Math F251 Final Exam Spring 2022

Name: Solutions

Section:
☐ F01 (Faudree)
☐ F02 (Gossell)
☐ UX1 (Gossell)

Rules:

You have 2 hours to complete the exam.

Partial credit will be awarded, but you must show your work.

No other aids are permitted.

Place a box around your FINAL ANSWER to each question where appropriate.

Turn off anything that might go beep during the exam.

Problem	Possible	Score
1	8	
2	8	
3	12	
4	8	
5	10	
6	8	
7	4	
8	10	
9	12	
10	10	
11	10	
Extra Credit	5	
Total	100	

1. (8 points)

Find the derivative of each of the following functions. You do not need to simplify your answer.

a. $g(x) = \left(\ln(x) + \frac{2x}{5}\right)^4$

$$g'(x) = 4\left(\ln x + \frac{2}{5}x\right)^4 \left(\frac{1}{x} + \frac{2}{5}\right)$$

b. $f(x) = \sqrt{x}e^{3x}$

$$f'(x) = \frac{1}{2}x^{-1/2}e^{3x} + x^{\frac{1}{2}}\left(3e^{3x}\right)$$

2. (8 points)

Evaluate the definite integrals below. **Simplify** your final answers.

a. $\int_1^2 6x - 5\,dx = 3x^2 - 5x\Big]_1^2 = (3\cdot 2^2 - 5\cdot 2) - (3-5)$

$$= (12 - 10) - (-2)$$

$$= 2 + 2 = 4$$

b. $\int 7(\sin x)^3 \cos x\,dx = \frac{7(\sin x)^4}{4} + C$

3. (12 points)

Evaluate the following limits. You must show your work to earn full credit. If you apply L'Hopital's Rule, you should indicate this.

a. $\lim_{x\to 2} \frac{2x^2-8}{x^2-3x+2} \overset{H}{=} \lim_{x\to 2} \frac{4x}{2x-3} = \frac{8}{4-3} = 8$

form $\frac{0}{0}$

b. $\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x} = \lim_{x\to 4} \frac{2-\sqrt{x}}{(2-\sqrt{x})(2+\sqrt{x})} = \lim_{x\to 4} \frac{1}{2+\sqrt{x}} = \frac{1}{2+\sqrt{4}} = \frac{1}{4}$

c. $\lim_{x\to\infty} \frac{45+x-4x^2}{x^2-16} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{45}{x^2}+\frac{1}{x}-4}{1-\frac{16}{x^2}} = \frac{-4}{1} = -4$

4. (8 points)

a. Find the linear approximation (also known as the linearization) of the function $f(x) = \sqrt{x}$ when $a = 4$.

$f(x) = x^{1/2}$ $\quad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(4) = \frac{1}{4} = 0.25$

$\boxed{L(x) = 2 + 0.25(x-4)}$

b. Use the linear approximation to estimate $\sqrt{4.04}$. Your answer must be in the form of a decimal.

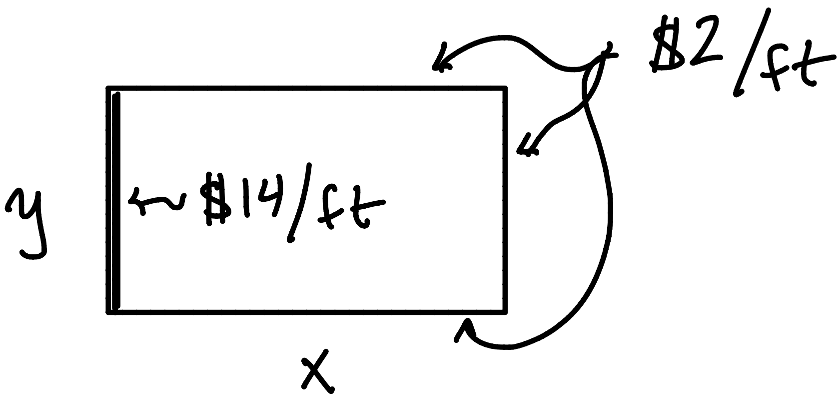
$L(4.04) = 2 + 0.25(4.04-4) = 2 + (0.25)(0.04)$

