NATURAL TRANSFORMATION

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1 Introduction

Natural transformation is a convenient notion that can be intuitively thought of as morphisms between functors in the sense of `"ref to def-funct-cat (WIP)"`. For this reason we often use diagrams

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "F_2"'] \arrow[r, phantom, "\phi \Downarrow"] & \mathsf{D} \end{tikzcd}$$

to denote natural transformation $\phi : F_1 \to F_2$.

Definition 1.1. Given two functors $F_1, F_2 : \mathsf{C} \to \mathsf{D}$, a *natural transformation* $\phi : F_1 \to F_2$ consists of the information

$$\{\phi_X \in \mathrm{Hom}(F_1X, F_2X) : X \in \mathrm{Obj}\,\mathsf{C}\},$$

such that the following diagram

$$\begin{tikzcd} F_1X \arrow[r, "\phi_X"] \arrow[d, "F_1f"'] & F_2X \arrow[d, "F_2f"] \\ F_1Y \arrow[r, "\phi_Y"'] & F_2Y \end{tikzcd}$$

commutes for all $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ for all $X, Y \in \mathrm{Obj}\,\mathsf{C}$.

2 Composition of natural transformations

We denote the head-to-tail composition of functors $\phi_1 : F_1 \to F_2$, $\phi_2 : F_2 \to F_3$ as $\phi_2 \circ \phi_1 : F_1 \to F_3$. This composition is illustrated in the diagram

$$\begin{tikzcd} C \arrow[r, bend left=50, "F_1"{name=A}] \arrow[r, "F_2"{description, name=B}] \arrow[r, bend right=50, "F_3"'{name=C}] \arrow[from=A, to=B, phantom, "\phi_1\Downarrow"] \arrow[from=B, to=C, phantom, "\phi_2\Downarrow"] & D \end{tikzcd} \rightsquigarrow \begin{tikzcd} C \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "F_3"'] \arrow[r, phantom, "\overset{\|}{\phi_2\circ\phi_1}\Downarrow"] & D \end{tikzcd}.$$

$\phi_2 \circ \phi_1$ consists of information $\{(\phi_2 \circ \phi_1)_X := (\phi_2)_X \circ (\phi_2)_X\}$ in the obvious sense. It is easy to show that $\circ$ preserves naturality. It is also associative.

There is also a way of defining ”parallel” composition of natural transformations. Given $\phi : F_1 \to F_2$ and $\psi : G_1 \to G_2$, a morphism $\psi * \phi : G_1F_1 \to G_2F_2$ given by the diagram

$$\begin{tikzcd} C \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "F_2"'] \arrow[r, phantom, "\phi\Downarrow"] & D \arrow[r, bend left=50, "G_1"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\psi\Downarrow"] & E \end{tikzcd} \rightsquigarrow \begin{tikzcd} C \arrow[r, bend left=50, "G_1F_1"] \arrow[r, bend right=50, "G_2F_2"'] \arrow[r, phantom, "\overset{\|}{\psi*\phi}\Downarrow"] & E \end{tikzcd}$$

is the parallel composition of the two natural transformation. Its definition is the information $\{(\psi * \phi)_X := \psi_{F_2X} \circ G_1\phi_X = G_2\phi_X \circ \psi_{F_1X}\}$ since

$$\begin{tikzcd} G_1F_1X \arrow[r, "\psi_{F_1X}"] \arrow[d, "G_1\phi_X"'] \arrow[dr, dashed, "(\psi*\phi)_X"] & G_2F_1X \arrow[d, "G_2\phi_X"] \\ G_1F_2X \arrow[r, "\psi_{F_2X}"'] & G_2F_2X \end{tikzcd}$$

commutes.

