Assignment 2

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Question 1

Let X be a topological space. For a sieve S on an open subset U of X define S covers U iff U is the union of the sets in S. Prove that this defines a Grothendieck topology on the partially ordered set $\mathcal{O}(X)$ of all open subsets of X.

Proof. We check the axioms in Definition 1 on page 110.

Observe that if $U \in \mathcal{O}(X)$, $h : V \subseteq U$ and S is a sieve on U, then $h^*(S) = \{W \mid W \subseteq V \subseteq U, W \in S\}$.

(i) The maximal sieve $t_C = \{f \mid cod(f) = C\}$ is in $J(C)$.
This is for all $U \in \mathcal{O}(X)$, $\bigcup\{V \mid V \subseteq U\} = U$. This is true because $U \subseteq U$.

(ii) (stability axiom) if $S \in J(C)$, then $h^*(S) \in J(D)$ for any arrow $h : D \to C$.
This is for all $U \in \mathcal{O}(X)$ and sieve S on U, if $\bigcup S = U$, then for any $V \subseteq U$, we need to show $\bigcup\{W \mid W \subseteq V \subseteq U, W \in S\} = V$.

We have $V = U \cap V$

$= (\bigcup S) \cap V$

$= \bigcup\{W \cap V \mid W \in S\}$

The last set is a subset of $\bigcup\{W \mid W \subseteq V \subseteq U, W \in S\}$, since S is a sieve and hence $W \in S$ implies $W \cap V \in S$. So $V \subseteq \bigcup\{W \mid W \subseteq V \subseteq U, W \in S\}$, clearly the inclusion for the other direction holds.

(iii) (transitivity axiom) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$, then $R \in J(C)$.

This is for all $U \in \mathcal{O}(X)$ and sieve S on U such that $\bigcup S = U$, and R is any sieve on U such that for any $V \in S$, $\bigcup\{W \mid W \subseteq V \subseteq U, W \in R\} = V$, then $\bigcup R = U$.

Obviously $\bigcup R \subseteq U$, we show $U \subseteq \bigcup R$.

$U = \bigcup S$

$= \bigcup\{V \mid V \in S\}$

$= \bigcup\{\bigcup\{W \mid W \subseteq V \subseteq U, W \in R\} \mid V \in S\}$

$\subseteq \bigcup R$

as desired. $\square$

Question 2

Let $\mathbf{T}$ be in §2, Example (b), with the open cover topology given by the basis K as defined there. Define K' by $\{f_i : Y_i \to X \mid i \in I\} \in K'(X)$ iff each f_i is etale, and moreover $X = \bigcup_i f_i(Y_i)$. Show that K and K' generates the same topology J on $\mathbf{T}$.

Proof. By definition on page 112, if K is a basis on $\mathbf{T}$, then K generated a topology J by $S \in J(C) \Leftrightarrow \exists R \in K(C), R \subseteq S$. Then our task is to show that for a space $X \in \mathbf{T}$ and a sieve S on X, then S contains a set $\{f_i : Y_i \to X \mid i \in I\}$ where each f_i is etale, and moreover $X = \bigcup_i f_i(Y_i)$ iff S contains a set $\{g_m : Y_m \to X \mid m \in M\}$ where $\{Y_m\}$ is an open cover of X and the $\{g_m\}_{m\in M}$ is the corresponding embedding.

If S contains a set $\{g_m : Y_m \to X \mid m \in M\}$ where $\{Y_m\}$ is an open cover of X and the $\{g_m\}_{m\in M}$ is the corresponding embedding, then as an inclusion of open set is an etale map, we also have $\{g_m : Y_m \to X \mid m \in M\} \in K'(X)$.

Conversely, if S contains a set $\{f_i : Y_i \to X \mid i \in I\}$ where each f_i is etale, and moreover $X = \bigcup_i f_i(Y_i)$, then for each Y_i, it is covered by open subsets $\{U_{i_m}\}$, each mapped homeomorphically to X, with its image denoted as $U_{i_m} \cong V_{i_m} \subseteq X$. As S is a sieve, all the maps $V_{i_m} \to U_{i_m} \hookrightarrow Y_i \to X$ are in S, and as the image of $\{Y_i\}_{i\in I}$ covers X, the open sets V_{i_m} indexed over i and m covers X as well. Hence S contains $\{V_{i_m} \to X\}_{i,m}$, which is a family of open sets that covers X.

