s(F) is a sheaf

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$$
\begin{tikzcd}
\mathbf{s}(F)(U) \arrow[r, "e"] \arrow[d, tail] & \Pi_{i\in I}\,\mathbf{s}(F)(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] \arrow[d, tail] & \Pi_{i,j}\,\mathbf{s}(F)(U_i\cap U_j) \arrow[d, tail] \\
\Pi_{a\in A}F(B_a) \arrow[r, dashed, "e"] \arrow[d, shift left] \arrow[d, shift right] & \Pi_{i\in I}\Pi_{m\in M_i}F(B_m) \arrow[r, dashed, shift left, "p"] \arrow[r, dashed, shift right, "q"'] \arrow[d, shift left] \arrow[d, shift right] & \Pi_{i,j}\Pi_{k\in K_{ij}}F(B_k) \arrow[d, shift left] \arrow[d, shift right] \\
\Pi_{a,b}F(B_a\cap B_b) & \Pi_{i\in I}\Pi_{m,n\in M_i}F(B_m\cap B_n) & \Pi_{i,j}\Pi_{k,g\in K_{ij}}F(B_k\cap B_g)
\end{tikzcd}
$$

Note:

- $\{U_i\}_{i\in I}$ is a covering family of U, indexed by i.
- A is the set indexing all the basic open sets contained in U.
- M is a function $M : I \to \mathcal{P}(A)$. M is defined by $M\ i := \{a \in A \mid B_a \subseteq U_i\}$.
- K is a function $K : I \times I \to \mathcal{P}(A)$. K is defined by $K\ i\ j := \{a \in A \mid B_a \subseteq U_i \cap U_j\}$.

Goal : Define a bijection between $\mathbf{s}(F)(U)$ and the equalizer of p and q.

Subgoal 1: Define a function $\phi : \mathbf{s}(F)(U) \to eq[\Pi_{i\in I}\,\mathbf{s}(F)(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \Pi_{i,j\in I}\,\mathbf{s}(F)(U_i\cap U_j)]$.
Definition of the function:
Given $\alpha \in \mathbf{s}(F)(U) \subseteq \Pi_{a\in A}F(B_a)$, define $\beta \in \Pi_{i\in I}\Pi_{m\in M_i}F(B_m)$ to be $\beta\ i\ m := \alpha\ m$.

It makes sense, since $m \in (M\ i) \subseteq A$ for all $i \in I$.

The function above is well defined:

Subgoal 1: $\beta \in \Pi_{i\in I}\,\mathbf{s}(F)(U_i)$

We want to show that $\forall(i \in I), \beta\ i \in \mathbf{s}(F)(U_i)$. Now fix any $i \in I$, we need to prove that $\beta\ i$ is in the equalizer of p_i and q_i in the following diagram (call it $(*_1)$):

$$\Pi_{m\in M_i}F(B_m) \underset{q_i}{\overset{p_i}{\rightrightarrows}} \Pi_{m,n\in M_i}F(B_m\cap B_n)$$

Note that the definitions of p_i and q_i are given by:
For any $b \in \Pi_{m\in M_i}F(B_m)$, denote $p_i(b)$ by $\gamma \in \Pi_{m,n\in M_i}F(B_m\cap B_n)$. Then γ is defined by for any $m, n \in M_i$, $\gamma\ m\ n := (b\ m)|_{B_m\cap B_n}$.

Similarly, denote $q_i(b)$ by $\gamma \in \Pi_{m,n\in M_i}F(B_m\cap B_n)$. Then γ is defined by for any $m, n \in M_i$, $\gamma\ m\ n := (b\ n)|_{B_m\cap B_n}$.

From the above definitions of p_i and q_i, for any $i \in I$, $b \in \Pi_{m\in M_i}F(B_m)$, we have $b \in \mathbf{s}(F)(U_i)$ iff for all $m, n \in M_i$, $(b\ m)|_{B_m\cap B_n} = (b\ n)|_{B_m\cap B_n}$.

