Assignment 5

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Question 1

Observe that a topos $\mathcal{E}$ satisfies the (internal) axiom of choice iff every object of $\mathcal{E}$ is internally projective, see Exercise 15 and 16 of Chapter IV. Rephrase some of the statements proved there, in terms of (internal) axiom of choice. Notice that if $\mathcal{E}$ is well-pointed then 1 is projective in $\mathcal{E}$, Conclude from Exercise IV.16(c) that, for well-pointed topoi, IAC and AC are equivalent. Prove that if $\mathcal{E}$ satisfied IAC, then so does $\mathcal{E}\,/E$ for any object E of $\mathcal{E}$. Is the same true for AC?

Proof. Conclude from Exercise IV.16(c) that, for well-pointed topoi, IAC and AC are equivalent:

If $\mathcal{E}$ is well-pointed then 1 is projective in $\mathcal{E}$. (SGL page 333 (iii)).

AC: Every object in $\mathcal{E}$ is projective.

IAC: Every object in $\mathcal{E}$ is internally projective.

Hence directly by IV.16(c), the result follows.

Prove that if $\mathcal{E}$ satisfied IAC, then so does $\mathcal{E}$ for any object E of $\mathcal{E}$:

???

Is the same true for AC?

Yes, as a corollary of 'IAC is preserved by slicing'.

Proof: AC = IAC + SS, as in: https://www.andrew.cmu.edu/user/jonasf/80-514-814/clive/more-topos-props.pdf

Assume $\mathcal{E}$ has AC, we need to prove that $\mathcal{E}\,/E$ has AC. As $\mathcal{E}$ has IAC and IAC is preserved by slicing, we conclude $\mathcal{E}\,/E$ has IAC. It suffices to prove that $\mathcal{E}\,/E$ has SS.

The terminal object in slice category is the identity $E \to E$. Let $f : A \to E$ be an object, then f has a mono-epi factorization as $A \overset{e}{\twoheadrightarrow} f(A) \overset{m}{\rightarrowtail} E$ in $\mathcal{E}$. The epi e has a section by AC in $\mathcal{E}$. Also, its section is indeed a map in slice category, as we can check. So AC is preserved by slicing.

□

Question 2

Prove that an arow $f : X \to Y$ in a topos $\mathcal{E}$ is monomorphism iff the sentence $\forall x \in X \forall x' \in X(fx = fx' \Rightarrow x = x')$ of the Mitchell-Benabou language holds in $\mathcal{E}$.

Proof. Direction 1: Suppose the composition:

$$1 \xrightarrow{\hat{f}} Y^X \xrightarrow{\forall x \forall x'(\hat{f}(x)=\hat{f}(x') \Rightarrow x=x')} \Omega$$

is true : $1 \to \Omega$, we want to prove if we have the copositions $ft_1 = f_t 2$:

$$T \overset{t_1}{\underset{t_2}{\rightrightarrows}} X \xrightarrow{f} Y$$

then we have $t_1 = t_2$.

Specializing Theorem 1 (vi), for the arrow $T \xrightarrow{!} 1, t_1 : T \to X$, we have the arrow $T \xrightarrow{!} 1 \xrightarrow{\hat{f}} Y^X$ such that the composition:

$$T \xrightarrow{\langle \hat{f}!, t_1 \rangle} Y^X \times X \xrightarrow{\forall x' ((\hat{f}!)(t_1) = (\hat{f}!)(x') \Rightarrow t_1 = x')} \Omega$$

is $\mathrm{true}_T : T \to \Omega$.

Specializing Theorem 1 (vi) again, for the arrow $1_T : T \to T, t_2 : T \to X$, we have the composition:

$$T \xrightarrow{\langle \langle \hat{f}!, t_1 \rangle, t_2 \rangle = \langle \langle \hat{f}!, t_1 \rangle \circ 1_T, t_2 \rangle} Y^X \times X \times X \xrightarrow{(\hat{f}!)(t_1) = (\hat{f}!)(t_2) \Rightarrow t_1 = t_2} \Omega$$

is $\mathrm{true}_T : T \to \Omega$.

