GIRSANOV THEORY AND ITS APPLICATIONS

U7110834

1. Introduction

Suppose $(\Omega, \Sigma, \mathbb{P})$ is given and $(X_t)_{t\geq 0}$ is a process defined by $dX_t = dB_t + \mu(\omega, t)dt$, where $\mu \neq 0$ is a process adapted to the standard Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$. We can easily check that $(X_t)_{t\geq 0}$ is not a martingale for the filtration $(\mathcal{F}_t)_{t\geq 0}$ because of the drift term $\mu(\omega, t)dt$, and this might put limitations on how we can approach problems involving such a process. However, by re-introducing a suitable measure on the measurable space (Ω, Σ), we may be able to ignore the drift and view the process as a standard Brownian motion under the new measure. Girsanov theory precisely tells us what measures we should use and under what conditions we can remove drift. It also tells us that instead of removing drift, we may swap the drift for another.

The theory provides a series of essential tools not only for removing or adding drift but also for transforming a non-martingale process into a martingale by a change of measure, the latter of which has been of great importance in mathematical finance (as demonstrated in Ranit's video). In this essay, we shall look at some of the critical theorems in the theory and one of its applications: the transformation of the discounted stock price into a martingale. We will also look at how Novikov's condition given in the textbook can be generalized.

Throughout the essay, when we say that a process is a martingale, we mean that it is a martingale for the standard Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$ on a given probability space.

2. Tilting Formula

Let $(X)_{t\geq 0}$ be given by $X_t = B_t + \mu t$, $\mu \in \mathbb{R}$. The tilting formula is an important tool in Girsanov theory, which states that when we compute the expectation of a function of X_{t_i}'s, $t_1 < \cdots < t_n$, we can ignore the drift term by introducing the process $(M_t)_{t\geq 0}$ defined below so that we would only need to compute the expectation of a function of B_{t_i}'s (and B_T).

Date: December 15, 2023.

Theorem 2.1 (Tilting formula). *For any* $0 = t_0 < t_1 < \cdots < t_n \leq T$ *and bounded Borel* $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathbb{E}(f(X_{t_1}, .., X_{t_n})) = \mathbb{E}(f(B_{t_1}, .., B_{t_n})M_T), \tag{1}$$

where $(M_t)_{t\geq 0}$, $M_t := e^{\mu B_t - \frac{1}{2}\mu^2 t}$ *is a martingale.*

Proof. It was already shown in example 5.6 in the lecture notes that $(M_t)_{t\geq 0}$ was a martingale. Since $B_{t_i} - B_{t_{i-1}}$ are independent, $X_{t_i} - X_{t_{i-1}}$ are also independent. Then noting $X_0 = 0$, the joint distribution ρ_X of $(X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}})$ is the product of the densities of the increments of the X_{t_i}'s. Thus, since $X_{t_i} - X_{t_{i-1}} \sim \mathcal{N}(\mu(t_i - t_{i-1}), t_i - t_{i-1})$, by letting $C := \prod_{i=1}^n 1/\sqrt{2\pi(t_i - t_{i-1})}$ and $D := exp\left(\sum_{i=1}^n (x_i - x_{i-1})^2/2(t_i - t_{i-1})\right)$ and taking $x_0 = 0$ as $X_0 = 0$,

$$\begin{aligned}\rho_X(x_1, \dots, x_n - x_{n-1}) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} exp\left(-\frac{((x_i - x_{i-1}) - \mu(t_i - t_{i-1}))^2}{2(t_i - t_{i-1})}\right) \\ &= C \cdot D \cdot exp(\mu x_n - \mu^2 t_n/2).\end{aligned}$$

