# **Natural Transformation**

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# 2 Natural transformation

#### 2.1 Definition

Natural transformation is a convenient notion that can be intuitively thought of as morphisms between functors in the sense of "ref to def-funct-cat (WIP)". For this reason we often use diagrams



to denote functors  $\phi: F_1 \to F_2$ .

**Definition 2.1.1.** Given two functors  $F_1, F_2: \mathcal{C} \to \mathcal{D}$ , a natural transformation  $\phi: F_1 \to F_2$  consists of the information

$$\{\phi_X\in \operatorname{Hom}(F_1X,F_2X): X\in \operatorname{Obj} \mathbf{C}\},$$

such that the following diagram

$$\begin{array}{ccc} F_1 X & \xrightarrow{\phi_X} & F_2 X \\ F_1 f \downarrow & & & \downarrow F_2 f \\ F_1 Y & \xrightarrow{\phi_Y} & F_2 Y \end{array}$$

commutes for all  $f \in \text{Hom}_{\mathcal{C}}(X,Y)$  for all  $X,Y \in \text{Obj }\mathcal{C}$ .

## 2.2 Composition of natural transformation

We denote the head-to-tail composition of functors  $\phi_1: F_1 \to F_2$ ,  $\phi_2: F_2 \to F_3$  as  $\phi_2 \circ \phi_1: F_1 \to F_3$ . This composition is illustrated in the diagram

$$C \xrightarrow{\phi_1 \Downarrow} D \Leftrightarrow C \xrightarrow{F_1} D.$$

$$F_3 \longrightarrow D \Leftrightarrow C \xrightarrow{\phi_2 \circ \phi_1} D.$$

 $\phi_2 \circ \phi_1$  consists of information  $\{(\phi_2 \circ \phi_1)_X := (\phi_2)_X \circ (\phi_2)_X\}$  in the obvious sense. It is easy to show that  $\circ$  preserves naturality. It is also associative.

There is also a way of defining "parallel" composition of natural transformations. Given  $\phi: F_1 \to F_2$  and  $\psi: G_1 \to G_2$ , a morphism  $\psi * \phi: G_1F_1 \to G_2F_2$  given by the diagram

$$C 

\downarrow F_1 \\
F_2$$
 $D 

\downarrow G_2$ 
 $E \Leftrightarrow C 

\downarrow G_1F_1 \\
\downarrow \psi * \phi \\
G_2F_2$ 
 $E$ 

is the parallel composition of the two natural transformation. Its definition is the information  $\{(\psi*\phi)_X:=\psi_{F_2X}\circ G_1\phi_X=G_2\phi_X\circ\psi_{F_1X}\}$  since

$$\begin{array}{ccc} G_1(F_1X) \xrightarrow{\psi_{F_1X}} G_2(F_1X) \\ & & & \downarrow \\ G_1\phi_X & & & \downarrow \\ G_1(F_2X) \xrightarrow{\psi_{F_2X}} G_2(F_2X) \end{array}$$

commutes.

## **Lemma 2.2.1.** \* preserves naturality and is associative

*Proof.* Choosing  $(\psi * \phi)_X := \psi_{F_2X} \circ G_1 \phi_X$  as a representation, we show that \* preserves naturality by verifying that the right and left squares of

commutes. Composing these morphisms gives the equality

$$G_2F_2f\circ(\psi_{F_2X}\circ G_1\phi_X)=(\psi_{F_2Y}\circ G_1\phi_Y)\circ G_1F_1f$$

as needed.

Next, we want to show that \* is associative. Given

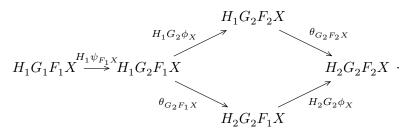
$$C \underbrace{ \left( \begin{array}{c} F_1 \\ \phi \end{array} \right) D \left( \begin{array}{c} G_1 \\ \psi \end{array} \right) E \underbrace{ \left( \begin{array}{c} H_1 \\ \theta \end{array} \right) F}_{H_2} F$$

we want to show that  $(\phi * \psi) * \theta = \phi * (\psi * \theta)$ .

 $((\phi * \psi) * \theta)_X$  and  $(\phi * (\psi * \theta))_X$  are given by the diagrams

To prove the equality, we first decompose the shaded morphisms into the two following diagrams

by the definition of parallel composition. The "middle ground" of  $H_1\psi_{F_1X}:H_1G_1F_1X\to H_1G_2F_1X$  reduces the problem to the commutativity of the diagram



The naturality of  $\theta$  ensures that the diagram commutes, which gives us the expected equality:

$$\begin{split} ((\phi * \psi) * \theta))_X &= \theta_{G_2F_2X} \circ H_1(\psi * \phi)_X \\ &= \theta_{G_2F_2X} \circ (H_1G_2\phi_X \circ H_1\psi_{F_1X}) \\ &= H_2G_2\phi_X \circ (\theta_{G_2F_1X} \circ H_1\psi_{F_1X}) \\ &= H_2G_2\phi_X \circ (\theta * \psi)_{F_1X} = ((\phi * \psi) * \theta))_X. \end{split}$$