By Theorem 1, it means that if $T \Vdash (\hat{f}!)(t_1) = (\hat{f}!)(t_2)$, then $T \Vdash t_1 = t_2$.

To prove our aim that $t_1 = t_2$, it suffices to have $T \Vdash t_1 = t_2$, since by page 298, the definition of equality means:

$$T \times T \xrightarrow{\langle t_1 \pi_1, t_2 \pi_2 \rangle = t_1 \times t_2} X \times X \xrightarrow{\delta_X} \Omega$$

is $\mathrm{true}_{T \times T} : T \times T \to \Omega$. By definition of δ_X, it means that $t_1 \times t_2$ factors throught the diagonal, as in the pullback diagram:

$$\begin{tikzcd}
T \times T \arrow[dr, "t"] \arrow[drr, "!"] \arrow[ddr, "t_1 \times t_2"'] & & \\
& X \arrow[r, "!"] \arrow[d, "\Delta_X = \langle 1_X, 1_X \rangle"] & 1 \arrow[d, "\mathrm{true}"] \\
& X \times X \arrow[r, "\delta_X"] & \Omega
\end{tikzcd}$$

So $t_1 = 1_X \circ t = t_2$.

Now it amounts to proving $T \Vdash (\hat{f}!)(t_1) = (\hat{f}!)(t_2)$. By the interpretation of terms, we need that the maps:

$$T \xrightarrow{\langle \hat{f}!, t_1 \rangle} Y^X \times X \xrightarrow{\mathrm{ev}} Y$$

and

$$T \xrightarrow{\langle \hat{f}!, t_2 \rangle} Y^X \times X \xrightarrow{\mathrm{ev}} Y$$

are equal.

By assumption, we have $ft_1 = ft_2$, so $f\langle !, t_1 \rangle = f\langle !, t_2 \rangle$. Hence by the diagram:

$$\begin{tikzcd} T \arrow[d, "\langle !, t_{1,2} \rangle"] & \\ 1 \times X \arrow[d, "\hat{f} \times 1_X"] \arrow[dr, "f"] & \\ Y^X \times X \arrow[r, "\mathrm{ev}"] & Y \end{tikzcd}$$

we have the desired equality.

direction 2: Suppose $f : X \to Y$ is a mono, we need the composition:

$$1 \xrightarrow{\hat{f}} Y^X \xrightarrow{\forall x \forall x' (\hat{f}(x) = \hat{f}(x') \Rightarrow x = x')} \Omega$$

is true : $1 \to \Omega$.

By Theorem 1 (vi'), it suffices to show that the composition:

$$1 \times X \xrightarrow{\hat{f}\pi_1, \pi_2\rangle} Y^X \times X \xrightarrow{\forall x' (\hat{f}\pi_1)(\pi_2) = (\hat{f}\pi_1)(x') \Rightarrow \pi_2 = x')} \Omega$$

is $\mathrm{true}_{1\times X} : 1 \times X \to \Omega$.

Again by Theorem 1 (vi'), it amounts to show that the composition:

$$1 \times X \times X \xrightarrow{\langle\langle \hat{f}\pi_1, \pi_2\rangle \circ \pi_{1\times X}, \pi_X\rangle} Y^X \times X \times X \xrightarrow{\forall x' ((\hat{f}\pi_1) \circ \pi_{1\times X})(\pi_2 \circ \pi_{1\times X}) = ((\hat{f}\pi_1) \circ \pi_{1\times X})(\pi_X) \Rightarrow \pi_2 \circ \pi_{1\times X} = \pi_X)} \Omega$$

is $\mathrm{true}_{1\times X\times X} \to \Omega$.

