Derivations of CAPM and APT

Alan Huang
University of Waterloo

1 CAPM

CAPM is an equilibrium model (i.e., involving the concept of demand equal supply). The key assumptions to begin with are: (i) perfect markets (perfect competition, no taxes, no transactions costs, no short-sale constraints), and (ii) investors have mean-variance utilities and common belief on the expected returns and variances of assets. Let's further assume there are N investors each of whom has \$1 to invest in the universe of a riskfree asset and K stocks.

We'll first argue that by the Modern Portfolio Theory, the optimal risky portfolio for the investors is the market portfolio. The reasons are: (1) by MPT, every investor holds some combination of the riskless asset and the tangency portfolio; (2) if there is no government (which we also assume), when we aggregate the portfolios of all individual investors, lending and borrowing will cancel out (since your lending must be somebody else's borrowing). In that case, the value of the aggregate risky portfolio will equate (1) tangency portfolio times some constant, and (2) the entire wealth of the economy, which is the market portfolio. Since scaling does not affect the portfolio composition, the tangency portfolio is the market portfolio.

Now let's look at individual investors's holdings and derive a market equilibrium condition (by equating supply to demand). By MPT, investors' risky portfolio weights are:

Investor	Risk Aversion	Portfolio Weight
1	A_1	$\frac{E(r_M)-r_f}{Var(r_M)A_1}$
2	A_2	$\frac{E(r_M)-r_f}{Var(r_M)A_2}$
...	...	...
N	A_N	$\frac{E(r_M)-r_f}{Var(r_M)A_N}$

Aggregating over all investors, the total wealth invested in the market portfolio (the DEMAND of the risky asset) is:

$$\$1 \cdot \frac{E(r_M)-r_f}{Var(r_M)}\left(\frac{1}{A_1}+\frac{1}{A_2}+...+\frac{1}{A_N}\right)$$

In equilibrium, the total wealth invested in the market portfolio (the SUPPLY of capital) must be:

$$\$1 \cdot N$$

Setting demand equals to supply, we have the (*market equilibrium*) condition:

$$E(r_M) - r_f = Var(r_M)\overline{A} \tag{1}$$

where $\overline{A}$ is the overall measure of risk aversion among the market participants:

$$\frac{1}{\overline{A}} \equiv \frac{1}{N}\left(\frac{1}{A_1} + \frac{1}{A_2} + \ldots + \frac{1}{A_N}\right)$$

Now let's go to individual risky assets pricing by looking at the preference of the average investor (who has risk aversion of $\overline{A}$). The average investor's utility function is

$$U(r) = E(r) - \frac{1}{2}\overline{A}Var(r) \tag{2}$$

and his portfolio holding is the market, so that:

$$r_M = w_1^* r_1 + w_2^* r_2 + \ldots + w_K^* r_K$$

where w_i^* is the weight of stock i in the market portfolio, which is also the optimal solution for the average investor. In equilibrium, we must have:

$$\frac{\partial U(r)}{\partial w_i} = 0$$

In other words, the investor cannot increase her utility by change the amount invested. The above partial derivative can be broken into two pieces from Equation (2):

$$\begin{aligned}\frac{\partial U(r)}{\partial w_i} &= 0 \\ &= \frac{\partial E(r)}{\partial w_i} - \frac{1}{2}\overline{A}\frac{\partial Var(r)}{\partial w_i}\end{aligned}$$

- First term: $\frac{\partial E(r)}{\partial w_i}$:
  if we invest a little bit more in risky asset i (by investing a bit less in the riskfree asset), at what rate will the expected portfolio return change? This is:

  $$E(r_i) - r_f$$

  This arises because if we invest a bit, say, δ more risky asset i by borrowing, our new portfolio will have a return of:

  $$r_M' = r_M + \delta r_i - \delta r_f$$

  Or:

  $$E(r_M') - E(r_M) = \delta\left(E(r_i) - r_f\right)$$

- Second term: $\frac{\partial Var(r)}{\partial w_i}$:
  if we invest a little bit more in risky asset i (by investing a bit less in the riskfree asset), at what rate will the portfolio variance change?

  $$2Cov(r_M, r_i)$$

  This is because of the following variance change:

  $$\begin{aligned} Var(r'_M) - Var(r_M) &= Var_m + 2\delta \cdot cov(r_M, r_i) + \delta^2 Var(r_i) - Var(r_M) \\ &= 2\delta \cdot cov(r_M, r_i) + \delta^2 Var(r_i) \\ &\approx 2\delta \cdot cov(r_M, r_i), \quad \text{since } \delta^2 Var(r_i) \text{ is very small} \end{aligned}$$

  So the change in variance is $2\delta \cdot cov(r_M, r_i)$ or at a rate of $2 \cdot cov(r_M, r_i)$

