

Assignment 0

Yiming Xu

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Question 1

Show that pullbacks of epis are epi for categories of each of the types (i)-(ix)

Proof. (i) **Sets**, the category of all small sets S, T , and functions $S \rightarrow T$ between them.

Consider the diagram below where X, Y, B are sets, by the last paragraph in page 29, the pullback is $P = \{\langle x, y \rangle \mid fx = gy\}$ and f', g' are projections.

$$\begin{array}{ccc} P = \{\langle x, y \rangle \mid fx = gy\} & \xrightarrow{f'} & Y \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & B \end{array}$$

Epis in **Sets** are set-theoretic surjections. So suppose g is surjective, we prove g' is surjective. This is proving for any $x \in X$, there exists $y \in Y$ such that $fx = gy$, this is because of the surjectiveness of g .

(ii) **Sets** \times **Sets**, the category of all pairs of sets, with morphisms pairs of functions.

An epi in **Sets**² is a pair of epis in **Sets**. Then the result follows from (i).

(iii) **Sets** ^{n} , the category of all n -tuples of sets with morphisms all n -tuples of functions. Here n is a fixed natural number.

By induction using (ii).

(iv) **BG**, or G -**Sets**, the category of all representations of a fixed group G .

An epi $X \rightarrow Y$ in **BG** is just a set-theoretic surjection $X \rightarrow Y$ which respects the G -action. For $X \xrightarrow{f} B \xleftarrow{g} Y$ in **BG**, the pullback is $P = \{\langle x, y \rangle \mid fx = gy\}$ with $P \rightarrow X, P \rightarrow Y$ projection maps. The action by G on P is coordinatewise. So the result follows from (i).

(v) **BM**, or M -**Sets**, the category of all representations $X \times M \rightarrow X$ of a fixed monoid M on a variable set X .

Same as (iv).

(vi) **Sets**², the category whose objects are all functions $\sigma : X \rightarrow X'$ from one set X' , with evident arrows between these objects.

An epi in **Sets** from $\sigma_X : X_1 \rightarrow X_2$ to $\sigma_Y : Y_1 \rightarrow Y_2$ is pair of functions f_1, f_2 such that the diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{\sigma_X} & X_2 \\ \downarrow f_1 & & \downarrow f_2 \\ Y & \xrightarrow{\sigma_Y} & Y' \end{array}$$

commutes where both f_1, f_2 are surjections. And for morphisms from $X_1 \rightarrow X_2, Y_1 \rightarrow Y_2$ to $B_1 \rightarrow B_2$, a pullback square looks like:

$$\begin{array}{ccccc}
& & P_2 & \xrightarrow{f'_2} & Y_2 \\
& \nearrow \sigma_P & \downarrow f'_1 & & \nearrow \sigma_Y \\
P_1 & \xrightarrow{f'_1} & Y_1 & & \\
& \downarrow g'_1 & \downarrow g'_2 & & \downarrow g_2 \\
& \nearrow \sigma_X & X_2 & \xrightarrow{f_2} & B_2 \\
& & \downarrow g_1 & & \nearrow \sigma_B \\
X_1 & \xrightarrow{f_1} & B_1 & &
\end{array}$$

We can check both the forth and back squares are pullback squares in **Sets**, the result follows by (i).

(vii) **Sets**^N, the category whose objects are all sequences X ,

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots$$

of sets X_n and functions $X_n \rightarrow X_{n+1}$, with evident arrows $X \rightarrow Y$.

Similar to (vi), a pullback in **Sets**^N looks like:

$$\begin{array}{ccccccc}
& & & & \dots & & \\
& & & & \nearrow & & \\
& & P_3 & \xrightarrow{\quad} & Y_3 & \nearrow & \dots \\
& & \downarrow & & \downarrow & & \\
& P_2 & \xrightarrow{\quad} & Y_2 & \nearrow & & \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
P_1 & \xrightarrow{\quad} & X_3 & \xrightarrow{\quad} & Y_1 & \xrightarrow{\quad} & B_3 \\
& \downarrow & \nearrow & & \downarrow & \nearrow & \\
& X_2 & \xrightarrow{\quad} & B_2 & \nearrow & & \\
& \downarrow & & \downarrow & & \downarrow & \\
X_1 & \xrightarrow{\quad} & B_1 & & & &
\end{array}$$

Each face consists with P_n, X_n, Y_n, B_n are pullback squares, so the result follows from (i).