$= 2 + 0.01 = \underline{\underline{2.01}}$

5. (10 points)

(Optimization Problem) You need to construct a 100 ft^2 rectangular pen for a dog. Three sides of the pen (north, east, and south) will be made of open fencing which costs \$2 per foot. To add privacy, the west side which faces the street will be made of closed fencing which costs \$14 per foot. Follow the steps below to find the dimensions of the pen that minimize the cost.

a. Draw a diagram and label the sides.



b. Write an equation for the cost of the fencing in terms of a single variable.

$$C = 14y + 2y + 2(2x) = 16y + 4x$$

replace y

$$A = 100 = xy$$

So $y = \frac{100}{x}$.

$$C(x) = 16\left(\frac{100}{x}\right) + 4x$$

$$= \frac{1600}{x} + 4x$$

c. Use Calculus to find the dimensions of the pen that **minimize** the cost. **Justify** your answer.

$$C'(x) = -1600x^{-2} + 4 = 0$$

So $4 = \frac{1600}{x^2}$ or $x^2 = \frac{1600}{4} = 400$

So $x = 20$. Thus $y = 5$.

Is it a min?

$C''(x) = 3200x^{-3}$ and $C''(20) = \frac{3200}{20^3} > 0$. So C is ccup.

So yes C has a min at $x = 20$.

Final Answer

dimensions:

$x = 20$ ft, $y = 5$ ft.

6. (8 points)

(Related Rates Problem) The volume V of a spherical snowball with radius r is given by the equation $V = \frac{4}{3}\pi r^3$. The surface area A is given by $A = 4\pi r^2$. Throughout the warm spring afternoon, the snowball melts at a constant rate of 36π cubic inches per hour.

a. At the moment that the radius is 6 inches, how fast is the radius decreasing? Include units in your answer.

$\frac{dV}{dt} = -36\pi$

Find $\frac{dr}{dt}$ when $r=6$.

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$-36\pi = 4\pi 6^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-36\pi}{4\pi \cdot 6^2} = \frac{-1}{4}$ in/hr

b. At the moment that the radius is 6 inches, how fast is the surface area decreasing? Include units in your answer.

Find $\frac{dA}{dt}$ when $r=6$ and $\frac{dr}{dt} = \frac{-1}{4}$

$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(6)\left(-\frac{1}{4}\right) = -12\pi$ in^2/hr

7. (4 points)

The number of subscribers to an internet streaming service is given by $s(t)$, where t is measured in months since the company started.

a. What does the statement $s'(36) = 4,580$ mean? Include units with your answer.

After 36 months, the company is gaining subscribers at a rate of 4580 subscribers per month.

b. Would the owners of the streaming service prefer $s''(36)$ to be positive or negative? Explain your reasoning.

Probably positive. If $s''(36) > 0$, then the rate at which their company is growing is increasing.

8. (10 points)

Suppose a particle moves along a straight line with **velocity** $v(t) = 3t^2 - 12t - 2$ m/s.

a. Find $s(t)$, the **position** of the particle at time t in seconds assuming that when $t = 1$ second the particle is at position $s = 10$ meters.

$s(t) = \int v(t)dt = \int (3t^2 - 12t - 2)\,dt = t^3 - 6t^2 - 2t + C$

$s(1) = 10 = 1^3 - 6(1)^2 - 2(1) + C = 1 - 6 - 2 + C = -7 + C$

So $C = 17$. $s(t) = t^3 - 6t^2 - 2t + 17$

b. Find $a(t)$, the **acceleration** of the particle at time t in seconds.

$a(t) = v'(t) = 6t - 12$

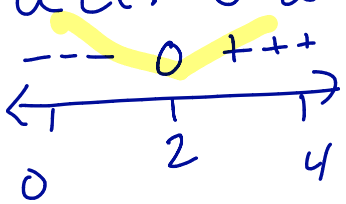
c. At time $t = 0$, is the particle speeding up or slowing down? Explain your answer.

at $t=0$, $a(0) = -12$, $v(0) = -2$

Since both are negative, the particle is speeding up.

d. Determine the minimum velocity of the particle.

