

WHAT IS A GROUP

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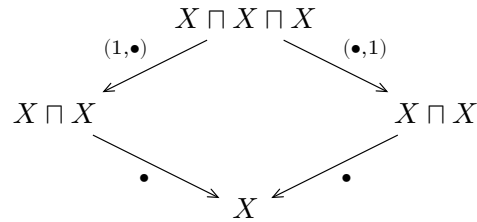
Concretely, (By Cayley's theorem) all groups are transformation groups (automorphism groups), that is, we can view groups as morphisms

$$_ \circ _ : \text{Aut}_{\mathcal{C}} X \sqcap \text{Aut}_{\mathcal{C}} X \rightarrow \text{Aut}_{\mathcal{C}} X$$

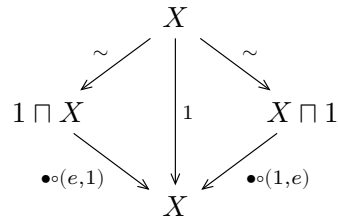
induced by the composition of morphisms in any category \mathcal{C} .

Abstractly, a group is a *group object* in **Set**. The concept of a group object models the properties of an automorphism group through categorical notions. It is by definition the information $_ \bullet _ : X \sqcap X \rightarrow X$, $e : 1 \rightarrow X$ and $\diagup : X \rightarrow X$, where 1 is the final object of the given category. It's axioms are illustrated in the following diagrams:

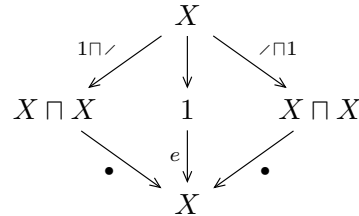
- Associativity:



- Identity:



- Inverse elements:



- An *abelian* group object satisfies an additional property of commutativity:

$$\begin{array}{ccc}
 X \sqcap X & \xrightarrow{p} & X \sqcap X \\
 & \searrow \bullet & \swarrow \bullet \\
 & X &
 \end{array}$$

where $p : X \sqcap Y \rightarrow Y \sqcap X$ is the "permutation" isomorphism between products.

In this sense, assume $(\bullet_X, e_X, \diagdown_X)$, $(\bullet_Y, e_Y, \diagdown_Y)$, are two group objects of the same category, $\phi : X \rightarrow Y$ is a group (object) homomorphism if and only if the diagrams

$$\begin{array}{ccccc}
 X \sqcap X & \xrightarrow{(\phi, \phi)} & Y \sqcap Y & & X & \xrightarrow{\phi} & Y \\
 \bullet_X \downarrow & & \downarrow \bullet_Y & & \diagdown_X \downarrow & & \downarrow \diagdown_Y \\
 X & \xrightarrow{\phi} & Y & & X & \xrightarrow{\phi} & Y
 \end{array}$$

commute.