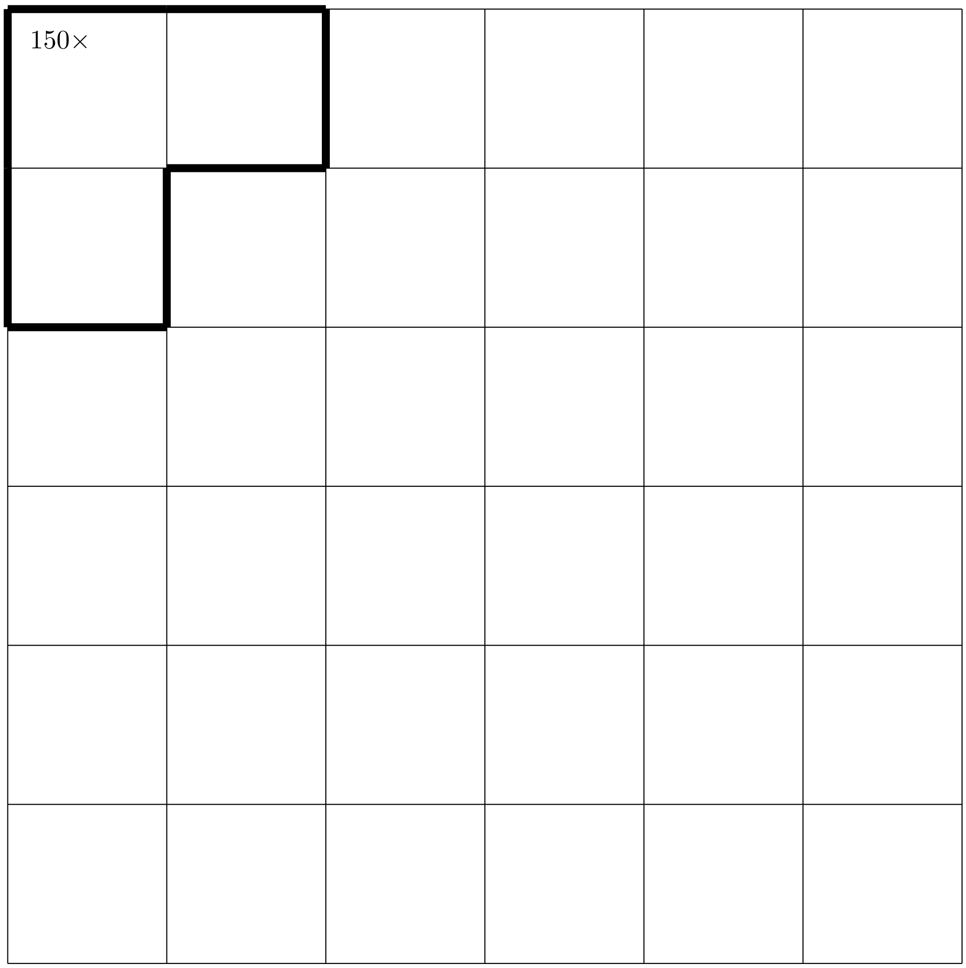
KenKen Strategies

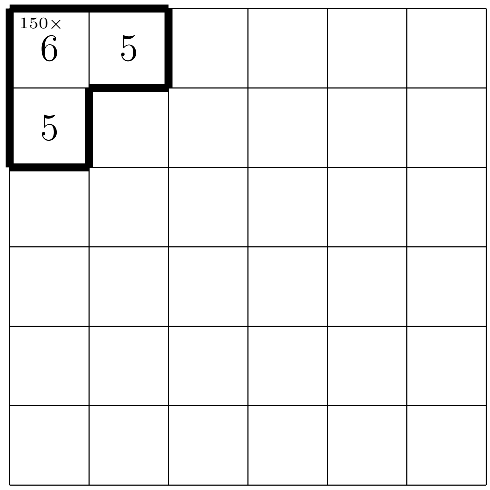
KenKen is a puzzle whose solution requires a combination of logic and simple arithmetic and combinatorial skills. The puzzles range in difficulty from very simple to incredibly difficult. Students who get hooked on the puzzle will find themselves practising addition, subtraction, multiplication and division facts.

Websites of interest:

My website: `http://www.math.uncc.edu/~hbreiter`
Tom Davis has a great tutorial intro for anyone who want to learn more KenKen:
`http://www.geometer.org/mathcircles/kenken.pdf`
The official KenKen website: `http://www.kenken.com/`
The New York Times KenKen daily puzzle
`http://www.nytimes.com/ref/crosswords/kenken.html`
Thomas Snyder's website: `http://www.stanford.edu/~tsnyder/kenken.htm`

Consider the 6×6 multiplicative Kenken® puzzle below. Find the digits that go in the three cells. Find all integers k for which the L-cage with clue $k\times$ has a unique solution.





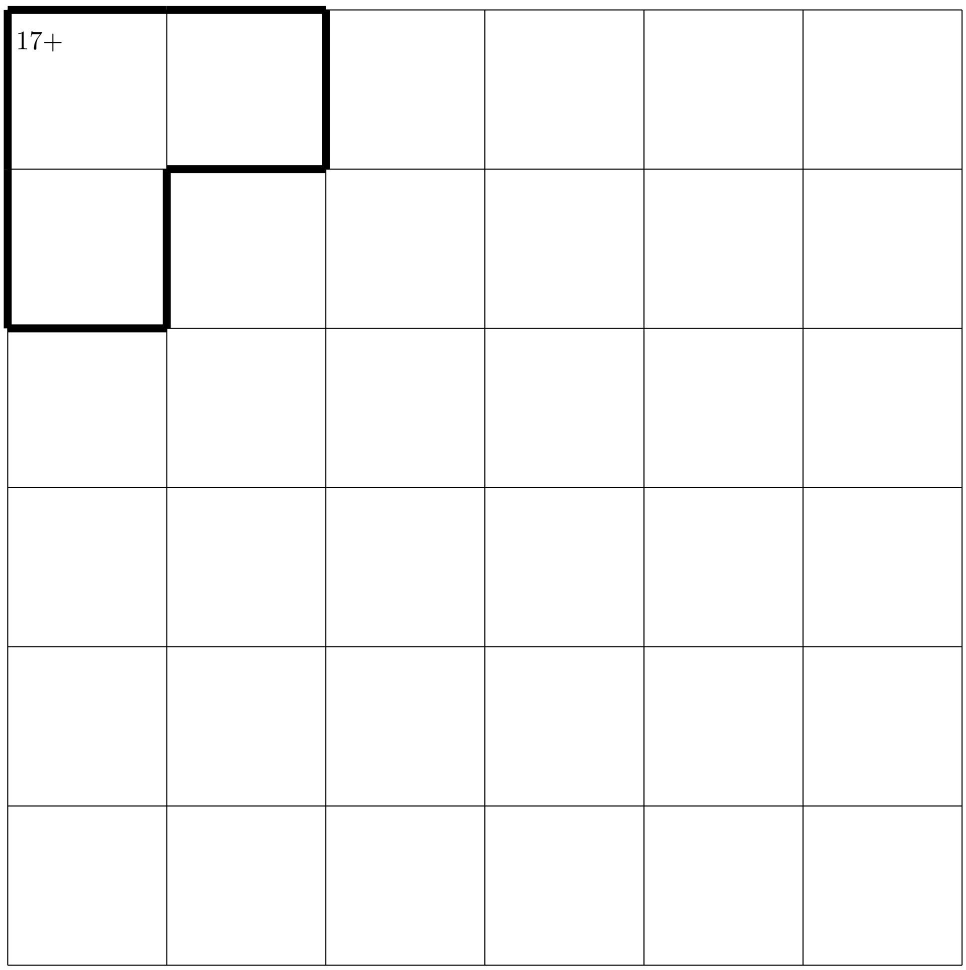
Solution:

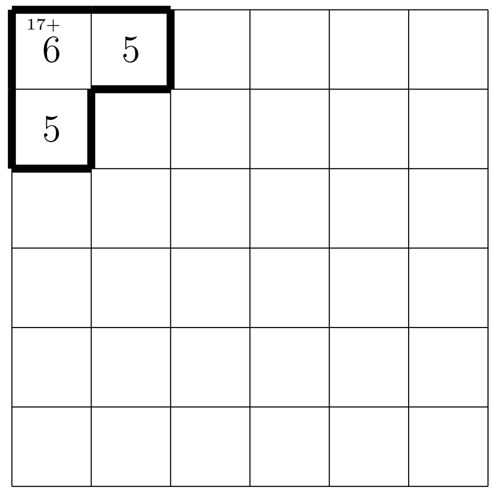
To answer this, build the 6×6 table of values of the form ab^2 with $a \in \{1, 2, 3, 4, 5, 6\}$ and $b \in \{1, 2, 3, 4, 5, 6\}$.

×	1	4	9	16	25	36
1	1	4	9	16	25	36
2	2	8	18	32	50	72
3	3	12	27	48	75	108
4	4	16	36	64	100	144
5	5	20	45	80	125	180
6	6	24	54	96	150	216

Now eliminate the numbers that appear more than once, like $16 = 4 \cdot 1 \cdot 4 = 2 \cdot 4 \cdot 2$, and those which can be represented as a product of three different digits like $24 = 2 \cdot 3 \cdot 4$. Finally, eliminate the perfect cubes. This leaves the 18 numbers $2, 3, 5, 9, 18, 45, 54, 32, 80, 96, 25, 50, 75, 100, 150, 108, 144, 180$.

Consider the 6×6 additive Kenken® puzzle below. Find the digits that go in the three cells. Find all integers k for which the L-cage with clue $k+$ has a unique solution.

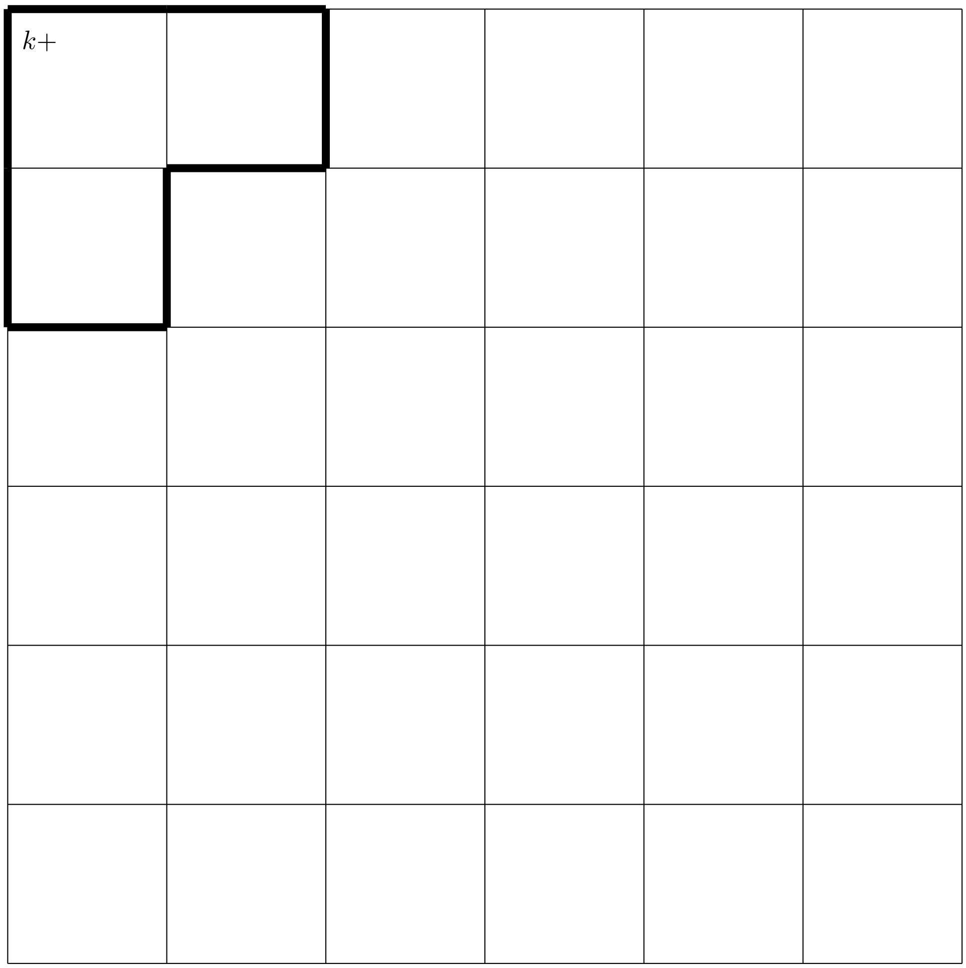




Solution:

To answer this, build the 6×6 table of values of the form $a + 2b$ with $a \in \{1, 2, 3, 4, 5, 6\}$ and $b \in \{1, 2, 3, 4, 5, 6\}$. You can see that only 17 and 4 work. Does the mapping $x \leftrightarrow 7 - x$ transform a 6×6 Kenken into an *isomorphic* one. Try taking the solution to a 6×6 KenKen puzzle and transforming it into an isomorphic one using the function $f(x) = 7 - x$.

KenKentorics. Consider the 6×6 additive KenKen® cage below. What are the possible values of k. For each such value, find the number of ways to fill the cells of the cage so that the clue is satisfied.

