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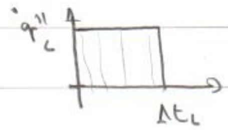
PME-EPUSP

PME 2361 – Projeto I



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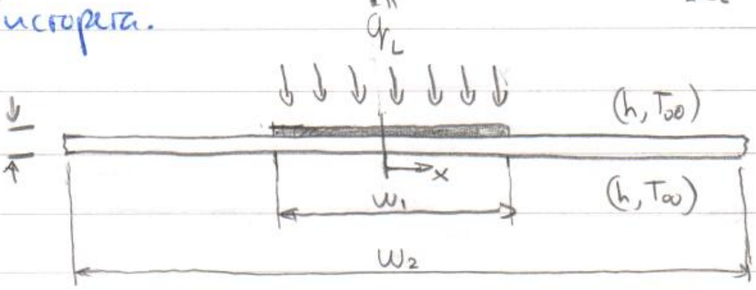


$\dot{q}''_L = 8.5 \times 10^4\ W/m^2$

$\Delta t_L = 10\,s$

$h = 10^2\ W/m^2 K$

$w_1 = 44\ mm$

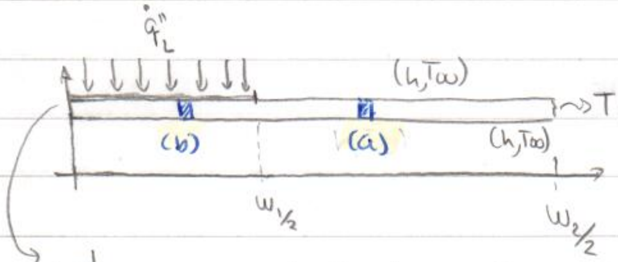
$\left.\begin{array}{l} w_2 = p\,w_1 \\ p = 3 \end{array}\right\} \Rightarrow w_2 = 132\ mm$

$\left.\begin{array}{ll} d = & mm \\ k = & W/m°C \\ c = & J/kg°C \\ \rho = & kg/m^3 \end{array}\right\}$ metal sheet.

adhesive Temperature: T_A

$T_A = T(x,t) \Big|\ -\frac{w_1}{2} \leq x \leq \frac{w_1}{2}$

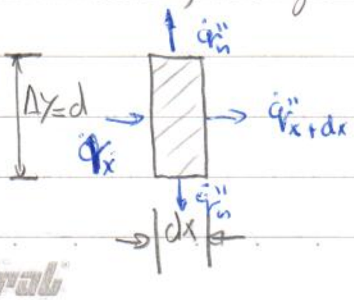
1-D transient heat transfer problem.



$\rightsquigarrow T\left(\frac{w_2}{2}, t\right) = T_\infty \Rightarrow$ constant temperature at the "tip".

$\left.\frac{dT}{dx}\right|_{x=0} = 0 \rightsquigarrow$ adiabatic wall condition at the symmetry plane.

On writing down the energy balance for "typical slices" of the metal sheet, we get:

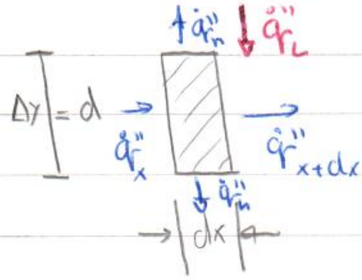
a) 

$$\Rightarrow \rho c d\, dx \frac{\partial T}{\partial t} = k d \frac{\partial^2 T}{\partial x^2} dx - 2h\,dx\,(T - T_\infty)$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - \frac{2h}{\rho c d}(T - T_\infty) \qquad (1)$$

where $\alpha \equiv k/\rho c$

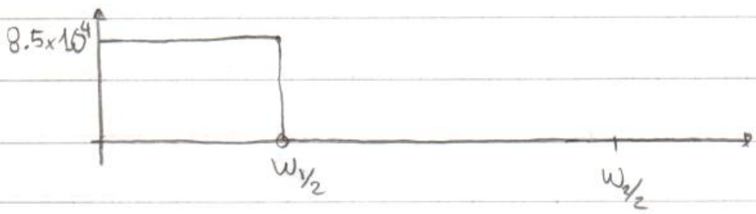
b)



$$\rho c d\, dx \frac{\partial T}{\partial t} = k d \frac{\partial^2 T}{\partial x^2} dx - 2h\, dx (T - T_\infty) + \dot{q}''_L\, dx$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - \frac{2h}{\rho c d}(T - T_\infty) + \frac{\dot{q}''_L}{\rho c d} \qquad (2)$$