(viii) **Sets**^{C^{op}}, where **C** is a fixed small category. Objects are all functors $P : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Sets}$ and arrows $P \rightarrow P'$ are all natural transformations $\theta : P \rightarrow P'$ between such functors.

An epi in **Sets**^{C^{op}} from P to P' is a natural transformation θ such that for any object C of **C**, $\theta_C : P(C) \rightarrow P'(C)$ is an epi. By page 30, the pullback of $X \rightarrow B \leftarrow Y$ for $X, Y, B : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Sets}$, the pullback $P : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Sets}$ is $(X \times_B Y)(C) \cong X(C) \times_{B(C)} Y(C)$. The result follows from (i).

(ix) **Sets**/ J , the comma category, with objects all sets over fixed set J .

By page 29, the comma category **Sets**/ J is equivalent to the functor category **Sets**^J, so the result follows from (viii). \square

Question 2

Prove $\mathbf{FinSets}^{\mathbf{N}}$ has no subobject classifier.

Proof. Suppose, in order to get a contradiction, that there exists $\Omega : \mathbf{N} \rightarrow \mathbf{FinSets} \subseteq \mathbf{Sets}$ such that Ω is a subobject classifier of $\mathbf{FinSets}^{\mathbf{N}}$. In particular, Ω must classify the subobjects of each representable functor $\mathrm{Hom}_{\mathbf{N}^{\mathrm{op}}}(-, n) : \mathbf{N} \rightarrow \mathbf{FinSets}$ for any object n of \mathbf{N} . Therefore,

$$\mathrm{Sub}_{\mathbf{FinSets}^{\mathbf{N}}}(\mathrm{Hom}_{\mathbf{N}^{\mathrm{op}}}(-, n)) \cong \mathrm{Hom}_{\mathbf{FinSets}^{\mathbf{N}}}(\mathrm{Hom}_{\mathbf{N}^{\mathrm{op}}}(-, n), \Omega) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{N}}}(\mathrm{Hom}_{\mathbf{N}^{\mathrm{op}}}(-, n), \Omega) \cong \Omega(n)$$

The first equivalence is by definition of subobject classifier, second equivalence is by the fact that $\mathbf{FinSets}^{\mathbf{N}}$ is a full subcategory of $\mathbf{Sets}^{\mathbf{N}}$, and the final equivalence is by Yoneda's lemma. So if such an Ω exist, it must be defined as $\Omega(n) = \{S \mid S \text{ is a subfunctor of } \mathrm{Hom}_{\mathbf{N}^{\mathrm{op}}}(-, n)\}$ for all $n \in \mathbf{N}$. By page 38, the right hand side is $\{S \mid S \text{ is a sieve on } n\}$. So to get the contradiction, it suffices to prove that the collection of sieves on some object $n \in \mathbf{N}$ is a infinite set.

For any $n \in \mathbf{N}$, there exists infinitely many natural numbers which is not less than it, and for each number $a \geq n$, $\{b \mid b \geq a\}$ is a sieve on n . So for each object $n \in \mathbf{N}$, the sieves on n is an infinite set. Hence such an subobject classifier does not exist. \square

Question 3

(a) In $\mathbf{BM} = \mathbf{Sets}^{M^{\mathrm{op}}}$ for M a monoid observe that an object X is a right action $X \times M \rightarrow X$ of M on a set X and that, Y being another object, $\mathrm{Hom}(X, Y)$ is the set of equivariant maps $e : X \rightarrow Y$ [maps with $e(xm) = (ex)m$ for all $x \in X, m \in M$]. Prove that the exponent Y^X is the set $\mathrm{Hom}(M \times X, Y)$ of equivariant maps $e : M \times X \rightarrow Y$, where M is the set M with right action by M , with the action $e \mapsto ek$ of $k \in M$ on e defined by $(ek)(g, x) = e(kg, x)$.

Proof. By definition of exponential, we are proving the natural bijection $\mathrm{Hom}(Z \times X, Y) \cong \mathrm{Hom}(Z, \mathrm{Hom}(M \times X, Y))$. We define the bijection explicitly.