□

Question 3

Let X be a topological space, and let G be a (discrete) group acting on X by a continuous map $G \times X \to X$, $(g, x) \mapsto g \cdot x$. An etale G-space over X is an etale map $p : E \to X$ (as in Chapter II, §6), where E is equipped with an action $G \times E \to E$ by G such that p is compactible with the two actions on E and on X.
(a) Use the correspondence between etale spaces and sheaves of §II.6 to show that the category of etale G-spaces is a Grothendieck topos, by explicitly describing a site.
(b) Prove that if the action og G on X is proper, then the category of etale G-space is equivalent to the category $\mathrm{Sh}(X/G)$ of sheaves on the orbit space X/G, where X/G is equipped with the quotient topology. (Recall that an action by G on X is called proper if for each point $x \in X$ there is a neighborhood U_x of x with the property that for any $g \in G$, if $g \cdot U_x \cap U_x \neq \emptyset$ then $g = e$.)

Proof. (a) Claim : $\mathrm{Etale}_G(X) \cong \mathrm{Sh}(\mathbf{C}, J)$ for $\mathbf{C}$ the category defined by :

- Object : $\mathcal{O}(X)$.
- Morphism : $\mathrm{Hom}_{\mathbf{C}}(U_1, U_2) = \mathrm{Hom}_{\mathrm{Etale}(X)}(U_1, G \times U_2)$. That is, maps q from U_1 to $G \times U_2$ such that

$$\begin{tikzcd} U_1 \arrow[rr, "q"] \arrow[dr, hook, "i_1"'] & & G \times U_2 \arrow[dl, "\alpha_X|_{G\times U_2}"] \\ & X & \end{tikzcd}$$

  commutes.

Note the composition of morphism is defined by: for $f : U_1 \to U_2, g : U_2 \to U_3$, $g \circ f : U_1 \to U_3$ is defined by

$$\begin{tikzcd} G \times U_2 \arrow[r, "\tilde{g}"] & G \times U_3 \\ U_1 \arrow[u, "f"] \arrow[ur, "g \circ f"] & \end{tikzcd}$$

Here $\tilde{g}(\sigma, u) := \sigma \cdot g(u)$, where the action of $\sigma \in G$ is given by $\sigma \cdot (\sigma_0, u_0) = (\sigma \cdot \sigma_0, u_0)$.

And the topology J on $\mathbf{C}$ is given by: a family of arrows $\{f_i : U_i \to U \mid i \in I\}$ covers U iff $\bigcup_{i \in I} U_i = U$. In other words, $\{f_i : U_i \to U \mid i \in I\} \in J(U)$ iff $\bigcup_{i \in I} U_i = U$.

Observe that for any open set of U and any sieve S on U, if there exists some arrow $f : U_0 \to G \times U$ in S, then the arrow $e_{U_0} : U_0 \to G \times U$ defined by $u_0 \mapsto (e, u_0)$ is also in S. Since it is the composition $f \circ g_{U_0}$ where $g_{U_0} : U_0 \to G \times U_0$ is defined by $u_0 \mapsto (f.1(u_0)^{-1}, u_0)$.

Also observe that for any matching family $(s_f)_{f \in S}$ for any covering sieve S on $U \in \mathcal{O}(X)$, the whole family are completely determined by the values s_{f_i} where $f_i : V \to G \times U$ is such that $f(v) = (e, v)$ for all $v \in V$.

Also observe that any sheaf F on $(\mathbf{C}, J)$ is a sheaf on X. For consider open $U \subseteq X$ and an open cover $\{U_i\}_{i \in I}$.

$$F(U) \xrightarrow{e} \Pi_{i \in I} F(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \Pi_{i,j \in I} F(U_i \cap U_j)$$

Pick $\alpha \in \Pi_{i \in I} F(U_i)$ such that $p(\alpha) = q(\alpha)$, we need to find a unique element in $F(U)$ which is mapped to it.

