# Construction of $\mathbb{Q}$

KeChao Chen
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The rational numbers are often defined to be

$$\mathbb{Q} := \{\frac{p}{q} : p, q \in \mathbb{Z}\}.$$

This definition is incomplete since we haven't defined what the notation $\frac{p}{q}$ really means. For instance, if one defines $\frac{p}{q}$ to be the ordered pairs $(p, q) \in \mathbb{Z} \sqcap \mathbb{Z}$, then $\frac{1}{2}$ and $\frac{2}{4}$ are two different elements in $\mathbb{Q}$ . This causes addition to fail on $\mathbb{Q}$. For instance, if one wants to compute the sum $\frac{1}{2} + \frac{1}{3}$, it fails to identify with $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$. What if the sum is defined to be $\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1+p_2}{q_1+q_2}$? Then we see that $\mathbb{Q}$ is really just a relabelling of $\mathbb{Z} \times \mathbb{Z}$ through the isomorphism $(p, q) \mapsto \frac{p}{q}$, which is not very interesting. Here is a more interesting definition of $\mathbb{Q}$:

The rational numbers are defined to be the quotient $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))/\sim$ through an equivalence relation $\sim$ such that the identification $(p, q) \sim (p', q')$ is true if and only if $pq' = qp'$. We prove that this relation is indeed an equivalence relation and denote the equivalence class $[(p, q)]_\sim$ by $\frac{p}{q}$ . Then we introduce the arithmetic on $\mathbb{Q}$ and show that there is a canonical injection that maps each integer $p$ to a unique equivalence class $\frac{p}{1}$ in $\mathbb{Q}$. (This is one instance of a more general construction called the *Grothendieck group*.)

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## $\mathbb{Q}$ as a quotient

**Proposition:** $\sim$ is an equivalence relation

Proof.

Reflexivity: $(p, q) \sim (p, q)$ because $pq = qp$ (commutativity of $\mathbb{Z}$);

Symmetry: $(p_1, q_1) \sim (p_2, q_2)$ implies $(p_2, q_2) \sim (p_1, q_1)$ because $p_1 q_2 = q_1 p_2$ implies $p_2 q_1 = q_2 p_1$ (commutativity of $\mathbb{Z}$ and symmetry of $=$).

Transitivity: $(p_1, q_1) \sim (p, q)$ and $(p_2, q_2) \sim (p_2, q_2)$ implies $(p_1, q_1) \sim (p_3, q_3)$.

From $p_1 q_2 = p_2 q_1$ and $p_2 q_3 = p_3 q_2$ we observe that

$$\begin{aligned} p_1 q_2 p_2 q_3 &= p_2 q_1 p_3 q_2 \\ p_1 q_3 (p_2 q_2) &= p_3 q_1 (p_2 q_2) \end{aligned}$$

Using the cancellation property of $\mathbb{Z}$, we see that $p_1 q_3 = p_3 q_1$. $\square$

We can now safely denote $[(p, q)]_\sim$ by the fraction notation $\frac{p}{q}$ that we are used to.

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**Lemma:** Cancellation property of $\mathbb{Z}$

The equality $ab = ac$ implies $b = c$ for all $a, b, c \in \mathbb{Z}$ such that $a \neq 0$.

Proof.

Notice that we can't derive $b = c$ from $ab = ac$ straight away because a multiplicative inverse of $a$, that is $\frac{1}{a}$, does not exist in $\mathbb{Z}$, otherwise we can directly multiply $\frac{1}{a}$ on both sides of the equation to get the result. In fact, one reason that a proper definition of $\mathbb{Q}$ is so important is that it allows us to construct the inverse of $a$. Because $\mathbb{Z}$ is a ring, we can use the distributive property to get:

$$\begin{aligned} ab &= ac \\ ab - ac &= ac - ac \\ ab - ac &= 0 \\ a(b - c) &= 0. \end{aligned}$$

Because we have assumed that $a \neq 0$, therefore $(b - c) = 0$. This is because $\mathbb{Z}$ is a principle domain. Thus we can conclude that:

$$\begin{aligned} b - c &= 0 \\ b - c + c &= 0 + c \\ b &= c. \end{aligned}$$

$\square$

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# Arithmetic on $\mathbb{Q}$

When dealing with arithmetic on equivalence classes, we have to first prove that our definitions are well defined. That is, computing two different representatives of the same equivalent class yields the same result.

## Addition on $\mathbb{Q}$

We have to bear in mind that we want $\mathbb{Q}$ to extend $\mathbb{Z}$. The identification $p \mapsto \frac{p}{1}$ seems the most natural to us, so we want to make sure the additive structure of $\mathbb{Z}$ is preserved through this injection, meaning $p + q \mapsto \frac{p}{1} + \frac{q}{1} = \frac{p+q}{1}$.

