

Assignment 3

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Question 1

In any cartesian closed category \mathbf{C} , prove that the internal composition $C^B \times B^A \rightarrow C^A$ defined in §2 is associative. Also show that for a given object C , C^C is a monoid object in \mathbf{C} with a two-side unit $e : 1 \rightarrow C^C$ which is the transpose of the identity $C \rightarrow C$. If, in addition, \mathbf{C} has pullbacks, construct the object $\text{Aut}(C)$ of automorphisms of a given object C of \mathbf{C} , and prove that $\text{Aut}(C)$ is a group object in \mathbf{C} .

Proof. (a) internal composition is associative.

Consider the diagram:

$$\begin{array}{ccccc}
 & D^C \times C^B \times B^A \times A & & & \\
 & \swarrow \quad \downarrow \quad \searrow & & & \\
 D^B \times B^A \times A & & D^C \times C^B \times B & & D^C \times C^A \times A \\
 \downarrow \quad \swarrow & & \downarrow & & \downarrow \\
 D^B \times B & & D^C \times C & \xrightarrow{=} & D^C \times C \\
 \searrow & \downarrow & \swarrow & & \searrow \\
 & D & & &
 \end{array}$$

(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, showing the relationships between various tensor products and evaluation maps.)

This diagram commutes. The parallelogram on the left commutes because it is just switch the order of applying ‘ ev_{AB} ’ and ‘ m_C ’. The triangle on the left corner commutes by definition of m_C . The triangle on the right upper corner commutes by the definition of m_B (multiply by D^C on the left of its definition.)

Alternative solution using generalized elements: (Using the result on SGL page 170, ‘The connection of internal composition ...’).

For the proofs of the discussion on the textbook.

(1) Given arrows $f : A \rightarrow B$ and $g : B \rightarrow C$, one can transpose these arrows to get global elements of the exponential objects $\hat{f} : 1 \rightarrow B^A$ and $\hat{g} : 1 \rightarrow C^B$. Then $m \circ (\hat{g}, \hat{f}) : 1 \rightarrow C^A$ is precisely the transpose of the ordinary composition $g \circ f : A \rightarrow C$.

$$\begin{array}{ccccc}
 1 \times A & \xrightarrow{\hat{f} \times 1_A} & 1 \times B^A \times A & \xrightarrow{\text{ev}_{AB}} & 1 \times B \\
 \downarrow (\hat{g}, \hat{f}) \times 1_A & \searrow & \downarrow \hat{g} \times 1_{B^A} & \searrow & \downarrow g \\
 C^B \times B^A \times A & \xrightarrow{m \times 1_A} & C^B \times B & \xrightarrow{\text{ev}_{BC}} & 1 \times C \\
 \downarrow m \times 1_A & & & & \\
 C^A \times A & \xrightarrow{\text{ev}_{AC}} & & &
 \end{array}$$

(2) Similarly, generalized elements can be composed using the map m . Given such generalised elements $f : X \rightarrow B^A$ and $g : X \rightarrow C^B$, one obtain their composition as a generalized element of C^A , simply by composing $(g, f) : X \rightarrow C^B \times B^A$ with $m : C^B \times B^A \rightarrow C^A$. The transpose of this map $m \circ (g, f) : X \rightarrow C^A$ is the (external) composite $X \times A \xrightarrow{(\pi_1, f')} X \times B \xrightarrow{g'} C$ where f' and g' are transposed maps of f and g .

$$\begin{array}{ccccc}
X(\times 1) \times A & & & & \\
\downarrow (g,f) \times 1_A & \searrow 1_X \times f \times 1_A \quad (\pi_1, f') & & & \\
C^B \times B^A \times A & \xleftarrow{g \times 1_B \times 1_A} & X \times B^A \times A & \xrightarrow{1_X \times \text{ev}_{AB}} & X \times B \\
\downarrow m \times 1_A & & \downarrow g \times 1_B & & \downarrow 1_X \times g' \\
C^A \times A & & C^B \times B & & X \times C \\
& \searrow \text{ev}_{AC} & \searrow \text{ev}_{BC} & & \\
& & & &
\end{array}$$

Return to the question, consider the diagram:

$$\begin{array}{ccccc}
& & T & & \\
& & \downarrow (h,g,f) & & \\
& D^C \times C^B \times B^A & & & \\
& \swarrow 1_{D^C} \times m_B & & \searrow m_C \times 1_{B^A} & \\
D^C \times C^A & & & & D^B \times B^A \\
& \searrow m_C & & \swarrow m_B & \\
& D^A & & &
\end{array}$$

Consider a generalized element of $D^C \times C^B \times B^A$, it is a tuple consists of maps $h : T \rightarrow D^C, g : T \rightarrow C^B, f : T \rightarrow B^A$. We take the notation of transpose as:

$$\begin{array}{ccc}
X \times A & & \\
\downarrow f \times 1_A & \searrow f' & \\
B^A \times A & \xrightarrow{\text{ev}_{AB}} & B
\end{array}$$

$$\begin{array}{ccc}
X \times A & & \\
\downarrow \hat{f} \times 1_A & \searrow f & \\
B^A \times A & \xrightarrow{\text{ev}_{AB}} & B
\end{array}$$

Hence we have $\hat{f}' = f, \hat{g}' = g$ under this notation. As in the textbook, for $f : X \rightarrow B^A, g : X \rightarrow C^B$, we have $m \circ (g, f) : X \rightarrow C^A$ is $\widehat{g' \circ (\pi_1, f')}$.

To show the diagram above commutes, firstly consider the left hand side. By above, we have $(h, g, f) \xrightarrow{1_{D^C} \times m_B} (h, \widehat{g' \circ (\pi_1, f')})$ in the first step, and $(h, \widehat{g' \circ (\pi_1, f')}) \xrightarrow{m_C} \widehat{h' \circ (\pi_1, g' \circ (\pi_1, f'))}$. For the right hand side, the first step is $(h, g, f) \xrightarrow{m_C \times 1_{B^A}} (\widehat{h' \circ (\pi_1, g')}, f)$, and the second step is $(\widehat{h' \circ (\pi_1, g')}, f) \mapsto (\widehat{h' \circ (\pi_1, g')}) \circ (\pi_1, f')$.

