INTEGRATION OF BANACH SPACE VALUED FUNCTIONS: BOCHER INTEGRAL AND PETTIS INTEGRAL

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1. Introduction

In this essay, we will look at a notion of integration that extends from the Lebesgue integral, the Bochner integral. It provides tools for the integration of Banach space-valued functions. We will then look at the Bochner spaces, which are analogs of the L^p spaces. We will also touch on the weaker version of the Bochner integral, the Pettis integral.

2. Strong Measurablity

There are three main notions of measurability for Banach space-valued functions: measurability, strong measurability, and weak measurability. Since the Bochner integral is defined for a class of strongly measurable functions on a measure space, we will briefly discuss strong measurability only.

We will assume that $(S, \mathcal{F}, \mu)$ is a fixed measure space, where S is a set, $\mathcal{F}$ is a σ-algebra of S, and μ is a measure on $(S, \mathcal{F})$. We will also assume that X is a fixed Banach space unless otherwise stated.

Definition 2.1 (σ-finite). *A measure μ is σ-finite if there exists a sequence $(S_k)_{k\in\mathbb{N}}$ in $\mathcal{F}$, where $\mu(S_k) < \infty$, such that $A_k \nearrow S$.*

Definition 2.2 ($\otimes$). *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For a function $f : S \to \mathbb{K}$ and $x \in X$, define $f \otimes x : S \to X$ by*

$$f \otimes x(s) = f(s)x.$$

Definition 2.3 (μ-simple functions). *A function $f : S \to X$ of the form $\sum_{i=1}^{n} \mathbb{1}_{A_k} \otimes x_k$, where $A_k \in \mathcal{F}$, $\mu(A_k) < \infty$ and $x_k \in X$, is called a μ-simple function.*

Since our measure μ is arbitrary, we shall use the phrase "μ-almost everywhere" instead of "almost everywhere".

Definition 2.4 (Strongly μ-measurable functions). *A function $f : S \to X$ is strongly μ-measurable if there exists a sequence (f_n) of μ-simple functions such that $f_n \to f$ μ-almost everywhere.*

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Note that any strongly μ-measurable function agrees with some measurable function μ-alsmost surely. If $X = \mathbb{K}$, then any measurable function is also strongly measurable (p.274 of [2]).

The following proposition lets us apply the dominated convergence theorem for scalar-valued functions to $s \mapsto \|f(s) - f_n(s)\|$ whenever we approximate a strongly μ-measurable function f by μ-simple functions (f_n).

Proposition 2.1 (Approximation by μ-simple functions). *If $f : S \to X$ is strongly μ-measurable, then there exists a sequence (f_n) of μ-simple functions such that $\|f_n(s)\| \leq \|f(s)\|$ and $f_n(s) \to f(s)$ for μ-almost every $s \in S$,*

Another important property of strongly μ-measurable functions is that the range of a strongly μ-measurable function is μ-almost surely contained in a closed separable subspace of X.

Proposition 2.2 (Separably valued). *Any strongly μ-measurable function $f : S \to X$ is (μ-essentially) separably valued, i.e. there exists a closed separable subspace $Y \subset X$ such that $f(s) \in X_0$ for μ-almost every $s \in S$. Moreover, $s \to \langle x^*, f(s)\rangle$ is (strongly) measurable (with respect to μ) for any x^*.*

Remark. The measurability of $s \to \langle x^*, f(s)\rangle$ implies that it is μ-strong measurable because it is scalar-valued.

Proof. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of μ-simple functions that approximates f μ-almost everywhere. Then $s \to \langle x^*, f(s)\rangle$ is measurable as an almost sure limit of measurable functions $s \to \langle x^*, f(s)\rangle$. Let $A \subset X$ be the set of all values taken by $(f_n)_{n\in\mathbb{N}}$. Then A is countable, and hence the closed subspace $Y := \overline{spanA}$ is separable and $f(s) \in Y$ for μ-almost every $s \in S$. $\square$

This property in particular allows us to assume the space X to be separable in many situations in which we consider the Bochner integral of a strongly μ-measurable function.

