

NATURAL TRANSFORMATION

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July 2021

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1 Introduction

Natural transformation is a convenient notion that can be intuitively thought of as morphisms between functors in the sense of "ref to def-funct-cat (WIP)". For this reason we often use diagrams

$$\begin{array}{ccc} & F_1 & \\ \curvearrowright & \Downarrow \phi & \curvearrowleft \\ C & & D \\ \curvearrowleft & \Uparrow & \curvearrowright \\ & F_2 & \end{array}$$

to denote natural transformation $\phi : F_1 \rightarrow F_2$.

Definition 1.1. Given two functors $F_1, F_2 : \mathcal{C} \rightarrow \mathcal{D}$, a *natural transformation* $\phi : F_1 \rightarrow F_2$ consists of the information

$$\{\phi_X \in \text{Hom}(F_1 X, F_2 X) : X \in \text{Obj } \mathcal{C}\},$$

such that the following diagram

$$\begin{array}{ccc} F_1 X & \xrightarrow{\phi_X} & F_2 X \\ F_1 f \downarrow & & \downarrow F_2 f \\ F_1 Y & \xrightarrow{\phi_Y} & F_2 Y \end{array}$$

commutes for all $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \text{Obj } \mathcal{C}$.

2 Composition of natural transformations

We denote the head-to-tail composition of functors $\phi_1 : F_1 \rightarrow F_2$, $\phi_2 : F_2 \rightarrow F_3$ as $\phi_2 \circ \phi_1 : F_1 \rightarrow F_3$. This composition is illustrated in the diagram

$$\begin{array}{ccc} & F_1 & \\ \curvearrowright & & \curvearrowright \\ C & \xrightarrow{\phi_1 \downarrow} & D \\ \curvearrowleft & & \curvearrowleft \\ & F_2 & \\ & \downarrow \phi_2 & \\ & F_3 & \end{array} \rightsquigarrow \begin{array}{ccc} & F_1 & \\ \curvearrowright & & \curvearrowright \\ C & \xrightarrow{\parallel} & D \\ \curvearrowleft & & \curvearrowleft \\ & \phi_2 \circ \phi_1 \downarrow & \\ & F_3 & \end{array} .$$

$\phi_2 \circ \phi_1$ consists of information $\{(\phi_2 \circ \phi_1)_X := (\phi_2)_X \circ (\phi_1)_X\}$ in the obvious sense. It is easy to show that \circ preserves naturality. It is also associative.

There is also a way of defining "parallel" composition of natural transformations. Given $\phi : F_1 \rightarrow F_2$ and $\psi : G_1 \rightarrow G_2$, a morphism $\psi * \phi : G_1 F_1 \rightarrow G_2 F_2$ given by the diagram

$$\begin{array}{ccccc} & F_1 & & G_1 F_1 & \\ \curvearrowright & & \curvearrowright & & \curvearrowright \\ C & \xrightarrow{\phi \downarrow} & D & \xrightarrow{\psi \downarrow} & E \\ \curvearrowleft & & \curvearrowleft & & \curvearrowleft \\ & F_2 & & G_2 F_2 & \end{array} \rightsquigarrow \begin{array}{ccc} & G_1 F_1 & \\ \curvearrowright & & \curvearrowright \\ C & \xrightarrow{\parallel} & E \\ \curvearrowleft & & \curvearrowleft \\ & \psi * \phi \downarrow & \\ & G_2 F_2 & \end{array}$$

is the parallel composition of the two natural transformation. Its definition is the information $\{(\psi * \phi)_X := \psi_{F_2 X} \circ G_1 \phi_X = G_2 \phi_X \circ \psi_{F_1 X}\}$ since

$$\begin{array}{ccc} G_1 F_1 X & \xrightarrow{\psi_{F_1 X}} & G_2 F_1 X \\ G_1 \phi_X \downarrow & \searrow (\psi * \phi)_X & \downarrow G_2 \phi_X \\ G_1 F_2 X & \xrightarrow{\psi_{F_2 X}} & G_2 F_2 X \end{array}$$

commutes.