**Definition:** addition to $\mathbb{Q}$ is defined by

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + q_1 p_2}{q_1 q_2}.$$

This definition certainly meets our requirement since

$$\frac{p_1}{1} + \frac{p_2}{1} = \frac{p_1 \times 1 + 1 \times p_2}{1 \times 1} = \frac{p_1 + p_2}{1}.$$

**Another possible solution may be $\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1+q_1}{q_1 q_2}$, but the reader should check on his/her own that this definition is not well defined on $\mathbb{Q}$.*

Now we prove that this definition is well defined on $\mathbb{Q}$.

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**Proposition:** Addition is well defined on $\mathbb{Q}$

Proof.

$\frac{p_1}{q_1} = \frac{p_1'}{q_1'}$ and $\frac{p_2}{q_2} = \frac{p_2'}{q_2'}$ implies $p_1 q_1' = q_1 p_1'$ and $p_2 q_2' = q_2 p_2'$.

We want to show that

$$\begin{aligned}\frac{p_1}{q_1}+\frac{p_2}{q_2}&=\frac{p_1'}{q_1'}+\frac{p_2'}{q_2'}\\ \frac{p_1q_2+q_1p_2}{q_1q_2}&=\frac{p_1'q_2'+q_1'p_2'}{q_1'q_2'}.\end{aligned}$$

This is by definition,

$$\begin{aligned}(p_1q_2+q_1p_2)(q_1'q_2')&=(p_1'q_2'+q_1'p_2')(q_1q_2)\\ p_1q_2q_1'q_2'+q_1p_2q_1'q_2'&=p_1'q_2'q_1q_2+q_1'p_2'q_1q_2\\ p_1q_1'q_2q_2'+q_1q_1'p_2q_2'&=q_1p_1'q_2q_2'+q_1q_1'q_2p_2'.\end{aligned}$$

We finish the proof by substituting $p_1q_1'=q_1p_1'$ and $p_2q_2'=q_2p_2'$ into the equation. □

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## Multiplication on $\mathbb{Q}$

The definition of multiplication seems much more natural on $\mathbb{Q}$. This should be no surprise since the principal motive for the construction of $\mathbb{Q}$ is to find multiplicative inverses for elements in $\mathbb{Z}$.

**Definition:** multiplication on $\mathbb{Q}$ is defined by

$$\frac{p_1}{q_1}\times\frac{p_2}{q_2}=\frac{p_1\times p_2}{q_1\times q_2}.$$

The routine proof to preserve multiplicative structure from $\mathbb{Z}$ to $\mathbb{Q}$ and the well-definedness of this definition are left to the readers as an exercise.

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# Order on $\mathbb{Q}$

A notion of order, that is the ability to compare the "size" of distinct elements is a very important concept in analysis, many important definitions rely on it.

In $\mathbb{Z}$, we say $a>b$ when $a-b$ is positive, that is $a-b>0$. We assume that we can compare the size of any integer with $0$. (A more formal proof traces down to the Peano axioms for the natural numbers). Let's extend this definition to elements in $\mathbb{Q}$ by setting $\frac{p_1}{q_1}>\frac{p_2}{q_2}$ whenever

$\frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{p_1q_2 - q_1p_2}{q_1q_2}$ is positive. Another problem occurs to us, that is, we don't know how to compare the size of a rational number to $\frac{0}{1}$. By noticing that $\frac{p}{q} = \frac{-p}{-q}$ and $\frac{p}{-q} = \frac{-p}{q}$ through the equivalence relation and that $\frac{p}{1}$ should be larger than $\frac{0}{1}$ in $\mathbb{Q}$, let's try to define $\frac{p}{q} > \frac{0}{1}$ when $p$ and $q$ have the same sign. This idea is polished in the following definition.

**Definition:** $\frac{p}{q}$ is positive ($\frac{p}{q} > \frac{0}{1}$) if and only if $pq > 0$.

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**Proposition:** Order is well-defined on $\mathbb{Q}$.

Suppose $\frac{p}{q} = \frac{p'}{q'}$, and $pq > 0$, then $p'q' > 0$.

Proof.

Since $pq' = qp'$ by definition, multiplying both sides by $pq'$ will give us

$$(pq')^2 = qp'pq' = (pq)(p'q')$$

Because $p, q' \neq 0$, $(pq')^2$ is positive. Since $pq$ is already positive, $p'q'$ must have the same sign, and therefore is positive too. □

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**Definition:** $\frac{p_1}{q_1} > \frac{p_2}{q_2}$ if and only if

$$(p_1q_2 - q_1p_2)(q_1q_2) > 0$$

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# $\mathbb{Q}$ is an extension to $\mathbb{Z}$

**Theorem:** There exists a canonical inclusion from $\mathbb{Z}$ to $\mathbb{Q}$.

Proof.

We had in mind the preservation of arithmetic structure from $\mathbb{Z}$ to $\mathbb{Q}$ when we came up with the definitions for addition and multiplication on $\mathbb{Q}$, so it should be no surprise that all integers $p$ can be identified with a unique rational number $\frac{p}{1}$ through an inclusion:

$$\begin{array}{ccc} \mathbb{Z} & \hookrightarrow & \mathbb{Q} \\ p & \mapsto & \frac{p}{1} \end{array}.$$