In a similar manner we can show that that the joint distribution $\rho_B(x_1, x_2 - x_1, \dots, x_n - x_{n-1})$ of $(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})$ is simply $C \cdot D$. Noting that there exists $g : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_1, \dots, x_n) = g(x_1, \dots, x_n - x_{n-1})$, the law of unconscious statistician tells us that

$$\begin{aligned}\mathbb{E}(f(X_{t_0}, ..X_{t_n})) &= \int_{\mathbb{R}^n} g(\tilde{x})\rho_X(\tilde{x})\ d\tilde{x} \\ &= \int_{\mathbb{R}^n} g(\tilde{x})\rho_B(\tilde{x})e^{\mu x_n - \mu^2 t_n/2}\ d\tilde{x} \\ &= \mathbb{E}(g(B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})M_{t_n}) \\ &= \mathbb{E}(f(B_{t_1}, \dots, B_{t_n})M_{t_n}),\end{aligned}$$

where $\tilde{x} = (x_1, \dots, x_n - x_{n-1})$. Finally, by the martingale property of $(M_t)_{t\geq 0}$, for any $T \geq t_n$, using the standard Brownian filtration $(\mathcal{F})_{t\geq 0}$ we obtain

$$\begin{aligned}\mathbb{E}(f(B_{t_1}, \dots, B_{t_n})M_T) &= \mathbb{E}(\mathbb{E}(f(B_{t_1}, \dots, B_{t_n})M_T|\mathcal{F}_{t_n})) \\ &= \mathbb{E}(f(B_{t_1}, \dots, B_{t_n})\mathbb{E}(M_T|\mathcal{F}_{t_n})) \\ &= \mathbb{E}(f(B_{t_1}, \dots, B_{t_n})M_{t_n}) \\ &= \mathbb{E}(f(X_{t_0}, ..X_{t_n})).\end{aligned}$$

□

3. Measure on $(C[0,T], \mathcal{B})$

The theorems which are to be introduced in the next few sections concern a particular measurable space, namely $(C[0,T], \mathcal{B})$, where $C[0,T]$ is the set of all continuous functions on [0,T], $\mathcal{B}$ is the Borel σ-algebra defined in accordance with the supremum norm $\|\cdot\|_\infty$, and its measure is to be determined.

Given a continuous process $(X_t)_{t\geq 0}$ on some $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mathbb{P}})$, for each $\tilde{\omega} \in \tilde{\Omega}$ the continuous path $t \mapsto X_t(\tilde{\omega})$ restricted to the interval $[0,T]$ is an element of $C[0,T]$. Thus the function $X : \tilde{\Omega} \to C[0,T]$ given by $X(\tilde{\omega}) = (t \mapsto X_t(\tilde{\omega}))|_{[0,T]}$ is well-defined, and we can use this function to define a measure on $(C[0,T], \mathcal{B})$.

Proposition 3.1 (Measure induced by $(X_t)_{t\geq 0}$). *Given a continuous process $(X_t)_{t\geq 0}$ on some $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mathbb{P}})$, the mapping $Q : \mathcal{B} \to [0, \infty)$ defined by*

$$Q(A) = \tilde{\mathbb{P}}(X^{-1}(A)) \quad \forall A \in \mathcal{B},$$

where X is defined as above, is a probability measure on the event space $(C[0,T], \mathcal{B})$ and is called the measure on $(C[0,T], \mathcal{B})$ induced by $(X_t)_{t\geq 0}$.

Proof. In [1], Steel did not mention that X is actually Borel measurable, so we must verify that $X^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}$ to check that Q is well-defined. Since the collection of all open balls in $C[0,T]$ generates $\mathcal{B}$, it suffices to check $X^{-1}(B_r(g)) \in \mathcal{B}$ for any open ball $B_r(g) \subset C[0,T]$ of radius $r > 0$ centered at $g \in C[0,T]$. Observe that $\tilde{\omega} \in X^{-1}(B_r(g))$ if and only if there exists $0 < p < r$ such that $|X_t(\tilde{\omega}) - g(t)| \leq p$ for all $t \in [0,T]$ by the definition of $\|\cdot\|_\infty$ and the continuity of $(X_t)_{t\geq 0}$. It follows from continuity and the density of $\mathbb{Q}$ in $\mathbb{R}$ that

$$X^{-1}(B_r(g)) = \bigcup_{p\in(0,r)\cap\mathbb{Q}} \bigcap_{q\in[0,T]\cap\mathbb{Q}} \{\tilde{\omega} : |X_q(\tilde{\omega}) - g(q)| \leq p\} = \bigcup_{p\in(0,r)\cap\mathbb{Q}} \bigcap_{q\in[0,T]\cap\mathbb{Q}} X_q^{-1}([g(q)-p, g(q)+p]).$$