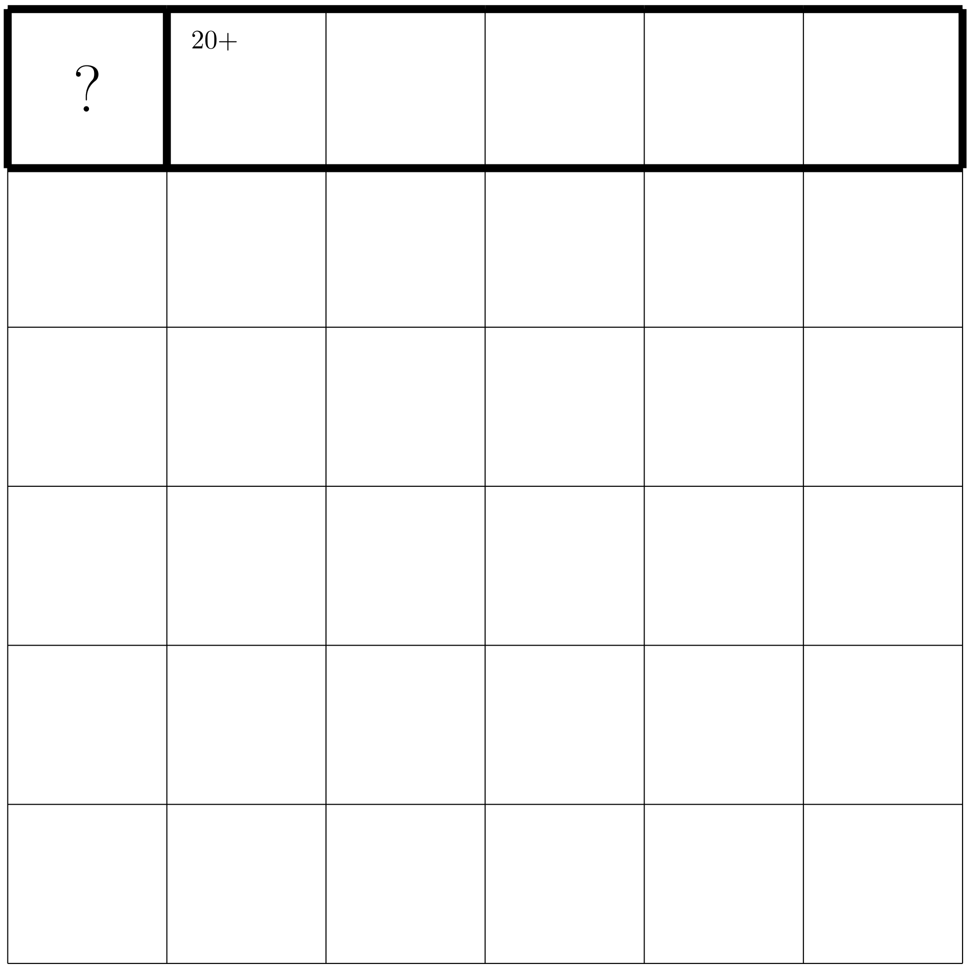


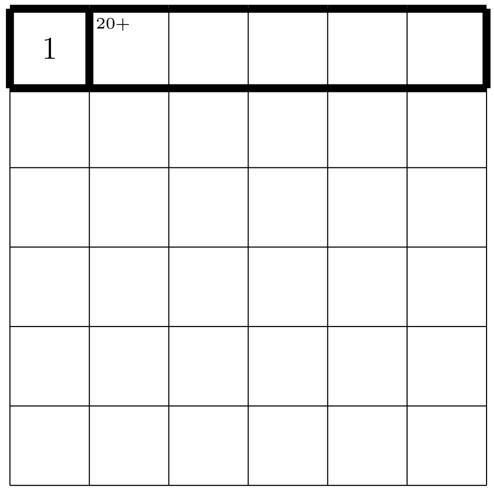
Solution: To answer this, notice that all the k values from 4 to 17 are possible. Only $k = 4$ and $k = 17$ have unique solutions. Let's build a table with the number of ways to solve the cage for each k. You can use the $x \to 7 - x$ transformation to cut your work in half. For example, once you find that there are 7 solutions to the L-cage clue 6+: $123, 132, 213, 231, 312, 321, 141$, where xyz means $\begin{array}{|c|c|}\hline y & z \\ \hline x & \\ \cline{1-1}\end{array}$ Now apply the transformation to the solution xyz to get the solution $7 - x7 - y7 - z$ for the cage $(21 - k)+$. For example the solution 123 for 6+ gets carried to 654 for the clue $(21 - 6)+ = 15+$. This transformation is one-to-one, so what you see in the second column is a palindrome.

k	$N(k)$
4	1
5	2
6	7
7	9
8	15
9	20
10	21
11	21
12	20
13	15
14	9
15	7
16	2
17	1

What is the sum of the entries of the second column? There are $\binom{6}{3} = 20$ different three-element subsets of $\{1, 2, 3, 4, 5, 6\}$, and each of these can be arranged in 6 ways resulting in 120 solutions to the L-cage problems $k+$. But there are other solutions with duplicated values in the x and z positions. There are 6 ways to pick the $x - z$ position and then 5 ways to fill the y position for a total of 30 solutions with duplicated values. So it is no surprise that the sum of the entries of the second column is 150.

Consider the 6×6 additive Kenken® puzzle below. Find the digit that goes in the cell with the ?. Find all integers k for which the rectangular 5-cage with clue $k+$ uniquely determines the other cell in that row.