Hence to show that the two maps are equal, it suffices to show $(\pi_1, g') \circ (\pi_1, f') = (\pi_1, (g' \circ (\pi_1, f')))$. The first one is the composition $T \times A \xrightarrow{(\pi_1, f')} T \times B \xrightarrow{(\pi_1, g')} T \times C$. And the second one is the map $T \times A \rightarrow T \times C$ with first component given by $\pi_1 : T \times A \rightarrow T$, and the second component given by $T \times A \xrightarrow{(\pi_1, f')} T \times B \xrightarrow{g'} C$. Hence the two maps are the same.

(b) For a given object C , C^C is a monoid object with a two-side unit $e : 1 \rightarrow C^C$, which is the transpose of the identity $C \rightarrow C$.

The multiplication is given by the internal composition, it is proved to be associate as in (a), it lefts to show that e is left and right unit.

Left unit: Stuck on diagram chasing for left unit.

Still do not have diagrammatic proof, but have one with generalized element.

The map $e : 1 \rightarrow C^C$ induces a map from generalized elements $1(T) := \{T \rightarrow 1\} \rightarrow C^C(T) := \{T \rightarrow C^C\}$ defined by sending $* \in 1(T)$ to the element in $\{T \rightarrow C^C\} \cong \{T \times C \rightarrow C\}$ corresponds to the projection $\pi_2 : T \times C \rightarrow C$. To see this, consider the only generalized T -point $*$ of 1 and consider the diagram:

$$\begin{array}{ccc} T \times C & & \\ \downarrow * \times 1_C & \searrow & \\ 1 \times C & & \\ \downarrow e \times 1_C & \searrow \cong & \\ C^C \times C & \xrightarrow{\text{ev}} & C \end{array}$$

The dashed arrow is the transpose of the map $e \circ *$, which is clearly $\pi_2 : T \times C \rightarrow C$.

Take a generalized element $f : T \rightarrow C^C$ of C^C , consider the diagram:

$$T \xrightarrow{f} C^C \xrightarrow{\cong} 1 \times C^C \xrightarrow{e \times 1_{C^C}} C^C \times C^C \xrightarrow{m} C^C$$

The procedure is $f \xrightarrow{\cong} (*, f) \xrightarrow{(e \times 1_{C^C})} (\hat{\pi}_2, f) \xrightarrow{m} \widehat{\pi_2 \circ (\pi_1, f')}$. To show the sequence of maps in the diagram above composes to the identity, it suffices to show $\pi_2 \circ (\pi_1, f') = f' : T \times C \rightarrow C$. It is true because the left hand side is just the composition $T \times C \xrightarrow{(\pi_1, f')} T \times C \xrightarrow{\pi_2} C$.

Right unit: Consider the diagram:

$$\begin{array}{ccc} C^C \times C & & \\ \downarrow \cong & \searrow & \\ C^C \times 1 \times C & & \\ \downarrow 1_{C^C} \times e \times 1_C & \searrow \cong^{-1} & \\ C^C \times C^C \times C & \xrightarrow{1_{C^C} \times \text{ev}} & C^C \times C \\ \downarrow m \times 1_C & & \searrow \text{ev} \\ C^C \times C & \xrightarrow{\text{ev}} & C \end{array}$$

The diagram above commutes. The bottom part commutes by definition of m (internal composition). The triangle commutes by definition of e (triangle of definition of e with C^C multiplied on the left hand side.)

Generalised element: Take a generalized element $g : T \rightarrow C^C$ of C^C , consider the diagram:

$$T \xrightarrow{g} C^C \xrightarrow{\cong} C^C \times 1 \xrightarrow{1_{C^C} \times e} C^C \times C^C \xrightarrow{m} C^C$$

We have $g \xrightarrow{\cong} (g, *) \xrightarrow{1_{C^C} \times e} (g, \hat{\pi}_2) \xrightarrow{m} \widehat{g' \circ (\pi_1, \pi_2)}$. In diagram, it is just the composition $T \times C \xrightarrow{(\pi_1, \pi_2)} T \times C \xrightarrow{g'} C$, which is g' , as desired.

(c) If \mathbf{C} has pullbacks, construct $\text{Aut}(C)$ and prove it is a group object.

There is a twist map $C^C \times C^C \rightarrow C^C \times C^C$ induced by switching the order of projection.

$$\begin{array}{ccccc} & & C^C \times C^C & & \\ & \swarrow \pi_1 & \downarrow \tau & \searrow \pi_2 & \\ C^C & \xleftarrow{\pi_2} & C^C \times C^C & \xrightarrow{\pi_1} & C^C \end{array}$$

Again, m is the internal composition, and $m\tau$ can be thought as switching the order, and then compose. The maps $m, m\tau : C^C \times C^C \rightarrow C^C$ induces a map $C^C \times C^C \rightarrow C^C \times C^C$. And the transpose of the identity 1_C , which is $e : 1 \rightarrow C^C$ induces a map $1 \rightarrow C^C \rightarrow C^C$. Then $\text{Aut}(C)$ is the pullback:

$$\begin{array}{ccc} \text{Aut}(C) & \xrightarrow{!} & 1 \\ \downarrow i & & \downarrow (e, e) \\ C^C \times C^C & \xrightarrow{(m, m\tau)} & C^C \times C^C \end{array}$$

