Assignment 1

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22 March 2019

Question 1

Prove theorem 3 of §1. [Hint: define a quasi-inverse $\mathbf{s} : \operatorname{Sh}(\mathcal{B}) \to \operatorname{Sh}(X)$ for \mathbf{r} as follows. Given a sheaf F on \mathcal{B} , and an open set $U \subset X$, consider the cover $\{B_i \mid i \in I\}$ of U by all basic open sets $B_i \in \mathcal{B}$ which are contained in U. Define $\mathbf{s}(F)(U)$ by the equalizer $\mathbf{s}(F)(U) \to \Pi_{i \in I}F(B_i) \rightrightarrows \Pi_{i,j}F(B_i \cap B_j)$

Proof. Claim 1: For any $B \in \mathcal{B}$, $\mathbf{s}(F)(B) = F(B)$.

The claim follows from the fact that F is a sheaf on the base, and the fact that equalizer is unique.

Claim 2: For any functor $F: \mathcal{O}(X) \to \mathbf{Sets}$, to check it is a sheaf, for any open set U, it suffices to check the sheaf condition for the cover $U = \bigcup_{i \in I} B_i$ where $B_i \in \mathcal{B}$ for all $i \in I$.

Proof: Suppose the sheaf condition holds for the bases, we check the sheaf condition holds for any arbitary cover. Consider an open set U covered by $U = \bigcup_{i \in I}$, we check $F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$ is an equalizer. For each U_i , we have $U_i = \bigcup_k B_{ik}$ for $B_{ik} \in \mathcal{B}$. We have a diagram:

$$F(U) \xrightarrow{e} \Pi_{i}F(U_{i}) \xrightarrow{q} \Pi_{i,j}F(U_{i} \cap U_{j})$$

$$\downarrow^{e'} \qquad \downarrow^{t_{1}} \qquad \downarrow^{t_{2}} \qquad \downarrow^{t_{2}} \qquad \qquad \downarrow^{t_{1}} \qquad \qquad \qquad \downarrow^{t$$

The map t_1, t_2 are injective, and the diagram commutes. Take $\alpha \in \Pi_{i \in I} F(U_i)$ such that $p(\alpha) = q(\alpha)$. Then for $t_1(\alpha) \in \Pi_{i,k}(B_{ik})$, we have $p'(t_1(\alpha)) = q'(t_1(\alpha))$. By the sheaf condition on the basis, there exists a unique $\alpha_0 \in F(U)$ such that $e'(\alpha_0) = t_1(\alpha)$. e' is a mono and hence is injective, e must be injective as well, α_0 is the unique element which is mapped to α .

There are three things to check:

(1) $\mathbf{s}(F)$ is indeed a sheaf:

By claim 2, it suffices to check the sheaf condition on basis, the sheaf condition on basis holds by claim 1 and definition of s.

(2) $\mathbf{r} \circ \mathbf{s} = id$:

We need to check for any $B \in \mathcal{B}$, $\mathbf{r}\mathbf{s}(F)(B) = F(B)$. By the claim, as $B \in \mathcal{B}$, $\mathbf{s}(F)(B) = F(B)$, hence $\mathbf{r}\mathbf{s}(F)(B) = \mathbf{r}(F)(B) = F(B)$ since \mathbf{r} is merely a restriction.

(3) $\mathbf{s} \circ \mathbf{r} = id$:

We need to check for a sheaf F_0 on X, $\mathbf{sr}(F_0)(U) = F_0(U)$ for any open set $U \subseteq X$. As F_0 is a sheaf, for any cover $\{U_i\}_{i\in I}$ of U, $F_0(U)$ is the equalizer: $F_0(U) \to \Pi_{i\in I}F_0(U_i) \rightrightarrows \Pi_{i,j}F_0(U_i\cap U_j)$. In particular, take the cover $U = \bigcup_{i\in I} B_i$ for $B_i \in \mathcal{B}$, then $F_0(U) \to \Pi_{i\in I}F_0(B_i) \rightrightarrows \Pi_{i,j}F_0(B_i\cap B_j)$ is a equalizer. As equalier is unique, it suffices to show that $\mathbf{sr}(U) \to \Pi_{i\in I}F_0(B_i) \rightrightarrows \Pi_{i,j}F_0(B_i\cap B_j)$ is a equalizer as well. But note that $F_0(B_i) = \mathbf{r}(B_i)$, so it holds by definition of \mathbf{s} .