Each $[g(q) - p, g(q) + p] \subset \mathbb{R}$ is Borel in $\mathbb{R}$. Hence $X_q^{-1}([g(q) - p, g(q) + p]) \in \tilde{\Sigma}$, and so $X^{-1}(B_r(g)) \in \tilde{\Sigma}$.

Finally, noting $X^{-1}(\bigcup_{k=1}^\infty A_k) = \bigcup_{k=1}^\infty X^{-1}(A_k)$ for any mutually disjoint $(A_k)_{k\in\mathbb{N}} \subset \mathcal{B}$, The fact that Q is a probability measure immediately follows from key properties of $\tilde{\mathbb{P}}$ as a probability measure. $\square$

From now on P denotes the measure on $(C[0,T], \mathcal{B})$ induced by a standard Brownian motion $(B_t)_{t\geq 0}$ unless otherwise stated.

4. Simplest Girsanov Theorem

The following theorem is a simple yet important theorem whose applications can be found in mathematical finance.

Theorem 4.1 (Simplest Girsanov Theorem). *If $(B_t^P)_{t\geq 0}$ is a Brownian motion on $(C[0,T], \mathcal{B}, P)$ and Q is the measure on $(C[0,T], \mathcal{B})$ induced by $X_t = B_t^P + \mu t$, then for every bounded Borel function $W : C[0,T] \to \mathbb{R}$,*

$$\mathbb{E}_Q(W) = \mathbb{E}_P(W M_T), \tag{2}$$

where $M_t := e^{\mu B_t^P - \mu^2 t/2}$ is a martingale on $(C[0,T], \mathcal{B}, P)$.

Remark. The statement of the theorem may sound stronger than the assertion that M_T is the Radon-Nikodym derivative $\frac{dQ}{dP}$. However, because any bounded Borel function can be approximated by a sequence of simple functions in the pointwise sense, due to the dominated convergence theorem, equation (2) would follow if we can show that the equation holds for $W = 1_B$, $B \in \mathcal{B}$, i.e. $\frac{dQ}{dP} = M_T$.

Lemma 4.2 (Dynkin's π-λ theorem [2])**.** *Suppose $\Omega \neq \emptyset$ and $P, D \subset \mathcal{P}(\Omega)$. If P is a π-system and D is a λ-system with $P \subset D$, then $\sigma(P) \subset D$.*

Proof of Theorem 4.1. We already know that $(M_t)_{t\geq 0}$ is a martingale. By the above remark, we only need to show that (2) holds for $W = 1_B$, $B \in \mathcal{B}$.

Let $\mathcal{I}$ be the collection of all sets of the form

$$A = \{f \in C[0,T] : f(t_i) \in [a_i, b_i] \text{ for } i = 1, \dots, n\}.$$

Each set A is called a cylinder set [3]. It is easy to see that $\mathcal{I}$ is nonempty and closed under nonempty finite intersections. Hence $\mathcal{I}$ is a π-system.

