WHAT IS A RING

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April 2021

A *ring* is an extension of the abelian group in the sense that a ring consists of both the commutative + ("addition") operation of an abelian group and a × ("multiplication") operation (whose symbol is often neglected in practice). These properties can be defined naturally on the endomorphisms of any abelian groups.

Addition on $\mathrm{End}_{\mathsf{Abel}}A$ is inherited from the addition of A in the sense that for all $\phi, \psi \in \mathrm{End}_{\mathsf{Abel}}A$, there is a

$$\phi + \psi : a \mapsto \phi\, a + \psi\, a$$

One can easily check that indeed $\phi + \psi \in \mathrm{End}_{\mathsf{Abel}}A$.

$$\begin{aligned}(\phi + \psi)(a + b) &= \phi(a + b) + \psi(a + b)\\ &= \phi\, a + \phi\, b + \psi\, a + \psi\, b\\ &= \phi\, a + \psi\, a + \phi\, b + \psi\, b\\ &= (\phi + \psi)\, a + (\phi + \psi)\, b\end{aligned}$$

Notice that the commutativity of A is necessary, otherwise $\phi+\psi$ would not have been a group homomorphism. That is, addition cannot be defined on $\mathrm{End}_{\mathsf{Grp}}G$. The additive identity of $\mathrm{End}_{\mathsf{Abel}}A$ is the trivial map 0 and the inverse of ϕ is $-\phi$ defined by

$$-\phi : a \mapsto -(\phi\, a)$$

Which is yet another map that is a homomorphism if and only if A is abelian.

The multiplication operation is defined by the natural composition of homomorphisms in $\mathrm{End}_{\mathsf{Abel}}A$.

$$\psi \times \phi := \psi \circ \phi$$

Therefore the multiplicative identity is the identity map $\mathrm{id} \in \mathrm{End}_{\mathsf{Abel}}A$.

The two operations of a ring corresponds naturally by the two distributive properties.

$$\theta \times (\phi + \psi) = \theta \times \phi + \theta \times \psi$$

and

$$(\phi + \psi) \times \theta = \phi \times \theta + \psi \times \theta$$

This is because for all $a \in A$,

$$\begin{aligned}\theta \times (\phi + \psi)\, a &= \theta \times (\phi\, a + \psi\, a)\\ &= \theta \times \phi\, a + \theta \times \psi\, a\\ &= (\theta \times \phi + \theta \times \psi)\, a.\end{aligned}$$

The right distributive property is given by the same reasoning.

The following commutative diagrams categorize the additional axioms of the *ring object*, making it an extension of the abelian group object. (A ring object is the information $(+, e_+, -, \times, e_\times)$.)

- Multiplicative associativity:

$$\begin{tikzcd}
& X \sqcap X \sqcap X \arrow[dl, "(1{,}\times)"'] \arrow[dr, "(\times{,}1)"] & \\
X \sqcap X \arrow[dr, "\times"'] & & X \sqcap X \arrow[dl, "\times"] \\
& X &
\end{tikzcd}$$

- Multiplicative identity:

$$\begin{tikzcd}
& X \arrow[dl, "\sim"'] \arrow[dd, "1"] \arrow[dr, "\sim"] & \\
1 \sqcap X \arrow[dr, "\times\circ(e_\times{,}1)"'] & & X \sqcap 1 \arrow[dl, "\times\circ(1{,}e_\times)"] \\
& X &
\end{tikzcd}$$

- Left distributivity:

$$\begin{tikzcd}
X \sqcap X \sqcap X \arrow[r, "((\_{}_1 \times \_{}_2){,}(\_{}_1 \times \_{}_3))"] \arrow[d, "(1{,}+)"'] & X \sqcap X \arrow[d, "+"] \\
X \sqcap X \arrow[r, "\times"'] & X
\end{tikzcd}$$

- Right distributivity:

$$\begin{tikzcd}
X \sqcap X \sqcap X \arrow[r, "((\_{}_1 \times \_{}_3){,}(\_{}_2 \times \_{}_3))"] \arrow[d, "(+{,}1)"'] & X \sqcap X \arrow[d, "+"] \\
X \sqcap X \arrow[r, "\times"'] & X
\end{tikzcd}$$

Some may question the choice of using $\mathrm{End}_{\mathsf{Abel}} A$ instead of $\mathrm{Aut}_{\mathsf{Abel}} A$ as the multiplicative structure for a ring. This is because addition is not well defined on the automorphisms of abelian groups.

A simple explanation is that $\mathrm{Aut}_{\mathsf{Abel}} A$ does not contain the trivial map 0 that is the identity of the abelian group. In fact, $\mathrm{Aut}_{\mathsf{Abel}} A$ is not even closed under addition. Given that $1 \in \mathrm{Aut}_{\mathsf{Abel}} A$, one can always construct maps

$$m1 := \underbrace{1 + \cdots + 1}_{m} : g \mapsto mg.$$

In particular, $\mathrm{lcm}(|a|, |b|)\,\mathrm{id}$ maps both a and b onto the identity, which is not an isomorphism.

A ring is a *division ring* if $\mathrm{End}_{\mathsf{Abel}} A \backslash \{0\} = \mathrm{Aut}_{\mathsf{Abel}} A$. One can then show that $\mathrm{End}_{\mathsf{Abel}} A$ is a division ring if and only if all elements other than the identity of A have the same order.

A commutative division ring is a *field.*

On the other hand, a weaker notion of a ring is that of a *pseudo-ring.* Take an arbitrary ring R, $mR := \{mr : m \in \mathbb{Z},\ r \in R\}$ is a ring if and only if for some $r \in R$, $mr = 1$, otherwise mR is a ring without multiplicative identity, making it a *pseudo-ring.*