$a(t) = 0$ when $t = 2$.



← sign $a = v'$

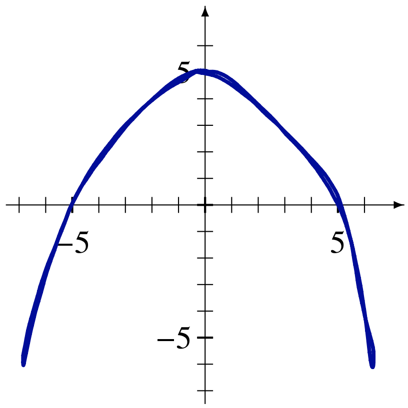
$v(2) = 3 \cdot 2^2 - 12(2) - 2 = 12 - 24 - 2 = -14$ m/s

9. (12 points)

Sketch graphs which satisfy the given conditions. **There are many correct answers.**

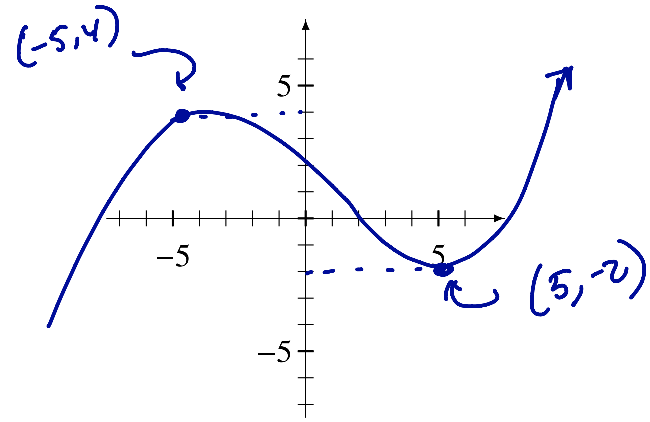
a. Sketch a graph of a function $k(x)$ such that

- $k'(-5) > 0$ ↑ at $x=-5$
- $k'(5) < 0$ ↓ at $x=5$



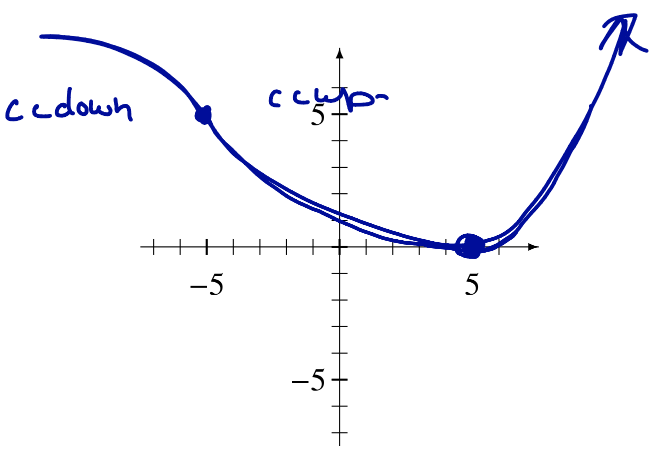
c. Sketch a graph of a function $g(x)$ that

- $\lim_{x\to 5^-} g(x) = 4$
- $\lim_{x\to 5^+} g(x) = -2$



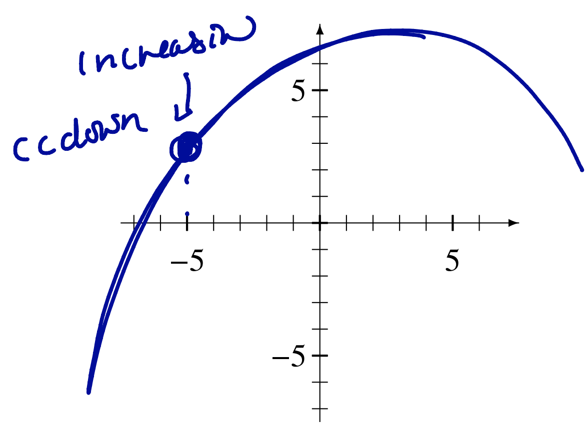
b. Sketch a graph of a function $f(x)$ that has

- an inflection point at $x = -5$, and
- a local minimum at $x = 5$.