That is, we want to show that once the compositions:

$$1 \times X \times X \xrightarrow{\langle \hat{f}\pi_1 \circ \pi_{1\times X}, \pi_2 \circ \pi_{1\times X}\rangle} Y^X \times X \xrightarrow{\mathrm{ev}} Y$$

$$1 \times X \times X \xrightarrow{\langle \hat{f}\pi_1 \circ \pi_{1\times X}, \pi_X\rangle} Y^X \times X \xrightarrow{\mathrm{ev}} Y$$

are equal, then the compositions:

$$1 \times X \times X \xrightarrow{\pi_{1\times X}} 1 \times X \xrightarrow{\pi_2} X$$

is equal to the map :

$$1 \times X \times X \xrightarrow{\pi_X} X$$

For $1 \times X \times X$, we have $\pi_{1\times X} \circ \pi_2$ is the projection on the middle X, denote it as pr_2, and π_X is the projection on the last X, denote it as pr_3, the π_1 is the projection on 1, denote it as pr_1, then under the assumption, we need to show $\mathsf{pr}_2 = \mathsf{pr}_3$. But the assumption says the vertical-horizontal composition in:

$$\begin{tikzcd}
1 \times X \times X \arrow[d, "\langle \mathsf{pr}_1, \mathsf{pr}_2 \rangle"] & \\
1 \times X \arrow[d, "\hat{f} \times 1_X"] \arrow[dr, "f"] & \\
Y^X \times X \arrow[r, "\mathrm{ev}"] & Y
\end{tikzcd}$$

is equal to that in:

$$\begin{tikzcd}
1 \times X \times X \arrow[d, "\langle \mathsf{pr}_1, \mathsf{pr}_3 \rangle"] & \\
1 \times X \arrow[d, "\hat{f} \times 1_X"] \arrow[dr, "f"] & \\
Y^X \times X \arrow[r, "\mathrm{ev}"] & Y
\end{tikzcd}$$

Hence since f is mono, we have $\mathsf{pr}_1 = \mathsf{pr}_3$, as desired.

$\square$

Question 3

Prove the sentense $\forall x \exists! y \phi(x, y) \Rightarrow \exists f \in Y^X \forall x \phi(x, f(x))$ where x and y are variables of types X and Y, holds for any two objects X and Y in any topos $\mathcal{E}$. [This formula expresses the "axiom of unique choice"; as usual, $\exists! \phi(x, y)$ is an abbreviation of $\exists y(\phi(x, y) \wedge \forall z(\phi(x, y) \Rightarrow y = z))$.]

Proof. This is a sketch of the proof idea, I can explain how intuitively does those things work. But I have big trouble figuring out a precise diagramatic argument.

Claim (which I cannot prove precisely): For any generalized element $R : U \to \Omega^{X \times Y}$, if it factor through Y^X, as in:

$$\begin{tikzcd}
U \arrow[dr, "r"] \arrow[drr, "!"] \arrow[ddr, "R"'] & & \\
& Y^X \arrow[r, "!"] \arrow[d, "m"] & 1 \arrow[d, "\widehat{\mathrm{true}_X}"] \\
& \Omega^{X \times Y} \arrow[r, "u"] & \Omega^X
\end{tikzcd}$$

Then we have $U \Vdash \forall x R(x, r(x))$.

For any $x : T \to X$, $r(x)$ is the evaluation:

$$U \times T \xrightarrow{r \times x} Y^X \times X \xrightarrow{e} Y$$

By SGL page 168, the map e is the one comes from the pullback:

$$\begin{tikzcd}
Y^X \times X \arrow[r, "m\times 1_X"] \arrow[dr, dashed, "e"] \arrow[ddr, "!"'] & \Omega^{X\times Y} \times X \arrow[dr, "v"] & \\
& Y \arrow[r, "\{\cdot\}_Y"] \arrow[d, "!"] & \Omega^Y \arrow[d, "\sigma_C"] \\
& 1 \arrow[r, "\text{true}"] & \Omega
\end{tikzcd}$$

That is, given a generalized element $r : U \to Y^X$ and $x : T \to X$, it is sent to the pair $(G(r), x)$ in $\Omega^{X\times Y} \times X$ by $m \times 1_X$, where $G(r)$ is the graph of the function r. v sends the pair $(G(r), x)$ to the subset of Y where its elements are related to x, as r is a function, such a set is a singleton, and will be sent to true by σ_C. The evaluation $Y^X \times X \to Y$ is defined by picking the element in Y which is the unique element which is in the set of the elements which is related to x.

Hence the claimed is supposed to hold by definition...

