Assignment 2

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Question 1

Let X be a topological space. For a sieve S on an open subset U of X define S covers U iff U is the union of the sets in S. Prove that this defines a Grothendieck topology on the partially ordered set $\mathcal{O}(X)$ of all open subsets of X.

Proof. We check the axioms in Definition 1 on page 110.

Observe that if $U \in \mathcal{O}(X)$, $h : V \subseteq U$ and S is a sieve on U, then $h^*(S) = \{W \mid W \subseteq V \subseteq U, W \in S\}$.

(i) The maximal sieve $t_C = \{f \mid cod(f) = C\}$ is in $J(C)$.
This is for all $U \in \mathcal{O}(X)$, $\bigcup\{V \mid V \subseteq U\} = U$. This is true because $U \subseteq U$.

(ii) (stability axiom) if $S \in J(C)$, then $h^*(S) \in J(D)$ for any arrow $h : D \to C$.
This is for all $U \in \mathcal{O}(X)$ and sieve S on U, if $\bigcup S = U$, then for any $V \subseteq U$, we need to show $\bigcup\{W \mid W \subseteq V \subseteq U, W \in S\} = V$.

We have $V = U \cap V$

$= (\bigcup S) \cap V$

$= \bigcup\{W \cap V \mid W \in S\}$

The last set is a subset of $\bigcup\{W \mid W \subseteq V \subseteq U, W \in S\}$, since S is a sieve and hence $W \in S$ implies $W \cap V \in S$. So $V \subseteq \bigcup\{W \mid W \subseteq V \subseteq U, W \in S\}$, clearly the inclusion for the other direction holds.

(iii) (transitivity axiom) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$, then $R \in J(C)$.

This is for all $U \in \mathcal{O}(X)$ and sieve S on U such that $\bigcup S = U$, and R is any sieve on U such that for any $V \in S$, $\bigcup\{W \mid W \subseteq V \subseteq U, W \in R\} = V$, then $\bigcup R = U$.

Obviously $\bigcup R \subseteq U$, we show $U \subseteq \bigcup R$.

$U = \bigcup S$

$= \bigcup\{V \mid V \in S\}$

$= \bigcup\{\bigcup\{W \mid W \subseteq V \subseteq U, W \in R\} \mid V \in S\}$

$\subseteq \bigcup R$

as desired. □

Question 2

Let $\mathbf{T}$ be in §2, Example (b), with the open cover topology given by the basis K as defined there. Define K' by $\{f_i : Y_i \to X \mid i \in I\} \in K'(X)$ iff each f_i is etale, and moreover $X = \bigcup_i f_i(Y_i)$. Show that K and K' generates the same topology J on $\mathbf{T}$.

Proof. By definition on page 112, if K is a basis on $\mathbf{T}$, then K generated a topology J by $S \in J(C) \Leftrightarrow \exists R \in K(C), R \subseteq S$. Then our task is to show that for a space $X \in \mathbf{T}$ and a sieve S on X, then S contains a set $\{f_i : Y_i \to X \mid i \in I\}$ where each f_i is etale, and moreover $X = \bigcup_i f_i(Y_i)$ iff S contains a set $\{g_m : Y_m \to X \mid m \in M\}$ where $\{Y_m\}$ is an open cover of X and the $\{g_m\}_{m \in M}$ is the corresponding embedding.

If S contains a set $\{g_m : Y_m \to X \mid m \in M\}$ where $\{Y_m\}$ is an open cover of X and the $\{g_m\}_{m \in M}$ is the corresponding embedding, then as an inclusion of open set is an etale map, we also have $\{g_m : Y_m \to X \mid m \in M\} \in K'(X)$.

Conversely, if S contains a set $\{f_i : Y_i \to X \mid i \in I\}$ where each f_i is etale, and moreover $X = \bigcup_i f_i(Y_i)$, then for each Y_i, it is covered by open subsets $\{U_{i_m}\}$, each mapped homeomorphically to X, with its image denoted as $U_{i_m} \cong V_{i_m} \subseteq X$. As S is a sieve, all the maps $V_{i_m} \to U_{i_m} \hookrightarrow Y_i \to X$ are in S, and as the image of $\{Y_i\}_{i \in I}$ covers X, the open sets V_{i_m} indexed over i and m covers X as well. Hence S contains $\{V_{i_m} \to X\}_{i,m}$, which is a family of open sets that covers X.

□