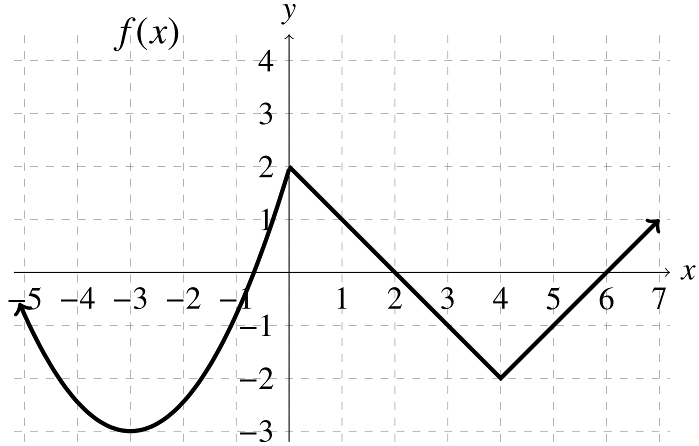
d. Sketch a graph of a function $h(x)$ such that

- $h'(-5) > 0$, and ↑
- $h''(-5) < 0$. ∩



10. (10 points)

Use the graph of the function $f(x)$ (on the right) to answer the questions below.



a. $\lim_{x \to 1} f(x) =$ 1

b. $\lim_{x \to 1} \frac{f(1+h) - f(1)}{h} =$ -1 ← slope @ $x=1$

c. At what values of x, if any, does the derivative, $f'(x)$, not exist?

$x = 0, 4$ (corners)

d. On what intervals, if any, is $f'(x) > 0$?

$(-3, 0) \cup (4, \infty)$

e. Does $f(x)$ have any local minimums? If so, state the location and the local minimum value.

min $y = -3$ at $x = -3$

min $y = -2$ at $x = 4$

The following questions concern $G(x) = \int_0^x f(s)\, ds$.

f. What is the value of $G(5)$?

$\int_0^5 f(x)dx = 2 - 2 - 1.5 = -1.5$

g. What is the value of $G'(5)$?

$G'(5) = f(5) = -1$

h. On the interval $[0, 7]$, does $G(x)$ have a local minimum? If so, state the location and the local minimum value.

local min at $x = 6$

min value is $\int_0^6 f(x)\, dx = 2 - 2 - 2 = -4$

11. (10 points)

A population of bacteria is growing at a rate of $p'(t) = 300e^{t/10}$ bacteria per day.

a. Compute $p'(0)$ and interpret its meaning in the context of the problem. Include units with your answer.

$p'(0) = 300e^{0/10} = 300$ bacteria per day.

When the experiment begins, the population is increasing at a rate of 300 bacteria per day.

b. Compute $\int_0^{10} p'(t)\,dt$.

$$\int_0^{10} 300e^{t/10}\,dt = 3000\,e^{t/10}\Big]_0^{10} = 3000\left(e^{10/10} - e^{0}\right)$$

$$= 3000(e-1)$$

c. Interpret your answer from part (b) in the context of the problem. Make sure to include units.

In the first 10 days, the net change in the population of bacteria is 3000 (e-1) bacteria.

12. (Extra Credit: 5 points)

Calculate $\frac{d}{dx}\left(\int_{\cos x}^{5} \frac{17^{-t}\ln(t+2)}{\sqrt{20-\sin^2 t}}\,dt\right)$. $= \frac{d}{dx}\left(-\int_5^{\cos x} \frac{17^{-t}\ln(t+2)}{\sqrt{20-\sin^2 t}}\,dt\right)$

$$= \sin(x)\left(\frac{17^{-\cos x}\ln(\cos x+2)}{\sqrt{20-\sin^2(\cos x)}}\right)$$