s}\,\mathbf{r}(F_0)(U) = F_0(U)$ for any open set $U \subseteq X$. As $F_0$ is a sheaf, for any cover $\{U_i\}_{i\in I}$ of $U$, $F_0(U)$ is the equalizer:$F_0(U) \to \Pi_{i\in I}F_0(U_i) \rightrightarrows \Pi_{i,j}F_0(U_i \cap U_j)$. In particular, take the cover $U = \bigcup_{i\in I} B_i$ for $B_i \in \mathcal{B}$, then $F_0(U) \to \Pi_{i\in I}F_0(B_i) \rightrightarrows \Pi_{i,j}F_0(B_i \cap B_j)$ is a equalizer. As equalier is unique, it suffices to show that $\mathbf{s}\,\mathbf{r}\,F_0(U) \to \Pi_{i\in I}F_0(B_i) \rightrightarrows \Pi_{i,j}F_0(B_i \cap B_j)$ is a equalizer as well. But note that $F_0(B_i) = \mathbf{r}\,F_0(B_i)$, so it holds by definition of $\mathbf{s}$.

$\square$

# Question 2

Let $f : X \to Y$ be an etale map. Show that $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ has a left adjoint. Give an example of map $f : X \to Y$ such that $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ cannot possibly have a left adjoint.

*Proof.* If $f$ is etale, then $f^*$ has a left adjoint: By the definition of $f^*$ as a pullback via eqivalence of Corollary 6.3, it suffices to show that $f^* : \mathrm{Etale}(Y) \to \mathrm{Etale}(X)$ has a left adjoint. We claim that the map $\sum_f := - \circ f : \mathrm{Etale}(X) \to \mathrm{Etale}(Y)$ defined by sending an etale map $p : F \to X$ to the composition $F \xrightarrow{p} X \xrightarrow{f} Y$ is its adjoint.

$$\textstyle\sum_f : \mathrm{Etale}(X) \rightleftarrows \mathrm{Etale}(Y) : f^*$$

Note that $\sum_f$ is well defined since composition of etale map is etale, so the image of $\sum_f$ indeed lies in $\mathrm{Etale}(Y)$. By definition of adjoint, it amounts to show that there is a natural bijection $\mathrm{Hom}_{\mathrm{Etale}(Y)}(\sum_f(p), q) \cong \mathrm{Hom}_{\mathrm{Etale}(X)}(p, f^*q)$. A map $p \circ f \to q$ in $\mathrm{Etale}(Y)$ is a commutative diagram:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "p"] & E \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

And a map $p \to f^*q$ is a commutative diagram:

$$\begin{tikzcd} F \arrow[rr] \arrow[dr, "p"] & & f^*E \arrow[dl, "f^*q"] \\ & X & \end{tikzcd}$$

It directly follows from definition (universal property) of pullback that the above diagrams are in bijection.

$$\begin{tikzcd} F \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right, "p"'] & & \\ & f^*E \arrow[r] \arrow[d, "f^*q"] & E \arrow[d, "q"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

Give an example of a map $f : X \to Y$ such that $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ cannot possibly have a left adjoint:

Claim: If $f^*$ have a left adjoint, then it must be $\sum_f := - \circ f$.
Proof: Suppose such a left adjoint $f_!$ : $\mathrm{Etale}(X) \to \mathrm{Etale}(Y)$ exists, then for any $p : F \to X$ in $\mathrm{Etale}(X)$ and $q : E \to Y$ in $\mathrm{Etale}(Y)$, $\mathrm{Hom}_{\mathrm{Etale}(Y)}(f_!(p), q) \cong \mathrm{Hom}_{\mathrm{Etale}(X)}(p, f^*q)$. But as shown in the diagram above, a map from $p$ to $f^*q$ is a map $F \to f^*E$ to the pullback, and hence is uniquely determined by a pair of maps $F \to X, F \to E$, which makes the outside of the diagram above commute. But such a pair is precisely the same thing as a map from $\sum_f(p)$ to $q$. As adjoint is unique up to isomorphism, if the left adjoint $f_!$ exists, then it must be $\sum_f(p)$.

By the claim above, to give a map $f$ without such a left adjoint, it suffices to give a map $f$ such that composing with it destroy etaleness. As discussed on SGL page 88, any projection $X \times R \to X$ can be taken as such an example.

□

# Question 3

(a) Prove that a map $p : Y \to X$ of topological spaces is etale iff both $p$ and the diagonal $Y \to Y \times_X Y$ are open maps.
(b) Prove that in the commutative diagram of continous maps

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

where $p$ and $q$ is etale, $f$ must be etale.