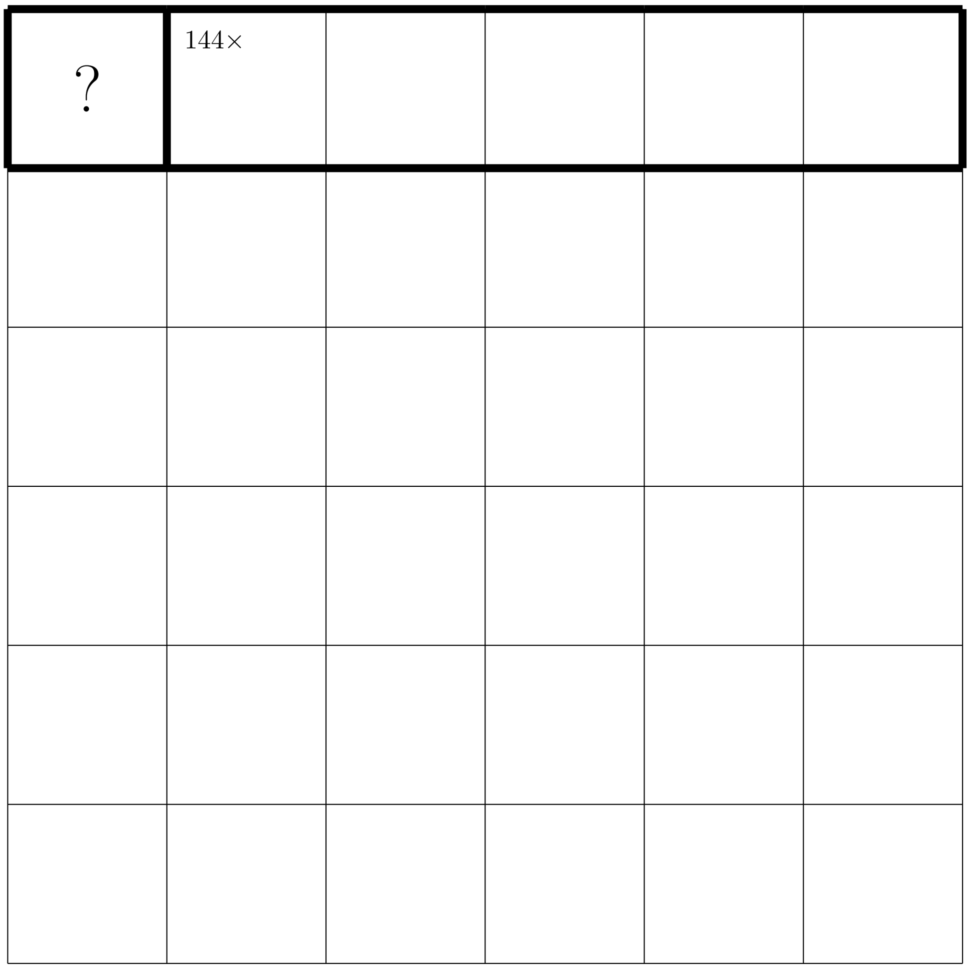


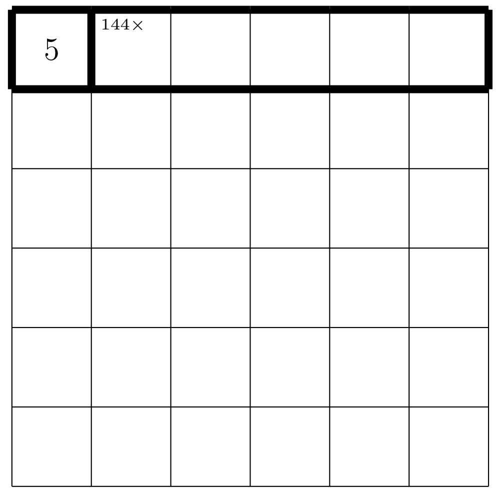


Solution:

Of course, the sum of the entries in each row is $1+2+\cdots+6=21$. So the cell with the ? must be exactly $21-k$.

Consider the 6×6 Kenken® puzzle below. Find the digit that goes in the cell with the ?. Find all integers k for which the rectangular 5-cage with clue $k\times$ uniquely determines the other cell in that row.

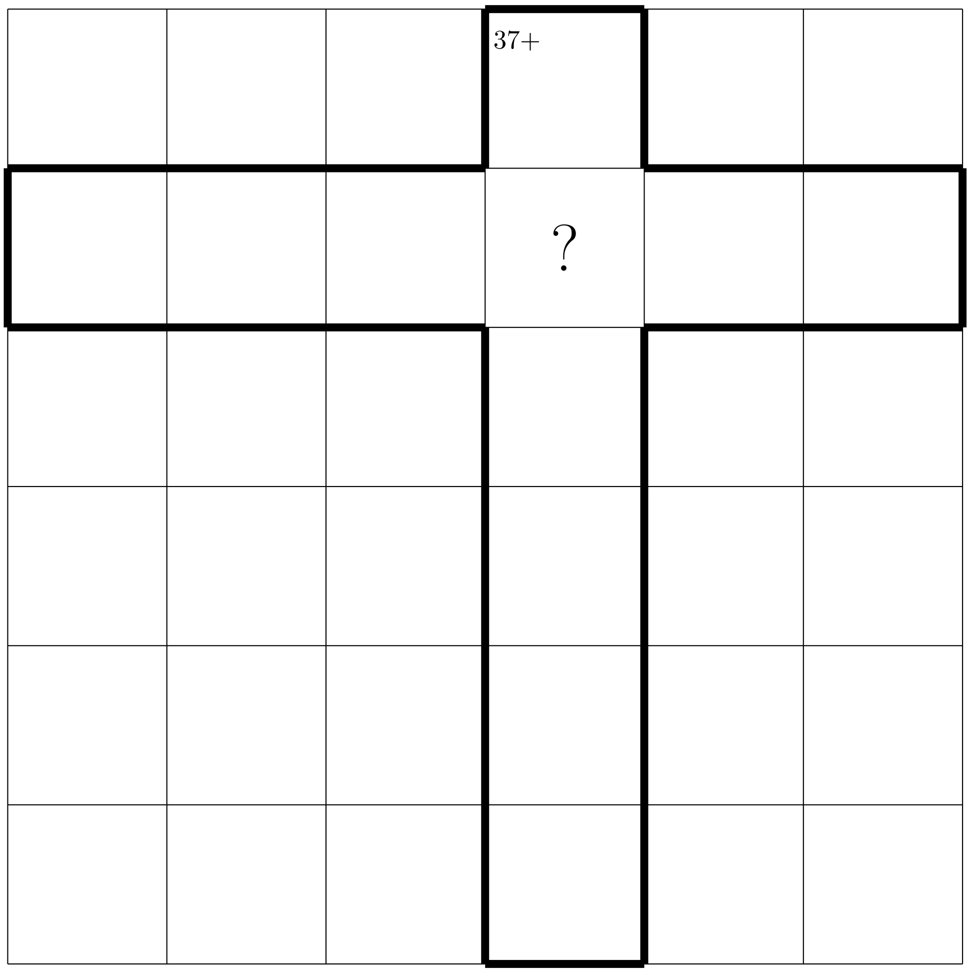


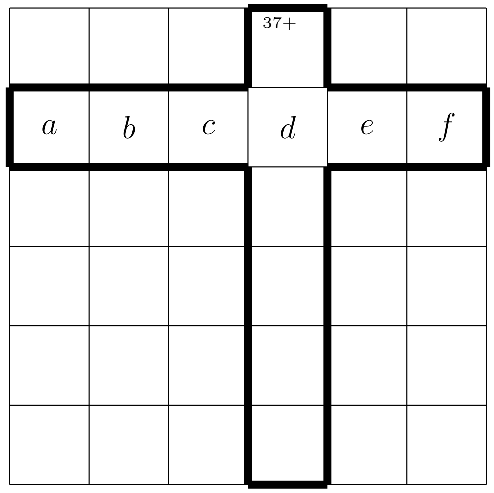


Solution:

Of course, the product of the entries in each row is $6! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 6 = 720$. So the cell with the ? must be exactly $6! \div k$.

Consider the 6×6 Kenken® puzzle below. Find the digit that goes in the cell with the ?. Find all integers k for which the rectangular 5-cage with clue $k+$ uniquely determines the other cell in that row.