3. Construction of The Bochner Integral

Let $(S, \mathcal{F}, \mu)$ be fixed. For the rest of this essay, we will assume that key properties of the integrals of scalar-valued functions on $(S, \mathcal{F}, \mu)$, such as the triangle inequality and the scalar dominated convergence theorem, hold (See chapter 6 of [3]).

The main difference between the Lebesgue integral and the Bochner integral is that the former is scalar-valued while the latter takes a vector $x \in X$, though the construction of the Bochner Integral is very similar to that of the Lebesgue integral.

We start with defining the Bochner integral of a μ-simple function.

Definition 3.1 (The Bochner Integral of a μ-simple function). *For a μ-simple function $f : S \to X$ given by $f = \sum_{k=1}^{N} \mathbb{1}_{A_k} \otimes x_k$, where $x_k \in X$ and $A_k \in \mathcal{F}$, ,set*

$$\int_S f \; d\mu := \sum_{k=1}^{N} \mu(A_k) x_k \in X.$$

Similar to the Lebesgue integral, the Bochner integral defined for μ-simple functions has the following properties.

Proposition 3.1. *For μ-simple functions f and g,*

(1) $\int_S f$ *is independent of the representation of f.*
(2) $\| \int_S f \| \leq \int_S \|f\|$.
(3) $\int_S f + \int_S g = \int_S f + g$.

While the linearity is trivial, the proof of the other properties is almost the same as the Lebesgue integral case given in Chapter 2 of [3] with the absolute value symbols replaced by $\| \cdot \|$ and relies on splitting $(A_k)_{k=1}^N$ into disjoint sets.

Next, we define the Bochner integral for a broader class of strongly μ-measurable functions using Definition 3.1.

Definition 3.2 (Integrability). *A strongly μ-measurable function $f : S \to X$ is (Bochner) integrable if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of μ-simple functions such that $\int_S \|f_n - f\| \; d\mu \to 0$ as $n \to \infty$.*

Since $l : S \to \mathbb{R}$ given by $l(s) = \|f(s) - f_n(s)\|_X$ is measurable (in the sense that $l^{-1}(B) \in \mathcal{F}$ for any borel set B in $\mathbb{R}$), the above definition is well-defined. If f is integrable, then by the triangle inequality for the integrals of scalar-valued functions and $\| \cdot \|$,

$$\| \int_S f_n - \int_S f_m \| \leq \int_S \|f_n - f + f - f_m\| \leq \int_S \|f_n - f\| + \int_S \|f - f_m\| \to 0$$

as $n, m \to \infty$. Hence $(\int_S f_n)_{n\in\mathbb{N}}$ is Cauchy, and by completeness, $\int_S f_n$ converges in X. This leads to the following definition.

Definition 3.3 (The Bochner integral for integrable functions). *For an integrable function $f : S \to X$, define $\int_S f \; d\mu := \lim_{n\to\infty} \int_S f_n \in X$.*

According to this definition, (3) in Proposition 3.1 clearly holds for integrable functions, and so does (2) in Proposition 3.1 by approximation by μ-simple functions. Moreover, by approximation by μ-simple functions and the scalar dominated convergence theorem (Proposition 2.1), it can be shown that a strongly μ-measurable function f is integrable if and only if $\int_S \|f\| < \infty$. This implies that whenever f is Bochner integrable, for any $A \in \mathcal{F}$ the function $1_A f : S \to X$, is Bochner integrable. Instead of $\int_S 1_A f$, we may write $\int_A f$.

All the functions mentioned in the next three sections are strongly μ-measurable unless otherwise stated.

4. Properties of The Bochner integral

Since the construction of the Bochner integral is similar to that of the Lebesgue integral, it shares some nice properties with the Lebesgue integral. Proofs of the following properties are mostly omitted as they either follow from their analogs for scalar-valued functions or make use of measure-theoretic tools, rather than those from functional analysis. The following theorem is an example of a property of the Bochner integral that immediately follows from its scalar analogue.