Lemma 2.1. ** preserves naturality and is associative*

Proof. Choosing $(\psi * \phi)_X := \psi_{F_2 X} \circ G_1 \phi_X$ as a representation, we show that $*$ preserves naturality by verifying that the right and left squares of

$$\begin{array}{ccccc} G_1 F_1 X & \xrightarrow{G_1 \phi_X} & G_1 F_2 X & \xrightarrow{\psi_{F_2 X}} & G_2 F_2 X \\ G_1 F_1 f \downarrow & & G_1 F_2 f \downarrow & & \downarrow G_2 F_2 f \\ G_1 F_1 Y & \xrightarrow{G_1 \phi_Y} & G_1 F_2 Y & \xrightarrow{\psi_{F_2 Y}} & G_2 F_2 Y \end{array}$$

commutes. Composing these morphisms gives the equality

$$G_2 F_2 f \circ (\psi_{F_2 X} \circ G_1 \phi_X) = (\psi_{F_2 Y} \circ G_1 \phi_Y) \circ G_1 F_1 f$$

as needed.

Next, we want to show that $*$ is associative. That is, given

$$C \begin{array}{c} \xrightarrow{F_1} \\ \Downarrow \phi \\ \xrightarrow{F_2} \end{array} D \begin{array}{c} \xrightarrow{G_1} \\ \Downarrow \psi \\ \xrightarrow{G_2} \end{array} E \begin{array}{c} \xrightarrow{H_1} \\ \Downarrow \theta \\ \xrightarrow{H_2} \end{array} F ,$$

$$(\phi * \psi) * \theta = \phi * (\psi * \theta).$$

$((\phi * \psi) * \theta)_X$ and $(\phi * (\psi * \theta))_X$ are identified by the diagrams

$$\begin{array}{ccc} H_1(G_1 F_1 X) & \xrightarrow{\theta_{G_1 F_1 X}} & H_2(G_1 F_1 X) \\ H_1(\psi * \phi)_X \downarrow & \searrow (\theta * (\psi * \phi))_X & \downarrow H_2(\psi * \phi)_X \\ H_1(G_2 F_2 X) & \xrightarrow{\theta_{G_2 F_2 X}} & H_2(G_2 F_2 X) \end{array} \quad \begin{array}{ccc} H_1 G_1(F_1 X) & \xrightarrow{(\theta * \psi)_{(F_1 X)}} & H_2 G_2(F_1 X) \\ H_1 G_1 \phi_X \downarrow & \searrow ((\phi * \psi) * \theta)_X & \downarrow H_2 G_2 \phi_X \\ H_1 G_1(F_2 X) & \xrightarrow{(\theta * \psi)_{(F_2 X)}} & H_2 G_2(F_2 X) \end{array} .$$

To prove the equality, we first decompose the shaded morphisms into the two following diagrams

$$\begin{array}{ccc} H_1(G_1 F_1 X) & \xrightarrow{H_1(\psi_{F_1 X})} & H_1(G_2 F_1 X) \\ & \searrow H_1(\psi * \phi)_X & \downarrow H_1(G_2 \phi_X) \\ & & H_1(G_2 F_2 X) \end{array} \quad \begin{array}{ccc} H_1 G_1(F_1 X) & & \\ H_1 \psi_{(F_1 X)} \downarrow & \searrow (\theta * \psi)_{(F_1 X)} & \\ H_1 G_2(F_1 X) & \xrightarrow{\theta_{G_2(F_1 X)}} & H_2 G_2(F_1 X) \end{array}$$

by the definition of parallel composition. The "middle ground" of $H_1 \psi_{F_1 X} : H_1 G_1 F_1 X \rightarrow H_1 G_2 F_1 X$ reduces the problem to the commutativity of the diagram

$$\begin{array}{ccccc} & & H_1 G_2 F_2 X & & \\ & H_1 G_2 \phi_X \nearrow & & \searrow \theta_{G_2 F_2 X} & \\ H_1 G_1 F_1 X & \xrightarrow{H_1 \psi_{F_1 X}} & H_1 G_2 F_1 X & & H_2 G_2 F_2 X \\ & \searrow \theta_{G_2 F_1 X} & & \nearrow H_2 G_2 \phi_X & \\ & & H_2 G_2 F_1 X & & \end{array} .$$

The naturality of θ ensures that the diagram commutes, which gives us the expected equality:

$$\begin{aligned} ((\phi * \psi) * \theta)_X &= \theta_{G_2 F_2 X} \circ H_1(\psi * \phi)_X \\ &= \theta_{G_2 F_2 X} \circ (H_1 G_2 \phi_X \circ H_1 \psi_{F_1 X}) \\ &= H_2 G_2 \phi_X \circ (\theta_{G_2 F_1 X} \circ H_1 \psi_{F_1 X}) \\ &= H_2 G_2 \phi_X \circ (\theta * \psi)_{F_1 X} = ((\phi * \psi) * \theta)_X. \end{aligned}$$

□

Work In Progress ...