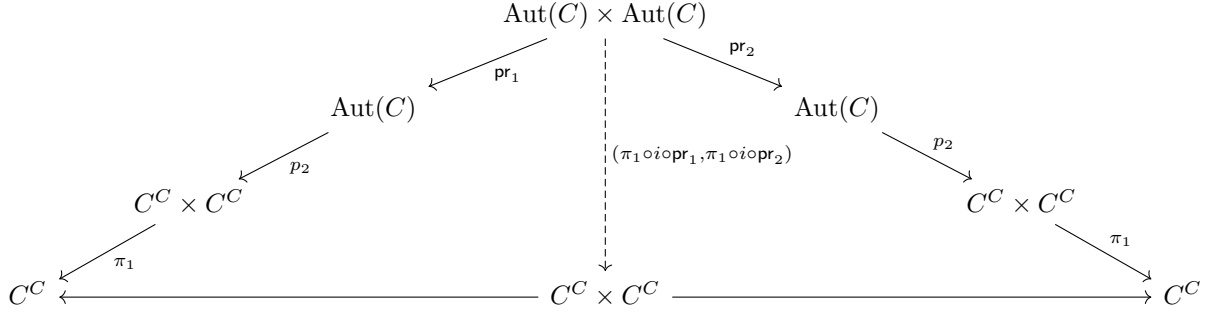
For the unit $1 \rightarrow \text{Aut}(C)$, it is the dashed arrow in the diagram:

$$\begin{array}{ccccc} 1 & & & & \\ & \searrow (e, e) & & \searrow ! & \\ & \text{Aut}(C) & \xrightarrow{!} & 1 & \\ & \downarrow i & & \downarrow (e, e) & \\ & C^C \times C^C & \xrightarrow{(m, m\tau)} & C^C \times C^C & \end{array}$$

For the composition $\text{Aut}(C) \rightarrow \text{Aut}(C) \rightarrow \text{Aut}(C)$, it is the dashed arrow in the diagram:

$$\begin{array}{ccccc} \text{Aut}(C) \times \text{Aut}(C) & & & & \\ & \searrow (a, b) & & \searrow ! & \\ & \text{Aut}(C) & \xrightarrow{!} & 1 & \\ & \downarrow i & & \downarrow (e, e) & \\ & C^C \times C^C & \xrightarrow{(m, m\tau)} & C^C \times C^C & \end{array}$$

Where a is given by the composing m after the dashed map in



And b is obtained similarly by composing τm with the map obtained from the diagram where the π_1 in the above diagram is replaced by π_2 .

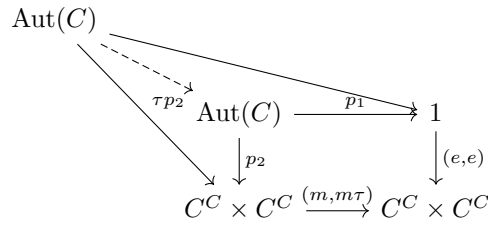
To check that there does exist the induced dashed arrow above, we need to prove the outside of the above diagram commutes, which amounts to test on an arbitrary generalized element $T \rightarrow \text{Aut}(C) \times \text{Aut}(C)$.

(not used here: To see what is a generalized element $T \rightarrow \text{Aut}(C)$. Firstly, it is a pair of arrows $f, g : T \rightarrow C^C$. And arrow $f : T \rightarrow C^C$ is an arrow $f' : T \times C \rightarrow C$, which is the same of an arrow $\tilde{f} := (\text{pr}_1, f) : T \times C \rightarrow T \times C$ where the component $T \times C \rightarrow T$ is the projection map. Such a map is a map in the slice category \mathbf{C}/T from the object $\text{pr}_1 : T \times C \rightarrow T$ to itself. Hence a generalized element $T \rightarrow \text{Aut}(C)$ is actually a pair of maps f, g from $\text{pr}_1 : T \times C \rightarrow T$ to itself such that $f \circ g = g \circ f = 1_{T \times C}$.)

A generalized element to $\text{Aut}(C)$ is a pair of maps $(f, g) : T \rightarrow C^C \times C^C$ such that $m \circ (f, g) = m\tau(f, g) = \hat{\pi}_2$. Consider a generalised element $((f, g), (h, k)) : T \rightarrow \text{Aut}(C) \times \text{Aut}(C)$. follow the big diagram above, the process on the left hand side goes as $((f, g), (h, k)) \mapsto (f, g) \mapsto (f, g) \mapsto f$. On right hand side, we have $((f, g), (h, k)) \mapsto (h, k) \mapsto (h, k) \mapsto h$. We aim to show $m \circ (m \circ (f, h), m \circ (k, g)) = \pi_2$, this is by using $m(f, g) = f' \circ (\pi_1, g')$ several times.

(Tedious! Any elegant way to do this?)

For inverse, it is the dashed arrow in



□

Question 2

If S and T are subobjects of some object E of a topos \mathcal{E} , show that the square of inclusion maps

$$\begin{array}{ccc}
S \cap T & \xrightarrow{\quad} & S \\
\downarrow & & \downarrow \\
T & \xrightarrow{\quad} & S \cup T
\end{array}$$

is both a pushout and a pullback. (Hint: Use Theorem 7.8.)

Proof. Recall the ‘intersection’ and union are defined by: intersection: Given two subobjects $S \rightarrowtail A$ and $T \rightarrowtail A$ we can form their intersection as their greatest lower bound (g.l.b.) in $\text{Sub}(A)$ simply by taking the pullback, as in

$$\begin{array}{ccc}
S \cap T & \xrightarrow{\quad} & T \\
\downarrow & & \downarrow \\
S & \xrightarrow{\quad} & A
\end{array}$$

To get their union or least upper bound (l.u.b.) in $\text{Sub}(A)$ we first form the coproduct (sum) $S + T$ in the topos, as in the square:

$$\begin{array}{ccc}
S + T & \xleftarrow{\quad} & T \\
\downarrow & \searrow & \downarrow \\
S & \xrightarrow{\quad} & A
\end{array}
\quad
\begin{array}{c}
M \\
\swarrow \quad \searrow \\
S \quad \quad T
\end{array}$$

by the definition of a coproduct, the monics $S \rightarrowtail A$ and $T \rightarrowtail A$ then determine uniquely an arrow $S + T \rightarrow A$ which need not be monic, but which by Proposition 1 (on SGL page 185) has an image M as displayed. This $M \rightarrowtail A$ is then a subobject of A which clearly contains both given subobjects S and T . The minimal property defining the image M then readily shows that M is a least upper bound of S and T in $\text{Sub}(A)$, so $M = S \cup T$.

Claim 1: The canonical map $S \rightarrow S + T, T \rightarrow S + T$ are monic.

