NATURAL TRANSFORMATION (WIP)

Kechao Chen

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1 Introduction

Natural transformation is a convenient notion that can be intuitively thought of as morphisms between functors in the sense of "ref to def-funct-cat". For this reason we often use diagrams



to denote natural transformation $\phi: F_1 \to F_2$.

Definition 1.1. Given two functors $F_1, F_2 : \mathbf{C} \to \mathbf{D}$, a natural transformation $\phi : F_1 \to F_2$ consists of the information

$$\{\phi_X \in \operatorname{Hom}(F_1X, F_2X) : X \in \operatorname{Obj} \mathbf{C}\},\$$

such that the following diagram

$$\begin{array}{ccc} F_1 X \stackrel{\phi_X}{\longrightarrow} F_2 X \\ F_1 f \!\!\! \downarrow & & \!\!\!\! \downarrow F_2 f \\ F_1 Y \stackrel{\phi_Y}{\longrightarrow} F_2 Y \end{array}$$

commutes for all $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$ for all $X,Y \in \operatorname{Obj}{\mathcal{C}}$.

2 Composition of natural transformations

We denote the head-to-tail composition of functors $\phi_1: F_1 \to F_2$, $\phi_2: F_2 \to F_3$ as $\phi_2 \circ \phi_1: F_1 \to F_3$. This composition is illustrated in the diagram

$$C \xrightarrow{F_1} D \Leftrightarrow C \xrightarrow{F_1} D .$$

$$C \xrightarrow{\phi_2 \Downarrow} D \Leftrightarrow C \xrightarrow{\phi_2 \circ \phi_1} D .$$

 $\phi_2 \circ \phi_1$ consists of information $\{(\phi_2 \circ \phi_1)_X := (\phi_2)_X \circ (\phi_2)_X\}$ in the obvious sense. It is easy to show that \circ preserves naturality and isomorphism. It is also associative.

There is also a way of defining "parallel" composition of natural transformations. Given $\phi: F_1 \to F_2$ and $\psi: G_1 \to G_2$, a morphism $\psi * \phi: G_1F_1 \to G_2F_2$ given by the diagram

$$C \underbrace{\phi \downarrow}_{F_2} D \underbrace{\psi \downarrow}_{G_2} E \Rightarrow C \underbrace{\psi * \phi}_{G_2 F_2} E$$

is the parallel composition of the two natural transformation. Its definition is the information $\{(\psi*\phi)_X:=\psi_{F_2X}\circ G_1\phi_X=G_2\phi_X\circ\psi_{F_1X}\}$ since

$$G_1F_1X \xrightarrow{\psi_{F_1X}} G_2F_1X$$

$$G_1\phi_X \downarrow \qquad (\psi * \phi)_X \qquad \downarrow G_2\phi_X$$

$$G_1F_2X \xrightarrow{\psi_{F_2X}} G_2F_2X$$

commutes.

Lemma 2.1. * preserves naturality, isomorphism and is associative.

Proof. Choosing $(\psi * \phi)_X := \psi_{F_2X} \circ G_1 \phi_X$ as a representation, we show that * preserves naturality by verifying that the right and left squares of

$$\begin{array}{ccc} G_1F_1X \stackrel{G_1\phi_X}{\longrightarrow} G_1F_2X \stackrel{\psi_{F_2X}}{\longrightarrow} G_2F_2X \\ G_1F_1f \Big| & G_1\overset{f}{F_2}f & \Big| G_2F_2f \\ G_1F_1Y \xrightarrow[G_1\phi_Y]{} G_1F_2Y \xrightarrow[\psi_{F_2Y}]{} G_2F_2Y \end{array}$$

commutes. Composing these morphisms gives the equality

$$G_2F_2f\circ (\psi_{F_2X}\circ G_1\phi_X)=(\psi_{F_2Y}\circ G_1\phi_Y)\circ G_1F_1f$$

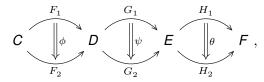
as needed.

Next, given that ϕ and ψ are both isomorphisms,

$$(\psi*\phi)_X^{-1} = (\psi_{F_2X} \circ G_1\phi_X)^{-1} = (G_1\phi_X)^{-1} \circ (\psi_{F_2X})^{-1} = G_1\phi_X^{-1} \circ \psi_{F_2X}^{-1}$$

implies $\psi * \phi$ is an isomorphism.

We also want to show that * is associative. That is, given

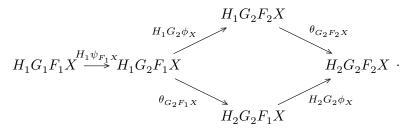


 $(\phi * \psi) * \theta = \phi * (\psi * \theta).$

 $((\phi * \psi) * \theta)_X$ and $(\phi * (\psi * \theta))_X$ are identified by the diagrams

To prove the equality, we first decompose the shaded morphisms into the two following diagrams

by the definition of parallel composition. The "middle ground" of $H_1\psi_{F_1X}:H_1G_1F_1X\to H_1G_2F_1X$ reduces the problem to the commutativity of the diagram



The naturality of θ ensures that the diagram commutes, which gives us the expected equality:

$$\begin{split} ((\phi * \psi) * \theta))_X &= \theta_{G_2F_2X} \circ H_1(\psi * \phi)_X \\ &= \theta_{G_2F_2X} \circ (H_1G_2\phi_X \circ H_1\psi_{F_1X}) \\ &= H_2G_2\phi_X \circ (\theta_{G_2F_1X} \circ H_1\psi_{F_1X}) \\ &= H_2G_2\phi_X \circ (\theta * \psi)_{F_1X} = ((\phi * \psi) * \theta))_X. \end{split}$$