In fact, eq. (2) can be adopted over the whole length of the sheet, if we take the function $\dot{q}''_L$ to be like:



$$\dot{q}''_L = \begin{cases} 8.5 \times 10^4 & \text{for } |x| \leq \frac{w_1}{2} \\ 0 & \text{for } |x| > \frac{w_2}{2} \end{cases}$$

Furthermore, we can scale the variables to the physical parameters of the problem, so as to have the evolution equation in nondimensional form. This is accomplished by defining:

$$\Theta \equiv \frac{T - T_\infty}{T_\infty} \Rightarrow d\Theta = \frac{dT}{T_\infty} \quad ; \quad T = T_\infty \Rightarrow \Theta = 0$$

$$\tilde{x} \equiv \frac{x}{w_2/2} = \frac{2x}{w_2} \Rightarrow d\tilde{x} = \frac{2}{w_2} dx$$

$$\tilde{t} = \frac{4\alpha}{w_2^2} t \Rightarrow d\tilde{t} = \frac{4\alpha}{w_2^2} dt$$

$$\begin{cases} x = 0 \Rightarrow \tilde{x} = 0 \\ x = \frac{w_1}{2} \Rightarrow \tilde{x} = \frac{1}{p} \\ x = \frac{w_2}{2} \Rightarrow \tilde{x} = 1 \end{cases}$$

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on substituting the above definitions for the corresponding terms in eq (2), we get:

$$\frac{T_\infty 4\alpha}{w_2^2}\frac{\partial\Theta}{\partial\tilde{t}} = \frac{\alpha T_\infty 4}{w_2^2}\frac{\partial^2\Theta}{\partial\tilde{x}^2} - \frac{2hT_\infty}{\rho c d}\Theta + \frac{\dot{q}_L''}{\rho c d}$$

$$\frac{\partial\Theta}{\partial\tilde{t}} = \frac{\partial^2\Theta}{\partial\tilde{x}^2} - \frac{w_2^2}{4\alpha}\frac{2h}{\rho c d}\Theta + \frac{\dot{q}_L''}{\rho c d T_\infty}\frac{w_2^2}{4\alpha}$$

$$\frac{\partial\Theta}{\partial\tilde{t}} = \frac{\partial^2\Theta}{\partial\tilde{x}^2} - 2\frac{L^2 h}{k d}\Theta + \frac{\dot{q}_L'' L^2}{k d T_\infty} \qquad (3)$$

where the length scale (L) has been defined as: $L \equiv W_2/2$. And, on accounting for the fact that the heat transfer is modelled per unit of width (perpendicular to the paper), we can recognize the Biot number of the problem as:

$$Bi \equiv \frac{h}{k}\frac{L^2 1}{d.1} \qquad (4)$$

Besides, the last term in eq. (3) can be easily shown to be dimensionless; thus defining the quantity

$$\tilde{q} \equiv \frac{\dot{q}'' L^2}{k d T_\infty} \qquad (5)$$

on introducing the above definitions into eq. (3), we are left with:

$$\frac{\partial\Theta}{\partial\tilde{t}} = \frac{\partial^2\Theta}{\partial\tilde{x}^2} - 2Bi\,\Theta + \tilde{q} \qquad (6)$$

where $\tilde{q} = \tilde{q}(\tilde{x},\tilde{t}) = \begin{cases} \dfrac{\dot{q}'' L^2}{k\, d\, T_\infty} & \text{for } |\tilde{x}| \leq \dfrac{1}{p} \; ; \; \tilde{t} \leq \dfrac{\Delta t_L 4\alpha}{\omega_2^2} \\ 0 & \text{for } |\tilde{x}| > \dfrac{1}{p} \; ; \; \forall \tilde{t} \\ 0 & \text{for } \forall \tilde{x} \; , \; \tilde{t} > \dfrac{4\alpha \Delta t_L}{\omega_2^2} \end{cases}$ (7)