*Proof.* For one direction. Suppose $p$ is etale, we prove $p$ is open and the diagonal $\Delta$ is open.

To show $p$ is open: Pick an open set $U \subseteq Y$, we show $p(U)$ is open in $X$. For each $x \in U$, by the fact that $p$ is etale, we have a a neighbourhood $U_x$ that contains $x$ and $U_x$ is homeomorphic to its image $p(U_x)$. By definition of induced topology on $U_x$, $U \cap U_x$ is open in $U_x$ and hence each $p(U \cap U_x)$ is open in $p(U_x)$ (any homeomorphism is open since it has continous inverse) and hence open in $Y$ (open subset of open subset is open). Hence the union $\bigcup_{x \in U} p(U \cap U_x) = p(U)$ is open in $Y$.

To show the diagonal is open, we need to write any set of the form $\{(a, a) \mid a \in U\}$ where $U$ is open in $Y$ as a union of a family of sets $\bigcup_{i \in I}\{(a, b) \mid a \in U_i, b \in V_i\} \cap \{(a, b) \mid p(a) = p(b)\}$. Given such an open set $U \subseteq Y$, for any point $a \in U$, since $p$ is etale, there is a neighbourhood $U_a$ of $a$ which is mapped homeophically to $X$, as $U$ is open in $Y$, each $U \cap U_a$ is open in $Y$. So $\{(a, a) \mid a \in U\} = \bigcup_{a \in U}\{(x, y) \mid x, y \in U \cap U_a\}$. Note that we have the equality since $p$ is homeomorphism on each $U_a$ and hence is bijective. So for $x, y \in U_a$, $p(x) = p(y)$ implies $x = y$.

For the other direction. Suppose both $p$ and the diagonal $y \mapsto (y, y)$ are open maps, we prove $p$ is etale. Note that it suffices to prove that $X$ is covered by open sets such that each of them are mapped homeomorphically to $X$. Let $\Delta$ denote the diagonal map, as $\Delta$ is open, in particular, $\{(y, y) \mid y \in Y\}$ is open in $Y \times_X Y = \{(y_1, y_2) \mid p(y_1) = p(y_2)\}$. According to the defintion of topology of the pullback, this means $Y \times_X Y = \bigcup_{i \in I}\{(y_1, y_2) \mid y_1 \in U_i, y_2 \in V_i, p(y_1) = p(y_2)\}$ for some $U$'s and $V$'s that are all open in $Y$. In particular, it means $\{(y_1, y_2) \mid y_1 \in U_i, y_2 \in V_i, p(y_1) = p(y_2)\} \subseteq \{(y, y) \mid y \in Y\}$ for all such $U_i$ and $V_i$. Hence for $y_1, y_2 \in U_i \cap V_i$, $p(y_1) = p(y_2)$ implies $y_1 = y_2$, in other words, $p|_{U_i \cap V_i}$ is a bijection from $U_i \cap V_i$ to its image. As $p$ is also an open map, its restriction to each such $U_i \cap V_i$ is an open map, by the fact that 'an open map which is a bijection is a homeomorphism', $p$ is a homeomorphism on such $U_i \cap V_i$.

(b) Pick a point $a \in Y$, we find an open neighbourhood of $a$ which is mapped homeomorphically to $Z$. As $p$ is etale, we have an open $Y_a$ such that $a \in Y_a$ and $p(Y_a)$ is homeomorphic to $Y_a$. As $q$ is etale, we have an open $Z_{f(a)}$ such that $f(a) \in Z_{f(a)}$ and $q(Z_{f(a)})$ is homeomorphic to $Z_{f(a)}$. As we have both $p(Y_a) \cap q(Z_{f(a)}) \subseteq p(Y_a)$ and $p(Y_a) \cap q(Z_{f(a)}) \subseteq q(Z_{f(a)})$, we know $p^{-1}(p(Y_a) \cap q(Z_{f(a)}))$ is homeomphic to $q^{-1}(p(Y_a) \cap q(Z_{f(a)}))$ which are both homeomorphic to $p(Y_a) \cap q(Z_{f(a)})$. By commutativity of the diagram, $f(p^{-1}(p(Y_a) \cap q(Z_{f(a)}))) = q^{-1}(p(Y_a) \cap q(Z_{f(a)}))$, which are homeomorphic to $p^{-1}(p(Y_a) \cap q(Z_{f(a)}))$. □