Question 2

Let $f: X \to Y$ be an etale map. Show that $f^*: \operatorname{Sh}(Y) \to \operatorname{Sh}(X)$ has a left adjoint. Give an example of map $f: X \to Y$ such that $f^*: \operatorname{Sh}(Y) \to \operatorname{Sh}(X)$ cannot possibly have a left adjoint.

Proof. If f is etale, then f^* has a left adjoint: By the definition of f^* as a pullback via eqivalence of Corollary 6.3, it suffices to show that f^* : $\operatorname{Etale}(Y) \to \operatorname{Etale}(X)$ has a left adjoint. We claim that the map $\sum_f := -\circ f : \operatorname{Etale}(X) \to \operatorname{Etale}(Y)$ defined by sending an etale map $p: F \to X$ to the composition $F \xrightarrow{p} X \xrightarrow{f} Y$ is its adjoint.

$$\sum_f : \operatorname{Etale}(X) \longleftrightarrow \operatorname{Etale}(Y) : f^*$$

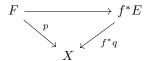
Note that \sum_f is well defined since composition of etale map is etale, so the image of \sum_f indeed lies in Etale(Y). By definition of adjoint, it amounts to show that there is a natural bijection $\operatorname{Hom}_{\operatorname{Etale}(Y)}(\sum_f(p),q)\cong \operatorname{Hom}_{\operatorname{Etale}(X)}(p,f^*q)$. A map $p\circ f\to q$ in $\operatorname{Etale}(Y)$ is a commutative diagram:

$$F \longrightarrow E$$

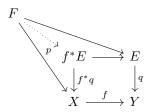
$$\downarrow^p \qquad \downarrow^q$$

$$X \stackrel{f}{\longrightarrow} Y$$

And a map $p \to f^*q$ is a commutative diagram:



It directly follows from definition (universal property) of pullback that the above diagrams are in bijection.



Give an example of a map $f: X \to Y$ such that $f^*: \operatorname{Sh}(Y) \to \operatorname{Sh}(X)$ cannot possibly have a left adjoint:

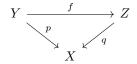
Claim: If f^* have a left adjoint, then it must be $\sum_f := - \circ f$.

Proof: Suppose such a left adjoint $f_!$: Etale $(X) \to \text{Etale}(Y)$ exists, then for any $p: F \to X$ in Etale(X) and $q: E \to Y$ in Etale(Y), $\text{Hom}_{\text{Etale}(Y)}(f_!(p), q) \cong \text{Hom}_{\text{Etale}(X)}(p, f^*q)$. But as shown in the diagram above, a map from p to f^*q is a map $F \to f^*E$ to the pullback, and hence is uniquely determined by a pair of maps $F \to X, F \to E$, which makes the outside of the diagram above commute. But such a pair is precisely the same thing as a map from $\sum_f(p)$ to q. As adjoint is unique up to isomorphism, if the left adjoint $f_!$ exists, then it must be $\sum_f(p)$.

By the claim above, to give a map f without such a left adjoint, it suffices to give a map f such that composing with it destroy etaleness. As discussed on SGL page 88, any projection $X \times R \to X$ can be taken as such an example.

Question 3

- (a) Prove that a map $p: Y \to X$ of topological spaces is etale iff both p and the diagonal $Y \to Y \times_X Y$ are open maps.
- (b) Prove that in the commutative diagram of continuous maps



where p and q is etale, f must be etale.

Proof. For one direction. Suppose p is etale, we prove p is open and the diagonal Δ is open.