Theorem 4.1 (Dominated Convergence Theorem). *Suppose $f_n : S \to X$ is Bochner integrable and $f_n \to f$ μ-almost everywhere. If there exists an integrable function $g : S \to \mathbb{R}$ with $\|f_n(s)\| \leq g(s)$ for μ-almost every $s \in S$, then f is Bochner integrable and*

$$\int_S \|f_n - f\| \to 0. \tag{1}$$

Proof. (1) immediately follows from applying the scalar dominated convergence theorem to the (measurable) function $s \to \|f_n(s) - f(s)\|$ as $\|f_n(s) - f(s)\| \leq 2g$. (1) then implies that $\int_S \|f_n\| \to \int_S \|f\|$, and since $\int_S \|f_n\| \leq \int_S \|g\|$ by monotonicity (for scalar-valued functions), $\int_S \|f\| \leq \int_S \|g\| < \infty$. Therefore f is Bochner integrable. □

We can show that vector-valued versions of Fubini's theorem (with the assumption that the involving measures are σ-finite) and the substitution rule functions hold. Their statements are omitted as they are almost identical to those of their scalar-valued analogs. (For the statements, see Propositions 1.2.6 and 1.2.7 in [2]) We can also derive Jensen's inequality for vector-valued functions, which states that if $\mu(S) = 1$ and $\phi : X \to \mathbb{R}$ is convex and lower-semicontinuous, then for any Bochner integrable function $f : S \to X$, $\phi(\int_S f) \leq \int_S \phi(f)$. The derivation of this inequality relies on the fact that ϕ can be written as the supremum of some affines functions on the condition that X is separable. But this condition is not an issue as X can be assumed to be separable by Proposition 2.2.

5. The Bochner Spaces

We can define an equivalence relation on μ-strongly measurable functions by saying that two functions are equivalent if they agree μ-almost surely. Using this equivalence relation, we can define the L^p spaces according to the definition of the Bochner integral. They are called the Bochner spaces.

Definition 5.1 (Bochner space L^p). *For $1 \leq p < \infty$ define $L^p(S, X)$ to be the linear space of all equivalence classes of $f : S \to X$ such that $\int_S \|f\|_X^p \, d\mu < \infty$. For $p = \infty$, define $L^\infty(S, X)$ to be the linear space of all equivalence classes of $f : S \to X$ such that there exists $M > 0$ for which $\|f(s)\|_X \leq M$ for μ-almost every $s \in S$.*

Remark. When equipped with the norms $\|f\|_{L^p(S,X)} = (\int_S \|f\|^p)^{1/p}$ and $\|f\|_{L^\infty(S,X)} = \inf\{M \geq 0 : \mu(\{x \in X : \|f(x)\|_X > M)\}) = 0\}$, respectively, they are Banach spaces. Moreover, for any convergent sequence in $L^p(S,K)$, we can extract a subsequence that converges to the L^p-limit μ-almost surely.

Remark. For the case where X is a scalar field ($X = \mathbb{R}, \mathbb{C}$), we may use the shorthand notation $L^p(S)$ to denote $L^p(S,X)$.

Remark. Similar to the usual L^p spaces, for $p \in [1,\infty)$, μ-simple functions are dense in $L^p(S,X)$, and bounded countably-valued μ-simple functions are dense in L^∞.

The Bochner spaces can be useful when studying second-order parabolic equations. For instance, when we look at a weak solution $u : \bar{U}_T \to \mathbb{R}$ to the problem

$$\begin{cases} Lu &= f \text{ in } U_T \\ u &= 0 \text{ on } \partial U \times [0,T] \\ u &= g \text{ on } U \times \{t = 0\}, \end{cases}$$

where $U \in \mathbb{R}^n$ is open and bounded, $U_T = U \times (0,T]$ and L is a uniformly parabolic operator, we may view the function u as an element of $L^2([0,T]; H^1(U))$ by associating it with the mapping $t \in [0,T] \to u(\cdot,t)$. This helps us to study energy estimates and the regularity of u [1].

