COMPLETELY POSITIVE MAPS AND STINESPRING'S THEOREM

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1. Introduction

Completely positive maps arise very often in the field of quantum physics, an example of which is quantum channels in quantum information theory. In this essay, We will probe into completely positive maps from a mathematical point of view; we shall look at how they can be viewed as a generalization of positive linear functionals, which was briefly talked about in sections 4.7 and 4.8 of Arveson. Completely positive maps have nice properties and related results such as complete boundedness and Stinespring's theorem, which is essentially a generalization of the GNS construction theorem for positive linear functionals. We will look into the proof of Stinespring's theorem to get a better understanding of its difference from the GNS construction theorem how the approximate unit serves as a substitute for the unit in genral. We shall also look at some applications of Stinespring's theorem: the Cauchy Schwartz inequality for completely positive maps and Arveson's extension theorem.

2. Basic Definitions and Properties

In this section, we will be looking at the definition of a completely positive map and its properties. We first look at the notion of matrix algebra.

Definition 2.1 (Matrix Algebra over C^*-algebra)**.** *Let A be a C^*-algebra, and denote by $M_n(A)$ the algebra of $n \times n$ matrices whose entries are elements of A. Then $M_n(A)$ is a C^*-algebra when equipped with the induced operator norm and the involution $(M^*)_{i,j} = (M)^*_{j,i}$.*

We have $M_n(B(H)) = B(H^n)$ in the sense that a bounded operator $T \in B(H^n)$ can be uniquely represented by a matrix whose entries are bounded operators on H. By the Gelfand-Naimark theorem (the unital case was proved in Arveson [1] but the nonunital case also works, see [2]), A can be represented by a faithful representation $\pi : A \to B(H)$ for some Hilbert space H. This induces a faithful representation $\pi^{(n)} : M_n(A) \to M_n(B(H)) = B(H^n)$ by applying π entrywise. Note that the injectivity of $\pi^{(n)}$ is clear from the injectivity of π. Using this representation we can define a norm on $M_n(A)$ by $\|M\| = \|\pi^{(n)}(M)\|_{B(H^n)}$ for $M \in B(H^n)$.

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Remark. The tensor product notation $A \odot M_n(\mathbb{C})$ for $M_n(A)$ is often convenient. We can see that these are isomorphic by identifying $M \in M_n(A)$, where $M_{i,j} = a$ for some i, j and $a \in A$ and the other entries set all zero, with $a \odot e_{i,j}$, where $e_{i,j}$ is the (i,j)-th matrix unit. For example [3] and [4] makes use of this notation in some of their proofs (see for instance the proof of Lemma 5.3 (ii) [4]).

Definition 2.2 (Induced function from $M_n(A)$ to $M_n(B)$)**.** *Let A, B be C^*-algebras, and $f : A \to B$ a function. f induces a function $f^{(n)} : M_n(A) \to M_n(B)$ defined by $(f^{(n)}(M))_{i,j} = f(M_{i,j})$ for each i, j-th entry of $M \in M_n(A)$.*

Definition 2.3 (Completely positive map)**.** *Let A, B be C^*-algebras, and $f : A \to B$ a linear function. f is called positive if for any positive element $a \in A$, $f(a) \geq 0$. Similarly, f is called n-positive if $f^{(n)}$ is positive. f is called completely positive if f is n-positive for all $n \geq 1$.*

Remark. Unlike sections 4.7 and 4.8 in Arveson, we will not restrict our attention to the case where the domain space is unital. Thus we will often be required to make use of the notion of (bounded) approximate unit. Note that when A is not unital, the spectrum $\sigma(x)$ of $x \in A$ is defined as the spectrum of its counterpart $(x, 0)$ in the unitization $\tilde{A} = A \oplus \mathbb{C}$ of A. For this reason, in some instances we can still assume A is unital without loss of generality.