Hence what we want to show is that for all $i \in I$ and $m, n \in M_i$, we have $(\beta\ i\ m)|_{B_m\cap B_n} = (\beta\ i\ n)|_{B_m\cap B_n}$. (a)

To prove this, we investigate the condition that $\alpha \in \mathbf{s}(F)(U)$, that is, α is in the equalizer of the diagram:

$$\Pi_{a\in A}F(B_a) \underset{q_0}{\overset{p_0}{\rightrightarrows}} \Pi_{a,b\in A}F(B_a \cap B_b)$$

Where the definitions of p_0 and q_0 are given by:
Denote $p_0(\alpha)$ as $\alpha' \in \Pi_{a,b\in A}F(B_a \cap B_b)$. Then for any $a, b \in A$, $\alpha'\ a\ b := (\alpha\ a)|_{B_a\cap B_b}$.

Denote $q_0(\alpha)$ as $\alpha' \in \Pi_{a,b\in A}F(B_a \cap B_b)$. Then for any $a, b \in A$, $\alpha'\ a\ b := (\alpha\ b)|_{B_a\cap B_b}$.

Hence $\alpha \in \mathbf{s}(F)(U)$ means for all $a, b \in A$, $(\alpha\ a)|_{B_a\cap B_b} = (\alpha\ b)|_{B_a\cap B_b}$. $(*_2)$

Return to the discussion of proving $\beta\ i$ is in the equalizer of $(*_1)$ for each $i \in I$, our goal is (a). By definition of β, for any $i \in I, m, n \in M_i$, we have $\beta\ i\ m := \alpha\ m$ and $\beta\ i\ n := \alpha\ n$. As $m, n \in M_i \subseteq A$, plug in a, b to be m, n in the sentence above gives:

$(\beta\ i\ m)|_{B_m\cap B_n} = (\alpha\ m)|_{B_m\cap B_n} = (\alpha\ n)|_{B_m\cap B_n} = (\beta\ i\ n)|_{B_m\cap B_n}$

as desired.

This completes Subgoal 1 for well-definedness.

Subgoal 2: $p(\beta) = q(\beta)$.

We aim to show that β as defined as before is in the equalizer of the maps p and q in the following diagram:

$$\Pi_{i\in I}\, \mathbf{s}(F)(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \Pi_{i,j\in I}\, \mathbf{s}(F)(U_i \cap U_j)$$

Note that the maps p and q are defined by:
For p:

For $\beta_0 \in \Pi_{i\in I}\, \mathbf{s}(F)(U_i) \subseteq \Pi_{i\in I}\Pi_{m\in M_i}F(B_m)$, denote $p(\beta_0)$ as γ, then $\gamma \in \Pi_{i,j\in I}\Pi_{k\in K_{ij}}F(B_k)$, and γ is defined as $\gamma\ i\ j\ k := \beta_0\ i\ k$.

Now we show that p is well-defined, that is, such a γ we defined above is in $\Pi_{i,j\in I}\, \mathbf{s}(F)(U_i \cap U_j)$. The aim is to show that for any $i, j \in I$, $\gamma\ i\ j \in \mathbf{s}(F)(U_i \cap U_j)$.

For any fixed pair of $i, j \in I$, consider the diagram:

$$\Pi_{k\in K_{ij}}F(B_k) \underset{q_{ij}}{\overset{p_{ij}}{\rightrightarrows}} \Pi_{k,g\in K_{ij}}F(B_k \cap B_g)$$

For any tuple $b_0 \in \Pi_{k\in K_{ij}}F(B_k)$, denote $p_{ij}(b_0)$ as c_0. Then $c_0 \in \Pi_{k,g\in K_{ij}}F(B_k \cap B_g)$ is defined by for any $k, g \in K_{ij}$, $c_0\ k\ g := (b_0\ k)|_{B_k\cap B_g}$.