Proof. As S and T are both subobject of A , consider the diagram:

$$\begin{array}{ccccc}
& & S + T & & \\
& \nearrow & \downarrow & \nwarrow & \\
S & \xrightarrow{\quad} & A & \xleftarrow{\quad} & T
\end{array}$$

So the canonical arrows $S \rightarrow S + T, T \rightarrow S + T$ compose with some other arrow to get a mono, then themselves are monos. \square

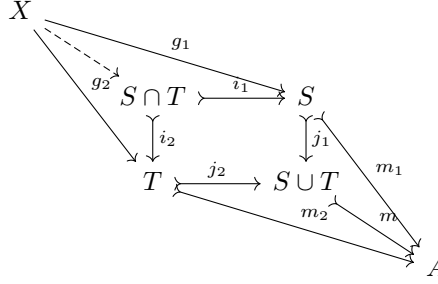
Claim 2: In the diagram above with $M = S \cup T$, the arrows on the diagonal is an epi followed by a mono. That is, $S + T \twoheadrightarrow S \cup T \rightarrowtail A$.

Proof. By Proposition 1 on page 185 and the comment below Proposition 2, $S + T \rightarrow A$ factorize uniquely through its image as a mono and a epi, so the diagram above is exactly ‘the’ factorization. \square

Claim 3: In the diagram for the definition of the union $M = S \cap T$, the compositions $T \rightarrow S + T \rightarrow S \cup T$ and $S \rightarrow S + T \rightarrow S \cup T$ are monos.

Proof. In the diagram, the bottom map and the right vertical map are monos, so both of these two maps compose with some other map to a mono, this implies these two maps themselves are monos. \square

To prove the diagram is a pullback, consider maps $g_1, g_2 : X \rightarrow T, X \rightarrow S$ such that $(X \rightarrow T \rightarrow S \cup T) = (X \rightarrow S \rightarrow S \cup T)$, we can augment the diagram as:



All the monos are proved in previous claims. Hence the dotted arrow is obtained from the pullback in the definition of intersection, which is the square with vertices $T, S \cap T, S, A$.

(Later I realised that I do not need to prove that the diagram is a pullback, since once I prove that it is a pushout, then the pullback is automatic by Corollary 4 on page 211 of SGL.)

To prove the diagram is a pushout:

Definition: A coherent category (also called a pre-logos) is a regular category in which the subobject posets $\text{Sub}(X)$ all have finite unions which are preserved by the base change functors $f^* : \text{Sub}(Y) \rightarrow \text{Sub}(X)$.

Claim: Any topos is coherent. (nLab page ‘coherent category-Examples’) Proof: By Proposition 3 on page 186 of SGL, a topos has binary and hence all finite unions. By Theorem 2 on SGL page 193, f^* has both right and left adjoint, and hence will preserve union, which is a colimit. It remains to show that any topos is a regular category.

Definition: A category \mathbf{C} is called regular if

- (1) It is finitely complete;
- (2) the kernel pair of any morphism $f : d \rightarrow c$ admits a coequalizer $d \times_c d \rightrightarrows d \rightarrow \text{coeq}(p_1, p_2)$;
- (3) the pullback of a regular epimorphism along any morphism is again a regular epimorphism.

Definition: A regular epimorphism is a morphism $f : c \rightarrow d$ (in a given category) that is the coequalizer of some parallel pair of morphisms.

(1) is by definition of a topos on SGL page 161. (2) is by Q14 in chapter IV. (By Q14(a), any kernel pair is an equivalence relation, the result follows from Q14(f).) (3) is by the hint (by Theorem 7.8) together with Proposition 3 on page 194 (pullbacks of an epi is an epi).

The proof is taken from Johnstone (Sketch of an elephant, A1.4, page 32, Proposition 1.4.3).

nLab page ‘coherent category-Examples’ says: Any topos is coherent.

Johnstone, Sketch of elephant, A1.4, Lemma 1.4.2: Let A_1, A_2, A_3 be three subobjects of an object A in a coherent category. The distributive law $A_1 \cap (A_2 \cup A_3) \cong (A_1 \cap A_2) \cup (A_1 \cap A_3)$ holds in $\text{Sub}(A)$.

Proof. Denote $m : A_1 \rightarrowtail A$, then the ‘intersecting with A_1 ’ operation is the composite $\text{Sub}(A) \xrightarrow{m^*} \text{Sub}(A_1) \xrightarrow{\exists_m} \text{Sub}(A)$. As we are working in a coherent category, m^* preserves union by definition of coherent category. Hence for any subobject A_2, A_3 of A , $(A_1) \cap (A_2 \cup A_3) = \exists_m(m^*(A_2 \cup A_3)) = \exists_m(m^*(A_2) \cup m^*(A_3))$. As \exists_m is a left adjoint, it preserves colimits and hence preserves union, $\exists_m(m^*(A_2) \cup m^*(A_3)) = \exists_m(m^*(A_2)) \cup \exists_m(m^*(A_3)) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$. \square

Consider maps $f_1 : S \rightarrow X, f_2 : T \rightarrow X$ such that the solid diagram commutes, we want a unique dashed arrow which makes the whole diagram commute:

$$\begin{array}{ccccc}
 S \cap T & \xrightarrow{n_1} & S & & \\
 \downarrow n_2 & & \downarrow p_1 & \searrow f_1 & \\
 T & \xrightarrow{p_2} & S \cup T & \xrightarrow{f_2} & X \\
 & & & \nearrow f_1 & \\
 & & & & X
 \end{array}$$

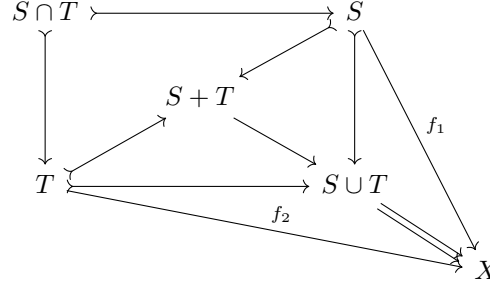
Consider the product $(S \cup T) \times X$, we have arrows $(p_1, f_1) : S \rightarrow (S \cup T) \times X$ and $(p_2, f_2) : T \rightarrow (S \cup T) \times X$, they are monic since p_1 and p_2 are. These two monos are subobjects of $C \times D$. Hence we can form their union in $(S \cup T) \times X$, denoted $(S \cup T)'$. The map $(S \cup T)' \rightarrow (S \cup T) \times X$ is a map to a product, write it as (g, h) , where $g : (S \cup T)' \rightarrow (S \cup T), h : (S \cup T)' \rightarrow X$.

$$\begin{array}{ccccc}
 S + T & \xleftarrow{\quad} & S & & \\
 \uparrow & \searrow & \downarrow u_1 & & \downarrow (p_1, f_1) \\
 & & (S \cup T)' & \xrightarrow{(g, h)} & (S \cup T) \times X \\
 \uparrow u_2 & \nearrow & \downarrow (p_2, f_2) & & \\
 T & \xrightarrow{\quad} & (S \cup T) \times X & &
 \end{array}$$

First reduction: It suffices to show that g is an isomorphism.