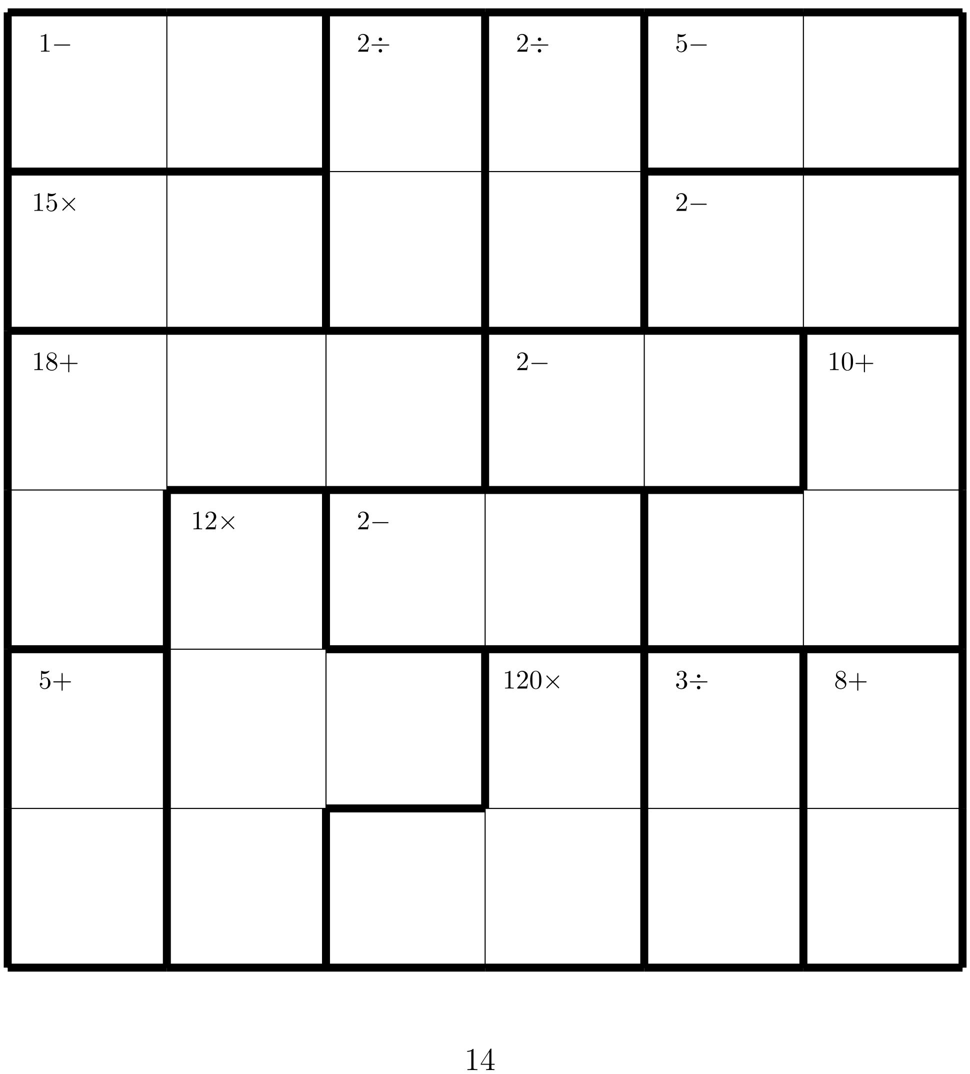


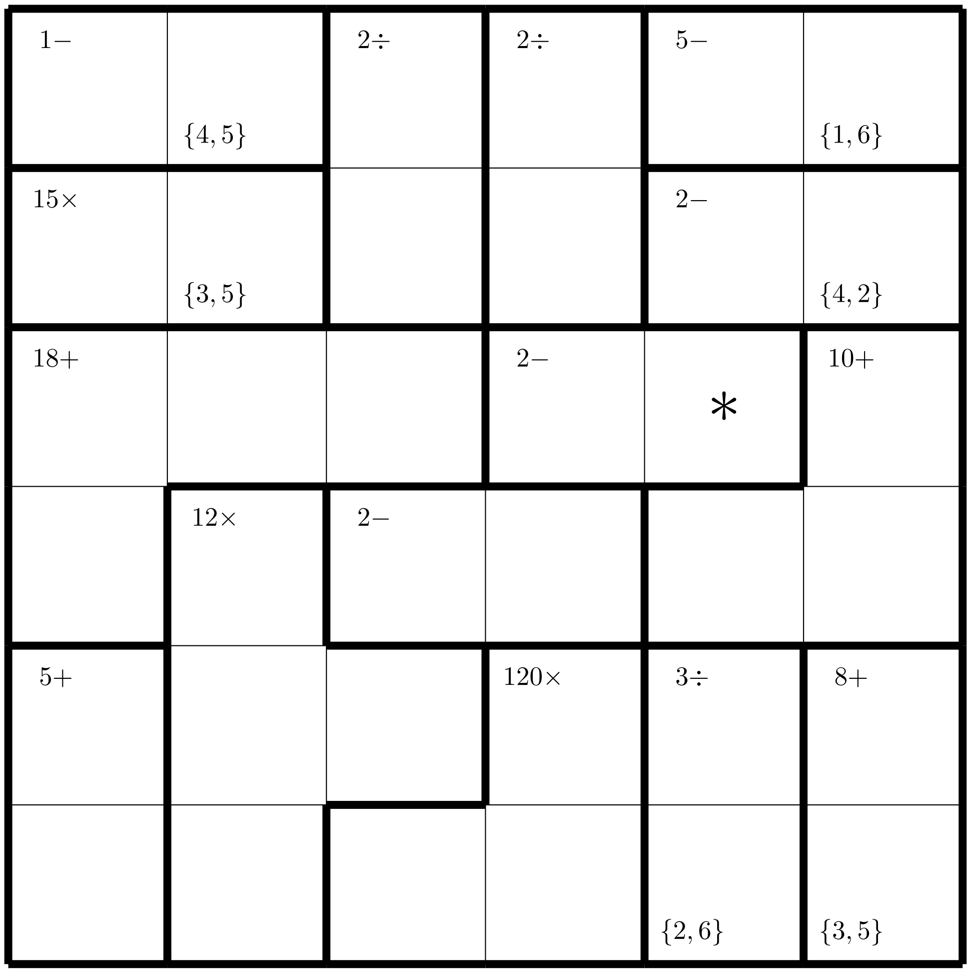


Solution:

The sum of the row entries, $a+b+c+d+e+f$ is 21, so the sum of the 5, non-d entries in the column must be $37-21=16$. Hence $d=21-16=5$.

Using Faultlines and Parity. Consider the 6×6 KenKen® puzzle below. A faultline is a horizontal or vertical cage line that extends across the whole KenKen grid. Numbering the seven lines $0, 1, 2, 3, 4, 5, 6$, horizontal line 2 is a faultline. The parity of a cage is either even or odd, depending on the parity of the sum of the entries in the cage. Initially each cage can be categorized into one of three categories: *odd*, *even*, or *undetermined* (that is, it could be odd or even). In the problem below, the odd cages are $1-, 5-, 5+, 120\times$ while the even cages are $15\times, 18+, 2-, 3\div, 8+$. The other three, $2\div, 2\div$ and $12\times$ are undetermined. Its worth spending a little time to determine the parity of cages of each type $\{+, -, \times, \div\}$. Of course the $+$ and $-$ cages are easy: $k+$ cages are even if k is even and odd if k is odd. Is the same true for $k-$ cages? Now for $k\times$, we have to consider case by case. What about the 3-cell cage $12\times$. Could be either even or odd, right: $\{1, 2, 6\}$ would be odd and $\{1, 3, 4\}$ would be even. What about a three cell cage $k\times$, like $5\times, 9\times$, or $15\times$. They must be odd cages. Why?





