Efficient Diversification via Mean-Variance Optimization

You can consult a standard investment textbook on mean-variance optimization for this content

A Little History

In March 1952, Harry Markowitz, a 25 year-old graduate student from the University of Chicago, published "Portfolio Selection" in The Journal of Finance.

The paper opens with:
"The process of selecting a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio".

Thirty eight years later, this paper would earn him a Nobel Prize in economic sciences.

Let's be more specific...

- "We next consider the rule that the investor does (or should) consider **expected return a desirable thing** and **variance of return an undesirable thing**". -Markowitz (1952).

  Or a mean-variance investor with the following "utility score" function over a portfolio:

  $$U(r) = E(r) - \frac{1}{2} \cdot A \cdot \sigma_r^2 \qquad (1)$$

  where A captures the investor's risk aversion

- Requirements for Portfolio A to dominate Portfolio B
  - $E(r_A) \geq E(r_B)$
  - $\sigma_A \leq \sigma_B$
  - At least one inequality is strict (to rule out indifference between the two portfolios)

How big is the Universe of Risky Assets

- As of the end of 2022, the Toronto Stock Exchange (excluding TSXV) had more than 2,000 listed companies
- Mutual funds: in 2022 there were more than 5,000 mutual funds in Canada (Link)
- Nadsdaq (Q3 2022): more than 3,700 stocks (Link)
- Their derivatives
- Other international markets
- ...

The Optimal Risky Portfolio Problem

- A mean-variance investor:

$$U(r) = E(r) - \frac{1}{2} \cdot A \cdot \sigma_r^2$$

- Choosing the best risky portfolio to maximize utility
- It turns out that one only needs to identify the portfolio with the highest return-to-risk ratio

Solving the problem

1. Identify the set of "best" risky portfolios—Mean-variance frontier
2. Pick the optimal portfolio from the above set

Two equivalent standards for the set of "best" portfolios:

- For a given level of expected return, how should I allocate my portfolio so that I have the smallest risk (standard deviation)? Or
- for a given level of risk, how should I allocate my portfolio so that I have the highest return?
- The former is easier to work with.

Starting example: Two risky assets

	Debt		Equity
Expected return, $E(r)$	8%		13%
Standard deviation, σ	12%		20%
Covariance, $\mathrm{Cov}(r_D, r_E)$		0.0072	
Correlation coefficient, ρ_{DE}		0.30	

Table 7.1
Descriptive statistics for two mutual funds

- The risky-asset portfolio: w in D, $1 - w$ in E:

Setting up the Problem for Portfolio P

- The problem:

$$\begin{aligned} & \min_{w} \sigma_p^2 \\ s.t. \quad & E(r_p) = k, \text{ some constant} \end{aligned}$$

- The two-asset case is trivial:

Graphically

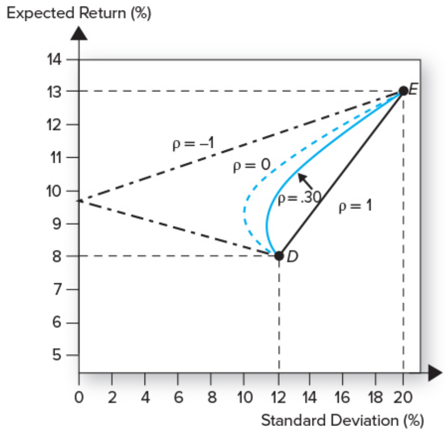

Figure 7.5 Portfolio expected return as a function of standard deviation

The global minimum-variance portfolio

- There is however, one question: What's the possible minimum risk that I can get from the combination of risky assets?

- Solution:

$$w_D = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

Global minimum-variance portfolio in our example

- $w_D = .82$
- $w_E = 1 - w_D = .18$
- Resulting portfolio moments:
  $E(r_p) = 8.9\%$, higher than the bond fund return,
  $\sigma_p = 11.45\%$, smaller than both assets' standard deviation
- What induces this?

Systematic vs. Idiosyncratic Risks

Each investment carries two distinct risks:

- Systematic risk is market-wide and pervasively influences virtually all security prices.
  - Examples are interest rates and the business cycle.
- Idiosyncratic risk involves unexpected events peculiar to a single security or a limited number of securities.
  - Examples are the loss of a key contract or a change in government policy toward a specific industry.