For any tuple $b_0 \in \Pi_{k\in K_{ij}}F(B_k)$, denote $q_{ij}(b_0)$ as c_0. Then $c_0 \in \Pi_{k,g\in K_{ij}}F(B_k \cap B_g)$ is defined by for any $k, g \in K_{ij}$, $c_0\ k\ g := (b_0\ g)|_{B_k\cap B_g}$.

Hence for any $i, j \in I$, $b_0 \in \Pi_{k\in K_{ij}}F(B_k)$, b_0 is in the equalizer of p_{ij} and q_{ij} iff for any $k, g \in K_{ij}$, we have $(b_0\ k)|_{B_k\cap B_g} = (b_0\ g)|_{B_k\cap B_g}$.

Hence we want to show that for any $i, j \in I$, $k, g \in K_{ij}$, we have $(\gamma\ i\ j\ k)|_{B_k \cap B_g} = (\gamma\ i\ j\ g)|_{B_k \cap B_g}$. By definition of γ, this is to show $(\beta_0\ i\ k)|_{B_k \cap B_g} = (\beta_0\ i\ g)|_{B_k \cap B_g}$. Recall $\beta_0 \in \Pi_{i \in I}\, \mathbf{s}(F)(U_i)$, that means the condition (a) holds for β_0, namely 'for all $i \in I$, $m, n \in M_i$, $(\beta_0\ i\ m)|_{B_m \cap B_n} = (\beta_0\ i\ n)|_{B_m \cap B_n}$'. As $k, g \in K_{ij} \subseteq M_i$, pluggin in k, g to be m, n gives us the result.

For q: For $\beta_0 \in \Pi_{i \in I}\, \mathbf{s}(F)(U_i) \subseteq \Pi_{i \in I} \Pi_{m \in M_i} F(B_m)$, denote $q(\beta_0)$ as γ, then $\gamma \in \Pi_{i,j \in I} \Pi_{k \in K_{ij}} F(B_k)$, and γ is defined as $\gamma\ i\ j\ k := \beta_0\ j\ k$. Similarly we can show q is well-defined.

Now we start proving that the β we defined in the begining of this direction satisfies $p(\beta) = q(\beta)$. By definition of p and q as above and function extensionality, we need to show for all $i, j \in I, k \in K_{ij}$, we have $\beta\ i\ k = \beta\ j\ k$. But by definition of β, we have $\beta\ i\ k = \alpha\ k$ and $\beta\ j\ k = \alpha\ k$, as desired.

Subgoal 2: Define a function $\psi : eq[\Pi_{i \in I}\, \mathbf{s}(F)(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \Pi_{i,j \in I}\, \mathbf{s}(F)(U_i \cap U_j)] \to \mathbf{s}(F)(U)$.

Given $\beta \in \Pi_{i \in I}\, \mathbf{s}(F)(U_i) \subseteq \Pi_{i \in I} \Pi_{m \in M_i} F(B_m)$ such that $p(\beta) = q(\beta)$, we construct an element $\alpha \in \mathbf{s}(F)(U)$, that is, an element in $\Pi_{a \in A} F(B_a)$ which satisfies the condition as in $(*_2)$.

Define S to be a function $A \to \mathcal{P}(A)$. For any $a \in A$, $S\ a := \{t \in A \mid \exists i.(i \in I \wedge t \in (M\ i) \wedge B_t \subseteq B_a)\}$. In words, $S\ a$ is the set indexing any basic open set which is contained in some U_i and is also contained in B_a.

Claim: For all $a \in A$, we have $\bigcup\{B_t \mid t \in S_a\} = B_a$.

Obviously $B_a \supseteq \bigcup\{B_t \mid t \in S_a\}$, it left to show that $B_a \subseteq \{B_t \mid t \in S_a\}$.

$B_a = U \cap B_a$

$= (\bigcup\{U_i \mid i \in I\}) \cap B_a$

$= (\bigcup\{\bigcup\{B_t \mid (M\ i)\} \mid i \in I\}) \cap B_a$.