Proof: If g is an isomorphism, then $hg^{-1} : S \cup T \rightarrow X$ is what we want. It satisfies $hg^{-1}p_1 = f_1$ since from the diagram above, $hu_1 = f_1$ and $g^{-1}p_1 = u_1$, similarly, $hg^{-1}p_2 = f_2$.

This decomposition is unique. Consider the diagram:



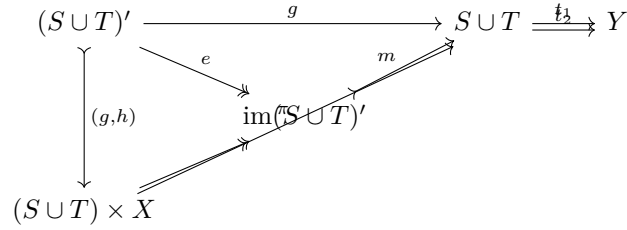
If we have two parallel arrows which makes the diagram commute, then they will compose with the map $S + T \rightarrow S \cup T$ to get the same map from $S + T$ from the universal property of coproduct. But here the map $S + T \rightarrow S \cup T$ is an epi, as discussed earlier. So composing with this map to get the same map implies the two parallel arrows themselves are equal.

This completes the first reduction.

By SGL page 167, Proposition 2, as \mathcal{E} is a topos, it suffices to show that g is both a mono and an epi.

To show g is an epi. Consider the projection $\pi : (S \cup T) \times X \rightarrow S \cup T$. By SGL page 186 Proposition 3, it has left adjoint \exists_π . As left adjoints preserve colimits, it preserves union. (Vague!) Hence $\exists_\pi(g, h) = \exists_\pi(p_1, f_1) \cup \exists_\pi(p_2, f_2) = p_1 \cup p_2 = 1_{S \cup T}$ by definition of \exists_π as in SGL page 186 Proposition 3. Hence g is a cover (Vague!).

Modified: To show that g is epi, consider arrows $r_1, r_2 : S \cup T \rightarrow Y$ such that $r_1 \circ g = r_2 \circ g$, we want $r_1 = r_2$. Under this assumption, let $\pi : (S \cup T) \times X \rightarrow S \cup T$ be the projection, consider the diagram:



Claim: Any mono is the image of itself.

Proof: By uniqueness of image.

Here $\text{im}(S \cup T)'$ is the image of $(S \cup T)'$ in $S \cup T$ under g , as $\pi \circ (g, h) = g$, we have these two equal maps has the same image, and hence the diagram commutes. As \exists_π is left adjoint of π^{-1} , it preserves colimits and hence preserves union, hence $\exists_\pi(g, h) = \exists_\pi(p_1, f_1) \cup (p_2, f_2) = \exists_\pi(p_1, f_1) \cup \exists_\pi(p_2, f_2)$. p_1, p_2 are both monos to $S \cup T$. Hence by the claim proved above, they are the images of themselves. That is, $\exists_\pi(p_1, f_1) = p_1$ and $\exists_\pi(p_2, f_2) = p_2$. So $\exists_\pi(g, h) = p_1 \cup p_2 = 1_{S \cup T}$.

If $t_1 g = t_2 g$, then $t_1 m e = t_2 m e$, as e is epi, it implies $t_1 m = t_2 m$, as $m = \exists_\pi(g, h) = 1_{S \cup T}$, we conclude $t_1 = t_2$, as desired.

It remains to show that g is monic. Consider two arrows $k, l : F \rightarrow (S \cup T)'$ such that the compositions: $F \xrightarrow[k]{l} (S \cup T)' \rightarrow (S \cup T)$ are equal. As $(S \cup T)'$ is covered by the inclusions from S and T , F is covered by

the pullbacks $k^*(S)$ and $k^*(T)$. By the same reason, it is covered by the pullbacks $l^*(S)$ and $l^*(T)$. That is $F = k^*(S) \cup k^*(T) = l^*(S) \cup l^*(T)$. By the lemma quoted from Johnstone, as in the begining of the question, this implies that F is the union of $k^*(S) \cap l^*(S)$, $k^*(S) \cap l^*(T)$, $k^*(T) \cap l^*(S)$ and $k^*(T) \cap l^*(T)$. We aim to show that $hk = hl$ as will, then together with the fact that $gk = gl$ and (g, h) is a mono, we can conclude $k = l$. It suffices to check that hk and hl agrees on the cover of F formed by the four intersections. We prove the one for $k^*(S) \cap l^*(T)$, the others are similar.

Consider the intersection of the pullbacks $k^*(S) \cap l^*(T)$, let v_1, v_2 be the arrows such that the diagram:

$$\begin{array}{ccccc} S & \xleftarrow{v_1} & k^*(S) \cap l^*(T) & \xrightarrow{v_2} & T \\ \downarrow u_1 & & \downarrow w & & \downarrow u_2 \\ (S \cup T)' & \xleftarrow{k} & F & \xrightarrow{l} & (S \cup T)' \end{array}$$

commutes. By the diagram defined (g, h) , we have $p_1 = gu_1, p_2 = gu_2$. Together with the commutivity of the diagram above, we have $p_1v_1 = gu_1v_1 = gkw = glw = gu_2v_2 = p_2v_2$. Hence by the pullback as proved in the first part in this question, there exists $x : k^*(S) \cap l^*(T) : S \cap T$ such that $v_1 = n_1x, v_2 = n_2x$.