Given a map $f \in \mathrm{Hom}(Z \times X, Y)$ in \mathbf{BM} , it is a function $f : Z \times X \rightarrow Y$ such that for all $z \in Z, x \in X, m \in M$, $f(z, x) \cdot m = f(zm, xm)$ ($*_1$), the map in $\mathrm{Hom}(Z, \mathrm{Hom}(M \times X, Y))$ corresponds to it is defined by $z \mapsto ((m, x) \mapsto f(zm, x))$. To check such an $f' \in \mathrm{Hom}(M \times X, Y)$, consider the diagram:

$$\begin{array}{ccc} (M \times X) \times M & \xrightarrow{f' \times 1} & Y \times M \\ \downarrow \mu & & \downarrow \mu \\ M \times X & \xrightarrow{f'} & Y \end{array}$$

For any $((m, x), a) \in (M \times X) \times M$, following the horizontal map first gives $f(zm, x) \cdot a$, which by ($*_1$) is $f(zma, xa)$. And note that the action of M on $M \times X$ is defined componentwise, so following the vertical map first gives the same result. Hence the diagram commutes.

A map $f \in \mathrm{Hom}(Z, \mathrm{Hom}(M \times X, Y))$ is a map $f : Z \rightarrow \mathrm{Hom}(M \times X, Y)$ such the diagram:

$$\begin{array}{ccc}
Z \times M & \xrightarrow{f \times 1} & \text{Hom}(M \times X, Y) \times M \\
\downarrow \mu & & \downarrow \mu \\
Z & \xrightarrow{f} & \text{Hom}(M \times X, Y)
\end{array}$$

commutes, that is, for all $z \in Z, m \in M, f(z) \cdot m = f(z \cdot m)$ ($*_3$). By the definition of the action of M on equivariant maps, this is saying that for all $k \in M, a \in X, (f(z) \cdot m)(k, a) = f(z)(mk, a) = f(z \cdot m)(k, a)$ ($*_2$). Given such a map, it corresponds to the map $f_0 : Z \times X \rightarrow Y$ defined as $(z, x) \mapsto f(z)(id_M, x)$. To check $f' \in \text{Hom}(Z \times X, Y)$, consider the diagram:

$$\begin{array}{ccc}
(Z \times X) \times M & \xrightarrow{f_0 \times 1} & Y \times M \\
\downarrow \mu & & \downarrow \mu \\
Z \times X & \xrightarrow{f_0} & Y
\end{array}$$

For any $z \in Z, x \in X, m \in M$, following the horizontal map first gives $f_0(z, m) \cdot m = (f(z)(id_M, x)) \cdot m$, and following the vertical map first gives $f_0(z \cdot m, x \cdot m) = f(z \cdot m)(id_M, x \cdot m)$. But by ($*_2$) we also have $f(z \cdot m)(id_M, x \cdot m) = f(z)(m, x \cdot m) = f(z)((id_M, x) \cdot m)$. So the diagram commutes because $(f(z)(id_M, x)) \cdot m = f(z)((id_M, x) \cdot m)$ by ($*_3$).

It left to show that the two maps are inverses. Given $f : Z \times X \rightarrow Y$, it is sent to $f' : Z \rightarrow \text{Hom}(M \times X, Y)$ defined by $z \mapsto ((m, x) \mapsto f(z \cdot m, x))$. And this map is then sent to the map $Z \times X \rightarrow Y$ defined by $(z, x) \mapsto f'(z)(id_M, x)$, which is $f(z, x)$ by definition of f' . So we get the map back. Also, start with a map $f : Z \rightarrow \text{Hom}(M \times X, Y)$, it is sent to $f_0 : Z \times X \rightarrow Y$ defined by $(z, x) \mapsto f(z)(id_M, x)$, and then sent to the map $f'_0 : Z \rightarrow \text{Hom}(M \times X, Y)$ defined by $z \mapsto ((m, x) \mapsto f_0(zm, id_M, x))$, which is $f(zm)(id_M, x)$ by definition of f_0 . But also $f(zm)(id_M, x) = f(z)(m, k)$ by ($*_2$). So the maps are inverses.