Consider the sieve S generated by the family $\{f_i : U_i \to G \times U\}$ defined by for each $i \in I$, $f_i(u) = (e, u)$, which is the minimal sieve contains this family, then this sieve is a covering sieve by the topology on $\mathbf{C}$. For each arrow $f \in S$, we have $f : V \xrightarrow{f_V} U_i \xrightarrow{f_i} U$ is a composition, let $s_f \in F(V)$ be $(\alpha\ i) \cdot f_V$, then $(s_f)_{f \in S}$ is a matching family by definition, hence there exists an element in $F(U)$ which is mapped to $(s_f)_{f \in S}$, and this is the element we want.

Proof: direction 1: Given an etale G-space $p : E \to X$, our claim is that it corresponds to the sheaf on $(\mathbf{C}, J)$ defined by sending $U \in \mathcal{O}(X)$ to the cross section over U, that it, the elements of $\Gamma_p(U)$ are maps $s : U \to E$ such that:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, hook] \arrow[d, "p|_{p^{-1}(U)}"] & E \arrow[d, "p"] \\ U \arrow[r, hook] \arrow[ur, "s"] & X \end{tikzcd}$$

commutes.

And for morphisms. Given a morphism from U_1 to U_2, that is, given $U_1 \subseteq U_2$ and a map $q : U_1 \to G \times U_2$ of etale spaces over X, we need a map $\Gamma_p(q) : \Gamma_p(U_2) \to \Gamma_p(U_1)$. $\Gamma_p(q)$ is given by sending $s \in \Gamma_p(U_2)$ to the map $s' : U_1 \to E$ defined by $u \in U_1 \mapsto q.1(u) \cdot s(u)$. This is indeed a cross section since $p(q.1(u) \cdot s(u)) = q.1(u) \cdot (p(s(u))) = q.1(u) \cdot u = u$ since q is a map of etale spaces over X.

To justify that it is indeed a sheaf on the site $(\mathbf{C}, J)$:

Consider an object $U \in \mathcal{O}(X)$ in $\mathbf{C}$ and a covering sieve S of U, and family $(s_f)_{f\in S}$ where $s_f \in \Gamma_p(\operatorname{dom} f)$ for each $f \in S$, we prove the diagram:

$$\Gamma_p(U) \xrightarrow{e} \Pi_{f\in S}\Gamma_p(\operatorname{dom} f) \overset{c}{\underset{a}{\rightrightarrows}} \Pi_{f,g,f\in S,\operatorname{dom} f=\operatorname{cod} g}\Gamma_p(\operatorname{dom} g)$$

is an equalizer of sets.

For a family $\alpha = \{s_f\}_{f\in S} \in \Pi_{f\in S}(\operatorname{dom} f)$ which agrees on c and a, we need a unique element in $\Gamma_p(U)$ which is mapped to it by e. Our claim is that such an element the cross section $s : U \to E$ defined by $u \mapsto \alpha\,\{\mathsf{CHOICE}\ \ f \mid f \in S \wedge f.1(u) = e \wedge u \in \operatorname{dom} f\}\ u$.

This definition is independence of choice. We will prove if $f \in S, f' \in S,\ f.1(u) = f'.1(u) = e$ and $u \in \operatorname{dom}(f), u \in \operatorname{dom} f'$, then $\alpha\ f\ u = \alpha\ f'\ u$. We have $\alpha\ f = s_f, \alpha\ f' = s_{f'}$. Suppose $\operatorname{dom} f = U_1, \operatorname{dom} f' = U_2$. The arrow $f_1 : U_1 \cap U_2 \to U_1$ defined by $u \mapsto (e,u) \in G \times U_1$ and the arrow $f_2 : U_1 \cap U_2 \to U_2$ defined by $u \mapsto (e,u) \in G \times U_2$. Then $f \circ f_1 = f' \circ f_2$. Hence $s_f\ u = s_f \cdot f_1\ u$ (by definition of behaviour of our functor on morphisms) $= s_{f\circ f_1}\ u = s_{f'\circ f_2}\ u = s_{f'} \cdot f_2\ u = s_{f'}\ u$, as desired.