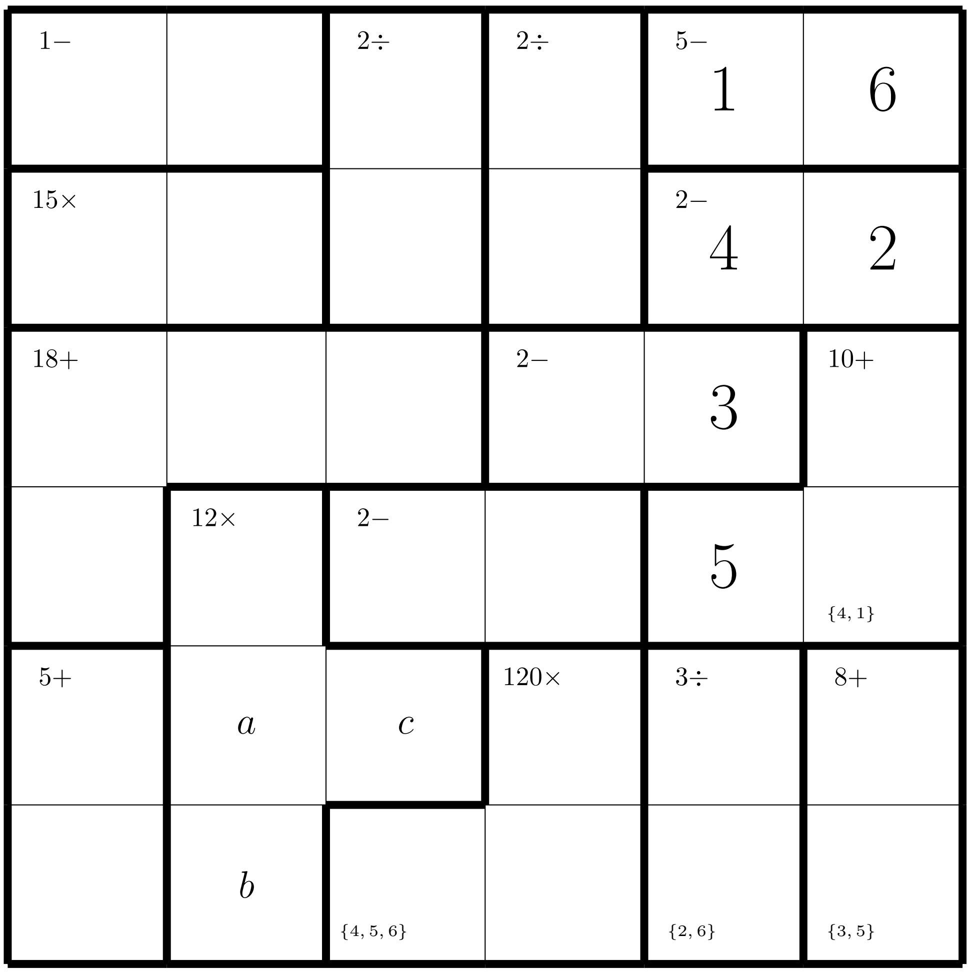
Solution:

We'll go through this solution one step at a time. For each cage $k\circ$, let $[k\circ]$ denote the sum of entries of the cage. For example if $15\times$ is a two-cell cage, $[15\times] = 8$. First focus on the top two rows. Put *candidates* in all the cages that you can. For example the 2-cell cage $5-$ must have the candidates $1, 6$, and the 5 in the top row must go in the cage $1-$. It must have the 4 with it? Why? Now that leaves the 2 and the 3 for top halves of the two $2\div$ cages. These cages must have the same parity because the top two rows must have an even number of odd cages: $1-, 5-$ are odd cages and $2-, 15\times$ are even cages. So they must be $3, 6$ and $1, 2$ Summing up the top two rows, we have $[1-] = 9, [2\div] + [2\div] = 3 + 9, [5-] = 7$, and $[15\times] = 8$. Since $2(1 + 2 + 3 + 4 + 5 + 6) = 42$, it follows that the cage $2-$ must have value $42 - (9 + 3 + 9 + 7 + 8) = 6$, whence it must be $\{2, 4\}$.

Next consider the rightmost two columns. The $3\div$ has value $1 + 3 = 4$ or $2 + 6 = 8$. Thus the five cells that are included in the rightmost two columns have value sum of $7 + 6 + 10 + 8 + L$, where L is 4 or 8. If $L = 4$, then the other cell in the 5th column would have to be 7, a contradiction. Thus, $L = 8$. So the sum is

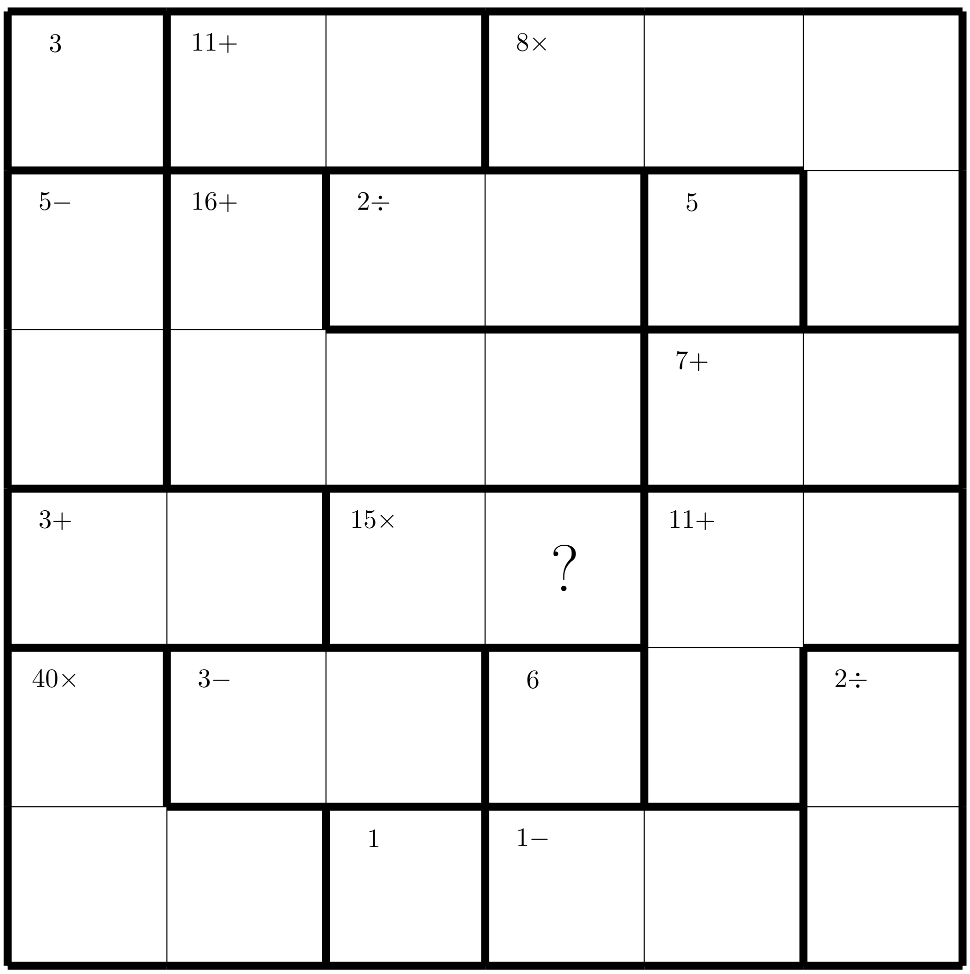
$7 + 6 + 10 + 8 + 8 = 39$. This means the cell marked $*$ is 3.