Now let $\mathcal{C}$ denote the collection of all $B \in \mathcal{B}$ such that $W = 1_B$ satisfies (2). If $B = C[0,T]$, then $1_B = 1$ and $B \in \mathcal{C}$. The linearity of $\mathbb{E}_P$ and $\mathbb{E}_Q$ implies that $\mathcal{C}$ is closed under complements. Suppose $(B_k)_{k\in\mathbb{N}} \subset \mathcal{C}$ is such that $B_k \nearrow \bigcup_{k\in\mathbb{N}} B_k =: S$. We want to show that $S \in \mathcal{C}$. Since $0 \leq 1_{B_k} \leq 1_{B_{k+1}}$ and $1_{B_k} \to 1_S$ a.s., the monotone convergence theorem implies that $\mathbb{E}_Q(1_{B_k}) \to \mathbb{E}_Q(1_S)$ and $\mathbb{E}_Q(1_{B_k} M_T) \to \mathbb{E}_Q(1_S M_T)$. Since $B_k \in \mathcal{C}$, we have $\mathbb{E}_P(1_{B_k} M_T) = \mathbb{E}_Q(1_{B_k})$. Hence $\mathbb{E}_P(1_S M_T) = \mathbb{E}_Q(1_S)$, and $\mathcal{C}$ is a λ-system.

The tilting formula implies that (2) holds for any $A \in \mathcal{C}$. Hence $\mathcal{I} \subset \mathcal{C}$, and Dynkin's π-λ theorem implies that $\sigma(\mathcal{I}) \subset C$, but according to Theorem 2.2 in [3], the collection of all cylinder sets generates $\mathcal{B}$. It follows that $\sigma(\mathcal{I}) \subset \mathcal{C}$, i.e. (2) holds for all $W = 1_A$, $A \in \mathcal{B}$. □

One nice thing about both the tilting formula and the simplest Girsanov theorem is that our choice of T can be arbitrary as long as T is large enough, which follows from the martingale property of their respective $(M_t)_{t\geq 0}$. It turns out that in the succeeding theorems, the martingale property of exponential processes in a more general form plays an important role in ensuring that a process is a martingale after the underlying measure has changed.

5. Removal of General Drift

In this section, we will take it for granted that the following statement holds.

Proposition 5.1 (Novikov Condition)**.** *If $\mu \in L^2_{LOC}[0,T]$ satisfies the Novkov condition*

$$\mathbb{E}[exp(\frac{1}{2}\int_0^T \mu^2(\omega, s)\ ds)] < \infty, \tag{3}$$

then the process $(M_t)_{t\geq 0}$ *given by*

$$M_t = exp(-\int_0^t \mu(\omega, s)\ dBs - \frac{1}{2}\int_0^t \mu^2(\omega, s)\ ds)$$

is a martingale.

Remark. If a process μ is bounded, then it immediately follows that μ satisfies the Novikov condition.

Theorem 13.2 in [1] tells us that we may view a Brownian motion (on $(C[0,T], \mathcal{F}, P)$, but we will see that the underlying probability space can be arbitrary) with a general drift can be viewed as a standard Brownian motion by introducing a new measure. The proof of the theorem given in [1] is straightforward, but some parts of the proof need more justification. This section focuses on some of the parts of the proof that were left ambiguous by the author. We may also generalize the statement by assuming that $\mu \in L^2_{LOC}$ satisfies the Novikov condition.

Theorem 5.1. *(Removal of general drift) Suppose* $(B_t)_{t\geq 0}$ *is a Brownian motion on* $(C[0,T], \mathcal{B}, P)$ *and* $\mu : C[0,T]\times[0,T] \to \mathbb{R}$ *is in* L^2_{LOC} *and satisfies the Novikov condition. Let* $X_t := B_t + \int_0^t \mu(\omega, s)\ ds$. *Then the process* $(M_t)_{t\geq 0}$ *given by*

$$M_t = exp(-\int_0^t \mu(\omega, s)\ dBs - \frac{1}{2}\int_0^t \mu^2(\omega, s)\ ds)$$

and $(M_t X_t)_{t\geq 0}$ *are martingales, and under the measure* Q *on* $(C[0,T], \mathcal{B})$ *defined by*

$$Q(A) = \mathbb{E}_P(1_A M_T)\ \forall A \in \mathcal{B},$$

$(X_t)_{t\geq 0}$ *is a Brownian motion on* $[0,T]$.

