Natural Transformation

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1 Introduction

Natural transformation is a convenient notion that can be intuitively thought of as morphisms between functors in the sense of "reference to def-funct-cat (WIP)". For this reason we often use diagrams



to denote functors $\phi: F_1 \to F_2$.

Definition 1.0.1. Given two functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$, a natural transformation $\phi: F_1 \to F_2$ consists of the information

$$\{\phi_X \in \operatorname{Hom}(F_1X, F_2X) : X \in \operatorname{Obj} \mathbf{C}\},\$$

such that the following diagram

$$\begin{array}{ccc} F_1 X & \xrightarrow{\phi_X} & F_2 X \\ F_1 f \downarrow & & \downarrow F_2 f \\ F_1 Y & \xrightarrow{\phi_Y} & F_2 Y \end{array}$$

commutes for all $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$ for all $X,Y \in \operatorname{Obj}{\mathcal{C}}$.

2 Composition of natural transformation

We denote the head-to-tail composition of functors $\phi_1: F_1 \to F_2, \ \phi_2: F_2 \to F_3$ as $\phi_2 \circ \phi_1: F_1 \to F_3$. This composition is illustrated in the diagram

$$C \xrightarrow{\phi_1 \Downarrow} D \Leftrightarrow C \xrightarrow{\phi_2 \circ \phi_1} D.$$

 $\phi_2 \circ \phi_1$ consists of information $\{(\phi_2 \circ \phi_1)_X := (\phi_2)_X \circ (\phi_2)_X\}$ in the obvious sense. It is easy to show that \circ preserves naturality. It is also associative.

There is also a way of defining "parallel" composition of natural transformations. Given $\phi: F_1 \to F_2$ and $\psi: G_1 \to G_2$, a morphism $\psi * \phi: G_1F_1 \to G_2F_2$ given by the diagram

$$C \underbrace{ \bigoplus_{F_2}^{F_1} D \bigoplus_{G_2}^{G_1} E}_{G_2} E \rightsquigarrow C \underbrace{ \bigoplus_{\psi * \phi \\ \psi * \phi}^{G_1 F_1} E}_{G_2 F_2}$$

is the parallel composition of the two natural transformation. Its definition is the information $\{(\psi*\phi)_X:=\psi_{F_2X}\circ G_1\phi_X=G_2\phi_X\circ\psi_{F_1X}\}$ since

$$\begin{array}{c|c} G_1(F_1X) \xrightarrow{\psi_{F_1X}} G_2(F_1X) \\ \hline G_1\phi_X & & \downarrow G_2\phi_X \\ \hline G_1(F_2X) \xrightarrow{\psi_{F_2X}} G_2(F_2X) \end{array}$$

commutes.

Lemma 2.0.1. * preserves naturality and is associative

Proof. Choosing $(\psi * \phi)_X := \psi_{F_2X} \circ G_1 \phi_X$ as a representation, we show that * preserves naturality by verifying that the right and left squares of

$$\begin{array}{cccc} G_1F_1X \xrightarrow{G_1\phi_X} G_1F_2X \xrightarrow{\psi_{F_2X}} G_2F_2X \\ G_1F_1f \Big| & G_1\overset{1}{F_2}f & \Big| G_2F_2f \\ G_1F_1Y \xrightarrow{G_1\phi_Y} G_1F_2Y \xrightarrow{\psi_{F_2Y}} G_2F_2Y \end{array}$$

commutes. Composing these morphisms gives the equality

$$G_2F_2f\circ (\psi_{F_2X}\circ G_1\phi_X)=(\psi_{F_2Y}\circ G_1\phi_Y)\circ G_1F_1f$$

as needed.

Next, we want to show that * is associative. Given

$$C \bigvee_{F_2}^{F_1} D \bigvee_{G_2}^{G_1} E \bigvee_{H_2}^{H_1} F ,$$

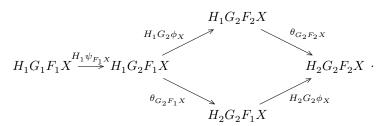
we want to show that $(\phi * \psi) * \theta = \phi * (\psi * \theta)$.

 $((\phi * \psi) * \theta)_X$ and $(\phi * (\psi * \theta))_X$ are given by the diagrams

To prove the equality, we first decompose the shaded morphisms into the two following diagrams

$$\begin{array}{cccc} H_1(G_1F_1X) & H_1(G_2F_1X) & & H_1G_1(F_1X) \\ & & & \downarrow \\ H_1(\psi*\phi)_X & & \downarrow \\ & & & \downarrow \\ H_1(G_2\phi_X) & & H_1\psi_{(F_1X)} \\ & & & & \downarrow \\ & & & & H_1G_2(F_1X) \\ & & & & & H_1G_2(F_1X) \\ & & & & & \\ \end{array}$$

by the definition of parallel composition. The "middle ground" of $H_1\psi_{F_1X}:H_1G_1F_1X\to H_1G_2F_1X$ reduces the problem to the commutativity of the diagram



The naturality of θ ensures it, which gives us the expected equality:

$$\begin{split} ((\phi * \psi) * \theta))_X &= \theta_{G_2F_2X} \circ H_1(\psi * \phi)_X \\ &= \theta_{G_2F_2X} \circ (H_1G_2\phi_X \circ H_1\psi_{F_1X}) \\ &= H_2G_2\phi_X \circ (\theta_{G_2F_1X} \circ H_1\psi_{F_1X}) \\ &= H_2G_2\phi_X \circ (\theta * \psi)_{F_1X} = ((\phi * \psi) * \theta))_X. \end{split}$$