□

(b) For objects X, Y in $\mathbf{Sets}^{G^{\text{op}}}$, for G a group, show that the exponent Y^X can be described as the set of all functions $f : X \rightarrow Y$, with the right action of $g \in G$ on such a function defined by $(fg)x = [f(xg^{-1})]g$ for $x \in X$.

Proof. We are proving that for all $X, Y, Z \in \mathbf{Sets}^{G^{\text{op}}}$, $\text{Hom}(Z \times X, Y) \cong \text{Hom}(Z, \text{Hom}(X, Y))$.

A map $f : \text{Hom}(Z \times X, Y)$ is a map $Z \times X \rightarrow Y$ such that for all $z \in Z, x \in X, g \in G, f(z, x) \cdot g = f(z \cdot g, x \cdot g)$ ($*_1$). Given such a map, it corresponds to the map $f' \in \text{Hom}(Z, \text{Hom}(X, Y))$ defined by $z \mapsto (x \mapsto f(z, x))$. To check such an f' is indeed a map in $\mathbf{Sets}^{\text{op}}$, consider the diagram:

$$\begin{array}{ccc}
Z \times G & \xrightarrow{f' \times 1} & \text{Hom}(X, Y) \times G \\
\downarrow \mu & & \downarrow \mu \\
Z & \xrightarrow{f'} & \text{Hom}(X, Y)
\end{array}$$

We should check that for all $z \in Z, g \in G, f'(z) \cdot g = f'(z \cdot g)$. That is, for all $x \in X, (f'(z) \cdot g)(x) = f'(z \cdot g)(x)$. By definition of action of G on functions $X \rightarrow Y$, the left hand side is $(f'(z)(xg^{-1})) \cdot g$. And by definition of f' , $(f'(z)(xg^{-1})) \cdot g = f(z, xg^{-1}) \cdot g \stackrel{(*_1)}{=} f(zg, x)$. Also the right hand side is $f(z \cdot g, x)$ by definition of f' . So the diagram above commutes.

A map $f \in \text{Hom}(Z, \text{Hom}(X, Y))$ is a map $f : Z \rightarrow \text{Hom}(X, Y)$ such that for all $g \in G, f(z) \cdot g = f(z \cdot g)$ ($*_2$). Given such a map, it corresponds to a map $f_0 : Z \times X \rightarrow Y$ defined by $(z, x) \mapsto f(z)(x)$. To check $f_0 \in \text{Hom}(Z \times X, Y)$, consider the diagram :

$$\begin{array}{ccc} (Z \times X) \times G & \xrightarrow{f_0 \times 1} & Y \times G \\ \downarrow \mu & & \downarrow \mu \\ Z \times X & \xrightarrow{f_0} & Y \end{array}$$

We need $(f_0(z, x)) \cdot g = f_0(z \cdot g, x \cdot g)$. By definition of f_0 , it amounts to check $(f(z)(x)) \cdot g = f(z \cdot g)(x \cdot g)$. By ($*_2$), the right hand side is $(f(z) \cdot g)(x \cdot g)$, which equals to $[f(z)((x \cdot g)g^{-1})] \cdot g = f(z)(x) \cdot g$.

The map we defined here is just currying and uncurrying, so it is obvious that the functions we defined above are inverses. \square

Question 4

Generalize Theorem 2 of Section 9 to presheaf categories. More precisely, prove that for a morphism (i.e., a natural transformation) $f : Z \rightarrow Y$ in $\widehat{\mathbf{C}} = \mathbf{Sets}^{\mathbf{C}^{\text{op}}}$, the pullback functor

$$f^* : \text{Sub}_{\widehat{\mathbf{C}}}(Y) \rightarrow \text{Sub}_{\widehat{\mathbf{C}}}(Z)$$

has both a left adjoint \exists_f and a right adjoint \forall_f . [Hint: the left adjoint can be constructed by taking the pointwise image. Define the right adjoint \forall_f on a subfunctor S of Z by $\forall_f(S)(C) = \{y \in Y(C) \mid \text{for all } u : D \rightarrow C \text{ in } \mathbf{C} \text{ and } z \in Z(D), z \in S(D) \text{ whenever } f_D(z) = yu\}.$]

Proof. By page 29 and 30, as T is a subfunctor of Y , for any $C \in \mathbf{C}$, $f^*T(C) = f_C^{-1}(T(C)) \subseteq Z(C)$.