Completely positive maps are indeed a generalization of positive linear functionals. To see this, we first note that for any positive element $a \in A$, the functional calculus implies that $a = b^2$ for some self-adjoint element $b \in A$. Indeed, let $\psi \in C(\sigma(a))$ be defined by $\psi(x) = \sqrt{x}$. This is well defined as a has a non-negative spectrum. If we denote by $b \in C^*(a)$ the inverse image of ψ under the Galfand transform, then $b^2 = a$. Moreover, b is self-adjoint because a is. Then for any $M \in M_n(A)$ positive, writing $M = B^2$ where B is self-adjoint, we have for any $v_i \in \mathbb{C}$ that

$$\begin{aligned}
0 &\leq BV(BV)^* \\
&= VMV^* \\
&= \begin{pmatrix} \bar{v}_1 & \dots & \bar{v}_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} M \begin{pmatrix} v_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i,j=1}^n \bar{v}_i v_j M_{i,j} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}
\end{aligned}$$

where the inequality follows from Theorem 4.8.3 in Arveson. It follows that $\sum_{i,j=1}^n \bar{v}_i v_j M_{i,j}$ is self-adjoint, and from the observation that $VMV^* - \lambda I$ is invertible if and only if $\sum_{i,j=1}^n \bar{v}_i v_j M_{i,j} - \lambda \mathbf{1} \in A^{-1}$ (by looking at the unitization $\tilde{A}$), we see that it has a nonnegative spectrum, thus positive. Hence for all $v = (v_i) \in \mathbb{C}^n$,

$$\langle f^{(n)}(M)v, v\rangle = f\left(\sum_{i,j=1}^{n} \bar{v}_i v_j M_{i,j}\right) \geq 0.$$

Then by another characterization of positivity, we conclude that $f^{(n)}(M) \geq 0$ is positive.

More generally, any positive linear map $f : A \to B$ is completely positive whenever B is commutative. The proof of this more general fact is similar to the above argument, but it relies on the fact that any commutative C^*-algebra can be identified with the space of continuous functions on a locally compact space that vanishing at infinity (Corollary 3.5 in [4]).

As is the case for positive linear functionals, positive linear maps are bounded.

Proposition 2.1 (Positive maps are bounded). *Let A, B be C^*-algebras, and let $f : A \to B$ be positive. Then f is bounded.*

The proof is essentially a slight modification of the proof that positive linear functionals are bounded. The key is to decompose an element of A into positive elements.

Proof. It is enough to show that f is bounded on the set A_+ of positive elements of A. To see why, note that any $a \in A$ can be written in the form $a = x+iy$ by taking the sel-adjoint elements $x = \frac{1}{2}(a + a^*)$ and $y = \frac{1}{2}(a - a^*)$). Furthermore, by the functional calculus, any self-adjoint element $\tilde{a} \in A$ can be uniquely decomposed into $\tilde{a} = x_+ - x_-$ where $x_\pm$ are positive (II 3.1.2 (vi) [5]). Hence a can be expressed as a linear combination of four positive elements $a = (x_+ - x_-) + i(y_+ - y_-)$. Hence, once we have shown that f is bounded on A^+, we would have that

$$\|f(a)\| \leq \|f(x_+)\| + \|f(x_-)\| + \|f(y_+)\| + \|f(y_-)\| \leq 4M\|a\|$$

for all $a \in A$ for some $M > 0$.

Assume for a contradiction that f is unbounded on A_+). Then we can find a sequence $x_n \geq 0$ in A with $\|x_n\| = 1$ and $\|f(x_n)\| \geq 4^n$. Then for $x = \sum_{n=1}^{\infty} 2^{-n} x_n$, we have $x - 2^{-n}x \geq 0$, and by the positivity of f it follows that $f(x) \geq f(2^{-n}x_n)$. Thus $\|f(x)\| \geq \|2^{-n}f(x_n)\| = 2^n \to \infty$ as $n \to \infty$, which is a contradiction. □

It follows that completely positive maps are bounded. Recalling that any C^*-algebra admits an approximate unit, one can show that for any completely positive map $f : A \to B$ and an approximate unit (e_λ) for A, the 2-positivity of f implies $\|f\| = \sup_\lambda \|f(e_\lambda)\|$ (Lemma 5.3 [3]). This particularly implies that when A is unital, $\|f\| = \|f(1)\|$. One can also show $\|f^{(n)}\| = \|f\|$ for all $n \geq 1$ by verifying that for any approximate unit (e_λ), $e_\lambda \odot 1 \in A \odot M_n(C) \cong M_n(A)$ form an approximate unit for $M_n(A)$. It follows that ρ is not only bounded but also completely bounded in that $\sup_n \|f^{(n)}\| < \infty$ with

$$\|f\| = \|f^{(n)}\| = \sup_{\lambda} \|f(e_\lambda)\| \tag{1}$$

Since any positive map preserves the self-adjointness of positive elements, it is $*$-preserving:

Proposition 2.2. *Every positive map $f : A \to B$ is $*$-preserving, i.e. $f(a^*) = f(a)^*$.*

Proof. Recalling the decomposition $a = (x_+ - x_-) + i(y_+ - y_-)$, we have

$$\begin{aligned} f(a^*) &= f((x_+ - x_-) + i(y_+ - y_-))^*) \\ &= f(x_+^*) - f(x_-^*) - if(y_+^*) - if(y_-^*) \\ &= f(x_+) - f(x_-) - if(y_+) - if(y_-) \\ &= f(x_+)^* - f(x_-)^* - if(y_+)^* - if(y_-)^* \\ &= (f(x_+ - x_-) + i(f(y_+) - f(y_-)))^* \\ &= f(a)^*. \end{aligned}$$

□

The composition of completely positive maps is completely positive. Suppose $f : A \to B$ and $g : B \to C$ are completely positive. By definition we have $(f \circ g)^{(n)} = f^{(n)} \circ g^{(n)}$. Since both $f^{(n)}$ and $g^{(n)}$ are positive, it immediately follows that $(f \circ g)^{(n)}$ is also positive.

3. Examples of Completely Positive maps

In this section, we look at a few examples of completely positive maps.

Example 3.1. Any $*$-homomorphism $f : A \to B$ is completely positive. We know that any positive $a \in A$ has a self-adjoint square root b. Hence $f(a) = f(b)^* f(b) \geq 0$ by Theorem 4.8.3 in [1]. By the definition of $*$ on matrix algebras, it is clear that $f^{(n)}$ is a $*$-homomorphism, hence positive by the previous argument. Thus any $*$-homomorphism is completely positive.

Another important example of completely positive maps is the compression. We will see later that any completely positive map is a compression of some $*$-homomorphism.

Definition 3.1 (Compression)**.** *Let $f : A \to B$ be n-positive and $b \in B$. Then the map $g : A \to B$ defined by $g(x) = bf(x)b^*$ is called the compression of f by b.*

Proposition 3.1. *Any compression of a $*$-homomorphism is completely positive.*

Proof. Let $f : A \to B$ be a $*$-homomorphism, and $g : A \to B$ the compression of f by some element $b \in B$. Then any $n \geq 1$, $f^{(n)}$ is positive. Observe that

$$g^{(n)}(M) = \begin{pmatrix} b & & 0 \\ & \ddots & \\ 0 & & b \end{pmatrix} f^{(n)}(M) \begin{pmatrix} b^* & & 0 \\ & \ddots & \\ 0 & & b^* \end{pmatrix} =: Bf^{(n)}(M)B^*$$

for any $M \in M_n(A)$. If $M \in M_n(A)$ is positive, then so is $f^{(n)}(M)$, so it has a self-adjoint square root $\sqrt{f^{(n)}(M)} \in M_n(B)$. It follows from Theorem 4.8.3 in [1] that

$$g^{(n)}(M) = Bf^{(n)}(M)B^* = B\sqrt{f^{(n)}(M)}(B\sqrt{f^{(n)}(M)})^* \geq 0.$$

Therefore g is completely positive. □

To get a better understanding of compressions, we restrict our attention to the space $B(\ell^2(\mathbb{N}))$ of bounded operators on $l^2(\mathbb{N})$. Given $n \geq 1$, consider the projection P of $\ell^2(\mathbb{N})$ onto the first n components, i.e. $Pe_j = e_j$ if $j \leq n$ and $Pe_j = 0$ otherwise. Note that $P^* = P$. If we represent $A \in B(\ell^2(\mathbb{N}))$ by the corresponding ("infintie dimensional") complex matrix $(a_{i,j})_{i,j\geq 1}$, then since P can be represented by the matrix

$$\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix},$$

we see that

$$PAP = \begin{pmatrix} a_{1\leq i,j \leq n} & 0 \\ 0 & 0 \end{pmatrix}$$

Hence the map on $B(l^2(\mathbb{N}))$ defined by $A \mapsto PAP$ "compresses" a bounded operator $B(l^2(\mathbb{N}))$ to a bounded operator that can be identified by some $n \times n$ matrix, that is we have $PB(\ell^2(\mathbb{N}))P \cong M_n(\mathbb{C})$ with

$$PAP = \begin{pmatrix} a_{1\leq i,j \leq n} & 0 \\ 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} a_{1\leq i,j\leq n} \end{pmatrix} \in M_n(\mathbb{C}).$$

More generally, one can see that for any Hilbert space H and its finite-dimensional subspace S, $P_S B(H) P_S, B(S)$ and $M_n(\mathbb{C})$ are isomorphic for some n [6].

