Fall 2024 **Math F251X**

Calculus 1: Midterm 1

Name: Solutions

Section: ☐ 9:15am (James Gossell)
☐ 11:45am (Kevin Meek)
☐ async (Deven Barnett)

Rules:

- Partial credit will be awarded, but you must show your work.
- You may have a single handwritten $3'' \times 5''$ notecard, both sides.
- Calculators are **not** allowed.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.

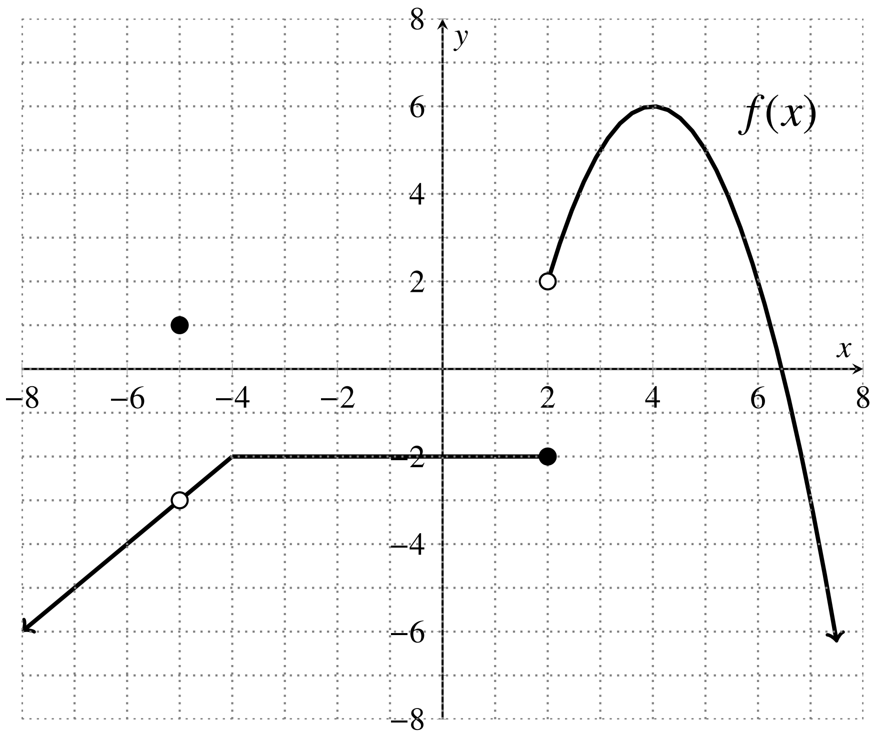
Good luck!

3:35
4:10

Problem	Possible	Score
1	12	
2	12	
3	6	
4	6	
5	10	
6	8	
7	6	
8	12	
9	12	
10	16	
Extra Credit	(5)	
Total	100	

1. (12 points)

Use the graph of $f(x)$ to answer the following questions.



a. Fill in the blanks below. If the value does not exist or is undefined, write DNE.

$\lim_{x\to -5} f(x) = \underline{-3}$ $\quad f(-5) = \underline{1}$ $\quad \lim_{x\to -4} f(x) = \underline{-2}$

$\lim_{x\to 2^+} f(x) = \underline{2}$ $\quad f(2) = \underline{-2}$ $\quad \lim_{x\to 2} f(x) = \underline{\text{DNE}}$

$f'(-6) = \underline{1}$ $\quad f'(4) = \underline{0}$ $\quad \lim_{x\to -4^+} f'(x) = \underline{0}$

b. State the x-values for which f is **not continuous**.

x = −5, 2

c. State the x-values for which f is **not differentiable** (where $f'(x)$ does not exist).

x = −5, −4, 2

2. (12 points)

Compute the following limits. Show your work and use limit notation where necessary. You will be graded both on your computation and your correct use of notation.

a. $\lim_{x\to3} \frac{x^2-5x+6}{3-x}$ $= \lim_{x\to3} \frac{(x-3)(x-2)}{-x+3} = \lim_{x\to3} \frac{x-2}{-1}$

$= -1$

b. $\lim_{x\to1} \frac{1-\frac{1}{x^2}}{x-1}$ $= \lim_{x\to1} \frac{x^2-1}{x^2(x-1)} = \lim_{x\to1} \frac{(x-1)(x+1)}{x^2(x-1)}$

$= \lim_{x\to1} \frac{(x+1)}{x^2} = 2$

c. $\lim_{x\to4} \frac{\sqrt{x}+2}{x+4}$ $= \frac{\sqrt{4}+2}{4+4} = \frac{4}{8} = \frac{1}{2}$

3. (6 points)