direction 2: Given a sheaf P on the site $(\mathbf{C}, J)$, we find a corresponding etale G-space. Our space is the space of all the germs of P. That is, the underlying set is $\coprod_x P_x = \{\text{all } \operatorname{germ}_x s \mid x \in X, s \in PU\}$, the topology is taken as in SGL page 85. The action is given by $g \cdot \operatorname{germ}_x s := \operatorname{germ}_{g\cdot x} g \cdot s$, where $g \cdot s \in P(g \cdot U)$ is the function given by $g \cdot u \mapsto g \cdot s(u)$, then $g \cdot s$ is indeed a cross section because s is. To show the projection $\operatorname{germ}_x s \mapsto x$ is indeed a map of G-spaces, consider the diagram:

$$\begin{tikzcd} G \times \coprod_x P_x \arrow[r] \arrow[d] & \coprod_x P_x \arrow[d] \\ G \times X \arrow[r] & X \end{tikzcd}$$

We have $(g, \operatorname{germ}_x s) \mapsto \operatorname{germ}_{g\cdot x}(g \cdot s) \mapsto g \cdot x$ and $(g, \operatorname{germ}_x s) \mapsto (g, x) \mapsto g \cdot x$. Hence the diagram commutes.

Also the projection map is etale by page 85 of SGL.

To show they are indeed inverses of each other.

direction 1: etale G-space $p : E \to X \mapsto$ sheaf on $(\mathbf{C}, J)$ given by cross-section $\Gamma_p \mapsto$ get back the etale G-space.

For an etale G-space $p : E \to X$, the sheaf we construct on the site $(\mathbf{C}, J)$ is in particular, also a sheaf on X, hence by the correspondence in chapter II, taking the germs get the etale space $E \to X$ back.

direction 2: sheaf on $(\mathbf{C}, J) \mapsto G$-space $\mapsto$ get back the sheaf on $(\mathbf{C}, J)$.

As a sheaf on $(\mathbf{C}, J)$ is also a sheaf on X, by the correspondence in chapter II, taking the germ and then the cross section get back the sheaf on X, but then as we get back a sheaf which is isomorphic to the original sheaf as a functor, the sheaf condition on the site is automatic.

(b) By the equivalence between etale G space and sheaves, it suffices to show the category of etale G-space is then equivalent to the category of etale bundle on X/G.

$\text{Etale}_G(X) \to \text{Etale}(X/G)$: Given an etale G space $p : E \to X$, we define an etale G-space over X/G. This is the space $p/G : E/G \to X/G$. By universal mapping property of E/G, a map $E/G \to X/G$ is a map $f : E \to X/G$ such that for all $g \in G, e \in E$, $f(g \cdot e) = f(e)$. Such an f is given by the composition of $\pi : X \to X/G$ and p. We have $\pi \circ p(g \cdot e) = \pi(g \cdot p(e)) = \pi(p(e))$.

Claim: If $p : E \to X$ is a map of G-spaces E and X, then if the action on X is proper action, then the action of G on E is also proper.

Proof: Let $e \in E$, then $p(e) \in X$, as the action on X is proper, there exists an open neighbourhood $U_{p(e)}$ such that $p(e) \in U_{p(e)}$ and for all $g \in G$, $g \cdot U_{p(e)} \cap U_{p(e)} \neq \emptyset \implies g = e$. We will prove $p^{-1}(U_{p(e)})$ is the neighborhood of e with the desired property. Let $g \in G$ and $g \cdot (p^{-1}(U_{p(e)})) \cap p^{-1}(U_{p(e)}) \neq \emptyset$, then there exists $a \in E$ such that $a \in g \cdot p^{-1}(U_{p(e)})$ and also $a \in p^{-1}(U_{p(e)})$. This means $g^{-1} \cdot a \in p^{-1}(U_{p(e)})$ and also $p(a) \in U_{p(e)}$, where the first conjunct implies $p(g^{-1} \cdot a) = g^{-1}(p(a)) \in U_{p(e)}$. Hence $p(a) \in g \cdot U_{p(e)} \cap U_{p(e)}$, which implies $g = e$ by the way we pick $U_{p(e)}$.