$= \bigcup\{\bigcup\{B_t \cap B_a \mid t \in (M\ i)\} \mid i \in I\}$

$= \bigcup_{t \in \bigcup\{M\ i \mid i \in I\}} (B_t \cap B_a)$

As B is closed under intersection, for any $i \in I, t \in M_i$, there exists an $s \in S_a$ such that $B_t \cap B_a = B_s$. Hence the the set above is a subset of $\bigcup_{t \in S_a} B_t$. This completes the proof of the claim.

For any $a \in A$, we can define a function $f_a : eq[\Pi_{i \in I}\, \mathbf{s}(F)(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \Pi_{i,j \in I}\, \mathbf{s}(F)(U_i \cap U_j)] \to \Pi_{t \in S_a} B_t$, as follows: (Implicitly, f is a function, takes an element $a \in A$ and give the function f_a.)

For any $\beta \in eq[\Pi_{i \in I}\, \mathbf{s}(F)(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \Pi_{i,j \in I}\, \mathbf{s}(F)(U_i \cap U_j)] \subseteq \Pi_{i \in I} \Pi_{m \in M_i}$, denote $f_a(\beta) \in \Pi_{t \in S_a} F(B_t)$ as β^{0a}, then for any $t \in S_a$, define $\beta^{0a}\ t := \beta\ (\mathsf{CHOICE}\ \{i \in I \mid t \in M_i\})\ t$. Here the application of choice function makes sense, since by definition of S_a, $t \in S_a$ implies $\{i \in I \mid t \in M_i\} \neq \emptyset$.

Claim: For any $a \in A$, the definition of f_a is independent of choice.

This is, for all $a \in A, t \in S_a$, if $t \in M_i$ and $t \in M_j$, then $\beta\ i\ t = \beta\ j\ t$. Recall $\beta \in eq[\Pi_{i \in I}\, \mathbf{s}(F)(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \Pi_{i,j \in I}\, \mathbf{s}(F)(U_i \cap U_j)]$. By definition of p and q as in Direction 1, for any $i, j \in I$ and $k \in K_{ij}$, we have $\beta\ i\ k = \beta\ j\ k$. As $t \in M_i$ and $t \in M_j$, by definition of M and K, we have $t \in K_{ij}$. Hence we have $\beta\ i\ t = \beta\ j\ t$, as desire.

Consider the diagram:

$$F(B_a) \overset{e_a}{\rightarrowtail} \Pi_{t \in S_a} F(B_t) \underset{q_a}{\overset{p_a}{\rightrightarrows}} \Pi_{t_1, t_2 \in S_a} F(B_{t_1} \cap B_{t_2})$$

As proved before, the family of basic open sets $\{B_t \mid t \in S_a\}$ covers B_a. Here p_a and q_a are defined by:

For $\delta \in \Pi_{t \in S_a} F(B_a)$, denote $p_a(\delta)$ as $\delta' \in \Pi_{t_1,t_2 \in S_a} F(B_{t_1} \cap B_{t_2})$. Then δ' is given by $\delta'\ t_1\ t_2 := (\delta\ t_1)|_{B_{t_1} \cap B_{t_2}}$.

For $\delta \in \Pi_{t \in S_a} F(B_a)$, denote $q_a(\delta)$ as $\delta' \in \Pi_{t_1,t_2 \in S_a} F(B_{t_1} \cap B_{t_2})$. Then δ' is given by $\delta'\ t_1\ t_2 := (\delta\ t_2)|_{B_{t_1} \cap B_{t_2}}$.

And the map e_a is defined by for $\delta_0 \in F(B_a)$, denote $e_a(\delta)$ by δ_0', then define $\delta_0'\ t := \delta_0|_{B_t}$.