Remark. Q is absolutely continuous with respect to P, i.e. for any $A \in \mathcal{B}$, $Q(A) = 0$ whenever $P(A) = 0$ because $1_A M_T = 0$ a.s. in P [4]. In fact, the Radon-Nikodim derivative dQ/dP is just the non-negative random variable M_T, which satisfies $\mathbb{E}(M_T) = \mathbb{E}(M_0) = 1$.

We first verify the following statements, which were assumed to hold trivially in [1].

Lemma 5.2. *For the measures* P *and* Q *in Theorem* 5.1, $\mathbb{E}_Q(1_A Y) = \mathbb{E}_P(1_A M_T Y)$, *where* $Y : C[0,T] \to \mathbb{R}$ *is a random variable with* $\mathbb{E}(|Y|) < \infty$.

Proof. The proof can be split into four parts: (1) Suppose $B \in \mathcal{B}$. Then with 1_B in place of Y, we have

$$\mathbb{E}_Q(1_A 1_B) = Q(A \cap B) = \mathbb{E}_P(1_A 1_B M_T).$$

(2) By linearity, (1) implies that the equation in the statement holds for any simple function Y. (3) If Y is just a non-negative function, then from p.274 of [5], there is sequence $(Y_n)_{n\in\mathbb{N}}$ of simple functions such that $Y_n \leq Y_{n+1}$ and $Y_n \to Y$

almost surely. Thus the monotone convergence theorem implies that the equation holds for Y. (4) For a general random variable Y, we can decompose Y into two non-negative random variables, and the equation follows from linearity. □

Lemma 5.3. *For the measures P and Q in Theorem* 5.1*, if a sequence $(Y_n)_{n\in\mathbb{N}}$ converges to a random variable Y in probability in P, then the sequence also converges to Y in probability in Q.*

Proof. For any $\epsilon > 0$, $P(|Y_n - Y| > \epsilon) \to 0$ as $n \to \infty$. Then $1_{\{|Y_n-Y|>\epsilon\}}M_t \to 0$ almost surely in P, and since $1_{\{|Y_n-Y|>\epsilon\}}M_t \leq M_t$ and $\mathbb{E}(|M_T|) = 1$, the dominated convergence theorem implies that $Q(|Y_n - Y| > \epsilon) = \mathbb{E}(1_{\{|Y_n-Y|>\epsilon\}}M_t) \to 0$. Hence $Y_n \to Y$ in probability in Q as well [6]. □

Let $Y_t = X_tM_t$. $(M_t)_{t\geq 0}$ is a martingale because μ satisfies the Novikov condition. By the stochastic product rule, $dY = X_tdM_t + M_tdX_t + dM_tdX_t$. Using box calculus, we obtain $dY_t = -(1 - \mu X_t)M_tdB_t$. This tells us that $(Y_t)_{t\geq 0}$ is a martingale [6].

Proof of Theorem 5.1. We can easily check that $(X_t)_{t\geq 0}$ is a martingale on $[0, T]$ under the measure Q by exploiting the martingale property of $(M_t)_{t\geq 0}$ and $(X_tM_t)_{t\geq 0}$ and Lemma 5.2 with Y in place of X_t. Then by Levy's characterization (Theorem 12.5 in [1]), to show $(X_t)_{t\geq 0}$ is a Brownian motion on $[0, T]$ under the measure Q, it is enough to show that the quadratic variation of $(X_t)_{t\geq 0}$ with respect to Q is t. From $dX_t = dB_t + \mu dt$, we see that the quadratic variation of $(X_t)_{t\geq 0}$ with respect to P is t by Theorem 5.16 in the lecture notes, and the definition of quadratic variation and Lemma 5.3 tell us that the quadratic variation of $(X_t)_{t\geq 0}$ does not change through the absolutely continuous change of measure. Therefore the quadratic variation of $(X_t)_{t\geq 0}$ with respect to Q is t. □

Sometimes we can remove the drift of a given process and add a new drift to it by an absolutely continuous change of measure.