The normal version of Fubini's theorem is concerned with σ-finite measures, but by constructing an isometric isomorphism between two Bochner spaces, we can at least say something similar (but weaker) about the case where the measures are not necessarily σ-finite.

Theorem 5.1. *Suppose $(S,\mathcal{F},\mu)$ and $(T,\mathcal{B},\nu)$ are measure spaces. For $p \in [1,p)$, map $(s \mapsto 1_A(s) \otimes (1_B(t) \otimes x)) \in L^p(S, L^p(T,X))$ to $(t \mapsto 1_B(t) \otimes (1_A(t) \otimes x)) \in L^p(T, L^p(S,X))$ for any $A \in \mathcal{F}$, $B \in \mathcal{B}$. Then we can extend the mapping to an isometric isomorphism between $L^p(S, L^p(T,X))$ and $L^p(T, L^p(S,X))$.*

In short, the above theorem tells us that $f \in L^p(S, L^p(T,X))$,

$$\|f\|^p_{L^p(S,L^p(T,X))} = \int_S \int_T \|f(s,t)\| \, d\nu(t) d\mu(s) = \int_T \int_S \|f(s,t)\| \, d\mu(t) d\mu(s) = \|f\|^p_{L^p(T,L^p(S,X))}.$$

Note that this theorem does not tell us if the above integrals are equal to $\int_{S\times T} \|f(s,t)\|^p \, d(\mu(s) \times \nu(t))$, where $\mu \times \nu$ is the product measure on $S \times T$. Hence it is a weaker version of Fubini's theorem. This theorem follows from the "continuous" version of Minkowski's inequality:

Proposition 5.1 (Minkowski's inequality)**.** *Suppose $(S,\mathcal{F},\mu)$ and $(T,\mathcal{B},\nu)$ are measure spaces. For $p,q \in [1,p)$ such that $p \leq q$, map $(s \mapsto 1_A(s) \otimes (1_B(t) \otimes x)) \in L^p(S, L^q(T,X))$ to $(t \mapsto 1_B(t) \otimes (1_A(t) \otimes x)) \in L^q(T, L^q(S,X))$. Then we can* *uniquely extend this mapping to a contraction embedding from* $v : L^p(S, L^q(T, X)) \to L^q(T, L^p(S, X))$*, in the sense that* $\|f\|_{L^q(S,L^p(T,X))} \leq \|f\|_{L^q(S,L^p(T,X))}$

To prove the above theorem, we can first show $\|f\|_{L^q(S,L^p(T,X))} \leq \|f\|_{L^q(S,L^p(T,X))}$ for any simple function $f = \sum_{i=1}^{M} 1_{A_i} \otimes \sum_{j=1}^{N} 1_{B_j} \otimes x_{i,j}$ using the definition of $\|\cdot\|_{L^q(S,L^p(T,X))}$ and $\|\cdot\|_{L^q(S,L^p(T,X))}$ and then use the density of the collection of all function of this form in $L^q(S, L^p(T, X))$.

We see that Theorem 5.1 is simply the special case where $p = q$. If μ and ν are σ-finite, then we can construct an isometric isomorphism between $L^p(S, L^q(T, X))$ and $L^p(S \times T, X)$, from which we can deduce that $\int_S \int_T \|f(s,t)\|^p d\nu(t) d\mu(s) = \int_{S\times T} \|f(s,t)\|^p \; d(\mu(s) \times \nu(t))$ for any $f \in L^p(S \times T, X)$

When we have $(S, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra of $\mathbb{R}^d$ and λ is the Lebsgue measure, we can define a convolution $f * \phi : \mathbb{R}^d \to \mathbb{R}$ of $f \in L^p(\mathbb{R}^d, X)$ with $\phi \in L^1(\mathbb{R}^d, \mathbb{R})$ by $f * \phi(x) = \int_{\mathbb{R}^d} \phi(x) f(x - y) \; dy$. We can also derive the vector-valued version of Young's inequality:$\|f * \phi\|_{L^p(\mathbb{R}^d,X)} \leq \|\phi\|_{L(\mathbb{R}^d,\mathbb{R})} \|f\|_{L^p(\mathbb{R}^d,X)}$. In this particular case, we can also introduce the vector-valued analog of mollifiers, which are often used in the theory of partial differential equations.