4. Stinespring's Theorem

The following theorem gives a generalization of the GNS construction. Note that the following theorem only looks at completely positive maps that take values in $B(H)$, but we can assume so since the GNS construction lets us view any C^* -algebra as a subalgebra of $B(H)$.

Theorem 4.1 (Stinespring's Theorem)**.** *Let A be a C^*-algebra and H a Hilbert space. Suppose that $f : A \to B(H)$ is completely positive. Then the following holds:*

(1) *There exists a Hilbert space H', a representation $\pi : A \to B(H')$, and $V \in B(H, H')$ with $\|V\|^2 = \|f\|$ such that $f(a) = V^*\pi(a)V$ for all $a \in A$.*
(2) *(H', π, V) are canonical and minimal in the sense that if $\tilde{H}$ is another Hilber t space with a representation ρ of A, and $W \in B(H, \tilde{H})$ with $f(a) = W^*\rho(a)W$ for all $a \in A$, then there exists an isometry $U : H' \to \tilde{H}$ onto a subspace invariant under ρ, $W\pi = \rho W$, and such that $W = UV$.*

Remark. This is indeed a generalization of the GNS construction. Given a positive linear functional $f : A \to \mathbb{C} \cong B(\mathbb{C})$, which we know is completely positive, by Stinespring's theorem we have $f = V^*\pi V$ where $\pi \in \text{rep}(A, B(H'))$, $V : \mathbb{C} \to H'$ and $\|V\|^2 = \|f\|$. Then $(\pi, V(1))$, $1 \in \mathbb{C}$, is a GNS pair for f

Remark. When A is unital, the construction of V is rather simple. For the non-unital case, we need to resort to the existence of a bounded approximate unit for arbitrary C^* algebras. The same can be said for the non-unital version of the GNS construction theorem (see Theorem 8.9 [2]).

The proof of this theorem is similar to the proof of the GNS theorem. In fact, the only part of the proof where we need to use the n-positivity of f is showing that the inner product we define to construct H' is positive semidefinite.

Before we get to the proof of the theorem, we look at this simple lemma:

Lemma 4.2 (*p.*193 [4])**.** *An element $M \in M_n(A)$ is positive if and only if it is a sum of matrices of the form $C_{i,j} = a_i^* a_j$ for some $a_1, \dots a_n \in A$.*

Proof. ($\Longrightarrow$) If $M_{i,j}$ is positive, then it has a unique self-adjoint square root $B \in M_n(A)$, i.e. $M_{i,j} = B^2$. Then recalling the definition of $*$ on $M_n(A)$ we have

$$M_{i,j} = \sum_{k=1}^{n} B_{i,k} B_{k,i} = \sum_{k=1}^{n} B_{k,i}^* B_{k,i}.$$

Now define $C^{(k)} \in M_n(A)$ by $C_{i,j}^{(k)} = B_{k,i}^* B_{k,j}$, Then $M = \sum_k C^{(k)}$.
($\Longleftarrow$) If $M_{i,j} = a_i^* a_j$ for some $a_1, \dots a_n \in A$, then $M = B^*B$ where $B_{1,j} = a_j$ and $B_{i,j} = 0$ for the other i. Hence $M \geq 0$. From Arveson we know that a sum of positive elements is also positive. $\square$

We will give a (fairly) brief proof of the theorem by filling in some of the details Blackadar [5] omitted in his proof by mainly referring to [2], [3] , and [4].

Proof of Theorem 4.1. We consider the algebraic tensor product $A \odot H$. Viewing this as a vector space, we claim that the following defines a pre-inner product on $A \odot H$: we first define

$$\langle a \odot h, b \odot g \rangle = \langle f(b^*a)h, g \rangle_H.$$

for $a, b \in A$ and $h, g \in H$, and since the collection of simple tensors $a \odot h$ span $A \odot H$, we can extend this definition linearly to define $\langle \cdot, \cdot \rangle$ on the whole space

$A \odot H$. Using the fact that f is $*$-preserving one readily sees that it is conjugate. Thus by construction, $\langle \cdot, \cdot \rangle$ is sesqlinear.

