SET

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This article introduces basic ideas in set theory (specifically the ZFC set theory). In particular, we will establish the arithmetic on $\mathbb{N}$ and prove some useful theorems along the way. The readers should be aware that ZFC is not the only solution for a foundation of mathematics, there are many other developing theories capable of handling the same task.

1 Ordinal numbers

Definition 1.1. An *ordinal number* is a set α such that the two following properties are satisfied:

O 1. transitivity: $\beta \in \alpha \implies \beta \subset \alpha$, in another word $\alpha \subset 2^{\alpha}$;

O 2. well-order: α is well-ordered under the relation $\in$.

The class of all ordinal numbers is denoted by $\boldsymbol{ON}$.

Lemma 1.2. *For all $S \subseteq \boldsymbol{ON}$ and $S \neq \emptyset$, $\bigcap S \in \boldsymbol{ON}$.*

Proof. We need to verify that $\bigcap S$ meets the requirements of an ordinal number.

1. transitivity: $\bigcap S$ is transitive by definition.

$$\alpha \in \bigcap S \implies \forall s \in S : \alpha \in s \implies \forall s \in S : \alpha \subset s \implies \alpha \subset \bigcap S.$$

2. well-order: Because $\bigcap S \subseteq s$, $\bigcap S$ is well-ordered.

□

Lemma 1.3. *Some basic properties of ordinal numbers:*

1. *For all $\alpha \in \boldsymbol{ON}$, $\beta \in \alpha \implies \beta \in \boldsymbol{ON}$.*
2. *For all $\alpha, \beta \in \boldsymbol{ON}$, $\beta \subset \alpha \implies \beta \in \alpha$.*
3. *For all $\alpha, \beta \in \boldsymbol{ON}$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

Proof.

1. We prove that β satisfies the transitive and well-order requirements of an ordinal number.

   transitivity: We claim that for all $\gamma \in \beta$, $\delta \in \gamma \implies \delta \in \beta$ because α is well-ordered under $\Subset$, $\delta \in \gamma \in \beta$ ensures $\delta \in \beta$.

   well-order: $\beta \in \alpha \implies \beta \subset \alpha$. For any $\gamma \subseteq \beta$, $\gamma \subset \alpha$, therefore β is well-ordered.

2. Assume $\gamma = \min \alpha \setminus \beta$. We claim that $\gamma = \beta$. For one direction, $\gamma \subseteq \beta$ because $\gamma \in \alpha \setminus \beta \implies \gamma \in \alpha \implies \gamma \subset \alpha$. This means for all $\delta \in \gamma$, $\delta \in \alpha$, yet γ is the least element of $\alpha \setminus \beta$, so $\delta \notin \alpha \setminus \beta$, thus $\delta \in \beta$, proving one side of the relation.

   Conversely, we want to show that $\beta \subseteq \gamma$. We verify that for all $\delta \in \beta$, $\delta \neq \gamma$ because otherwise $\gamma \in \beta \implies \gamma \notin \alpha \setminus \beta$, which contradicts with $\gamma \in \alpha \setminus \beta$. For the same reason, because $\delta \in \beta \implies \delta \subset \beta$, $\gamma \in \delta$ would imply $\gamma \in \beta$, so $\gamma \notin \delta$. Since $\gamma \neq \delta$ and $\gamma \notin \delta$, $\delta \in \gamma$.

3. Assume $\gamma = \alpha \cap \beta$, $\gamma \in \boldsymbol{ON}$ is ensured by 1.2. We claim that either $\gamma = \alpha$ or $\gamma = \beta$, otherwise 2. implies $\gamma \in \alpha$ and $\gamma \in \beta$, which means $\gamma \in \alpha \cap \beta = \gamma$, contradicting with the trichotomy of $\Subset$.

□

Theorem 1.4. *$\boldsymbol{ON}$ is well-ordered under $\Subset$.*

Proof. It is easy to verify that $\boldsymbol{ON}$ is totally ordered, simply observe that 2. and 3. from 1.3 implies either $\alpha \Subset \beta$ or $\beta \Subset \alpha$. To show that $\boldsymbol{ON}$ is well-founded. We prove that for any subset $S \subseteq \boldsymbol{ON}$, $\min S = \bigcap S$.