6. Pettis Integral

Finally, we look at the Pettis integral, the weaker version of the Bochner integral. The following proposition provides a tool for defining a new notion of integration: the weak integral.

Proposition 6.1. *Suppose* Y *is a closed subspace of* X^* *and* $f : S \to X$ *is such that* $s \mapsto \langle x^*, f(s) \rangle$ *is in* $L^1(S)$ *for all* $x^* \in Y$. *Then the function* $T_f : Y \to L^1(S)$ *given by* $T_f(y) = s \mapsto \langle x^*, f(s) \rangle$ *is a bounded, linear operator.*

Proof. The linearity is clear. Observe that Y is a Banach space with the norm $\|\cdot\|_{X^*}$. Suppose $x_n^* \to x^*$ in Y and $T_f(x_n^*) \to g$ in $L^1(S)$ for some g. Then we may assume that $T_f(x_n^*) \to g$ almost surely. Since $x_n^* \to x^*$ strongly, we have $x_n^* \to x^*$ in the w^*-topology. It follows that $T_f(x_n^*)(s) = \langle x_n^*, f(s) \rangle \to \langle x^*, f(s) \rangle$ for any $s \in S$. Thus $g \equiv s \mapsto \langle x^*, f(s) \rangle$ and the closed graph theorem implies that $T_f \in B(Y, L^1(S))$. □

Given a function f satisfying the condition in Proposition 6.1, using $T_f : Y \to L^1(S)$, we can introduce the following notion.

Definition 6.1 ($\tau(X, Y)$-integral)**.** *Suppose* $f : S \to X$ *satisfies the condition in Proposition 6.1. Note that* $L^\infty(S) \subset (L^1(S))^*$ *in the sense that for each* $l \in L^\infty(S)$, *there exists a unique* $x^* \in (L^1(S))^*$ *such that* $\langle l, s \rangle = \langle s, x^* \rangle$ *for all* $s \in S$ *and*

$\|l\|_{L^\infty} = \|x^*\|_{(L^1)^*}$. *For each* $A \in \mathcal{F}$, *viewing* $\mathbb{1}_A \in L^\infty(S)$ *as an element of* $(L^1(S))^*$, *define*

$$\tau(X,Y)\text{-}\int_S f\ du := T_f^*(\mathbb{1}_A) \in Y^*,$$

where $T_f^* : (L^1(S))^* \to Y^*$ *is the adjoint of* T_f. *This integral is called the* $\tau(X,Y)$-*integral of* f *over* A.

For the case in which $Y = X^{**}$, the function f (such that $s \mapsto \langle x^*, f(s)\rangle \in L^1(S)$ for all $x^* \in X^*$) is called weakly integrable, and $\tau(X,X^*)$- $\int_S f\ du \in X^{**}$ is called the weak integral of f over A.

For a weakly integrable function $f : S \to X$, if X is reflexive, then the function T_f^* may be viewed as a function that takes values in X, and eventually, each weak integral of f may be thought of as an element of X. Although $X \subset X^{**}$, this is not always the case for non-reflexive Banach spaces. This and the fact that the weak integral is defined using elements of $L^\infty(S)$ lead to the following definition.

Definition 6.2 (Pettis Integral)**.** *A weakly integrable function* $f : S \to K$ *is Pettis integrable if* $T_f^*(g) \in X \subset X^{**}$ *for all* $g \in L^\infty(S)$ $(*)$, *where* g *is viewed as an element of* $(L^1(S))^*$. *The Pettis integral* (P)- $\int_A f\ d\mu$ *of* f *over* A *is defined to be*

$$(P)\text{-}\int_A f\ d\mu := T_f^*(\mathbb{1}_A).$$

Next, we verify that any Bochner integral function is Pettis integral. To do so, we first prove the following lemmas.

