

SOLUTIONS

Math F251: Midterm 2 Review

1. Consider the function $f(x)$ and its derivatives:

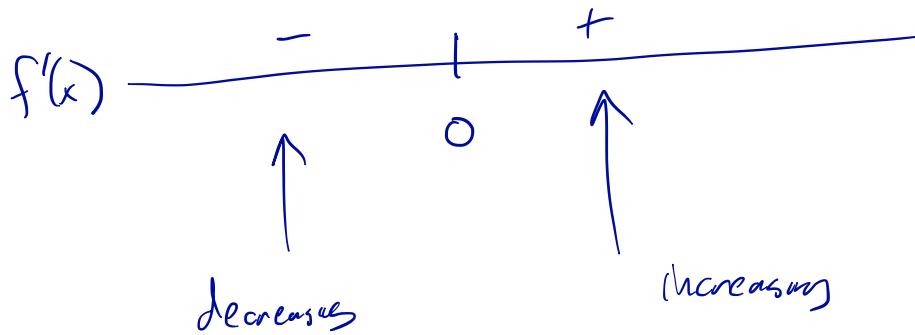
$$f(x) = \frac{e^x}{1+x}$$
$$f'(x) = \frac{xe^x}{(1+x)^2}$$
$$f''(x) = \frac{e^x(x^2+1)}{(1+x)^3}.$$

- a. Find the critical numbers of $f(x)$.

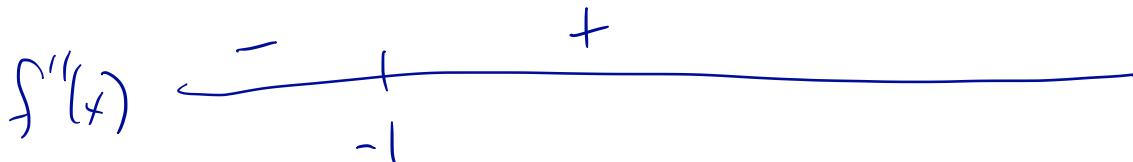
$$f'(x) = 0 : x = 0 \text{ only}$$

(function not defined at $x=-1$)

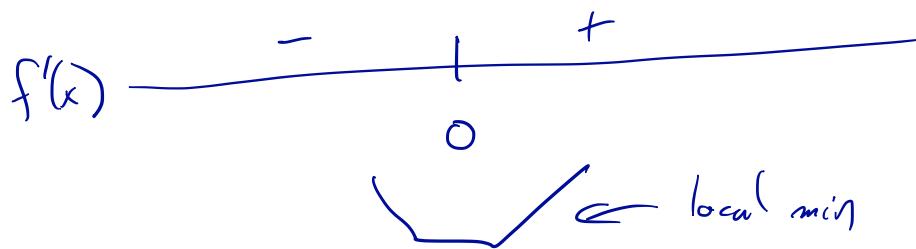
- b. Find the open intervals on which the function is increasing or decreasing.



- c. Find the open intervals on which the function is concave up or concave down.



- d. Classify all critical points – using the first derivative test.



- e. Classify all critical points – using the second derivative test.

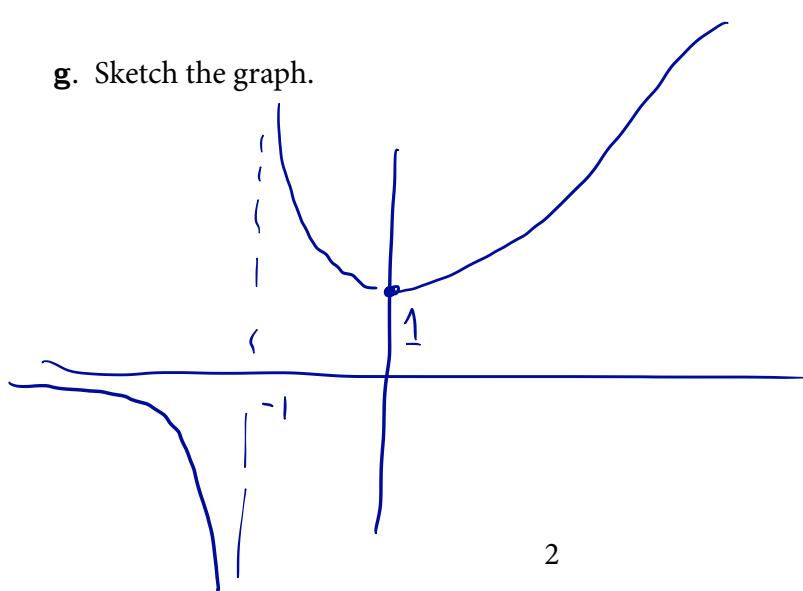
$$f''(0) = 1 > 0$$

local min

- f. Find the inflection points.