Now we prove $p_{/G} : E/G \to X/G$ is etale.

Let $e \in E/G$, pick $\tilde{e} \in E$ such that $\pi_E(\tilde{e}) = e$. By etaleness of p, exists neighbourhood $U_{\tilde{e}}$ of $\tilde{e}$ such that $U_{\tilde{e}} \cong p(U_{\tilde{e}})$. As the action on X is proper, $p(\tilde{e})$ has neighbourbood $V_{p(\tilde{e})} \subseteq X$ such that for all $g \in G$, $g \cdot V_{p(\tilde{e})} \cap V_{p(\tilde{e})} \neq \emptyset \implies g = e$. The neighbourhood of e we need is $(U_{\tilde{e}} \cap p^{-1}(V_{p(\tilde{e})}))/G$. We need $p_{/G}|_{(U_{\tilde{e}} \cap p^{-1}(V_{p(\tilde{e})}))/G}$ is a homeomorphism. It is open by induces topology on quotient, use the fact that $p_{U_{\tilde{e}}}$ is an open map, and also it is surjective on its image, it lefts to show that it is injective.

Given $[x_1], [x_2] \in (U_{\tilde{e}} \cap p^{-1}(V_{p(\tilde{e})}))/G$, pick lifts $x_1, x_2 \in U_{\tilde{e}}$ such that $p(x_1), p(x_2) \in V_{p(\tilde{e})}$. We have $p_{/G}([x_1]) = p_{/G}([x_2])$ implies $p(x_1) = g \cdot p(x_2)$, then $g \cdot V_{p(\tilde{e})} \cap V_{p(\tilde{e})} \neq \emptyset$, thus $g = e$, so $p(x_1) = p(x_2)$. As p is local homeomorphism in $U_{\tilde{e}}$, this implies $x_1 = x_2$, as desired.

$\text{Etale}(X/G) \to \text{Etale}_G(S)$: Given an etale map $p_0 : E_0 \to X/G$, we need to define an etale G-space over X. Consider the pullback p of p_0 along the quotient map $\pi : X \to X/G$:

$$\begin{tikzcd}
E = \{(x, e_0) \mid x \in X, e_0 \in E_0, \pi(x) = p_0(e_0)\} \arrow[r] \arrow[d, "p"] & E_0 \arrow[d, "p_0"] \\
X \arrow[r, "\pi"] & X/G
\end{tikzcd}$$

The $p : E \to X$ is the etale map we want. It is indeed etale since pullback of etale map is etale.

To equip $p : E \to X$ with an action, define $G \times E \to E$ by for $x \in X, e_0 \in E_0$, $g \cdot (x, e_0) = (g \cdot x, e_0)$. Hence consider the maps:

$$\begin{tikzcd}
G \times E \arrow[r, "\alpha_E"] \arrow[d, "id \times p"] & E \arrow[d, "p"] \\
G \times X \arrow[r, "\alpha_X"] & X
\end{tikzcd}$$

We have $p \circ \alpha_E(g, (x, e_0)) = p(g \cdot (x, e_0)) = p(g \cdot x, e_0) = g \cdot x$ and also $\alpha_X \circ id \times p(g, (x, e_0)) = \alpha_X(g, x) = g \cdot x$. Hence the diagram commutes.

To see they are indeed inverses:

direction 1: Under assumptions as given in the question:

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"] & E/G \arrow[d, "p_{/G}"] \\
X \arrow[r] & X/G
\end{tikzcd}$$

is a pullback square.

We claim that the map $m : e \mapsto (p(e), \pi_E(e))$ from E to the pullback $P = \{(x, [e]) \mid p(x) = p_{/G}([e])\}$ is an isomorphism of etale G-spaces over X.