Theorem 5.4. *Suppose $(X_t)_{t\geq 0}$ is given by $dX_t = \mu(\omega, t)dt + \sigma(\omega, t)dB_t$, where $\sigma \neq 0$. If $\theta = \frac{\mu-\nu}{\sigma}$ is in $L^2_{LOC}[0, T]$ and satisfies the Novikov condition, then*

$$M_t = exp(-\int_0^t \theta(\omega, s)\ dBs - \frac{1}{2}\int_0^t \theta^2(\omega, s)ds)$$

is a martingale with respect to P, and under the measure Q on C[0,T] given by $Q(A) = \mathbb{E}_P(1_AM_T)$,

$$\tilde{B}_t = B_t + \int_0^t \theta(\omega, s)\ ds$$

is a Brownian motion. Moreover,

$$dX_t = \nu(\omega, t)dt + \sigma(\omega, t)d\tilde{B}_t. \tag{4}$$

Proof. Everything in the statement, except for (4), follows from the proof of the last theorem, but

$$dX_t = \mu dt + \sigma dB_t = \nu dt + \sigma(dB_t + \theta dt) = \nu dt + \sigma d\tilde{B}_t.$$

□

Note that the proofs of Theorem 5.1 and Theorem 5.4 do not rely on the fact that the underlying event space is $(C[0,T],\mathcal{B})$, nor do they rely on how the measure P is defined. Therefore these theorems also work for arbitrary probability spaces in general. Generalized versions of the theorems can be found in [6].

6. Discounted Stock Price

In this section, we take a digression and look at an example in which a theorem in Girsanov theory can be used to transform a non-martingale process into a martingale.

Suppose $(\Omega, \Sigma, \mathbb{P})$ is given. In the Balck-Scholes model, the stock price S_t and the bond price β_t are given by

$$S_t = S_0 e^{t(\mu-\sigma^2/2+\sigma B_t)} \text{ and } \beta_t = e^{rt}.$$

The discounted stock price is the process $D_t = S_t/\beta_t = e^{-rt}S_t$ whose SDE is given by $dD_t = (\mu - r)D_t dt + D_t dB_t$. We are interested in the case where $\mu - r \neq 0$. In this case, the dt term is nonzero and hence $(D_t)_{t\geq 0}$ is not a martingale. By introducing $X_t = B_t + at$, where $a = \frac{\mu - r}{\sigma}$, the SDE can be simplified to $dD_t = \sigma D_t dX_t$. Then under the measure Q on (Ω, Σ) defined by

$$Q(A) = \mathbb{E}_{\mathbb{P}}(1_A e^{aB_T - a^2T/2}) \ \forall A \in \Sigma,$$

$(X_t)_{t\geq 0}$ is a standard Brownian motion on $[0,T]$. This follows from Theorem 5.1 with $(\Omega, \Sigma, \mathbb{P})$ in place of $(C[0,T],\mathcal{B},P)$. Hence D_t is a martingale under Q. As mentioned in Ranit's video, this choice of measure, together with the representation theorem, lets us construct a martingale that models a self-financing replicating portfolio of a derivative.

7. Novikov Condition

For the theorems in section 6, the martingale property of $(M_t)_{t\geq 0}$ was essential for ensuring that $(X_t)_{t\geq 0}$ was a martingale under the new measure. Furthermore, we had $Q(C[0,T]) = \mathbb{E}(M_T)$ by the definition of Q. If $(M_t)_{t\geq 0}$ is a martingale on $[0,T]$, then because $\mathbb{E}(M_0) = 1$ we would have $\mathbb{E}(M_T) = 1$, and consequently Q would be well-defined as a probability measure.

Conversely, as the process is a non-negative local martingale (assuming $\mu, \theta \in L^2_{LOC}[0,T]$ so that their Ito integrals would be well-defined), Proposition 7.11 in [1] tells us that $(M_t)_{t\geq 0}$ is a martingale on $[0,T]$ whenever $\mathbb{E}(M_T) = \mathbb{E}(M_0) = 1$. This means that the process $(M_t)_{t\geq 0}$ being a martingale on $[0,T]$ is actually equivalent to having $\mathbb{E}(M_T) = 1$. We can make use of this fact to prove Proposition 5.1, i.e. the Novikov condition is a sufficient condition for $(M_t)_{t\geq 0}$ to be a martingale.

