NATURAL TRANSFORMATION

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Contents

1 Introduction 1

2 Composition of natural transformations

2

1 Introduction

Natural transformation is a convenient notion that can be intuitively thought of as morphisms between functors in the sense of "ref to def-funct-cat (WIP)". For this reason we often use diagrams



to denote natural transformation $\phi: F_1 \to F_2$.

Definition 1.1. Given two functors $F_1, F_2 : \mathbf{C} \to \mathbf{D}$, a natural transformation $\phi : F_1 \to F_2$ consists of the information

$$\{\phi_X\in \operatorname{Hom}(F_1X,F_2X): X\in \operatorname{Obj} \mathbf{C}\},$$

such that the following diagram

$$\begin{array}{ccc} F_1 X & \xrightarrow{\phi_X} & F_2 X \\ F_1 f \downarrow & & \downarrow F_2 f \\ F_1 Y & \xrightarrow{\phi_Y} & F_2 Y \end{array}$$

commutes for all $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$ for all $X,Y \in \operatorname{Obj}{\mathcal{C}}$.

2 Composition of natural transformations

We denote the head-to-tail composition of functors $\phi_1: F_1 \to F_2$, $\phi_2: F_2 \to F_3$ as $\phi_2 \circ \phi_1: F_1 \to F_3$. This composition is illustrated in the diagram

$$C \xrightarrow{F_1} D \Leftrightarrow C \xrightarrow{F_1} D .$$

$$C \xrightarrow{\phi_2 \Downarrow} D \Leftrightarrow C \xrightarrow{\phi_2 \circ \phi_1} D .$$

 $\phi_2 \circ \phi_1$ consists of information $\{(\phi_2 \circ \phi_1)_X := (\phi_2)_X \circ (\phi_2)_X\}$ in the obvious sense. It is easy to show that \circ preserves naturality. It is also associative.

There is also a way of defining "parallel" composition of natural transformations. Given $\phi: F_1 \to F_2$ and $\psi: G_1 \to G_2$, a morphism $\psi * \phi: G_1 F_1 \to G_2 F_2$ given by the diagram

$$C

\downarrow F_1 \\
F_2$$
 $D

\downarrow G_2$
 $E \implies C

\downarrow G_2F_2$
 G_1F_1
 $\psi * \phi$
 G_2F_2

is the parallel composition of the two natural transformation. Its definition is the information $\{(\psi*\phi)_X:=\psi_{F_2X}\circ G_1\phi_X=G_2\phi_X\circ\psi_{F_1X}\}$ since

$$G_1F_1X \xrightarrow{\psi_{F_1X}} G_2F_1X$$

$$G_1\phi_X \downarrow \qquad (\psi*\phi)_X \qquad \downarrow G_2\phi_X$$

$$G_1F_2X \xrightarrow{\psi_{F_2X}} G_2F_2X$$

commutes.

Lemma 2.1. * preserves naturality and is associative

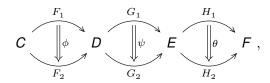
Proof. Choosing $(\psi * \phi)_X := \psi_{F_2X} \circ G_1 \phi_X$ as a representation, we show that * preserves naturality by verifying that the right and left squares of

commutes. Composing these morphisms gives the equality

$$G_2F_2f\circ (\psi_{F_2X}\circ G_1\phi_X)=(\psi_{F_2Y}\circ G_1\phi_Y)\circ G_1F_1f$$

as needed.

Next, we want to show that * is associative. That is, given

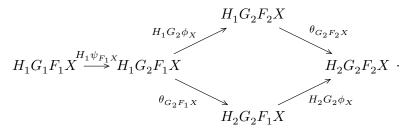


 $(\phi * \psi) * \theta = \phi * (\psi * \theta).$

 $((\phi * \psi) * \theta)_X$ and $(\phi * (\psi * \theta))_X$ are identified by the diagrams

To prove the equality, we first decompose the shaded morphisms into the two following diagrams

by the definition of parallel composition. The "middle ground" of $H_1\psi_{F_1X}:H_1G_1F_1X\to H_1G_2F_1X$ reduces the problem to the commutativity of the diagram



The naturality of θ ensures that the diagram commutes, which gives us the expected equality:

$$\begin{split} ((\phi * \psi) * \theta))_X &= \theta_{G_2 F_2 X} \circ H_1 (\psi * \phi)_X \\ &= \theta_{G_2 F_2 X} \circ (H_1 G_2 \phi_X \circ H_1 \psi_{F_1 X}) \\ &= H_2 G_2 \phi_X \circ (\theta_{G_2 F_1 X} \circ H_1 \psi_{F_1 X}) \\ &= H_2 G_2 \phi_X \circ (\theta * \psi)_{F_1 X} = ((\phi * \psi) * \theta))_X. \end{split}$$

Work In Progress ...