Lemma 6.1. *If* $f : S \to X$ *is Bochner integrable and* $T \in B(X,Y)$, *where* Y *is another Banach space, then* $T(f) : S \to Y$ *is Bochner integrable and*

$$T\left(\int_S f\ d\mu\right) = \int_S T(f)\ d\mu. \tag{2}$$

Proof. Since f is Bochner integrable, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of μ-simple functions such that $\int_S \|f_n - f\|\ d\mu \to 0$. Then

$$\int_S \|T(f_n) - T(f)\|_Y\ d\mu \le \int_S \|T\|_{B(X,Y)}\|f_n - f\|_X\ d\mu \to 0.$$

Hence $T(f)$ is integrable, and since $f_n = \sum_{i=1}^{N^{(n)}} \mu(A_i^{(n)})x_i^{(n)}$,

$$T\left(\int_S f\ d\mu\right) = \lim_{n\to\infty} T\left(\sum_{i=1}^{N^{(n)}} \mu(A_i^{(n)})x_i^{(n)}\right) = \lim_{n\to\infty} \sum_{i=1}^{N^{(n)}} \mu(A_i^{(n)})T(x_i^{(n)}) = \lim_{n\to\infty} \int_S T(f_n)\ d\mu = T(f).$$

□

Lemma 6.2. *(Equivalent condition) A weakly integrable function $f : S \to K$, is Pettis integrable if and only if for any $A \in \mathcal{F}$, there exists $x_A \in X$ such that $\langle x^*, x_A \rangle = \int_A \langle f(s), x^* \rangle \ d\mu(s)$ for all $x^* \in X^*$, in which case,*

$$(P)\text{-}\int_A f \ d\mu = x_A.$$

Proof. (Sketch) ($\Longrightarrow$) follows from setting $x_A = T^*(\mathbb{1}_A)$. The proof of ($\Longleftarrow$) can be split into three parts. (1) If μ is a finite measure, we can verify the statement by just checking $T^*(g) \in X$ for each bounded countably-valued μ-simple g because of the density of such functions in $L^\infty(S)$. (2) For the case where μ is σ-finite, we can exploit the first case by introducing the finite measure ν with $d\nu = \omega d\mu$, where $\omega : S \to (0,1)$ is integrable. (3) For a general measure μ, we can use the fact that any strongly μ-measurable function has induces a disjoint decomposition $S = S_0 \cup S_1$, where $f = 0$ on S_0 almost surely and μ restricted to the sub-sigma algebra of $\mathcal{F}$ corresponding to S_1 is σ-finite. □

Proposition 6.2. *Any Bochner integrable function $f : S \to X$ is Pettis integrable, and $(P)\text{-}\int_A f \ d\mu = \int_A f \ d\mu$.*

Proof. By setting $Y = \mathbb{R}$ in Lemma 6.1, with $f\mathbb{1}_A$ in place of f, we have that f is weakly integrable and $\langle \int_A f \ d\mu, x^* \rangle = \int_A \langle T(f(s)), x^* \rangle \ d\mu$ for any $x^* \in X^*$. Therefore Lemma 6.2 implies that f is Pettis integrable and $(P)\text{-}\int_A f \ d\mu = \int_A f \ d\mu$. □

Conversely, a Pettis integrable function is not necessarily Bochner integrable (see 1.2.37 in [2]).

A weakly integrable function is not necessarily Pettis integrable. Suppose λ is the Lebesgue measure on $(0,1)$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets in $(0,1)$ with $\lambda(A_n) > 0$ and c_0 the set of all sequences of scalars that converge to 0. Then $(c_0, \|\cdot\|_\infty)$ is a Banach space and denote by $(e_n)_{n\in\mathbb{N}}$ the canonical basis of c_0. Since $(A_n)_{n\in\mathbb{N}}$ are disjoint, for each $x \in (0,1)$, there exists at most one A_n that contains x. Using this property, we can check that $f(s) := \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n}(s) \otimes e_n/\lambda(A_n)$ is strongly measurable and weakly integrable. However, f is not Pettis integrable because its weak integral (over $(0,1)$) is the sequence of ones, which is not in c_0.