CAPM and risks

- Recall earlier that we measure the total risk of a stock as the standard deviation of its returns.
- β is our "market" risk of the stock.
- To link total risk with beta risk, note the following risk decomposition for observed returns from CAPM of a single stock (e.g., when estimating the stock's beta):

$$r_i - r_f = \beta_i (r_M - r_f) + \varepsilon_i$$

$$Var(r_i) = \beta_i^2 Var(r_M) + \sigma_e^2$$

  - Total risk = systematic risk + idiosyncratic risk

Linear Factors Models (or Multi-Index Model)

- More generally, assuming that returns are determined by multiple indexes (or use today's term, multiple factors), a simple model to capture these two types of risks is the linear-factor model:

$$r_i = r_f + \beta_{1,i} \cdot F_1 + \beta_{2,i} \cdot F_2 + ... + \beta_{N,i} \cdot F_N + \varepsilon_i$$

- Systematic risks: F's that are common to all securities
- idiosyncratic risk: ε

Diversification

- Let's use a single-factor (a single-index) model as an example:

$$r_i = r_f + \beta_i \cdot F + \varepsilon_i$$

- Consider an equally weighted portfolio P consisting of N assets:

$$\begin{aligned} r_p &= \sum_{i=1}^{N} \frac{1}{N} r_i \\ &= \frac{1}{N} \sum_{i=1}^{N} \left[r_f + \beta_i \cdot F + \varepsilon_i \right] \\ &= r_f + \frac{1}{N} \sum_{i=1}^{N} \beta_i F + \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \\ &= r_f + \beta_p F + \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \end{aligned}$$

Diversifiable risks

- What is the variance of the portfolio? What if N gets large?

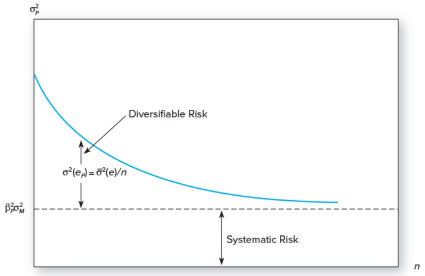

Figure 8.2 The variance of an equally weighted portfolio with risk coefficient β_P in the single-factor economy

Diversification revisited—what's driving the reduction in volatility when you form portfolio?

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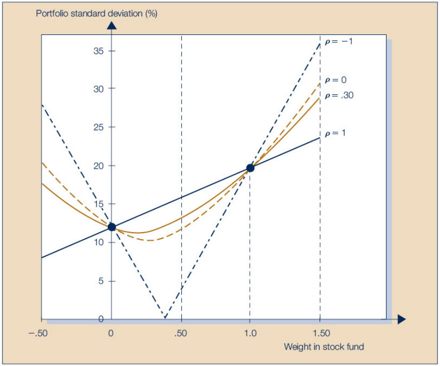
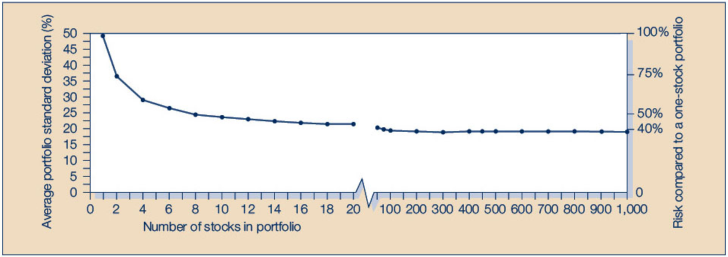
Table 7.1
Descriptive statistics for two mutual funds



Empirical evidence

Figure 7.2 Portfolio diversification. The average standard deviation of returns of portfolios composed of only one stock was 49.2 percent. The average portfolio risk fell rapidly as the number of stocks included in the portfolio increased. In the limit, portfolio risk could be reduced to only 19.2 percent.



Source: Meir Stratman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* 22 (September 1987). Reprinted by permission.

Extension to Multiple Risky Assets

- The problem is not so straightforward when you extend to 3 or more assets. The general problem is:

$$\min_{\{w_1,..,w_n\}} \sigma_p^2 \tag{2}$$

$$s.t. \qquad E(r_p) = k, \text{ some constant} \tag{3}$$

$$\sum_i^n w_i = 1 \tag{4}$$

- Equations (3) and (4) do not provide a unique solution of $\{w_1, .., w_n\}$

An Example of 3 Risky Assets

You invest in three tech stocks:

	$E(r_i)$	Covariance Matrix		
		Apple	Microsoft	HP
Apple (a)	0.20	0.09	0.045	0.05
Microsoft (m)	0.12	0.045	0.07	0.04
HP (h)	0.15	0.05	0.04	0.06