For positive semidefiniteness, due to linearity we only need to check for the elements of $A \odot H$ of the form $x = \sum_{i=1}^n a_i \odot h_i$ for some $a_i \in A$, $h_i \in H$ and $n \geq 1$. Define $M \in M_n(A)$ by $M_{i,j} = a_j^* a_i$, i.e. Then by the above lemma, M is positive, hence so is $f^{(n)}(M)$. Viewing $f^{(n)}(M)$ as an operator on H^n, it is not hard to see that $\langle x, x \rangle = \langle f^{(n)}(M)h, h \rangle_{H^n} \geq 0$ where $h = (h_1, \ldots h_n)$.

Finally, take the quotient space $(A \odot H)/N$, where $N = \{x \in A \odot H : \langle x, x \rangle = 0\}$, and complete it with respect to the inner product to obtain a Hilbert space H'. Now define $\pi(a)$ on $A \odot H$ by $\pi(a)(b \odot g) = ab \odot g$ and extending linearly. We claim that $\pi(a)(N) \subset N$. For $x = \sum_{i=1}^n b_i \odot g_i \in N$, we have

$$\langle \pi(a)x, \pi(a)x \rangle = \langle f^{(n)}(\tilde{x}^* \tilde{a}^* \tilde{a} \tilde{x})g, g \rangle_{H^n},$$

where $\tilde{a}, \tilde{x} \in M_n(A)$ with $\tilde{a}_{i,j} = \delta_{i,j} a$ and $\tilde{x}_{i,j} = \delta_{1,i} b_j$. $\tilde{x}^* \tilde{a}^* \tilde{a} \tilde{x}$ is obviously positive, and we also have that $\|\tilde{a}\|^2 \tilde{x}^* \tilde{x} - \tilde{x}^* \tilde{a}^* \tilde{a} \tilde{x} \geq 0$. This follows from that $\|\tilde{a}\|^2$ is the spectral radius of $\tilde{a}^* \tilde{a}$ by the C^*-identity (for more detail see the proof of Theorem 8.9 in [2]). Therefore by the positivity of $f^{(n)}$ we have

$$\langle \pi(a)x, \pi(a)x \rangle = \langle f^{(n)}(\tilde{x}^* \tilde{a}^* \tilde{a} \tilde{x})g, g \rangle_{H^n} \leq \|\tilde{a}\|^2 \langle f^{(n)}(\tilde{x}^* \tilde{x})g, g \rangle_{H^n}.$$

Recalling the definition of $\tilde{x}$, we readily see that $\langle f^{(n)}(\tilde{x}^* \tilde{x})g, g \rangle_{H^n} = \langle x, x \rangle$. Since $x \in N$, we conclude $\pi(a)x \in N$. It follows that $\pi(a)$ induces a linear map on $A \odot H/N$. Furthermore, by the above inequality, $\pi_\phi(a)$ is continuous. Thus we can extend this function continuously to H', which is the completion of $A \odot H/N$.

Let $(e_\lambda)_{\lambda \in I}$ be an approximate unit for A and $h \in H$ given. then for λ_1, λ_2, we have

$$\|e_{\lambda_1} \odot h - e_{\lambda_2} \odot h\|^2 = \langle f((e_{\lambda_1} - e_{\lambda_2})^2)h, h \rangle \leq \langle f((e_{\lambda_1} - e_{\lambda_2}))h, h \rangle.$$

By the positivity of f, $f((e_\lambda))$ is a bounded increasing directed net of positive elements in $B(H)$ (boundedness follows from the continuity of f and that $\|e_\lambda\| \leq 1$). But according to [7], every bounded increasing directed net of positive elements in $B(H)$ is strongly convergent, implying $f((e_{\lambda_1} - e_{\lambda_2})) \to 0$. It follows that $(e_\lambda \odot h)_{\lambda \in I}$ is Cauchy, and we define the operator $V : H \to H'$ that compresses $\pi(a)$ to be the limit $V(h) = \lim_\lambda (e_\lambda \odot h + N)$. Note that when A is unital, we can just take $V(h) = (1 \odot h) + N$ instead. We can easily check that $\|V\|^2 \leq \|f\|$ using the Cauchy-Schwartz inequality, and conversely, the C^*-identity implies $\|f\| \leq \|V\|^2$. Thus $\|V\|^2 = \|f\|$. Finally, for any $h, g \in H$. $\langle V^* \pi(a) V h, g \rangle_H = \lim_\lambda \langle \pi(e_\lambda a e_\lambda)h, g \rangle_H$ by the continuity of $\langle \cdot, \cdot \rangle_H$ and the definition of $\langle \cdot, \cdot \rangle_{H'}$.