It turns out that if X does not contain a closed subspace isomorphic to c_0, the weak integrability is equivalent to the Pettis integrability. This tells us that the above case is the only example of a weakly integrable function that is not Pettis integrable. The next theorem guides us in proving the aforementioned statement.

Theorem 6.3. *Let be $(x_n)_{n\in\mathbb{N}}$ a sequence in X with $\inf_{n\in\mathbb{N}} \|x_n\|_{x\in\mathbb{N}} > 0$ and $(\epsilon_n)_{n\in\mathbb{N}} \in (-1,1)^{\mathbb{N}}$. If $\|\sum_{i=n}^N \epsilon_n x_n\| < C$ for all $N \in \mathbb{N}$ for some $C > 0$ independent of N, then there exists a subspace Y of $\overline{span\{x_n : n \in \mathbb{N}\}}$ such that $Y \cong c_0$.*

Proof. (Sketch) The conditions on $(x_n)_{n\in\mathbb{N}}$ imply that $(x_n)_{n\in\mathbb{N}}$ is bounded and we may assume $(x_n)_{n\in\mathbb{N}}$ to be normalized. We proceed by explicitly constructing a sequence $(y_n)_{n\in\mathbb{N}}$ whose closed linear span is isomorphic to c_0. Define $a_k := \sup\left\{\left\|\sum_{i=k}^{l} b_i x_i\right\| : k \leq l, b_i \in \mathbb{K}, |b_i| \leq 1\right\}$ for $k \in \mathbb{N}$. Then $(a_k)_{k\in\mathbb{N}}$ is bounded from below and non-increasing. Hence it is convergent and we denote its limit by a. With an appropriate choice of $(b_k)_{k\in\mathbb{N}}$ we can construct an increasing sequence $(k_i)_{\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and another sequence $(y_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ given by $y_i = \sum_{j=k_i}^{k_{i+1}-1} b_j x_j$ with $\|y_i\| \geq \frac{3}{4}a$ and $|b_i| \leq 1$. Then for any $(c_i)_{i=1}^N \in \mathbb{K}^N$, $\|\sum_{i=1}^N c_i y_i\|_X \leq \frac{4}{3}a \max_{i=1\ldots N} |c_i|$ and $\|\sum_{i=1}^N c_i y_i\|_X \geq \frac{1}{6}a \max_{i=1\ldots N} |c_i|$. This shows that the two norms $\|\cdot\|_X$ on Y and $\|\cdot\|_\infty$ on c_0 are equivalent, and consequently $Y \cong c_0$. □

Proposition 6.3. *If X does not contain a subspace isomorphic to c_0, then any weakly integrable function $f : S \to K$ is Pettis integrable.*

We can prove the above proposition by contradiction. Assuming a weakly integrable function $f : S \to X$ is not Pettis integrable, we can find a sequence $(y_i)_{i\in\mathbb{N}}$ in X that satisfies the conditions in Theorem 6.3 and apply the theorem to $(y_i)_{i\in\mathbb{N}}$ to conclude that X contains a subspace isomorphic to c_0, which is a contradiction.

Although the Pettis integral is weaker than the Bochner integral, There are some properties they have in common, especially the properties concerning convexity. One example is Jensen's inequality, and another example is given below.

Proposition 6.4. *Suppose $f : S \to X$ is Bochner integrable and $A \subset X$ is the closure of the convex hull of the range of f, then $\int_S f \in A$. Similarly, if $f : S \to X$ is Pettis integrable, then (P)-$\int_S f \in A$.*

Proofs of both examples only rely on approximating a convex, lower semi-continuous function by affine functions, in which case how the integrals are defined does not matter as long as they are linear.

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