By definition, for all $s \in S$, $\bigcap S \subseteq s$, this implies $\bigcap S \Subset s$ by 1.2 and 2. from 1.3, showing that $\bigcap S$ is indeed a lower-bound of S. Now we need to show that $\bigcap S \in S$. If $\bigcap S \in s$ for all $s \in S$, then by definition $\bigcap S \in \bigcap S = \bigcap S$ causing contradiction. Therefore $\bigcap S = s$ for some $s \in S$. □

Proposition 1.5. *For any $\alpha \in \boldsymbol{ON}$, $\alpha \sqcup \{\alpha\}$ is the successor of α (smallest ordinal number $\ni \alpha$).*

Proof. To show that $\alpha \sqcup \{\alpha\}$ is an ordinal number, we need to check that the axioms of ordinal number holds for $\alpha \sqcup \{\alpha\}$. Indeed, $\alpha \in \alpha \sqcup \{\alpha\} \implies \alpha \subset \alpha \sqcup \{\alpha\}$ and α is the maxima in $\alpha \sqcup \{\alpha\}$. To show that $\alpha \sqcup \{\alpha\}$ is the successor of α, observe that the existence of β in the following statement leads to a contradiction.

$$\alpha \in \beta \in \alpha \sqcup \{\alpha\}$$

β is an ordinal number according to 1. from 1.3. Since $\alpha \sqcup \{\alpha\}$ is well-ordered under $\Subset$, the trichotomy of $\Subset$ must hold. $\beta \in \alpha \sqcup \{\alpha\}$ implies either $\beta \in \alpha$ or $\beta = \alpha$, which contradicts with $\alpha \in \beta$. $\square$

Because of this property, $\alpha \in \beta$ is often denoted by $\alpha + 1$. Notice that while all ordinal numbers have successors, not all of them are successors.

Definition 1.6 (Limit ordinal)**.** A limit ordinal is an ordinal α such that for all $\beta \in \alpha$, $\beta + 1 \neq \alpha$.

One can extract the following set

$$\omega := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \cdots\} = \{0, 1, 2, \cdots\}$$

from the axiom of infinity. It is not the successor of any $\alpha \in \omega$, because this ordinal number has the property $\alpha \in \omega \implies \alpha + 1 \in \omega$. In fact, it is one representation of $\mathbb{N}$, the set of natural numbers.

Theorem 1.7 (Transfinite induction)**.** *If $S \subseteq \boldsymbol{ON}$ has the following properties, then $S = \boldsymbol{ON}$.*

1. *$\emptyset \in S$;*
2. *$\alpha \in S \implies \alpha + 1 \in S$;*
3. *$(\forall \beta \in \alpha : \beta \in S) \implies \alpha \in S$.*

Proof. Suppose not, then $\alpha \in \min \boldsymbol{ON} \setminus S$ exists by 1.4. Then whether γ is $\emptyset$, a successor or a limit ordinal will all result in contradictions. $\square$

Theorem 1.8 (Transfinite recursion)**.** *For a given function $G : \bigsqcup_{\beta \in \alpha} \boldsymbol{V}^\beta \to \boldsymbol{V}$, there exists a unique function $F_\alpha : \alpha \to \boldsymbol{V}$ such that*

$$F_\alpha\, \beta = G\, F_\alpha|_\beta$$

for all $\beta \in \alpha$.

Proof. To show that F_α is unique, we assume that F_α and F'_α are both functions satisfying the preceding requirements and construct $P := \{\beta \in \alpha : F_\alpha\, \beta \neq F'_\alpha\, \beta\}$ accordingly. Take $\min P = \beta_0$, then $F_\alpha\, \beta = F'_\alpha\, \beta$ for all $\beta \in \beta_0$. This means $F_\alpha|_{\beta_0} = F'_\alpha|_{\beta_0}$, yet $F_\alpha\, \beta_0 = G\, F_\alpha|_{\beta_0} = G\, F'_\alpha|_{\beta_0} = F'_\alpha\, \beta_0$, showing contradiction.

To show that F_α does exist, we construct $F : \boldsymbol{ON} \to \boldsymbol{V}$ by extending the domain α of F_α to $\boldsymbol{ON}$ using transfinite induction and conclude that $F|_\alpha = F_\alpha$ by the uniqueness of F_α.

1. For $\beta = 0$, we have
$$F_1\, 0 = G\, F_1|_0 = G\, 0.$$
2. Then we assume
$$F_\alpha\, \beta = G\, F_\alpha|_\beta$$
for all $\beta \in \alpha$. One can always make the extension $F_{\alpha+1} : \alpha + 1 \to \boldsymbol{V}$ since $F_{\alpha+1}$ is determined by F_α and $F_{\alpha+1}\, \alpha = G\, F_{\alpha+1}|_\alpha = G\, F_\alpha$. Now assume α is a limit ordinal.
3. If for all $\beta \in \gamma \in \alpha$, we have
$$F_\gamma\, \beta = G\, F_\gamma|_\beta$$
then one can deduce F_α from $F_\gamma\, \beta$ for all $\beta \in \alpha$, thus $F_\alpha\, \beta = G\, F_\alpha|_\beta$.

□

`Work In Progress ...`