It suffices to prove that there is a cover $\{U_i\}_{i\in I}$ of X such that for each U_i, let E_i be the pullback:

$$\begin{tikzcd}
E_i = p^{-1}(U_i) \arrow[r, hook] \arrow[d, "p|_{E_i}"] & E \arrow[d, "p"] \\
U_i \arrow[r, hook] & X
\end{tikzcd}$$

And P_i be the pullback:

$$\begin{tikzcd}
P_i = \mathsf{pr}_1^{-1}(U_i) \arrow[r, hook] \arrow[d, "p|_{E_i}"] & P \arrow[d, "\mathsf{pr}_1"] \\
U_i \arrow[r, hook] & X
\end{tikzcd}$$

then $m|_{E_i}$ is an isomorphism from E_i to P_i for each $i \in I$.

As the action on X is proper, X is covered by neighbourhood U_x of $x \in X$, each satisfies $\forall g \in G, g \cdot U_x \cap U_x \neq \emptyset \implies g = e$. Let the cover be $\{U_x\}_{x\in X}$. For each U_x,we need to prove $p^{-1}(U_x)$ is homeomorphic to $\{(a, [e]) \mid a \in U_x \wedge \pi_X(a) = p_{/G}([e])\}$ via the map $e \mapsto (p(e), \pi_E(e))$. Clearly the map has the correct codomain. By the claim proved earlier, π_X is an isomorphism on U_x implies π_E is an isomorphism on $p^{-1}(U_x)$. For $e \in E$ satisfies $p_{/G}([e]) = \pi_X(a)$ for $a \in U_x$, we have $[p(e)] = p_{/G}([e]) = \pi_X(a)$. As π_X is an isomorphism on U_x, we conclude $p(e) = a$, which says $e \in p^{-1}(U_x)$. Hence for all element $(a, [e]) \in \{(a, [e]) \mid a \in U_x \wedge \pi_X(a) = p_{/G}([e])\}$, $e \mapsto [e]$ is identity. Hence we have a continous inverse defined by forgeting the first coordinate and remove the bracket on the second coordinate.

direction 2: If $p_0 : E_0 \to X/G$ is etale, then pulling back and then quotient get back the map p_0.

Pullback of p_0 is the set $\{(x, e_0) \mid x \in X, e_0 \in E_0, [x] = p_0(e_0)\}$, and it is an etale bundle over X with projection map. The action is given by $g \cdot (x, e_0) := (g \cdot x, e_0)$. Then identifying elements in the same orbit gives the set $\{(x, e_0) \mid x \in X, e_0 \in E_0, [x] = p_0(e_0)\}/(x, e_0) \sim (g \cdot x, e_0)$, it is just the set of elements of form $(p_0(e_0), e_0)$, which is the same as E_0 itself.

□

Question 4

Let X be a topological space. Recall for a set S, $\Delta(S) \in \mathrm{Sh}(X)$ is the associated sheaf of the constant presheaf $\mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$ with value S. [cf. (4) of §6]
(a) Show that $\Delta(S)$ is the sheaf of continuous S-valued functions on X, where S is given the discrete topology.
(b) Show that if X is locally connected, then Δ has left adjoint $\pi_0 : \mathrm{Sh}(X) \to \mathbf{Sets}$. [Hint: What is $\Delta(S)$ as an etale space over X]
(c) Show that if X is locally conected, then the functor $\Delta : \mathbf{Sets} \to \mathrm{Sh}(X)$ commutes with exponentials (meaning that for any two sets S and T, the canonical morphism $\Delta(T^S) \to \Delta(T)^{\Delta(S)}$ of sheaves on X is an isomophism).

Proof. (a) Using the definition in the textbook 'Grothendieck Topologies', for a presheaf $P : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$ and $V \in \mathcal{O}(X)$, the plus construction is given by $P^+(V) := \mathrm{colim}_{\mathcal{U} \text{ covering } V}\, \check{H}^0(\mathcal{U}, P)$, where $\mathcal{U} = (U_i)_{i \in I}$ and $\check{H}^0(\mathcal{U}, P) = \mathrm{eq}[\Pi_{i \in I} P(U_i) \rightrightarrows \Pi_{j,k \in I} P(U_j \cap U_k)]$. Start with the presheaf $\mathcal{O}(\mathcal{X}) \to \mathbf{Sets}$ defined by $U \mapsto S$, and sending all the arrows to identity on S, we claim $S^+(U) = S$ for $U \neq \emptyset$, and $S^+(\emptyset) = \{*\}$. It suffices to check the universal propserty of equalizer.