Also by the diagram defining (g, h) , we have $hu_1 = f_1, hu_2 = f_2$. Hence $hu_1n_1 = f_1n_1 = f_2n_2 = hu_2n_2$. And thus $hkw = hu_1v_1 = hu_1n_1x = hu_2n_2x = hu_2v_2 = hlw$. This shows $hk = kl$ agrees on this intersection. \square

Question 3

Let \mathcal{E} be a topos, and A and object of \mathcal{E} . An equivalence relation on A is a monomorphism $(p, q) : R \rightarrow A \times A$ such that R is reflexive, symmetric and transitive (expressed by appropriate diagrams). One also says that the pair $p, q : R \rightrightarrows A$ “is” an equivalence relation.

(a) Let $f : A \rightarrow B$ be a map in \mathcal{E} . Prove that the kernel pair $A \times_B A \rightrightarrows A$ of f is an equivalence relation on A . (The propose of the rest of this exercise is to show the converse; see also the Appendix, §4.)

Proof. Firstly we need the definition of equivalence relation. That is, for a pair $p, q : R \rightrightarrows A$, we need suitable concepts of reflexivity, symmetry and transitivity.

Reflexivity: A pair of arrows $p, q : R \rightrightarrows A$ is reflexive if the diagonal $\Delta : A \rightarrow A \times A$ factor through p, q .

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \times A \\ & \searrow (p, q) & \uparrow \\ & & R \end{array}$$

Symmetry: A pair of arrows $p, q : R \rightrightarrows A$ is symmetric if there exists a ‘twist map’ $\tau : R \rightarrow R$ such that the diagram below commutes:

$$\begin{array}{ccc} R & \xrightarrow{\tau} & R \\ & \searrow (p, q) & \swarrow (q, p) \\ & & A \times A \end{array}$$

A pair of arrows $p, q : R \rightrightarrows A$ is transitive if for the pullback:

$$\begin{array}{ccc} R * R & \xrightarrow{\pi_1} & R \\ \downarrow \pi_2 & & \downarrow q \\ R & \xrightarrow{p} & A \end{array}$$

The arrow $(p\pi_2, q\pi_1) : R * R \rightarrow A \times A$ factors through R .

Consider the kernel pair of f :

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\text{pr}_2} & A \\ \downarrow \text{pr}_1 & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

Reflexivity: As the kernel pair is a pullback and the diagonal is induced by a pair of identity maps of A , by the commutativity of:

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & & \searrow & \\ & A \times_B A & \xrightarrow{\text{pr}_2} & A & \\ & \downarrow \text{pr}_1 & & \downarrow f & \\ & A & \xrightarrow{f} & B & \end{array}$$

(Note: A dashed arrow labeled '1' goes from the top A to $A \times_B A$, and a solid arrow labeled '1' goes from the top A to A .)

We have the dashed arrow is exactly the arrow that makes:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \times A \\ & \searrow (\text{pr}_1, \text{pr}_2) & \uparrow \\ & & A \times_B A \end{array}$$

commute.

Symmetry: Consider the diagram:

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & & \searrow & \\ & A \times_B A & \xrightarrow{\text{pr}_2} & A & \\ & \downarrow \text{pr}_1 & & \downarrow f & \\ & A & \xrightarrow{f} & B & \end{array}$$

(Note: A dashed arrow labeled τ goes from the top A to $A \times_B A$, and a solid arrow labeled pr_1 goes from the top A to A .)

The dashed arrow is exactly the map τ which makes the diagram below commutes:

$$\begin{array}{ccc}
 A \times_B A & \xrightarrow{\tau} & A \times_B A \\
 & \searrow (p,q) & \swarrow (q,p) \\
 & A \times A &
 \end{array}$$

Transitivity: Consider the pullback diagram:

$$\begin{array}{ccc}
 (A \times_B A) * (A \times_B A) & \xrightarrow{\pi_1} & A \times_B A \\
 \downarrow \pi_2 & & \downarrow p \\
 A \times_B A & \xrightarrow{p} & A
 \end{array}$$

We need the dashed arrow:

$$\begin{array}{ccccc}
 & & (A \times_B A) * (A \times_B A) & & \\
 & \swarrow \pi_2 & \downarrow & \searrow \pi_1 & \\
 A \times_B A & & A \times_B A & & A \times_B A \\
 \swarrow p & \searrow p & \downarrow & \swarrow q & \searrow q \\
 A & & A \times A & & A
 \end{array}$$

this is the dashed arrow in the following diagram:

$$\begin{array}{ccccc}
 & & (A \times_B A) * (A \times_B A) & & \\
 & \searrow p\pi_2 & \downarrow & \searrow q\pi_1 & \\
 & & A \times_B A & \xrightarrow{p\pi_2} & A \\
 & \searrow & \downarrow \text{pr}_1 & & \downarrow f \\
 & & A & \xrightarrow{f} & B
 \end{array}$$

This completes the proof □

(b) Prove that if $p, q : R \rightrightarrows A$ is the kernel pair of some map $A \rightarrow B$, then it is also the kernel pair of the coequalizer of p and q .

Proof. Let $R = A \times_B A$ be the kernel pair:

$$\begin{array}{ccc}
 R = A \times_B A & \xrightarrow{q} & A \\
 \downarrow p & & \downarrow f \\
 A & \xrightarrow{f} & B
 \end{array}$$

Consider the coequalizer:

$$A \times_B A \rightrightarrows A \xrightarrow{e} E$$

It suffices to prove that the square in the diagram below is a pullback:

$$\begin{array}{ccccc}
X & & & & \\
& \searrow^{g_2} & & \searrow^{pr_2} & \\
& & A \times_B A & \xrightarrow{\quad} & A \\
& \swarrow_{g_1} & \downarrow pr_1 & & \downarrow e \\
& & A & \xrightarrow{\quad e \quad} & E
\end{array}$$

As $R = A \times_B A$ is the pullback square, we have $fp = fq$, so we can consider the diagram:

$$\begin{array}{ccc}
A \times_B A & \xrightarrow{\eta} & A \xrightarrow{e} E \\
& & \searrow f \quad \downarrow t \\
& & B
\end{array}$$

Here t is the unique arrow $E \rightarrow B$ that makes the right hand side commute.