Lemma 2.1. *$*$ preserves naturality and is associative*

Proof. Choosing $(\psi * \phi)_X := \psi_{F_2X} \circ G_1\phi_X$ as a representation, we show that $*$ preserves naturality by verifying that the right and left squares of

$$\begin{tikzcd} G_1F_1X \arrow[r, "G_1\phi_X"] \arrow[d, "G_1F_1f"'] & G_1F_2X \arrow[r, "\psi_{F_2X}"] \arrow[d, dashed, "G_1F_2f"] & G_2F_2X \arrow[d, "G_2F_2f"] \\ G_1F_1Y \arrow[r, "G_1\phi_Y"'] & G_1F_2Y \arrow[r, "\psi_{F_2Y}"'] & G_2F_2Y \end{tikzcd}$$

commutes. Composing these morphisms gives the equality

$$G_2F_2f \circ (\psi_{F_2X} \circ G_1\phi_X) = (\psi_{F_2Y} \circ G_1\phi_Y) \circ G_1F_1f$$

as needed.

Next, we want to show that $*$ is associative. That is, given

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathsf{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathsf{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "H_2"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathsf{F}
\end{tikzcd},$$

$(\phi * \psi) * \theta = \phi * (\psi * \theta)$.

$((\phi * \psi) * \theta)_X$ and $(\phi * (\psi * \theta))_X$ are identified by the diagrams

$$\begin{tikzcd}
H_1(G_1F_1X) \arrow[r, "\theta_{G_1F_1X}"] \arrow[d, "H_1(\psi*\phi)_X"'] \arrow[dr, dashed, "(\theta*(\psi*\phi))_X" description] & H_2(G_1F_1X) \arrow[d, "H_2(\psi*\phi)_X"] \\
H_1(G_2F_2X) \arrow[r, "\theta_{G_2F_2X}"'] & H_2(G_2F_2X)
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1G_1(F_1X) \arrow[r, "(\theta*\psi)_{(F_1X)}"] \arrow[d, "H_1G_1\phi_X"'] \arrow[dr, dashed, "((\phi*\psi)*\theta))_X" description] & H_2G_2(F_1X) \arrow[d, "H_2G_2\phi_X"] \\
H_1G_1(F_2X) \arrow[r, "(\theta*\psi)_{(F_2X)}"'] & H_2G_2(F_2X)
\end{tikzcd}.$$

To prove the equality, we first decompose the shaded morphisms into the two following diagrams

$$\begin{tikzcd}
H_1(G_1F_1X) \arrow[r, "H_1(\psi_{F_1X})"] \arrow[dr, dashed, "H_1(\psi*\phi)_X"'] & H_1(G_2F_1X) \arrow[d, "H_1(G_2\phi_X)"] \\
& H_1(G_2F_2X)
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1G_1(F_1X) \arrow[d, "H_1\psi_{(F_1X)}"'] \arrow[dr, dashed, "(\theta*\psi)_{(F_1X)}"] & \\
H_1G_2(F_1X) \arrow[r, "\theta_{G_2(F_1X)}"'] & H_2G_2(F_1X)
\end{tikzcd}$$

by the definition of parallel composition. The "middle ground" of $H_1\psi_{F_1X} : H_1G_1F_1X \to H_1G_2F_1X$ reduces the problem to the commutativity of the diagram

$$\begin{tikzcd}
& & H_1G_2F_2X \arrow[dr, "\theta_{G_2F_2X}"] & \\
H_1G_1F_1X \arrow[r, "H_1\psi_{F_1X}"] & H_1G_2F_1X \arrow[ur, "H_1G_2\phi_X"] \arrow[dr, "\theta_{G_2F_1X}"'] & & H_2G_2F_2X \\
& & H_2G_2F_1X \arrow[ur, "H_2G_2\phi_X"'] &
\end{tikzcd}.$$

The naturality of θ ensures that the diagram commutes, which gives us the expected equality:

$$\begin{aligned}
((\phi * \psi) * \theta))_X &= \theta_{G_2F_2X} \circ H_1(\psi * \phi)_X \\
&= \theta_{G_2F_2X} \circ (H_1G_2\phi_X \circ H_1\psi_{F_1X}) \\
&= H_2G_2\phi_X \circ (\theta_{G_2F_1X} \circ H_1\psi_{F_1X}) \\
&= H_2G_2\phi_X \circ (\theta * \psi)_{F_1X} = ((\phi * \psi) * \theta))_X.
\end{aligned}$$

$\square$

`Work In Progress ...`