Claim: For any $\beta \in eq[\Pi_{i \in I}\, \mathbf{s}(F)(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \Pi_{i,j \in I}\, \mathbf{s}(F)(U_i \cap U_j)] \subseteq \Pi_{i \in I} \Pi_{m \in M_i}$, and any $a \in A$, the image $\beta^{0a} := f_a(\beta) \in \Pi_{t \in S_a} F(B_t)$ under f_a is in the equalizer of p_a and q_a.

Under the above conditions, we need to prove that for all $t_1, t_2 \in S_a$, $(\beta^{0a}\ t_1)|_{B_{t_1} \cap B_{t_2}} = (\beta^{0a}\ t_2)|_{B_{t_1} \cap B_{t_2}}$. By definition of β^{0a}, it amounts to show $(\beta\ (\mathsf{CHOICE}\ \{i \in I \mid t_1 \in M_i\})\ t_1)|_{B_{t_1} \cap B_{t_2}} = (\beta\ (\mathsf{CHOICE}\ \{i \in I \mid t_2 \in M_i\})\ t_2)|_{B_{t_1} \cap B_{t_2}}$.

By definition of S_a, there exists $i_1, i_2 \in I$, such that $t_1 \in M_{i_1}, t_2 \in M_{i_2}$, $B_{t_1} \subseteq B_a$ and $B_{t_2} \subseteq B_a$. As we have assumed the base is closed under intersection, there exists $c \in A$ such that $B_{t_1} \cap B_{t_2} = B_c$. As $B_c \subseteq B_{t_1}$ and $t_1 \in M_{i_1}$, by definition of M, we have $B_{t_1} \subseteq U_{i_1}$, and hence $B_c \subseteq U_{i_1}$ as well. Again by definition of M, it follows that $c \in M_{t_1}$ as well.

We have $(\beta\ (\mathsf{CHOICE}\ \{i \in I \mid t_1 \in M_i\})\ t_1)|_{B_{t_1} \cap B_{t_2}} = (\beta\ i_1\ t_1)|_{B_{t_1} \cap B_{t_2}}$ by independence of choice, as proved earlier. By definition of B_c, we have $B_{t_1} \cap B_{t_2} = B_{t_1} \cap B_c$, so $(\beta\ i_1\ t_1)|_{B_{t_1} \cap B_{t_2}} = (\beta\ i_1\ t_1)|_{B_{t_1} \cap B_c}$. Recall $\beta \in \Pi_{i \in I}\, \mathbf{s}(F)(U_i)$, as discussed in last direction (labeled condition (a)), it means for all $i \in I$, $m, n \in M_i$, $(\beta\ i\ m)|_{B_m \cap B_n} = (\beta\ i\ n)|_{B_m \cap B_n}$. In particular, we can plug in i_1 to be the i, t_1 to be m and c to be n, and hence conclude $(\beta\ i_1\ t_1)|_{B_{t_1} \cap B_c} = (\beta\ i_1\ c)|_{B_{t_1} \cap B_c}$.

Similarly $(\beta\ (\mathsf{CHOICE}\ \{i \in I \mid t_2 \in M_i\})\ t_2)|_{B_{t_1} \cap B_{t_2}} = (\beta\ i_2\ c)|_{B_{t_2} \cap B_c}$.

By definition of B_c, we have $B_{t_1} \cap B_c = B_{t_2} \cap B_c = B_c$. So the task reduces to show $(\beta\ i_1\ c)|_{B_c} = (\beta\ i_2\ c)|_{B_c}$. Note that for all $i \in I, m \in M_i$, we have $\beta\ i\ m \in F(B_m)$, hence the restrictions are both identities. It remains to show $\beta\ i_1\ c = \beta\ i_2\ c$. But recall β is in the equalizer of p and q, hence for any $i, j \in I, c \in K_{ij}$, we have $\beta\ i\ k = \beta\ j\ k$. We do have $c \in K_{ij}$ by definition of K. Hence $\beta\ i_1\ c = \beta\ i_2\ c$, as desired.

Hence β^{0a} is in the equalizer of p_a and q_a. As F is a sheaf on the base, there exists a unique element $\beta_a^0 \in F(B_a)$ such that $e_a(\beta_a^0) = \beta^{0a}$.