We have that $e_\lambda a e_\lambda \to a$ as

$$\|e_\lambda x e_\lambda - x\| \leq \|e_\lambda x e_\lambda - e_\lambda x\| + \|e_\lambda x - x\| = \|e_\lambda\| \|x e_\lambda - x\| + \|e_\lambda x - x\| \to 0$$

by the boundedness of (e_λ). Therefore $\langle V^* \pi(a) V h, g \rangle_H = \langle \pi(a)h, g \rangle$, which shows that $V^* \pi(a) V = \pi(a)$.

For minimality, we can set $U(a \odot h) = \rho(a)Wh$. We can easily verify that U is isometric using the definition of the norm on H', and the fact that $U(H')$ is invariant under ρ, $U\pi = \rho U$ follow from that ρ is a homomorphism. $\square$

Remark. When A and f are unital, then V would be an isometry. Moreover, for $h \in H$, $1 \odot h \in N$ if and only if $h = 0$. So by the correspondence $h \leftrightarrow 1 \odot h + N$ we have $H \cong V(H) \subset H'$, and $f(a) = P_H \pi(a) P_H$,

Note that according to the above proof, if H is finite-dimensional with $\dim H = n < \infty$, then the assumption that ϕ is completely positive is redundant. We may instead assume ϕ is n-positive as we only need to look at the elements $x = \sum_{i=1}^{n} a_i \odot e_i$ where $a_i \in A$, and $\{e_i\}_{i=1}^{n}$ is some basis of H. Moreover, from the results of Stingberg's theorem, we can infer that n-positivity implies complete positivity in the finite-dimensional case.

5. Applications of Stinespring's Theorem

In this section, we look at some applications of Stinespring's Theorem. This section is mainly based on [5] and [6]. We start with a fairly simple application. In section 4.7 in Arveson we saw the Cauchy-Schwartz inequality for positive linear functionals on a Banach $*$-algebra with normalized unit: $|f(y^*x)|^2 \leq f(x^*x)f(y^*y)$. We also have the Cauchy Schwartz inequality for completely positive maps $\rho : A \to B$ between C^*-algebras:

Proposition 5.1. *Let $f : A \to B$ be a completely positive map between C^*-algebras. Then*

$$f(a^*)f(a) \leq \|f\| f(a^*a), \;\; a \in A.$$

The proof of this inequality was demonstrated in Lemma 5.3 [3] by identifying B with some subalgebra of $B(H)$ thanks to the GNS construction and using equality (1) for $\|f\|$ in terms of some approximate unit of A. However, we can give a nice alternate proof using Stinespring's theorem.

Proof. If we view B as a subalgebra of $B(H)$, then by Stinespring's theorem there exists a Hilbert space H' with $f(a) = V^*\pi(a)V$ for some $V \in B(H, H')$, $\pi \in \operatorname{rep}(A, H')$ with $\|VV^*\| = \|f\|$. Then since $\|VV^*\| 1_{H'} \geq VV^*$, for any $a \in A$ we have

$$f(a)^*f(a) = V^*\pi(a)^*VV^*\pi(a)V \leq V^*\pi(a^*)(\|VV^*\|1)\pi(a)V = \|f\|V^*\pi(a^*a)V = \|f\|f(a^*a).$$

$\square$

When f is a contraction, i.e. $\|f\| \leq 1$, the inequality becomes $f(a^*)f(a) \leq f(a^*a)$, which is often referred to as Kadison's inequality.

One important application is Arveson's extension theorem, which states that any completely positive map defined on a smaller space of some C^*-algebra can be extended to the ambient space.

Since completely positive maps send positive elements to positive elements, which are self-adjoint, it is worth looking at completely positive maps that are defined on a self-adjoint subspace of C^*-algebra. Particularly, when the C^*-algebra we are interested in is unital, we can look at operator systems.