Firsty, consider an open set $U \subseteq X$ and $U \neq \emptyset$. Consider the diagram below, here $(U_i)_{i \in I}$ is the limit of the covering families of U.

$$\begin{tikzcd}
S^+(U) = S \arrow[r, "e"] & \Pi_{i \in I} S(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \Pi_{i,j \in I} S(U_i \cap U_j) \\
A \arrow[u, dashed, "\exists ! t"] \arrow[ur, "f"] & &
\end{tikzcd}$$

The map $e : S \to \Pi_{i \in I} S(U_i)$. is given by for $s \in S$, sending it to $\alpha \in \Pi_{i \in I} S(U_i)$ such that for all $i \in I$, $\alpha\ i = s$. The maps p is given by sending $\alpha \in \Pi_{i \in I} S(U_i)$ to $\alpha_p \in \Pi_{i,j \in I} S(U_i \cap U_j)$ defined by for any $i, j \in I$, $\alpha_p\ i\ j := (\alpha\ i)|_{U_i \cap U_j} = \alpha\ i$ (since restriction is identity). And the map q is given by sending $\alpha \in \Pi_{i \in I} S(U_i)$ to $\alpha_p \in \Pi_{i,j \in I} S(U_i \cap U_j)$ defined by for any $i, j \in I$, $\alpha_p\ i\ j := (\alpha\ j)|_{U_i \cap U_j} = \alpha\ j$.
Given a map $f : A \to \Pi_{i \in I} S(U_i)$ such that $p \circ f = q \circ f$, we claim we have a unique map $t : A \to S$ such that $f = e \circ t$, defined by for each $a \in A$, sending a to $f\ a\ (\mathsf{CHOICE}\ I)$. The choice here makes sense since U is nonempty, so any cover cannot be indexed by an empty set.
Claim 1: The definition of t is independent of choice.

Proof. This is, for any $i \in I, j \in I$, we need $f\ a\ i = f\ a\ j$. This is directly implied by the fact that $p \circ f = q \circ f$ and the definitions of p and q. □

Claim 2: $f = e \circ t$.

Proof. By function extensionality, we need to show for all $a \in A, i \in I$, we have $f\ a\ i = (e \circ t)\ a\ i$. We have $(e \circ t)\ a\ i = (e(f\ a\ (\mathsf{CHOICE}\ I)))\ i = f\ a\ (\mathsf{CHOICE}\ i)$ by definition of t and e, which is the same as $f\ a\ i$ by independence of choice which is just proved above. □

Claim 3: Such a map t is unique.

For any map $t : A \to S$ which satisfies $f = e \circ t$, it means that for all $a \in A, i \in I$, we have $f\ a\ i = (e \circ t)(a)\ i$. By definition of e, it means for all $a \in A, i \in I$, we have $t(a) = f\ a\ i$. Hence such t is uniquely determined.

For $U = \emptyset$, since the limit of the covering family is indexed over empty set, the case is trivial.

By conclusion S^+ is given by $S^+(U) = S$ for $U \neq \emptyset$, and $S^+(\emptyset) = \{*\}$.

Having known that $S^+(\emptyset) = \{*\}$ and $S^+(U) = S$ for $U \neq \emptyset$, we will show $S^{++}(\emptyset) = \{*\}$ and $S^{++}(U) = \mathrm{colim}_{\mathcal{U} \text{ covering } U} \check{H}^0(\mathcal{U}, S^+)$, where for each cover $\mathcal{U}$ of U, $\check{H}^0(\mathcal{U}, S^+)$ is the equalizer $\mathrm{eq}[\Pi_{i\in I} S^+(U_i) \rightrightarrows \Pi_{j,k\in I} S^+(U_j \cap U_k)]$. We claim that for each cover, this set is the set of functions $U \to S$ which is constant over $\mathcal{U}$, more precisely, it is the set $\{g : U \to S \mid \forall i \in I, \forall x, y \in U_i, g(x) = g(y)\}$. To see this, consider the diagram:

$$\begin{tikzcd} \{g : U \to S \mid g \text{ is constant over } U_i\} \arrow[r, "e"] & \Pi_{i\in I} S^+(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \Pi_{i,j\in I} S^+(U_i \cap U_j) \\ A \arrow[u, dashed, "\exists! t"] \arrow[ur, "f"] & & \end{tikzcd}$$