None

- g. Sketch the graph.



$$\lim_{x \rightarrow \infty} \frac{e^x}{1+e^x} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x} = \infty$$

$$\lim_{x \rightarrow -\infty} \frac{e^x}{1+e^x} = \frac{0}{1} = 0$$

2. Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$ and use it to estimate $\sqrt{4.1}$.

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4) \\ &= 2 + \frac{1}{4}(x-4) \end{aligned}$$

$$\sqrt{4.1} \approx L(4.1) = 2 + \frac{1}{4}(0)$$

3. Show that the point $(2, 3)$ lies on the curve $x^2 + xy - y^2 = 1$. Then find the slope of the tangent line to the curve at that point.

$$2^2 + 2 \cdot 3 - 3^2 = 4 + 6 - 9 = 1 \checkmark$$

$$2x + y + xy' - 2y y' = 0$$

$$y'(x-2y) = -2x-y$$

$$y' = \frac{-2x-y}{x-2y} = \frac{-2 \cdot 2 - 3}{2 - 2 \cdot 3} = \frac{-7}{-4} = \frac{7}{4}$$

4. A ball of metal is being heated in an oven, and its radius is increasing at a rate of 0.1 cm/min. At what rate is the ball's volume increasing when its radius is 3 cm?

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi \cdot 9 \cdot \frac{1}{10} = \frac{36}{10} \pi \text{ cm}^3/\text{min}$$

5. Evaluate the following limits.

$$\lim_{x \rightarrow 0} \frac{1+x-e^x}{\sin x}$$

$$\lim_{x \rightarrow 0^+} (1+2x)^{1/x}$$

$$\stackrel{0}{=} \lim_{x \rightarrow 0} \frac{1-e^x}{\cos x} = \frac{0}{1} = 0$$

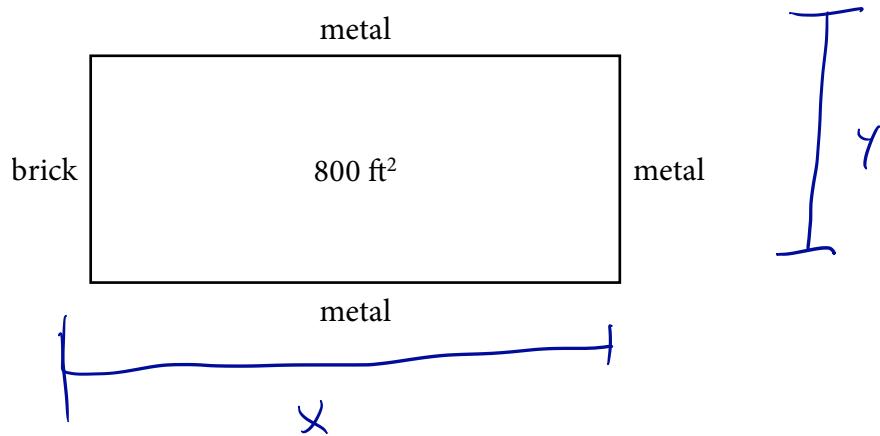
$$y = (1+2x)^{1/x}$$

$$\ln(y) = \frac{1}{x} \ln(1+2x)$$

$$\lim_{x \rightarrow 0^+} \frac{\ln(1+2x)}{x} \stackrel{0}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+2x} \cdot 2}{1} = 2$$

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln(y)} = e^2$$

6. A landscape architect wishes to enclose a rectangular garden on one side by a brick wall costing \$30 per foot and on the other three sides with a metal fence costing \$10 per foot. The area of the garden is to be 800 ft^2 . What are the dimensions of the garden that minimize the cost of the fencing?



$$\text{Cost: } C = 30y + 20x + 10y = 40y + 20x$$

$$\text{Area: } A = xy = 800$$

$$y = \frac{800}{x}$$

$$C = \frac{40 \cdot 800}{x} + 20x$$

$$C' = -\frac{32000}{x^2} + 20$$

$$C'' = \frac{2 \cdot 32000}{x^3} > 0 \text{ on } (0, \infty)$$

$$C' = 0: \frac{1600}{x^2} = 1$$

$$\begin{aligned} x &= \sqrt{1600} \\ &= 4 \cdot 10 \\ &= 40. \end{aligned}$$

$$y = \frac{800}{x} = \frac{800}{40} = 20$$

7.

- a. State the Mean Value Theorem and draw a picture to illustrate it.

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) , then there is c in (a,b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- b. Suppose $f(x)$ is continuous on $[-1,1]$ and has a derivative at each x in $(-1,1)$. If $f(-1) = 7$ and $f(1) = 5$, what does the Mean Value Theorem let you conclude?

There is a c in $(-1,1)$ where

$$f'(c) = \frac{5 - 7}{(-1) - 1} = -\frac{2}{2} = -1$$