Suppose $\mu \in L^2_{LOC}[0,T]$ satisfies the Novikov condition and $(M_t(c\mu))_{t\geq 0}$ is the exponential martingale given in Proposition 5.1 with the μ's in the exponent replaced by $c\mu$'s, $|c| \leq 1$. According to the above argument, to show that $(M_t(\mu))_{t\geq 0}$ is a martingale on $[0,T]$, we only need to check that $\mathbb{E}(M_T(\mu)) = 1$. Instead of computing the expectation directly, we can take a detour and proceed through steps similar to those we usually follow in an induction argument. The following is an outline of the proof of Proposition 5.1:

Step 1. Show $\mathbb{E}(M_{\tau_a}(c\mu)) = 1$ for $c \leq 0$.
Let $Y_t = \int_0^t \mu dB_s + \int_0^t \mu^2 ds$ and $\tau_a = \inf\{t : Y_t = -a \text{ or } t \geq T\}$, $a > 0$. We can check that τ_a is a stopping time using the continuity of Y_t and the density of $\mathbb{Q}$. We then proceed to show $\mathbb{E}(M(c\mu)_{\tau_a}) = 1$ for $c \leq 0$. Since $M(c\mu)_{\tau_a} = 1 + \int_0^{\tau_a} c\mu(\omega,s)M_{\tau_a}(c\mu)dB_s$, we only need to check $\mathbb{E}(\int_0^{\tau_a} \mu^2 M(c\mu)^2_{\tau_a} ds) < \infty$, but this can be done by estimating M_s from above by e^{-ac} for $s \leq \tau_a$ and applying the Novikov condition.

Step 2. Show $\mathbb{E}(M_{\tau_a}(\mu)) = 1$.
Viewing $e^a M_{\tau_a}(c\mu)$ as a function of $x \in [-1,1]$, where $c \leq 1$ and $x = -(1-c)^2 + c$ (or equivalently, $c = 1 - \sqrt{1-x}$), we can write it as a power series based at $x = 0$: $e^a M_{\tau_a}(c\mu) = \sum_{n=0}^{\infty} c_k(\omega)x^n$, where $c_k(\omega) \geq 0$ for all $k \in \mathbb{N}$. We will find that the expectation of $e^a M_{\tau_a}(c\mu)$ is controlled by some function $g(x)$ such that $g(1) = e^a$ and $\mathbb{E}(e^a M_{\tau_a}(c\mu)) = g(x)$ when $x \in (-1,0]$ (by Step 1). This implies that $\mathbb{E}(e^a M_{\tau_a}(\mu)) = g(1) = e^a$, i.e. $\mathbb{E}(M_{\tau_a}(\mu)) = 1$. (See Lemma 13.1 in [1].)

Step 3. Show $\mathbb{E}(M_T(\mu)) = 1$ by sending $a \to \infty$.
From Step 2, we have

$$1 = \mathbb{E}(M_{\tau_a}(\mu)) = \mathbb{E}(M_{\tau_a}(\mu)1_{\{\tau_a<T\}}) + \mathbb{E}(M_T(\mu)1_{\{\tau_a=T\}}).$$

Similarly,

$$\mathbb{E}(M_T(\mu)) = \mathbb{E}(M_T(\mu)1_{\{\tau_a=T\}}) + \mathbb{E}(M_T(\mu)1_{\{\tau_a<T\}}).$$

This means that to derive the desired equality, we only need to check that $\mathbb{E}(M_{\tau_a}(\mu)1_{\{\tau_a<T\}}) \to 0$ and $\mathbb{E}(M_T(\mu)1_{\{\tau_a<T\}}) \to 0$ as $a \to \infty$.