The boundary and intial conditions for eq. (6) are given in dimensionless form by:

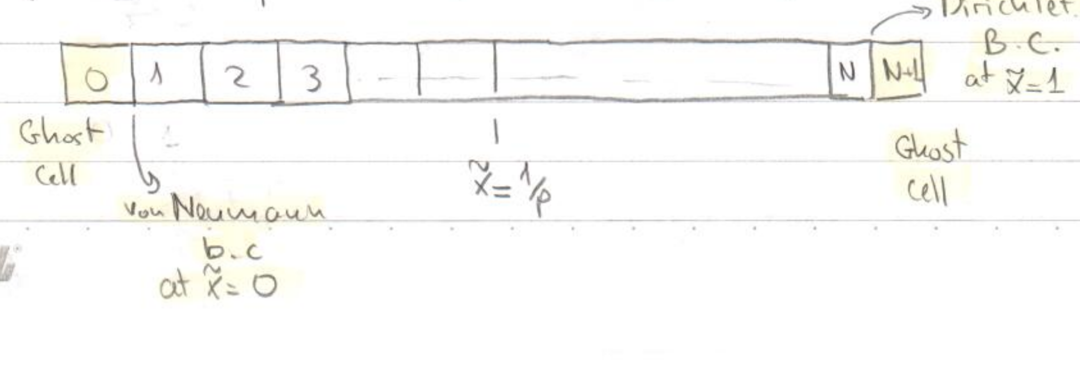
$$\left.\frac{\partial \Theta}{\partial \tilde{x}}\right|_{\tilde{x}=0} = 0 \quad ; \quad \Theta\Big|_{\tilde{x}=1} = 0 \quad ; \quad \Theta\Big|_{\tilde{t}=0} = 0 \tag{8}$$

and it is understood that the forcing term in eq (6) is $\tilde{q}$, which meets the conditions that are given in eq. (7).

The simplest way to tackle eq. (6) is by finite differences. In this approach, the derivatives are approximated by the following expressions:

$$\begin{cases} \dfrac{\partial \Theta}{\partial \tilde{t}} = \dfrac{\Theta_i^{p+1} - \Theta_i^{p}}{\Delta \tilde{t}} + O(\Delta \tilde{t}) \\ \dfrac{\partial^2 \Theta}{\partial \tilde{x}^2} = \left\{ \dfrac{\Theta_{i-1}^{p+1} - 2\Theta_i^{p+1} + \Theta_{i+1}^{p+1}}{(\Delta \tilde{x})^2} \right\} + O(\Delta \tilde{x})^2 \end{cases} \tag{9}$$

The grid is represented (schematically) by:



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Von Neumann bc. at $\tilde{x}=0$.

$$\left.\frac{\partial\theta}{\partial\tilde{x}}\right|_0 = \left.\frac{\partial\theta}{\partial\tilde{x}}\right|_{\tilde{x}=0} = 0$$

From Lomax and Pulliam's book, on page 44, we get:

$$(\delta_{\tilde{x}\tilde{x}}\theta)_1 = \frac{a}{\Delta\tilde{x}}\left(\frac{\partial\theta}{\partial\tilde{x}}\right)_0 + \frac{1}{\Delta\tilde{x}^2}(b\theta_1 + c\theta_2)$$