Left adjoint: Define $\exists_f : \text{Sub}_{\widehat{\mathbf{C}}}(Z) \rightarrow \text{Sub}_{\widehat{\mathbf{C}}}(Y)$ by for any subfunctor S of $Z : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Sets}$, $\exists_f S(C) = f_C(S(C))$ is the image of $S(C)$ in $Y(C)$ under f . To prove

$$\exists_f : \text{Sub}_{\widehat{\mathbf{C}}}(Z) \rightleftarrows \text{Sub}_{\widehat{\mathbf{C}}}(Y) : f^*$$

is a pair of adjoints is to prove for any subfunctor S of Z and subfunctor T of Y , $\exists_f S$ is a subfunctor of T iff S is a subfunctor of f^*T .

Saying S is a subfunctor of f^*T is saying that for all $C \in \mathbf{C}$, $S(C) \subseteq f^*(T(C)) = \{y \in Y(C) \mid f_C(y) \in T(C)\}$, and it is clear that this is equivalent to saying all $C \in \mathbf{C}$, $f_C(S(C)) \subseteq T(C)$.

Right adjoint: Define \forall_f as in the hint, proving the adjunction:

$$f^* : \text{Sub}_{\widehat{\mathbf{C}}}(Y) \rightleftarrows \text{Sub}_{\widehat{\mathbf{C}}}(Z) : \forall_f$$

is to prove that for all subfunctor $S \in \text{Sub}_{\mathbf{C}}(Z), T \in \text{Sub}_{\mathbf{C}}(Y)$, f^*T is a subfunctor of S iff T is a subfunctor of $\forall_f S$.

This amounts to show that $\forall C \in \mathbf{C}, f^{-1}(T(C)) \subseteq S(C)$ iff $\forall C \in \mathbf{C}, T(C) \subseteq \forall_f S(C) = \{y \in Y(C) \mid \forall u : D \rightarrow C, \forall z \in Z(D), f_D(z) = yu \implies z \in S(D)\}$.

Suppose f^*T is a subfunctor of S , we prove for any $C \in \mathbf{C}, t \in T(C) \implies \forall u : D \rightarrow C, \forall z \in Z(D), f_D(z) = tu \implies z \in S(D)$. Fix such C, t, u, z . As f^*T is a subfunctor of S , the fact that $f_D(z) = tu \in T(D)$ implies $z \in S(D)$. Conversely, suppose for any $C \in \mathbf{C}, t \in T(C) \implies \forall u : D \rightarrow C, \forall z \in Z(D), f_D(z) = tu \implies z \in S(D)$, let $A \in \mathbf{C}$, we prove $f^{-1}(T(A)) = \{z \in Z(A) \mid f_A(z) \in T(A)\} \subseteq S(A)$. Let $z \in Z(A)$ such that $f_A(z) \in T(A)$, plug in A for C , $f_A(z)$ for t , $1 : A \rightarrow A$ for u , z for z gives $z \in S(A)$. \square

Question 5

Prove Proposition 5.1, that every functor P to sets is representable, by constructing for each $P : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Sets}$ a coequalizer.

$$\coprod_{\substack{C' \xrightarrow{u} C \\ p \in P(C)}} \mathbf{y}(C') \xrightarrow[\tau]{\theta} \coprod_{\substack{C \in \mathbf{C} \\ p \in P(C)}} \mathbf{y}(C) \xrightarrow{\epsilon} P$$

where \coprod denotes the coproduct and for each object B the maps are defined for each $v : B \rightarrow C$ or C' as follows

$$\epsilon_B(C, p; v) = P(v)p, \theta_B(u, p; v) = (C, p; uv), \tau_B(u, p; v) = (C', pu; v)$$

Proof. Key idea: If there is a surjective map $A \twoheadrightarrow X$, then X can be recovered from A by identifying points in A that is mapped to the same point in X .