So from $\frac{\partial U(r)}{\partial w_i} = 0$, it must be that:

$$E(r_i) - r_f = \overline{A} \cdot Cov(r_M, r_i) \tag{3}$$

Recall from market equilibrium (Equation (1)) that:

$$E(r_M) - r_f = \overline{A} \cdot Var(r_M) \tag{4}$$

Dividing Equation (3) by Equation (4), we must have:

$$\frac{E(r_i) - r_f}{E(r_M) - r_f} = \frac{Cov(r_M, r_i)}{Var(r_M)}$$

Or our familiar Capital Asset Pricing Model:

$$E(r_i) - r_f = \beta_i \cdot [E(r_M) - r_f]$$

where

$$\beta_i = \frac{Cov(r_M, r_i)}{Var(r_M)}$$

2 Derviation of APT and why idiosyncratic risk is not priced

Let's use a single-index model for illustration:

$$r_i = r_f + \beta_i \cdot F + \varepsilon_i$$

Now form Portfolio P, with w_i invested in stock i, and $\sum w_i = 1$

$$r_P = r_f + \beta_P F + \varepsilon_P$$

where $\beta_P = \sum w_i \beta_i$, and $\varepsilon_P = \sum w_i \varepsilon_i$. For the portfolio variance, we have

$$\sigma_P^2 = \beta_P^2 \sigma_F^2 + \sigma_{\varepsilon_P}^2$$

where

$$\sigma_{\varepsilon_P}^2 = Var(\sum w_i \varepsilon_i) = \sum w_i^2 \sigma_{\varepsilon_i}^2$$

which approaches zero for a well-diversified portfolio. For example, in an equally-weighted portfolio, when $w_i = \frac{1}{N}$. We now know that $E(\varepsilon_p) = 0$; if in addition, $Var(\varepsilon_p) = 0$, then each realization of ε_p must be 0. Hence:

$$r_P = r_f + \beta_P F$$

or the APT — that (well-diversified portfolio) return is linear in factor returns.

- Idiosyncratic risk is not priced. In other words, if you hold idiosyncratic risk, you will not be compensated in returns. Equivalently, if you are given a security, you would only pay for the part of risk that's systematic, but not for the idiosyncratic risk part. In this sense, we say that idiosyncratic risk is not priced.

If idiosyncratic risk is priced, then what can you do? Consider the following example. Suppose there is a well diversified portfolio P, whose return, by diversification, is determined by systematic risk. Now assume there exists an individual stock X, for which idiosyncratic volatility requires some positive return. Assume that a fund manager tracks the index consisted of P and X, and he will buy a small amount w_x of X so that his portfolio is well diversified. Let's call this fund P'. By diversification, we know that this fund only display systematic return, since all idiosyncratic risks are diversifiable. Compare that return of fund P' with the return on separate purchases of $(1 - w_x)$ in P and w_x in X, we find that the separate purchases will have a higher return, since idiosyncratic return in X requires positive return. We know that P' and the separate purchases are the same portfolio; so if their returns are different, this is an arbitrage opportunity. By no-arb., this cannot happen; i.e., idiosyncratic volatility in X should require zero return.

A math proof is on the next page if you want to see it.

Proof: Assume a single index model. Without loss of generality, I can write the return on a well-diversified portfolio as:

$$R_P = \beta_P F$$

and the return on an individual stock X as:

$$R_X = \beta_X F + \varepsilon_X$$

Note that in the above equations, there is no idio. risk in a well-diversified portfolio; but there is idio. risk for individual stocks (ε is firm-specific shocks).[1]

If idiosyncratic volatility requires some positive return, then $E(\varepsilon_X) > 0$. We know that for the index fund manager,

$$E(R_{P'}) = \beta_{P'} E(F)$$

since all the idiosyncratic risks are diversified away. However, a replicating portfolio of P' as constructed in the former slide have the following expected return:

$$E\left[(1-w_x)R_P + w_x R_x\right] = E\left[(1-w_x)\beta_P F + w_x \beta_X F + w_x \varepsilon_X\right] = \beta_{P'} E(F) + w_x E(\varepsilon_X) > \beta_{P'} E(F)$$

which is a contradiction. Done.

[1] Just to answer some questions raised here. Without loss of generality, I can assume shocks across firms are not related; for example, CEO turnover of Microsoft is probably not related to CFO turnover at Barrick Gold. I can relax this assumption to include some correlatedness of shocks (e.g. industry-wide news affects firms in the industry); this makes the algebra a bit more involving, but does not change at all the nature that idiosyncratic risk is diversifiable—what's being changed is that now you need more stocks in a portfolio for it to be well-diversified.