Next, because the list of candidates for the rightmost two columns includes two 2's, two 3's and two 6's, the 10+ cage must have the three digits $1, 4$, and 5. We fill the cages in the two right columns as much as possible:

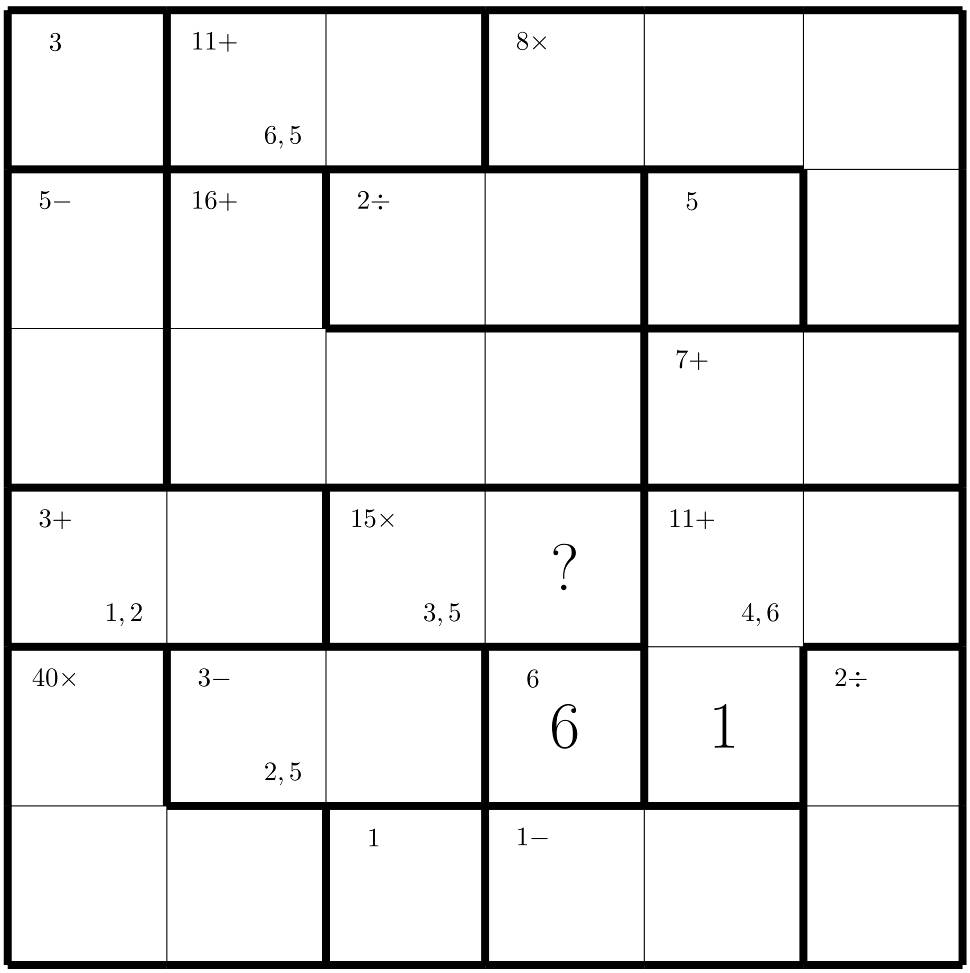


At this point we ask, what is the sum $a + b + c$? Since the sum of each row is 21, it follows that $a + b + c = 42 - (5 + 15 + 8 + 8) = 7$.

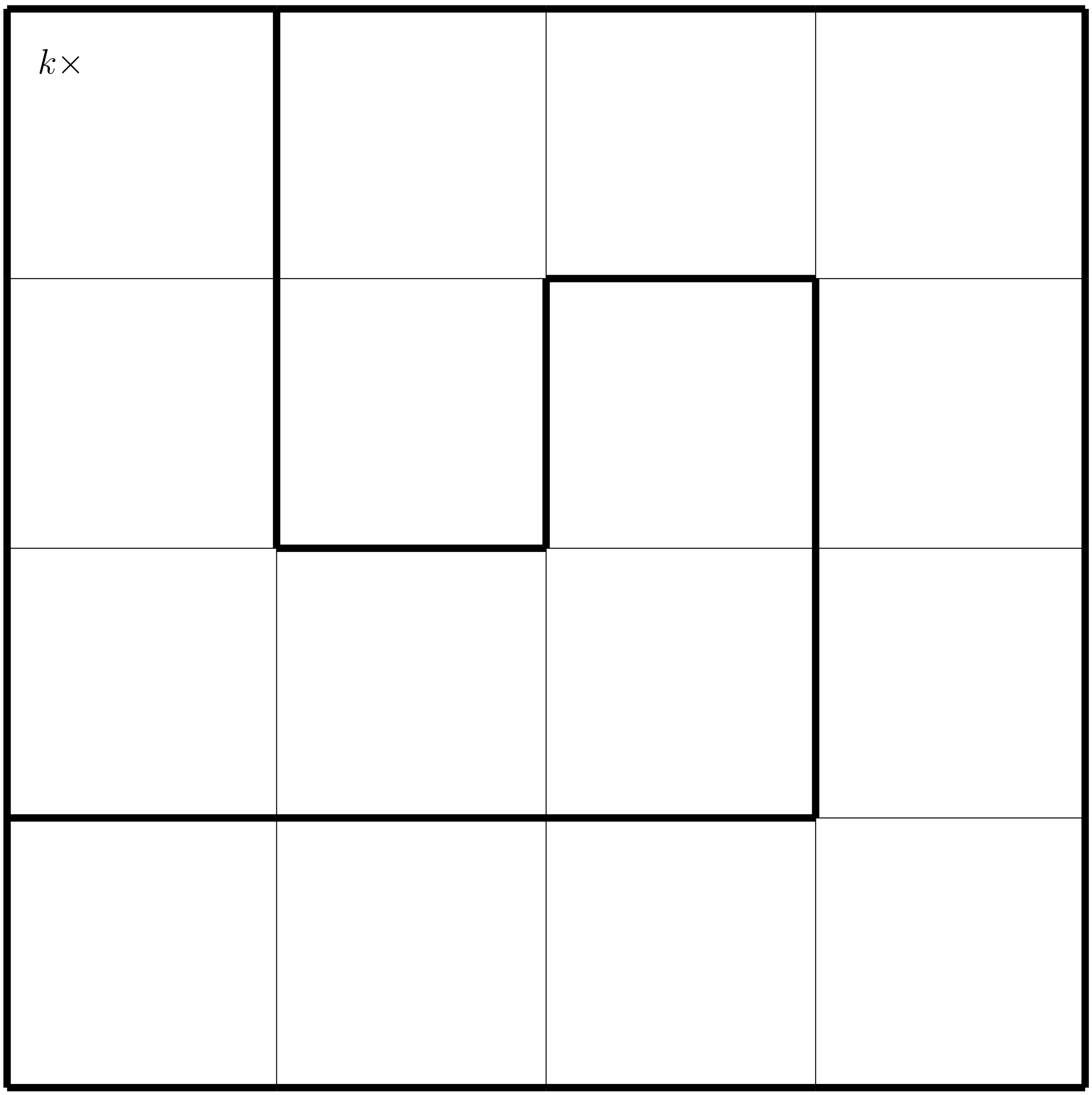
The X-wing strategy. Consider the 6×6 additive Kenken® puzzle below. Find the digit that goes in the cell marked ?. The reasoning we use here is called the $X - wing$ strategy: suppose a candidate can only be in two squares in a row (or column), and the same candidate can only be in the same two squares of a different row (or column). Then that candidate is eliminated from any other positions in the column (or row) containing either of the two squares.