By Yoneda's lemma, an element in $P(C)$ is a natural transformation $\mathbf{y}(C) \rightarrow P$, and a map $C' \rightarrow C$ is a natural transformation $\mathbf{y}(C') \rightarrow \mathbf{y}(C)$. From this point of view, this amounts to prove P is a equalizer:

$$\coprod_{\substack{C' \xrightarrow{u} C \\ \mathbf{y}(C') \rightarrow \mathbf{y}(C) \\ \mathbf{y}(C) \rightarrow P}} \mathbf{y}(C') \xrightarrow[\tau]{\theta} \coprod_{\substack{C \in \mathbf{C} \\ \mathbf{y}(C) \rightarrow P}} \mathbf{y}(C) \xrightarrow{\epsilon} P$$

To make sense of it, we consider affine schemes. The analogue claim in language of schemes is that we can prove every scheme X is a colimit of some affine schemes by proving we have the coequalizer:

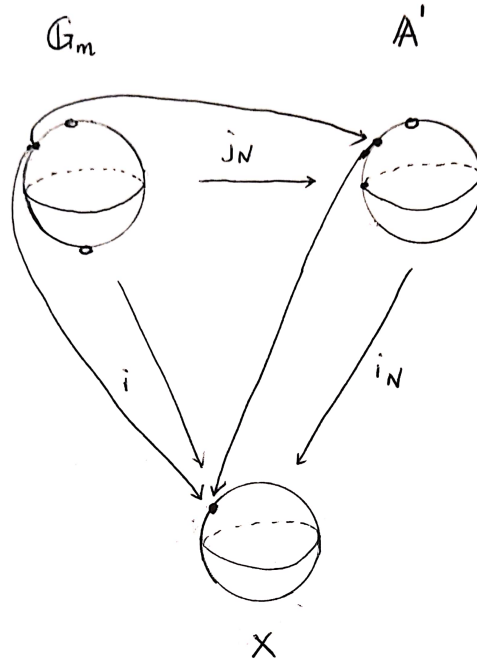
$$\coprod_{u: V' \rightarrow V} \coprod_{c: V \rightarrow X} V' \xrightarrow[\tau]{\theta} \coprod_{a: U \rightarrow X} U \xrightarrow{\epsilon} X$$

where the U, V, V' 's are affine.

Let X be the projective space \mathbb{P}^1 , to visualise it, just consider $\mathbb{C}\mathbb{P}^1$ which is the Riemann sphere. Note that X itself is not affine, but it can be covered by two copies of affine 1-space \mathbb{A}^1 . With one copy of \mathbb{A}^1 covering everything in the sphere except for the north pole, and another copy $\tilde{\mathbb{A}}^1$ covering everything in the sphere except for the south pole. Also there is an inclusion map from the affine scheme \mathbb{G}_m of multiplicative group to X , which covers every point on the sphere except for the poles.

Just to keep things simple, let $i_N : \mathbb{A}^1 \rightarrow X$ be the only map we are considering here from $\mathbb{A}^1 \rightarrow X$, and same for $i_S : \tilde{\mathbb{A}}^1 \rightarrow X, i : \mathbb{G}_m \rightarrow X$ which are the canonical inclusion maps, so we will have $\coprod_{a:U \rightarrow X} U$ has one copy of each of $\mathbb{A}^1, \tilde{\mathbb{A}}^1, \mathbb{G}_m$. Now we want to recover X from the disjoint union. We have a canonical surjection $\mathbb{A}^1 \coprod \tilde{\mathbb{A}}^1 \coprod \mathbb{G}_m \rightarrow X$, so what we want is to glue the points in $\mathbb{A}^1, \tilde{\mathbb{A}}^1, \mathbb{G}_m$ which are mapped to the same point.

We ask when can we get two points in the disjoint union are mapped to the same point in X : Analyse the $\coprod_{u:V' \rightarrow V, c:V \rightarrow X} V'$. Given an element in it, the element consists the following information: two affine schemes V and V' , a map $c : V \rightarrow X$, a map $u : V' \rightarrow V$ and a point in V' . Using these information, it is two ways to construct an element in $\coprod_{a:U \rightarrow X} U$: We can either take the map $V' \rightarrow X$ which is obtained by composing u and c , with the point in V' , or we can take the map $V \rightarrow X$ with the point the image in V under u of the given point in V' . And we have a canonical way to get a point in X given a point in U and a map $a : U \rightarrow X$, namely sending the give point to X using the map.