Determine whether or not the given function is continuous at $x = 3$. **Justify your answer using limits**

$$f(x) = \begin{cases} x^2 - 4 & \text{if } x > 3 \\ 2x + 1 & \text{if } x \leq 3 \end{cases}$$

$f(3) = 2(3)+1 = 7$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} x^2 - 4 = 5$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3} 2x+1 = 7$

Left and right hand limits do not equal, f(x) is not continuous at $x=3$

4. (6 points)

Find the value(s) of k that make the function continuous at $x = -2$.

$$g(x) = \begin{cases} \frac{x^2+3x+2}{x+2} & \text{if } x \neq -2 \\ k & \text{if } x = -2 \end{cases}$$

$f(-2) = k$

$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2+3x+2}{x+2} = \lim_{x \to -2} \frac{(x+2)(x+1)}{(x+2)}$

$= -1$

For continuity, we need

$f(-2) = \lim_{x \to -2} f(x)$

So $k = -1$

5. (10 points)

Use the limit definition (given below) of the derivative to find the derivative of

$$f(x) = 3x^2 - 4x.$$

Show all your work clearly, step by step, using correct notation. **No credit will be awarded for a solution that does not use the definition below.**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 4(x+h) - (3x^2 - 4x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 4x - 4h - 3x^2 + 4x}{h}$$

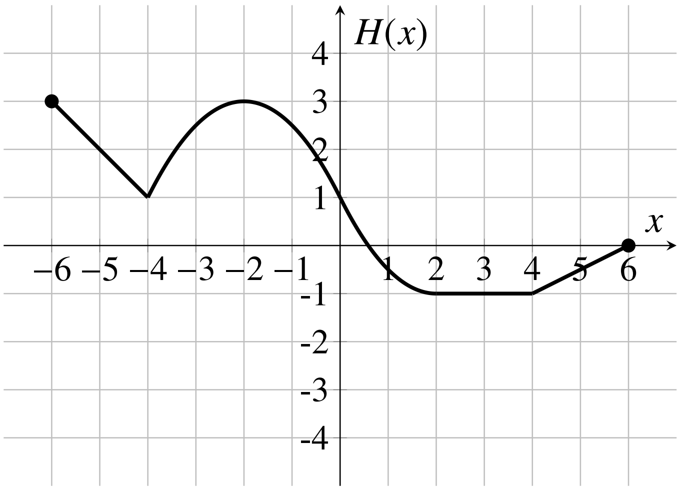
$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 4h}{h}$$

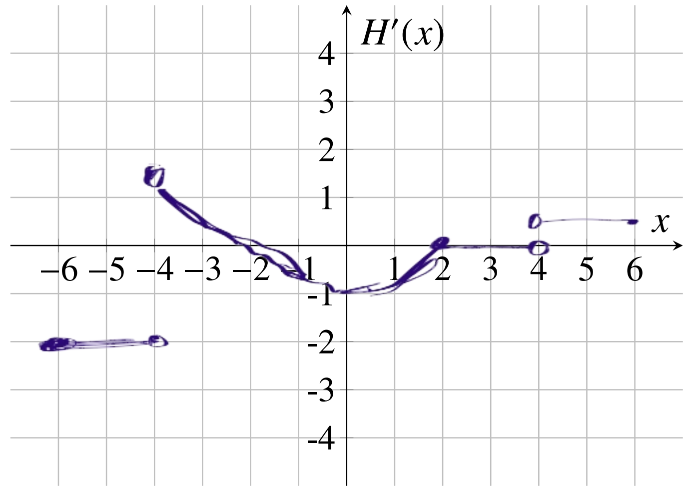
$$= \lim_{h \to 0} 6x + 3h - 4$$

$$= 6x - 4$$

6. (8 points)

The function $y = H(x)$ is graphed below. Sketch the graph of $H'(x)$ on the blank set of axes provided.





7. (6 points)

$S(t)$ is a function that describes the human population of Lewisburg, West Virginia t years after 1970 when the population was first recorded.

a. Interpret the meaning of $\boxed{S(10) = 2570}$ in the context of the problem. Write a sentence or two including appropriate units.

In 1980, the population of Lewisburg was 2,570.

b. Interpret the meaning of $\boxed{S'(10) = -135}$ in the context of the problem. Write a sentence or two including appropriate units.

In 1980, the population of Lewisburg was decreasing 135 people per year.

c. Given $S(10) = 2570$ and $S'(10) = -135$, **estimate** $S(9)$ including units. Write a sentence explaning how you arrived at your estimation.

$S(9) \approx 2570 + 135 = 2705$

In 1980, the population is decreasing at 135 people/year, so we can estimate the pop. was 135 people greater the year prior.