	θ_j	$\left.\frac{\partial\theta}{\partial\tilde{x}}\right\|_j \Delta\tilde{x}$	$\left.\frac{\partial^2\theta}{\partial\tilde{x}^2}\right\|_j \Delta\tilde{x}^2$	$\left.\frac{\partial^3\theta}{\partial\tilde{x}^3}\right\|_j \Delta\tilde{x}^3$	$\left.\frac{\partial^4\theta}{\partial\tilde{x}^4}\right\|_j \Delta\tilde{x}^4$
$\left.\frac{\partial^2\theta}{\partial\tilde{x}^2}\right\|_j \Delta\tilde{x}^2 \Rightarrow$	0	0	1	0	0
$a\left.\frac{\partial\theta}{\partial\tilde{x}}\right\|_{j-1} \Delta\tilde{x} \Rightarrow$	0	a	$\frac{a(-1)}{1}$	$\frac{a(-1)^2}{2}$	$\frac{a(-1)^3}{6}$
$b\,\theta_j \Rightarrow$	b	0	0	0	0
$c\,\theta_{j+1} \Rightarrow$	c	$\frac{c(+1)^1}{1}$	$\frac{c(+1)^2}{2}$	$\frac{c(+1)^3}{6}$	$\frac{c(+1)^4}{24}$

$$\left.\frac{\partial^2\theta}{\partial\tilde{x}^2}\right|_j \Delta\tilde{x}^2 + a\left.\frac{\partial\theta}{\partial\tilde{x}}\right|_{(j-1)} \Delta\tilde{x} + \frac{(b\theta_j + c\theta_{j+1})}{\Delta\tilde{x}^2} = O(\Delta x)^k$$

The first three columns yield the set:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (10)$$

which yields: $a = b = -\frac{2}{3}$; $c = \frac{2}{3}$

$$\left(\delta_{\tilde{x}\tilde{x}}\Theta\right)_1 = -\frac{2}{3\Delta\tilde{x}}\left(\frac{\partial\Theta}{\partial\tilde{x}}\right)_0 + \frac{2}{3\Delta\tilde{x}^2}\left(\Theta_2 - \Theta_1\right) \qquad (11)$$

and the error is given by the first column for which the summation of coefficients is non-zero. In this case, the fourth column yields:

$$-\frac{a}{2} + \frac{c}{6} = +\frac{\cancel{2}}{3\cdot\cancel{2}} + \frac{\cancel{2}}{3\cdot\cancel{6}_{3}} = \frac{1}{3} + \frac{1}{9} \Rightarrow \frac{4}{9\Delta\tilde{x}^2}\frac{\partial^3\Theta}{\partial\tilde{x}^3}\bigg|_J \Delta\tilde{x}^3$$

$$\frac{\partial^2\Theta}{\partial\tilde{x}^2}\bigg|_J = +\frac{2}{3\Delta\tilde{x}}\frac{\partial\Theta}{\partial\tilde{x}}\bigg|_{(J-1)} + \frac{2}{3\Delta\tilde{x}^2}\left(\Theta_J - \Theta_{J+1}\right) + \frac{4}{9}\frac{\partial^3\Theta}{\partial\tilde{x}^3}\bigg|_J \Delta\tilde{x}$$

Hence, if this formula were used to approximate the 2nd derivative of Θ, it would be only first order accurate. However, when it comes to approximating the first derivative at the boundary, we get:

$$\frac{\partial\Theta}{\partial\tilde{x}}\bigg|_{J-1} = \frac{3}{2}\Delta\tilde{x}\left\{\frac{\partial^2\Theta}{\partial\tilde{x}^2}\bigg|_J - \frac{2}{3\Delta\tilde{x}^2}\left(\Theta_J - \Theta_{J+1}\right) - \frac{4}{9}\frac{\partial^3\Theta}{\partial\tilde{x}^3}\bigg|_J \Delta\tilde{x}\right\}$$

$$\frac{\partial\Theta}{\partial\tilde{x}}\bigg|_{J-1} = \frac{3}{2}\frac{\partial^2\Theta}{\partial\tilde{x}^2}\bigg|_J \Delta\tilde{x} - \frac{\left(\Theta_J - \Theta_{J+1}\right)}{\Delta\tilde{x}} - \frac{2}{3}\frac{\partial^3\Theta}{\partial x^3}\bigg|_J \Delta\tilde{x}^2 \Rightarrow \underline{\underline{O(\Delta\tilde{x}^2)}}$$