Given $g_1, g_2 : X \rightarrow A$ such that $eg_1 = eg_2$, we then have $fg_1 = fg_2$, then the required dashed arrow is the dashed arrow in:

$$\begin{array}{ccccc}
A & & & & \\
& \searrow^{g_2} & & \searrow^{pr_2} & \\
& & A \times_B A & \xrightarrow{\quad} & A \\
& \swarrow_{g_1} & \downarrow pr_1 & & \downarrow e \\
& & A & \xrightarrow{\quad e \quad} & E
\end{array}$$

□

(c) Let R be an equivalence relation on A , Show, using symmetry and transitivity, that $(p \times 1)^{-1}(R) = (q \times 1)^{-1}(R)$; i.e., that the pullbacks $(p, q) : R \rightrightarrows A \times A$ along $p \times 1$, respectively along $q \times 1 : R \times A \rightarrow A \times A$, are isomorphic as subobject of $R \times A$.

Proof. Let $R * R$ be the pullback:

$$\begin{array}{ccc}
R * R & \xrightarrow{\pi_1} & R \\
\downarrow \pi_2 & & \downarrow q \\
R & \xrightarrow{p} & A
\end{array}$$

We claim the pullbacks in the question are given by:

$$\begin{array}{ccc}
R * R & \xrightarrow{\pi_2} & R \\
\downarrow (\pi_1, q\pi_2) & & \downarrow (p, q) \\
R \times A & \xrightarrow{q \times 1} & A \times A
\end{array}$$

$$\begin{array}{ccc}
R * R & \xrightarrow{\tau \pi_1} & R \\
\downarrow (\pi_2, p \pi_1) & & \downarrow (p, q) \\
R \times A & \xrightarrow{p \times 1} & A \times A
\end{array}$$

To show the two squares are indeed pullbacks:

For the first one, given $(g_1, g_2) : X \rightarrow R \times A$ and k satisfying $(p, q)k = (q \times 1)(g_1, g_2)$, we can augment the diagram as:

$$\begin{array}{ccccccc}
X & & & & & & \\
& \searrow^{k} & & & & & \\
& & R * R & \xrightarrow{\pi_2} & R & \xrightarrow{1} & R \\
& \searrow^{(g_1, g_2)} & \downarrow (\pi_1, q \pi_2) & & \downarrow (p, q) & & \downarrow p \\
& & R \times A & \xrightarrow{q \times 1} & A \times A & & \\
& & \downarrow p_1 & & \searrow \text{pr}_1 & & \\
& & R & \xrightarrow{q} & A & &
\end{array}$$

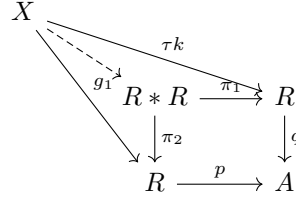
The required dashed arrow is the dashed arrow in:

$$\begin{array}{ccccc}
X & & & & \\
& \searrow^{g_1} & & & \\
& & R * R & \xrightarrow{\pi_1} & R \\
& \searrow^{k} & \downarrow \pi_2 & & \downarrow q \\
& & R & \xrightarrow{p} & A
\end{array}$$

For the second one, for k and (g_1, g_2) satisfies $(p, q)k = (p \times 1)(g_1, g_2)$, augment the diagram into:

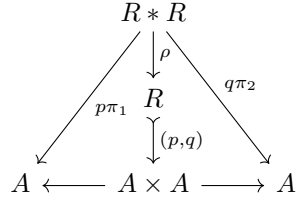
$$\begin{array}{ccccccc}
X & & & & & & \\
& \searrow^{k} & & & & & \\
& & R * R & \xrightarrow{\tau \pi_1} & R & \xrightarrow{\tau} & R \\
& \searrow^{(g_1, g_2)} & \downarrow (\pi_2, p \pi_1) & & \downarrow (p, q) & & \downarrow q \\
& & R \times A & \xrightarrow{p \times 1} & A \times A & & \\
& & \downarrow p_1 & & \searrow \text{pr}_2 & & \\
& & R & \xrightarrow{p} & A & &
\end{array}$$

The dashed arrow required is the dashed arrow in:



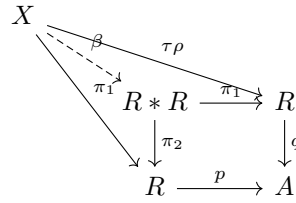
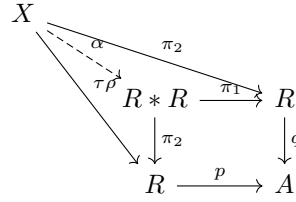
Hence the above two squares are indeed pullbacks. The next claim is that $(\pi_1, p\pi_1) \cong (\pi_1, q\pi_2)$.

As R is transitive, we have arrow $\rho : R * R \rightarrow R$ such that:

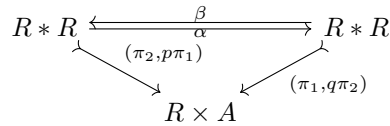


commutes, that is, $p\rho = p\pi_1$, $q\rho = q\pi_2$.

Consider arrows α, β given by the below diagrams respectively:



We will prove $\beta\alpha = \alpha\beta = 1$. We spell out the proof of $\beta\alpha =$, then $\alpha\beta = 1$ is symmetric. Consider the diagram:



The diagram commutes, since $q\pi_2\alpha \stackrel{\text{left hand side of definition of } \alpha}{=} q\tau\rho = p\rho = p\pi_1$, $\pi_1\alpha = \pi_2$ by upper part of definition of α . And $p\pi_1\beta \stackrel{\text{upper part of definition of } \beta}{=} p\tau\rho = q\rho = q\pi_2$.

