WHAT IS A GROUP

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Concretely, (By Cayley's theorem) all groups are transformation groups (automorphism groups), that is, we can view groups as morphisms

$$\_\circ\_ : \mathrm{Aut}_{\mathcal{C}}X \sqcap \mathrm{Aut}_{\mathcal{C}}X \to \mathrm{Aut}_{\mathcal{C}}X$$

induced by the composition of morphisms in any category $\mathcal{C}$.

Abstractly, a group is a *group object* in Set. The concept of a group object models the properties of an automorphism group through categorical notions. It is by definition the information $\_\bullet\_ : X \sqcap X \to X$, $e : 1 \to X$ and $\diagup : X \to X$, where 1 is the final object of the given category. It's axioms are illustrated in the following diagrams:

- Associativity:

$$\begin{tikzcd}
 & X \sqcap X \sqcap X \arrow[dl, "(1,\bullet)"'] \arrow[dr, "(\bullet,1)"] & \\
X \sqcap X \arrow[dr, "\bullet"'] & & X \sqcap X \arrow[dl, "\bullet"] \\
 & X &
\end{tikzcd}$$

- Identity:

$$\begin{tikzcd}
 & X \arrow[dl, "\sim"'] \arrow[dd, "1"] \arrow[dr, "\sim"] & \\
1 \sqcap X \arrow[dr, "\bullet\circ(e,1)"'] & & X \sqcap 1 \arrow[dl, "\bullet\circ(1,e)"] \\
 & X &
\end{tikzcd}$$

- Inverse elements:

$$\begin{tikzcd}
 & X \arrow[dl, "1\sqcap\diagup"'] \arrow[d] \arrow[dr, "\diagup\sqcap 1"] & \\
X \sqcap X \arrow[dr, "\bullet"'] & 1 \arrow[d, "e"'] & X \sqcap X \arrow[dl, "\bullet"] \\
 & X &
\end{tikzcd}$$

- An *abelian* group object satisefies an additional property of commutativity:

$$\begin{tikzcd}
X \sqcap X \arrow[rr, "p"] \arrow[dr, "\bullet"'] & & X \sqcap X \arrow[dl, "\bullet"] \\
& X &
\end{tikzcd}$$

where $p : X \sqcap Y \to Y \sqcap X$ is the "permutation" isomorphism between products.

In this sense, assume $(\bullet_X, e_X, \diagup_X)$, $(\bullet_Y, e_Y, \diagup_Y)$, are two group objects of the same category, $\phi : X \to Y$ is a group (object) homomorphism if and only if the diagrams

$$\begin{tikzcd}
X \sqcap X \arrow[r, "{(\phi,\phi)}"] \arrow[d, "\bullet_X"'] & Y \sqcap Y \arrow[d, "\bullet_Y"] \\
X \arrow[r, "\phi"'] & Y
\end{tikzcd}
\quad
\begin{tikzcd}
& 1 \arrow[dl, "e_X"'] \arrow[dr, "e_Y"] & \\
X \arrow[rr, "\phi"'] & & Y
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\diagup_X"'] & Y \arrow[d, "\diagup_Y"] \\
X \arrow[r, "\phi"'] & Y
\end{tikzcd}$$

commute.