For any $f : A \to \Pi_{i\in I} S^+(U_i)$ such that $p \circ f = q \circ f$, the unique t will be given by sending $a \in A$ to the function g defined by $g(u) := f\ a\ (\mathsf{CHOICE}\ \ i \in I \mid u \in U_i)$.

The colimit over these $\check{H}^0(\mathcal{U}, S^+)$ is the set $\{f : U \to S \mid \exists \text{ a cover } \mathcal{U} \text{ of } U \text{ such that } f \text{ is constant over } \mathcal{U}\}$, which ia precisely the set of locally connected functions $U \to S$.

(b) We need an adjunction:

$$\pi_0 : \mathrm{Sh}(X) \rightleftarrows \mathbf{Sets} : \Delta$$

For any $S \in \mathbf{Sets}$, $\Delta(S) \in \mathrm{Sh}(X)$ is the etale bundle $\mathsf{pr}_1 : X \times S \to X$ where the map is projecting to the first coordinate. Hence what we want is that for any etale bundle $p : E \to X$, $[\pi_0(p), S \times X]_{\mathbf{Sets}} \cong [p, \mathsf{pr}_1 : X \times S \to X]_{\mathrm{Etale}(X)}$.

By definition of product and definition of map between etale bundles, $[p, \mathsf{pr}_1 : X \times S \to X]_{\mathrm{Etale}(X)}$ is continuous map from E to the discrete set S, which is locally constant functions $E \to S$. A function $E \to S$ is locally constant iff it sends each connected component of E to the same point in S, hence Π_0 is the functor that taking the connected components of a space.

(c) If X is locally connected, $\bigcup_{x\in X} U_x$ where for each $x \in X$, U_x is a connected component of X forms a cover of X. For any such U_x, $\Delta(T^S)(U_x) = \{f : U_x \to (S \to T) \mid f \text{ is locally constant}\}$, As U_x is connected, each such f is just a function $S \to T$ between sets. And $\Delta(T)^{\Delta(S)}(U_x)$ is the set of functions from $\{f : U_x \to S \mid f \text{ is locally constant}\} \cong S$ to $\{f : U_x \to T \mid f \text{ is locally constant}\} \cong T$.

The canonical morphism $\Delta(S^T) \to \Delta(S)^{\Delta(T)}$. is defined by: For $U \in \mathcal{O}(X)$, sending a locally constant map $g : U \to (T \to S)$ to the map $g : (U \to T) \to (U \to S)$ defined by for a locally connected function $t : U \to T$, send it to the locally connected function $U \to S$ defined by $u \mapsto g(u, t(u))$. In the case that U is connected, this map takes a constant function $U \to (T \to S)$ to a function $c : T \to S$ and give a function that sending a constant function $U \to T$ at $t_0 \in T$ to a constant function $U \to S$ at $c(t_0) \in S$.

Hence for each of U_x, both $\Delta(T^S)(U_x)$ and $\Delta(T)^{\Delta(S)}(U_x)$ are equal to the set of functions $S \to T$ between sets, and the canonical morphism translates into the identity under this characterisation. Hence $\Delta(T^S)(U_x) \cong \Delta(T)^{\Delta(S)}(U_x)$ for each of U_x.

By Proposition 1 on page 136 in SGL, the functor $\Delta(T)^{\Delta(S)}$ is a sheaf since $\Delta(T)$ is. Hence the behaviour on the covering $\bigcup_{x \in X} U_x$ determines the behaviour of both $\Delta(T^S)$ and $\Delta(T)^{\Delta(S)}$ on the whole $\mathcal{O}(X)$. The result follows.

$\square$