Start with the β at the start of this direction, denote the element we get from β as $\alpha \in \Pi_{a \in A} F(B_a)$. Then α is defined by $\alpha\ a := \beta_a^0$ as constructed above.

Now we check $\alpha \in \mathbf{s}(F)(U)$. This is, for any $a, b \in A$, we need to show $(\alpha\ a)|_{B_a \cap B_b} = (\alpha\ b)|_{B_a \cap B_b}$. As we have assumed that the base is closed under intersection, there exists $l \in A$ such that $B_a \cap B_b = B_l$. It suffices to prove that $(\alpha\ a)|_{B_a \cap B_b} = \alpha\ l$ and $(\alpha\ b)|_{B_a \cap B_b} = \alpha\ l$.

We prove $(\alpha\ a)|_{B_a \cap B_b} = \alpha\ l$, then the other equation will hold by a symmetric argument.

Consider the diagram:

$$F(B_l) \overset{e_l}{\rightarrowtail} \Pi_{t\in S_l} F(B_t) \underset{q_l}{\overset{p_l}{\rightrightarrows}} \Pi_{t_1,t_2\in S_l} F(B_{t_1} \cap B_{t_2})$$

By definition of α, $\alpha\ l$ is the unique element in $F(B_l)$ which is mapped to the element $\beta_l \in \Pi_{t\in S_l} F(B_t)$ defined by for all $t \in S_l$, $\beta_l\ t = (\alpha\ l)|_{B_t}$. Hence to show $(\alpha\ a)|_{B_a\cap B_b} = \alpha\ l$, it suffices to show that for all $t \in S_l$, $((\alpha\ a)|_{B_a\cap B_b})|_{B_t} = (\alpha\ l)|_{B_t}$. By definition of S, we have for any $t \in S_l$, $B_t \subseteq B_l$. As F is a functor, $((\alpha\ a)|_{B_a\cap B_b})|_{B_t} = (\alpha\ a)|_{B_t}$. Therefore, it amounts to show that for all $t \in S_l$, $(\alpha\ a)|_{B_t} = (\alpha\ l)|_{B_t}$.

By definition of α, the above amounts to show $\beta^0_a|_{B_t} = \beta^0_l|_{B_t}$. Recall how we picked β^0_a, it is the unique element in $F(B_a)$ such that $e_a(\beta^0_a) = \beta^{0a}$. By definition of e_a, as spelled out before, it means for all $t \in S_a$, $\beta^0_a|_{B_t} = \beta^{0a}\ t$. But we know that $\beta^{0a}\ t = \beta\ (\mathsf{CHOICE}\ \{i \in I \mid t \in M_i\})\ t$ by definition of β^{0a}. As $S_l \subseteq S_a$, by conclusion, for all $t \in S_l$, $\beta^0_a|_{B_t} = \beta\ (\mathsf{CHOICE}\ \{i \in I \mid t \in M_i\})\ t$.

Also consider $\beta^0_l|_{B_t}$, by the construction of β^0_l, it is the unique element in $F(B_l)$ such that for all $t \in S_l$, $\beta^0_l|_{B_t} = \beta\ (\mathsf{CHOICE}\ \{i \in I \mid t \in M_i\})\ t$.

Thus $\alpha\ a = \alpha\ l$.

Thus we have the two maps, it lefts to show that these two maps $\phi : \mathbf{s}(F)(U) \rightleftarrows eq[\Pi_{i\in I}\, \mathbf{s}(F)(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \Pi_{i,j\in I}\, \mathbf{s}(F)(U_i \cap U_j)] : \psi$ are indeed inverses. Recall the definitions of these maps:

ϕ is defined by: Given $\alpha \in \mathbf{s}(F)(U) \subseteq \Pi_{a\in A} F(B_a)$, define $\beta \in \Pi_{i\in I}\Pi_{m\in M_i} F(B_m)$ to be $\beta\ i\ m := \alpha\ m$.