(Sketch of) main proof:

Unwind the anticedent $\forall x \exists y(R(x,y) \wedge \forall z R(x,z) \Rightarrow y = z)$:

The anticedent holds iff the composition:

$$U \xrightarrow{R} \Omega^{X\times Y} \xrightarrow{\forall x\exists y(R(x,y)\wedge\forall zR(x,z)\Rightarrow y=z)} \Omega$$

is $\text{true}_U : U \to \Omega$.

If an only if the composition:

$$U \times X \xrightarrow{\langle R\pi_1, \pi_2\rangle} \Omega^{X\times Y} \times X \xrightarrow{\exists y(R\pi_1(\pi_2,y)\wedge\forall zR\pi_1(\pi_2,z)\Rightarrow y=z)} \Omega$$

is $\text{true}_{U\times Z} : U \times Z \to \Omega$.

If and only if there exists an api $p : V \twoheadrightarrow U \times X$ and generalized element $\beta : V \to Y$ such that the composition:

$$V \xrightarrow{\langle R\pi_1 p, \pi_2 p, \beta\rangle} \Omega^{X\times Y} \times X \times Y \xrightarrow{R\pi_1 p(\pi_2 p,\beta)\wedge(\forall zR\pi_1 p(\pi_2 p,z)\Rightarrow\beta=z)} \Omega$$

is $\text{true}_V : V \to \Omega$.

If and only if the composition:

$$V \xrightarrow{\langle R\pi_1 p, \pi_2 p, \beta\rangle} \Omega^{X\times Y} \times X \times Y \xrightarrow{\text{ev}} \Omega$$

is $\text{true}_V : V \to \Omega$ and

$$V \times Y \xrightarrow{\langle\langle R\pi_1 p, \pi_2 p, \beta\rangle\circ\pi_V, \pi_Y\rangle} \Omega^{X\times Y} \times X \times Y \times Y \xrightarrow{(R\pi_1 p\pi_V(\pi_2 p\pi_V,\pi_Y)\Rightarrow\beta\pi_V=\pi_Y)} \Omega$$

is $\text{true}_{V\times Y} : V \times Y \to \Omega$.

Unwinding the conclusion $\exists f \forall x R(x, f(x))$, it is true if and only if the composition:

$$U \xrightarrow{R} \Omega^{X\times Y} \xrightarrow{\exists f \forall x R(x,f(x))} \Omega$$

is $\mathrm{true}_U : U \to \Omega$.

If and only if there exists an epi $q : W \to U$ and a generalized element $r : W \to Y^X$, such that the composition:

$$W \xrightarrow{\langle Rq, r\rangle} \Omega^{X\times Y} \times Y^X \xrightarrow{\forall x Rp(x, r(x))} \Omega$$

is $\mathrm{true}_W : W \to \Omega$.

If and only if the composition:

$$W \times X \xrightarrow{\langle\langle Rq, r\rangle \circ \pi_1, \pi_2\rangle} \Omega^{X\times Y} \times Y^X \times X \xrightarrow{Rp\pi_1(\pi_2, r\pi_1(\pi_2))} \Omega$$

is $\mathrm{true}_{W\times X} : W \times X \to \Omega$.

I think the answer is supposed to be:

As $p : V \twoheadrightarrow U \times X$ is epi, and the projection maps from product are epis, the U-component $p_U : V \to U$ is a composition of epis and such is an epi. So the W we require in the conclusion is V, and the epi $V \to U$ is p_U. For the required generalized element $V \to Y^X$, from the antecedent, we have a map $R\pi_1 p : V \to \Omega^{X\times Y}$, and the 'unique existence of y' is SUPPOSED TO imply that the transpose of the composition of u and $R\pi_1 p$ is $\mathrm{true}_{V\times X}$, hence we will have a map r as in:

$$\begin{tikzcd}
V \arrow[dr, dashed, "r"] \arrow[drr, "!"] \arrow[ddr, "R\pi_1 p"'] & & \\
& Y^X \arrow[r, "!"] \arrow[d, "m"] & 1 \arrow[d, "\widehat{\mathrm{true}_X}"] \\
& \Omega^{X\times Y} \arrow[r, "u"] & \Omega^X
\end{tikzcd}$$

And this r is SUPPOSED TO satisfy the require condition as we needed for the conclusion.

$\square$