- The diagonal cells of the covariance matrix hold variances

$$cov(r_a, r_a) = \sigma^2(r_a)$$

- The covariance matrix is symmetric

Example cont'd

- Computing $E(r_p)$ and σ_p:

  $E(r_p) = w_a E(r_a) + w_m E(r_m) + w_h E(r_h)$

  $$\begin{aligned} \sigma_p^2 &= w_a^2 \sigma_a^2 + w_m^2 \sigma_m^2 + w_h^2 \sigma_h^2 \\ &\quad + 2 w_a w_m cov(r_a, r_m) + 2 w_a w_h cov(r_a, r_h) + 2 w_m w_h cov(r_m, r_h) \end{aligned}$$

- For example, investing 1/3 in each stock:

  $E(r_p) = 15.7\%$, $\sigma_p = 23.3\%$

3-risky-asset problem

The general problem now becomes:

$$\min_{\{w_a, w_m, w_h\}} \sigma_p^2 \tag{5}$$

$$s.t. \quad E(r_p) = k, \text{ some constant} \tag{6}$$

$$w_a + w_m + w_h = 1 \tag{7}$$

- The problem is mathematically solved by the constrained optimization technique. Basically, you take partial derivatives of w's and equate the partials to zero.
  - Extra constraints in this case, just reduce the number of unknowns.
- Let's do Excel solution first.
- Your first Python tutorial will illustrate how to solve this in Python (invoking package such as `pypfopt` and `efficient_frontier` therein).

The mean-variance frontier

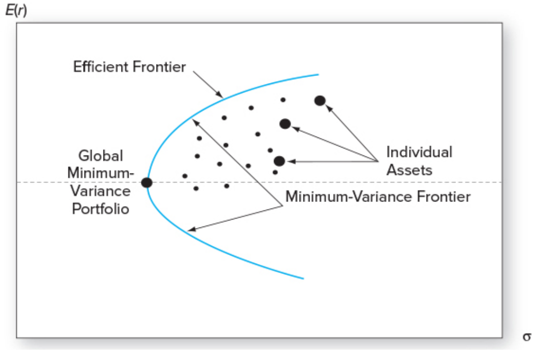


Figure 7.10 The minimum-variance frontier of risky assets

Choosing The Optimal Risky Portfolio

Now suppose there exists a riskfree asset...

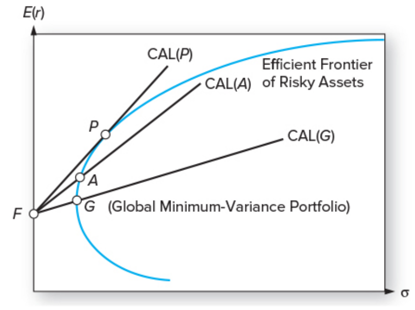


Figure 7.13 Capital allocation lines with various portfolios from the efficient set

- The best CAL is the one with the steepest slope, or the highest Sharpe ratio
- The **optimal portfolio is the Tangency portfolio**: the unique portfolio with the highest Sharpe ratio

The optimal portfolio in the two-risky-asset example

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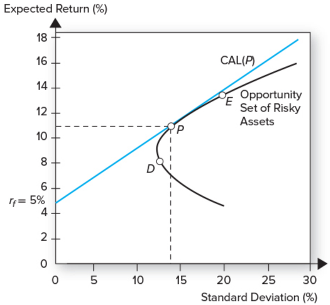

Figure 7.7 The opportunity set of the debt and equity funds with the optimal CAL and the optimal risky portfolio

Solving the Tangency portfolio in the two-asset example

- It has the highest slope, i.e. the highest reward-to-variability ratio; therefore, an alternative way to derive p is to solve the following problem:

$$\max_{w_D} S_p$$

  where

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

- Write the expression for the Sharpe ratio out as $f(w_D)$, set first derivative to zero

$$w_D = \frac{[E(r_D) - r_f]\sigma_E^2 - [E(r_E) - r_f]Cov(r_D, r_E)}{[E(r_D) - r_f]\sigma_E^2 + [E(r_E) - r_f]\sigma_D^2 - [E(r_D) + E(r_E) - 2r_f]Cov(r_D, r_E)}$$

Example

Let $r_f = 5\%$. The optimal risky portfolio for our two-funds risky assets example is:

The Optimal Portfolio in Practice

- In practice, however, different investors might have very different ideas about their Optimal Risky Portfolio. Why?

Summary

Optimal portfolio choice:

1. Use all of the risky assets to determine the tangency portfolio;
2. Depending on the degree of risk aversion, allocate wealth between the tangency portfolio and the risk free asset.