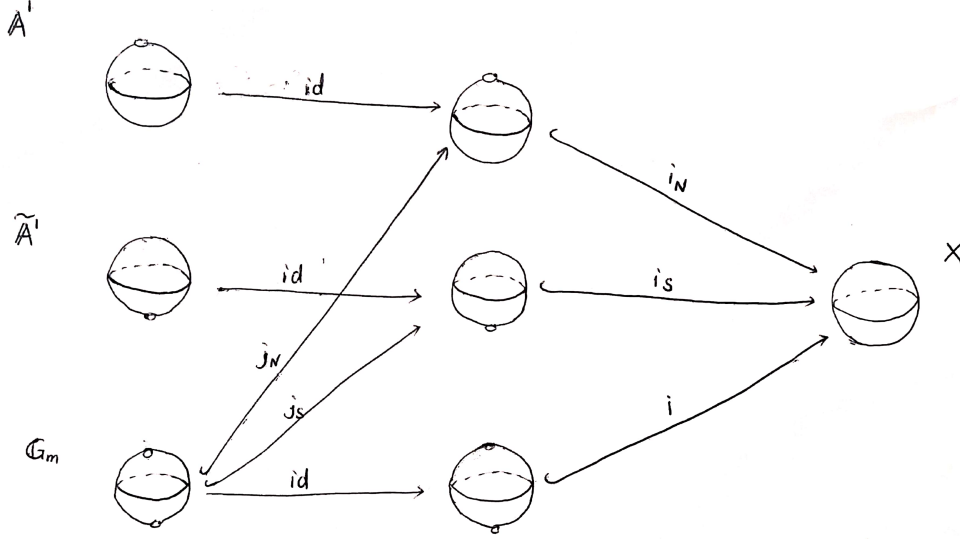


In our case, the inclusion $\mathbb{G}_m \rightarrow \mathbb{A}^1$, the canonical map $\mathbb{A}^1 \rightarrow X$ and the point in \mathbb{G}_m as shown in the picture consists an element in $\coprod_{u:V' \rightarrow V, c:V \rightarrow X} V'$. If we use the first way described above to construct element

in $\coprod_{a:U \rightarrow X} U$, we will get the pair consists of the same point we start with and the map $i_N \circ j_N : \mathbb{G}_m \rightarrow X$.

If we use the second way, we will get the point \mathbb{A}^1 corresponds to the point we start with together with the map i_N . Both of these two elements will be sent to the same point in X , so we should identify them in the disjoint union $\mathbb{A}^1 \amalg \tilde{\mathbb{A}}^1 \amalg \mathbb{G}_m$. So each pair of elements in $\mathbb{A}^1 \amalg \tilde{\mathbb{A}}^1 \amalg \mathbb{G}_m$ comes from applying the different map on the same element should be identified. And after identifying all such elements, we will get X .

This is saying X is a colimit of the diagram on the left part of:



This diagram is a diagram of affine schemes, so we can say X is a colimit of affine schemes.

Note that it is not the canonical way of covering the X using the three spaces, the canonical way is to consider any map from each of these schemes to X . For instance, we do not only consider the canonical inclusion $\mathbb{A}^1 \rightarrow X$, but the map $x \rightarrow x^{-1}$ which will cover each point of X except for the south pole (instead of north pole, which is not covered by the canonical inclusion). For the canonical cover, the uncovered point will range over all points in X . So for the canonical cover, the diagram in the left hand side will be much more complicated.

Return to our question, we want to prove P is the equalizer:

$$\coprod_{\substack{y(C') \xrightarrow{u} y(C) \\ y(C) \rightarrow P}} y(C') \xrightarrow[\tau]{\theta} \coprod_{\substack{C \in \mathbf{C} \\ y(C) \rightarrow P}} y(C) \xrightarrow{\epsilon} P$$

By the Yoneda's lemma, $\text{Hom}(y(C), P) \cong P(C)$ for each $C \in \mathbf{C}$. From the coproduct $\coprod_{\substack{C \in \mathbf{C} \\ y(C) \rightarrow P}} y(C)$, we can get the information $\text{Hom}(y(C), P) \cong P(C)$ for each C , in this sense, the coproduct covers P . So we can use

the surjection $\coprod_{\substack{C \in \mathbf{C} \\ \mathbf{y}(C) \rightarrow P}} \mathbf{y}(C) \rightarrow P(C)$ and identify the overlapping point together to recover P . Hence P is a colimit of representables in the same sense as X is a colimit of affine schemes.

□