Definition 5.1 (Operator system). *Let A be a unital C^*-algebra. An operator system B of A is a closed self-adjoint subspace of A containing a unit.*

For a completely positive map f defined on an operator system, similar to the characterization of the norm for completely positive maps between C^*-algebras, we have f is bounded with $\|f\| = \|f(1)\|$ [6]. Hence we have the following unital version of Arveson's extension theorem.

Theorem 5.1 (Arveson's extension theorem, unital version). *Let X be an operator system of a unital C^*-algebra A and $f : X \to B(H)$ a completely positive map. Then f can be extended to a completely positive map $g : A \to B(H)$ of the same norm.*

By the Hann-Banach theorem, any positive linear functional defined on an operator system $X \subset A$ can be extended to A while preserving its norm. In addition to this, for any C^*-algebras A, B there is a one-to-one correspondence between completely positive maps from $M_n(A)$ to B and completely positive mas from $M_n(B)$ to A. From these, we obtain the following results:

Theorem 5.2 (Arveson's extension theorem, finite-dimensional version). *Let X be an operator system of a unital C^*-algebra A and $f : X \to M_n(\mathbb{C})$ a completely positive map. Then f can be extended to a completely positive map $g : A \to M_n(\mathbb{C})$ of the same norm.*

Remark. The reason we regard this as the finite-dimensional version is that $B(S) \cong M_n(\mathbb{C})$ for some n for any finite-dimensional Hilbert space S.

For the proof of Theorem 5.2, we can use the fact that $P_S B(H) P_S, B(S)$ and $M_n(\mathbb{C})$ are isomorphic and apply the above finite-dimensional version (with the help of the weak bounded topology).

We have a more general version of Arveson's extension theorem.

Theorem 5.3 (Arveson's extension theorem, nonunital version). *Let $B \subset A$ be C^*-algebras and $f : B \to B(H)$ a completely positive map. Then f can be extended to a completely positive map $g : A \to B(H)$ of the same norm.*

With the unital version of the theorem at hand, the proof of this version can be simplified if we use Stinespring's theorem. Namely, Stinespring's theorem can be used to prove the existence of an extension of a completely positive map defined on a nonunital C^* algebra to one defined on its unitization, which is the key to proving the general version. Recall that any C^* algebra can be embedded into its unitization as an ideal of codimension 1 (Proposition 1.15 [2]).

Lemma 5.4. *Suppose A is nonunital, and let $\tilde{A}$ denote its unitization. Let $f : A \to B(H)$ be completely positive. Then there exists a completely positive map $g : \tilde{A} \to B(H)$ with $g|_A = f$ and $\|g\| = \|f\|$.*

Proof. By Stinespring's theorem, there exists $\pi \in \mathrm{rep}(A, B(H'))$, $V \in B(H, H')$ with $f(a) = V^*\pi(a)V$ for all $a \in A$ and $\|V\|^2 = \|f\|$. Viewing $A \subset \tilde{A}$, define $\tilde{\pi} : \tilde{A} \to B(H')$ by $\tilde{\pi}(a + \lambda 1) = \pi(a) + \lambda I_{B(H')}$. Since π is a $*$-homomorphism, $\tilde{\pi}$ is obviously $*$-preserving and linear. Moreover, one can easily check using the definition of $\tilde{\pi}$ that

$$\tilde{\pi}(a + \lambda 1)\tilde{\pi}(b + \eta 1) = \tilde{\pi}(ab + \eta a + \lambda b + \eta\lambda 1).$$

Thus $\tilde{\pi} \in \mathrm{rep}(\tilde{A}, B(H'))$. Define $g : \tilde{A} \to B(H')$ by

$$g(a + \lambda 1) = V^*\tilde{\pi}(a + \lambda 1)V = f(a) + \lambda V^*V.$$

Then g is completely positive as a compression of the representation $\tilde{\pi}$. Clearly, $g(a + 0 \cdot 1) = f(a)$. So $g|_A = f$. Finally since $f(0) = 0$ equation (1) from Section 2 implies:

$$\|g\| = \|g(1)\| = \|V^*V\| = \|f\|.$$

□

The proof of Theorem 5.3 looks at two separate cases: the case where B is unital and the case where it is not. In either case, we take the unitization of A if necessary, and then pick a suitable unital subalgebra D of $\tilde{A}$ that contains B. We first extend f to this D (for the case B is nonunital we use the above lemma). We then further extend f to $\tilde{A}$ using the unital version of Arveson's extension theorem, and simply restrict the extended function to A.

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