(1) Recalling $M_{\tau_a}(\mu) = exp(Y_{\tau_a} + \int_0^T \frac{1}{2}\mu^2 ds)$ and the definition of τ_a, the Novikov condition tells us that
$$\mathbb{E}(M_{\tau_a}(\mu)1_{\{\tau_a<T\}}) \leq e^{-a}\mathbb{E}(\int_0^T \frac{1}{2}\mu^2 ds) \to 0.$$

(2) We have $\mathbb{E}(M_T(\mu)1_{\{\tau_a<T\}}) \leq \mathbb{E}(M_T(\mu) \leq 1$, where the last inequality follows from $\mathbb{E}(M_0(\mu)) = 1$ and that $(M_T(\mu))_{t\geq 0}$ is a supermartingale (Prop 7.11 in [1]). Moreover, $t \mapsto Y_t$ is almost surely bounded on $[0,T]$ by continuity. Thus $1_{\{\tau_a<T\}} = 0$ almost surely. The dominated convergence thus implies that $\mathbb{E}(M_T(\mu)1_{\{\tau_a<T\}}) \to 0$.

8. Generilization of Proposition 5.1

In Proposition 5.1, we assumed that the exponent of M_t takes the specific form $-\int_0^t \mu(\omega, s)\ dBs - \frac{1}{2}\int_0^t \mu^2(\omega, s)\ ds$, where $\mu \in L^2_{LOC}[0, T]$. If we define $Z_t := \int_0^t \mu dBs$, then $(Z_t)_{t\geq 0}$ is a continuous local martingale and we can write $M_t = exp(Z_t - \frac{1}{2}\langle Z\rangle_t)$, where $\langle Z\rangle_t$ is the quadratic variation of Z_t with respect to $\mathbb{P}$. We can generalize the proposition by taking $(Z_t)_{t\geq 0}$ to be any continuous local martingale such that $Z_0 = 0$ and setting $M_t := exp(Z_t - \frac{1}{2}\langle Z\rangle_t)$, and consequently equation (3) in the proposition can be replaced by $\mathbb{E}(exp(\frac{1}{2}\langle Z\rangle_t)) < \infty$ $(*)$ [7]. Note that the condition $\mathbb{E}(exp(\frac{1}{2}\langle Z\rangle_t)) < \infty$ is weaker than $\mathbb{E}(exp(\frac{1}{2}Z_t)) < \infty$. This follows from the Cauchy-Swartz inequality and $\mathbb{E}(M_t) \leq 1$ (by the supermartingale property of $(M_t)_{t\geq 0}$) [8].

One might ask whether or not $\tilde{M}_t := exp(-Z_t - \frac{1}{2}\langle Z\rangle_t)$ is a martingale when $(Z_t)_{t\geq 0}$ satisfies $(*)$, but this is not always the case. [8] provides a counterexample by introducing a certain stopping time θ_t for each $t \geq 0$ and setting $Z_t = B_{\theta_t}$.

References

[1] J. M. Steele, *Stochastic calculus and Financial Applications.* Springer, 2011.

[2] Wikipedia. Dynkin system. [Online]. Available: https://en.wikipedia.org/wiki/Dynkin_system

[3] Notes for math6050d. [Online]. Available: https://www.math.hkust.edu.hk/ mazhu/Math6050_16/math6050-note1.pdf

[4] Absolutely continuous measures. [Online]. Available: https://encyclopediaofmath.org/wiki/Absolutely_continuous_measures

[5] E. M. Stein and R. Shakarchi, *Real analysis: Measure theory, integration, and Hilbert spaces.* Princeton University Press, 2013.

[6] J. C. M. Andrea Agazzi. Introduction to stochastic calculus. [Online]. Available: https://services.math.duke.edu/ agazzi/notesSDE.pdf

[7] N. V. Krylov, "A few comments on a result of a. novikov and girsanov's theorem," *Stochastics*, vol. 91, no. 8, p. 1186–1189, 2019.

[8] N. Kazamaki, "On a problem of girsanov," *Tohoku Mathematical Journal*, vol. 29, no. 4, pp. 597–600, 1977.