ψ is defined by: Given $\beta \in eq[\Pi_{i\in I}\, \mathbf{s}(F)(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \Pi_{i,j\in I}\, \mathbf{s}(F)(U_i \cap U_j)] \subseteq \Pi_{i\in I}\Pi_{m\in M_i} F(B_m)$, define $\psi(\beta)$ to be $\alpha \in \Pi_{a\in A} F(B_a)$ such that for all $a \in A$, $\alpha\ a := \beta^0_a$.

$\phi \circ \psi = id$:

We want for all $\beta \in eq[\Pi_{i\in I}\, \mathbf{s}(F)(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \Pi_{i,j\in I}\, \mathbf{s}(F)(U_i \cap U_j)]$, Let α denote $\psi(\beta)$, then $\phi(\alpha) = \beta$. It amounts to show that for any $i \in I, m \in M_i$, we have $\phi(\alpha)\ i\ m = \beta\ i\ m$. By definition of ϕ, $\phi(\alpha)\ i\ m = \alpha\ m$. it remains to show $\alpha\ m := \psi(\beta)\ m = \beta\ i\ m$. By definition of ψ, it is to show $\beta^0_m = \beta\ i\ m$. By construction of β^0_m, for all $x \in S_m$, $\beta^0_m|_{B_x} = \beta^{0m}\ x$. In particular, by definition of S, we have $m \in S_m$, so $\beta^0_m|_{B_m} = \beta^{0m}\ m$. But we already have $\beta^0_m \in F(B_m)$, as F is a functor, the restriction is the identity map, hence $\beta^0_m = \beta^{0m}\ m$. By construction of β^{0m}, we have $\beta^{0m}\ m = \beta\ (\mathsf{CHOICE}\ \{i_0 \in I \mid m \in M_{i_0}\})\ m$. As we have already know $i \in I$ and $m \in M_i$, by independence of choice as proved before, $\beta\ (\mathsf{CHOICE}\ \{i_0 \in I \mid m \in M_{i_0}\})\ m = \beta\ i\ m$. This completes the proof.

$\psi \circ \phi = id$:

We want for all $\alpha \in \mathbf{s}(F)(U) \subseteq \Pi_{a\in A} F(B_a)$, $\psi(\phi(\alpha)) = \alpha$. Let β denote $\phi(\alpha)$, we prove for all $a \in A$, $\psi(\beta)\ a = \alpha\ a$. By definition of ψ, this is to prove $\beta^0_a = \alpha\ a$. By definition of β^0_a, it is the unique element in $F(B_a)$ such that for all $t \in S_a$, its restriction to B_t is $\beta^0_a|_{B_t}$. Hence it suffices to prove that for all $t \in S_a$, $\beta^0_a|_{B_t} = (\alpha\ a)|_{B_t}$. By construction of β^0_a, $\beta^0_a|_{B_t} = \beta^{0a}\ t$. And $\beta^{0a}\ t = \beta\ (\mathsf{CHOICE}\ \{i \in I \mid t \in M_i\})\ t$. By definition of ϕ, $\beta\ (\mathsf{CHOICE}\ \{i \in I \mid t \in M_i\})\ t = \alpha\ t$. Then the task reduced to showing $\alpha\ t = (\alpha\ a)|_{B_t}$. As $\alpha \in \mathbf{s}(F)(U)$, for any $a_1, a_2 \in A$, $(\alpha\ a_1)|_{B_{a_1}\cap B_{a_2}} = (\alpha\ a_2)|_{B_{a_1}\cap B_{a_2}}$. In particular, plug in t and a as a_1 and t_2. Note that as $t \in S_a$, $B_t \subseteq B_a$ and hence $B_t \cap B_a = B_t$. Hence $(\alpha\ t)|_{B_t} = (\alpha\ a)|_{B_t}$. As the first restriction map is identity, the result follows.