We have $\pi_1\beta = \tau\rho$, hence $\pi_1\beta\alpha = \tau\rho\alpha = \tau\rho(\pi_2, \tau\rho)$. And hence

$$p(\pi_1\beta)\alpha = p(\tau\rho(\pi_2, \tau\rho)) = q\rho(\pi_2, \tau\rho) = q\pi_2(\pi_2, \tau\rho) = q\tau\rho = p\rho = p\pi_1.$$

$$q(\pi_1\beta)\alpha = q(\tau\rho(\pi_2, \tau\rho)) = p\rho(\pi_2, \tau\rho) = p\pi_1(\pi_2, \tau\rho) = p\pi_2 = q\pi_1.$$

As (p, q) is mono, by the diagram:

$$\begin{array}{ccccc} & & & A & \\ & & & \uparrow & \\ & & & A \times A & \\ & & & \downarrow & \\ & & & A & \\ R * R & \xrightarrow{\pi_1\beta\alpha} & R & \xrightarrow{p} & A \\ & & & \downarrow q & \\ & & & A & \end{array}$$

implies $\pi_1 = \pi_1(\beta\alpha)$. We have $\pi_2 = \pi_2(\beta\alpha)$ since $\pi_2\beta = \pi_1$ by definition of β and $\pi_1\alpha = \pi_2$ by definition of α . Hence $\beta\alpha = (\pi_1, \pi_2) = 1$.

(d) Let $\hat{r} : A \rightarrow \Omega^A$ be the transpose of the characteristic map $r : A \times A \rightarrow \Omega$ of $R \rightarrow A \times A$. Deduce from c that $\hat{r}p = \hat{r}q : R \rightarrow \Omega$.

Proof. Putting pullbacks horizontally side by side gives another pullback, so we can put the pullbacks together like:

$$\begin{array}{ccccc} R * R & \xrightarrow{\pi_2} & R & \xrightarrow{!} & 1 \\ \downarrow (\pi_1, q\pi_2) & & \downarrow (p, q) & & \downarrow \text{true} \\ R \times A & \xrightarrow{q \times 1} & A \times A & \xrightarrow{r} & \Omega \end{array}$$

Hence the characteristic map of $(\pi_1, q\pi_2) : R * R \rightarrow R \times A$ is given by $r \circ (q \times 1)$. Similarly, the characteristic map of $(\pi_2, p\pi_1) : R * R \rightarrow R \times A$ is $r \circ (p \times 1)$. Since the two subobjects are isomorphic, they have the same characteristic map, so $r \circ (q \times 1) = r \circ (p \times 1)$.

We have $\hat{r}p$ is the transpose of $r \circ (p \times 1)$, as shown in:

$$\begin{array}{ccc} A \times \Omega^A & \xrightarrow{\quad} & \Omega \\ 1 \times \hat{r} \uparrow & \nearrow r & \\ A \times A & \xrightarrow{r \circ (p \times 1)} & \Omega \\ 1 \times p \uparrow & \nearrow & \\ A \times R & & \end{array}$$

Similarly, $\hat{r}q$ is the transpose of $r \circ (q \times 1)$. As the maps are same, we conclude they have the same transpose, hence $\hat{r}p = \hat{r}q$. \square

(e) Prove that the square

$$\begin{array}{ccc} R & \xrightarrow{p} & A \\ \downarrow q & & \downarrow \hat{r} \\ A & \xrightarrow{\hat{r}} & \Omega^A \end{array}$$

which commutes by (d), is a pullback (use the symmetry of R).

Proof. Consider maps $t_1, t_2 : X \rightarrow A$ such that $\hat{r}t_1 = \hat{r}t_2$, we prove there is a unique map $t : X \rightarrow R$ such that $qt = t_1, pt = t_2$. As $(p, q) : R \rightarrow A \times A$ is monic, if such a map exists, then it must be unique. Hence it suffices to prove existence of such t .

Consider the pullbacks along $1 \times t_1, 1 \times t_2$, as shown in:

$$\begin{array}{ccccc} A \times X & \xrightarrow{1 \times t_1} & A \times A & \xleftarrow{1 \times t_2} & A \times X \\ \uparrow a_{t_1} & & \uparrow (p, q) & & \uparrow a_{t_2} \\ P_{t_1} & \xrightarrow{b_{t_1}} & R & \xleftarrow{b_{t_2}} & P_{t_2} \end{array}$$

Putting pullbacks together:

$$\begin{array}{ccccc} P_{t_1} & \xrightarrow{b_{t_1}} & R & \xrightarrow{!} & 1 \\ \downarrow a_{t_1} & & \downarrow (p, q) & & \downarrow \text{true} \\ A \times X & \xrightarrow{1 \times t_1} & A \times A & \xrightarrow{r} & \Omega \end{array}$$

We obtain a pullback which allows us to conclude that the characteristic map of P_{t_1} is $r \circ (1 \times t_1)$. As $\hat{r}t_1 = \hat{r}t_2$ are equal by assumption, so does their transposes. That is, $r \circ (1 \times t_1) = r \circ (1 \times t_2)$.

Hence $P_{t_1} \cong P_{t_2}$ as subobjects of $A \times X$, and there exists an isomorphism $\theta : P_{t_1} \rightarrow P_{t_2}$, such that $a_{t_2}\theta = a_{t_1}$.

Consider the map $(t_1, 1) : X \xrightarrow{(t_1, 1)} A \times A$, as $(t_1, t_1) : X \rightarrow A$ factor through the diagonal and R is reflexive, then (t_1, t_1) factor through $(p, q) : R \rightarrow A \times A$.

Hence by the pullback diagram of P_{t_1} , consider:

$$\begin{array}{ccc} X & \xrightarrow{(t_1, t_1)} & R \\ \downarrow (t_1, 1) & \searrow b_{t_1} & \downarrow (p, q) \\ P_{t_1} & \xrightarrow{b_{t_1}} & R \\ \downarrow a_{t_1} & & \downarrow (p, q) \\ A \times X & \xrightarrow{1 \times t_1} & A \times A \end{array}$$

Hence we have $(t_1, t_2) = (1 \times t_2)(t_1, 1) = (1 \times t_2)(a_{t_1} \circ k) = (1 \times t_2)(a_{t_2}\theta \circ k) = (p, q)b_{t_2}\theta k$.

The map $b_{t_1}\theta k$ is what we want. □

(f) Wrap up by the statement that every equivalence relation is the kernel pair of its coequalizer. □

Proof. By (e), every equivalence relation is the kernel pair of some map. The result follows from (b). □

□