8. (12 points)

For each of the following functions, compute the derivative. **You do not need to simplify your answers.** Your answer must begin with $f'(x)$ or similar notation, as appropriate to the problem.

a. $f(x) = \dfrac{x^3 + 3x^2 + 2}{\sqrt{x}}$

$$f'(x) = \frac{\sqrt{x}(3x^2 + 6x) - (x^3 + 3x^2 + 2)\left(\frac{1}{2}x^{-1/2}\right)}{x}$$

b. $g(y) = y^{-6}\cos(y) + 5^2 - 4\pi y$

$$g'(y) = y^{-6}(-\sin(y)) + -6y^{-7}\cos(y) - 4\pi$$

$$= -y^{-6}\sin(y) - 6y^{-7}\cos(y) - 4\pi$$

c. $g(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

$$g'(\theta) = \frac{\sin(\theta)(-\sin\theta) - \cos\theta(\cos\theta)}{\sin^2(\theta)}$$

$$= \frac{-\sin^2(\theta) - \cos^2(\theta)}{\sin^2\theta}$$

$$= \frac{-1}{\sin^2\theta} = -\csc^2\theta$$

9. (12 points)

Consider the function $g(x) = \dfrac{1}{x^2} + \dfrac{x^4}{4}$.

a. Find $g'(x)$. (Use differentiation rules, not the definition of the derivative.)

4pts

$g(x) = x^{-2} + \frac{1}{4}x^4$

$g'(x) = -2x^{-3} + x^3$

$= -\frac{2}{x^3} + \frac{x^6}{x^3}$

1/2 set up

b. Determine the x-values for which the function has a horizontal tangent line.

3pts

$g'(x) = 0$

$0 = -2x^{-3} + x^3$ (multiply by x^3)

$0 = -2 + x^6$

$2 = x^6$

$x = \pm\sqrt[6]{2}$

1 work, no statement

c. Is the function increasing or decreasing at $x = -2$? Show your work.

2pts

$g'(-2) = -2(-2)^{-3} + (-2)^3$

$= -2\left(-\frac{1}{8}\right) + -8$

$= \frac{1}{4} - 8 < 0$ Decreasing

d. Find the equation of the tangent line to $g(x)$ at $x = -2$.

3pts

$g'(-2) = \frac{1}{4} - \frac{\cancel{8}\,32}{4} = -31/4$

$g(-2) = \frac{1}{4} + \frac{16}{4} = \frac{17}{4}$

$y - \frac{17}{4} = -\frac{31}{4}(x+2)$

$y = -\frac{31}{4}x - \frac{62}{4} + \frac{17}{4}$

$y = -\frac{31}{4}x - \frac{45}{4}$

10. (16 points)

The position of a lady bug on a branch is given by $P(t) = t^3 - \frac{11}{2}t^2 + 6t + 8$ where time t is in minutes between 0 and 20 and position $P(t)$ is in inches.

a. Find the function describing the **velocity** of the lady bug.

4pts

$v(t) = 3t^2 - 11t + 6$

b. At what times did the lady bug stop moving?

4pts

$v(t) = 0$

$0 = 3t^2 - 11t + 6$

$0 = 3t^2 - 9t - 2t + 6$

$0 = 3t(t-3) - 2(t-3)$

$0 = (3t-2)(t-3)$

at times $t = \frac{2}{3}, 3$ minutes

c. $P(t)$ is measured from the trunk of the tree meaning $P(t) = 0$ indicates that the lady bug is at the trunk, the base of the branch. At time $t = 2$ minutes, is the lady bug moving away from the trunk or towards the trunk? Show your work.

4pts

$v(2) = 3(2)^2 - 11(2) + 6$

$= 3 \cdot 4 - 22 + 6$

$= 12 - 22 + 6$

$= -4$ towards the trunk "negative" direction towards $P(t) = 0$

d. At time $t = 2$ minutes, is the lady bug speeding up or slowing down? Explain.

4pts

$a(t) = 6t - 11$

$a(2) = 12 - 11 = 1$

Since $v(2) < 0$ and $a(2) > 0$, the lady bug is slowing down.

11. (Extra Credit: 5 points)

Find a point on the graph of $f(x) = x^2 + x - 1$ such that the tangent line at that point has a y intercept at $(0, -2)$.

$f'(x) = 2x + 1$ 1 pt

Need
$y - -2 = f'(x)(x - 0)$

Use $y = f(x)$ 2 pt

$(x^2 + x - 1) + 2 = (2x+1)(x)$

$x^2 + x + 1 = 2x^2 + x$

$0 = x^2 - 1$

$x = \pm 1$ 4 pts

$f(1) = (1)^2 + (1) - 1$
$= 1$

$f(-1) = (-1)^2 + (-1) - 1$
$= -1$

Points $(1, 1)$, $(-1, -1)$ 5 pts