With the second eq. in set (9) and eq. (11), the second order spatial derivative can be conveniently put in matrix form:

$$\delta_{\tilde{x}\tilde{x}}\underline{\Theta} = \frac{1}{\Delta\tilde{x}^2}\begin{pmatrix} -2/3 & 2/3 & 0 & 0 & & \\ 1 & -2 & 1 & 0 & & 0 \\ 0 & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & & & \\ & & & & 1 & -2 \end{pmatrix}\begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \vdots \\ \Theta_{N-1} \\ \Theta_N \end{pmatrix} + \frac{1}{\Delta\tilde{x}^2}\begin{pmatrix} -\frac{2\Delta\tilde{x}}{3}\left.\frac{\partial\Theta}{\partial\tilde{x}}\right|_0 \\ \vdots \\ 0 \\ 0 \\ \Theta_{N+1} \end{pmatrix}$$

$$\delta_{\tilde{x}\tilde{x}}\underline{\Theta} = A\,\underline{\Theta} + \vec{bc} \qquad (12)$$

where A corresponds to the tri-diagonal matrix in eq. (12) and the vector $\vec{bc}$ automatically accounts for both boundary conditions: von Neumann at $\tilde{x}=0$ and Dirichlet at $\tilde{x}=1$.

On the basis of these results, we can cast eq. (6) in the finite differences matrix form, as well.

$$\frac{\Theta_i^{p+1} - \Theta_i^{p}}{\Delta\tilde{t}} = A_{ij}\Theta_j^{p+1} + bc_i - 2B_i\Theta_i^{p+1} + \tilde{q}_i \qquad (13)$$

$$\Theta_i^{p+1} - \Theta_i^{p} = \Delta t\left(A_{ij} - 2B_i\delta_{ij}\right)\Theta_j^{p+1} + \left(bc_i + \tilde{q}_i\right)\Delta\tilde{t}$$

$$\left[\delta_{ij} + \Delta\tilde{t}\left(2B_i\delta_{ij} - A_{ij}\right)\right]\Theta_j^{p+1} = \left(bc_i + \tilde{q}_i\right)\Delta\tilde{t} + \Theta_i^{p}$$

$$\left[\left(1 + 2B_i\Delta\tilde{t}\right)\delta_{ij} - \Delta\tilde{t}A_{ij}\right]\Theta_j^{p+1} = \Delta\tilde{t}\left(bc_i + \tilde{q}_i\right) + \Theta_i^{p}$$

$$\boxed{\Theta_j^{p+1} = \left[\left(1 + 2B_i \Delta\tilde{\tau}\right) \delta_{ij} - \Delta\tilde{\tau} A_{ij} \right]^{-1} \cdot \left[\left(bc_i + \tilde{q}_i\right) \Delta\tilde{\tau} + \Theta_i^p \right]} \tag{14}$$

And eq (14) is the implicit form for time integration of the $\underline{\Theta}$ vector, and the transient temperature profile thereof. On accounting for the fact that both boundary conditions are homogeneous, the vector bci can simply be omitted.

$$\left.\frac{\partial\Theta}{\partial\tilde{x}}\right|_{\tilde{x}=0} = \Theta\Big|_{\tilde{x}=1} = 0 \quad \Rightarrow \vec{bc} = 0$$

$$\Theta\Big|_{\tilde{\tau}=0} = 0$$

$$B_i = \frac{h}{k}\frac{L^2}{d} \quad ; \quad \tilde{q} = \frac{\dot{q}'' L^2}{k\, d\, T_\infty} \quad ; \quad \tilde{x} = \frac{2x}{w_2} \quad ; \quad \tilde{\tau} = \frac{4\alpha t}{w_2^2}$$

$$w_1 = \frac{1}{p} w_2 \quad ; \quad \Theta = \frac{T - T_\infty}{T_\infty}$$

A short matlab code was written for the purpose of solving eq. (14). It's called proj1_11.m

C /classes/proj1_11.m