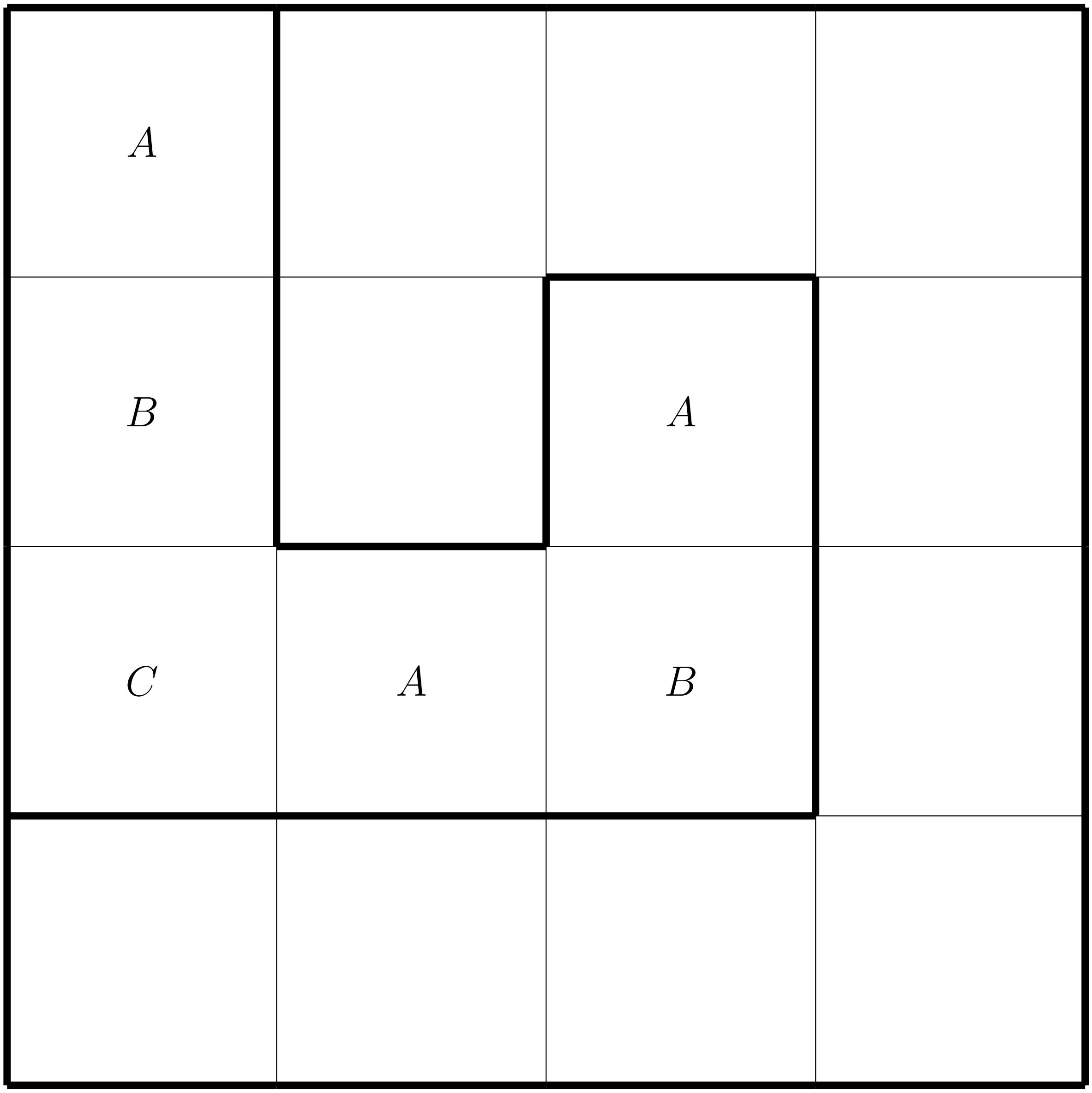


Solution: First note that the only candidates for the 3+ cage are $1, 2$ and for the $15\times$ cage $3, 5$. So our ? cell has a 3 or a 5 in it. Since the entries $1, 2, 3, 5$ are in the first four positions of the row, the fifth and sixth positions must be 4 and 6, which requires a 1 in the cell of the 11+ cage in the second row. Now focus on the 3− cage in the second row. It can't be $3, 6$ and it can't be $1, 4$ so it must be $2, 5$. Note also that the 11+ cage in the top row must be $5, 6$. Now the $X - wing$ strategy applies to the left half of the $15\times$ cage in the third row: it cannot be a 5. Thus a 5 goes in the ? cell.



Consider the 4×4 KenKen® puzzle below. Instead of using the digits $1, 2, 3$ and 4, use the letters A, B, C and D. Can you arrange it so that the cage $k\times$ is the multiset $\{A, A, A, B, B, C\}$? If so, can you find k such that the two-cage KenKen puzzle with clues $k\times$ has a unique solution.





Next, is it possible to complete the pseudo-KenKen above using the four symbols A, B, C and D? Is the solution unique? Note, all KenKen puzzles have unique solutions, so, strictly speaking, just calling these KenKen puzzles implies uniqueness. Finally, can you assign values to A, B, C and D so that