To show p is open: Pick an open set $U \subseteq Y$, we show p(U) is open in X. For each $x \in U$, by the fact that p is etale, we have a neighbourhood U_x that contains x and U_x is homeomorphic to its image $p(U_x)$. By definition of induced topology on U_x , $U \cap U_x$ is open in U_x and hence each $p(U \cap U_x)$ is open in $p(U_x)$ (any homeomorphism is open since it has continous inverse) and hence open in Y (open subset of open subset is open). Hence the union $\bigcup_{x \in U} p(U \cap U_x) = p(U)$ is open in Y.

To show the diagonal is open, we need to write any set of the form $\{(a,a) \mid a \in U\}$ where U is open in Y as a union of a family of sets $\bigcup_{i \in I} \{(a,b) \mid a \in U_i, b \in V_i\} \cap \{(a,b) \mid p(a) = p(b)\}$. Given such an open set $U \subseteq Y$, for any point $a \in U$, since p is etale, there is a neighbourhood U_a of a which is mapped homeophically to X, as U is open in Y, each $U \cap U_a$ is open in Y. So $\{(a,a) \mid a \in U\} = \bigcup_{a \in U} \{(x,y) \mid x,y \in U \cap U_a\}$. Note that we have the equality since p is homeomorphism on each U_a and hence is bijective. So for $x,y \in U_a$, p(x) = p(y) implies x = y.

For the other direction. Suppose both p and the diagonal $y\mapsto (y,y)$ are open maps, we prove p is etale. Note that it suffices to prove that X is covered by open sets such that each of them are mapped homeomorphically to X. Let Δ denote the diagonal map, as Δ is open, in particular, $\{(y,y)\mid y\in Y\}$ is open in $Y\times_XY=\{(y_1,y_2)\mid p(y_1)=p(y_2)\}$. According to the defintion of topology of the pullback, this means $Y\times_XY=\bigcup_{i\in I}\{(y_1,y_2)\mid y_1\in U_i,y_2\in V_i,p(y_1)=p(y_2)\}$ for some U's and V's that are all open in Y. In particular, it means $\{(y_1,y_2)\mid y_1\in U_i,y_2\in V_i,p(y_1)=p(y_2)\}\subseteq \{(y,y)\mid y\in Y\}$ for all such U_i and V_i . Hence for $y_1,y_2\in U_i\cap V_i,p(y_1)=p(y_2)$ implies $y_1=y_2$, in other words, $p|_{U_i\cap V_i}$ is a bijection from $U_i\cap V_i$ to its image. As p is also an open map, its restriction to each such $U_i\cap V_i$ is an open map, by the fact that 'an open map which is a bijection is a homeomorphism', p is a homeomorphism on such $U_i\cap V_i$.

(b) Pick a point $a \in Y$, we find an open neighbourhood of a which is mapped homeomorphically to Z. As p is etale, we have an open Y_a such that $a \in Y_a$ and $p(Y_a)$ is homeomorphic to Y_a . As q is etale, we have an open $Z_{f(a)}$ such that $f(a) \in Z_{f(a)}$ and $q(Z_{f(a)})$ is homeomorphic to $Z_{f(a)}$. As we have both $p(Y_a) \cap q(Z_{f(a)}) \subseteq p(Y_a)$ and $p(Y_a) \cap q(Z_{f(a)}) \subseteq q(Z_{f(a)})$, we know $p^{-1}(p(Y_a) \cap q(Z_{f(a)}))$ is homeomorphic to $q^{-1}(p(Y_a) \cap q(Z_{f(a)}))$ which are both homeomorphic to $p(Y_a) \cap q(Z_{f(a)})$. By commutativity of the diagram, $f(p^{-1}(p(Y_a) \cap q(Z_{f(a)}))) = q^{-1}(p(Y_a) \cap q(Z_{f(a)}))$, which are homeomorphic to $p^{-1}(p(Y_a) \cap q(Z_{f(a)}))$.