Your Name

Solutions

Your Signature

Instructor Name

End Time

Desk Number

- The total time allowed for this exam is 175 minutes. There are 110 points possible on this exam.
- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question** where appropriate.
- Raise your hand if you have a question.

This exam is printed double-sided.

There are problems on both sides of the page!

If you need more space, you may use extra sheets of paper. If you use extra pages:

- Put your name on each extra sheet
- Label your work with the problem you're working on
- Write on the exam problem that there is additional work at the end
- Turn in your additional pages at the end of your exam.

$\boxed{1}$ (12 points)

(a) (4 points) If $f(t) = \cos(t)\sqrt{1-t^2} + \sin(\pi t)$, compute $f'(t)$. You do not need to simplify your answer.

$$f'(t) = -\sin(t)\left[\sqrt{1-t^2}\right] + \cos(t)\left(\tfrac{1}{2}(1-t^2)^{-1/2}(-2t)\right) + \cos(\pi t)\cdot\pi$$

(b) (4 points) Let $g(x) = \int_x^2 \frac{\arctan(t)}{1+t^2}\, dt$. Compute $g'(x)$.

Note $g(x) = -\int_2^x \frac{\arctan(t)}{1+t^2}\, dt$, so

$$g'(t) = -\frac{\arctan(t)}{1+t^2}$$

(by FTC 1)

(c) (4 points) Find $\frac{dy}{dx}$ by implicit differentiation: $y^3 - x^3 = xy$. Your answer must be of the form $\frac{dy}{dx} = \ldots$ or $y' = \ldots$

$$3y^2 \frac{dy}{dx} - 3x^2 = x \cdot \frac{dy}{dx} + y(1) \Rightarrow$$

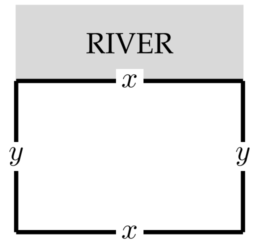
$$3y^2 \frac{dy}{dx} - x\frac{dy}{dx} = y + 3x^2 \Rightarrow$$

$$\frac{dy}{dx}\left(3y^2 - x\right) = y + 3x^2 \Rightarrow$$

$$\frac{dy}{dx} = \frac{y+3x^2}{3y^2-x}$$

2 (8 points) A rectangular field adjacent to a river is to be enclosed. Fencing along the river costs \$ 5 per meter, and the fencing for the other sides costs \$ 3 per meter. The area of the field is to be 1200 square meters. Find the dimensions of the field that is the least expensive to enclose. What is the minimum cost?

(a) Write a formula that determines the **cost** of the fencing as a function of x and y.



$$\text{Cost} = 5x + 3x + 2(3y) = 8x + 6y$$

(b) Solve the problem. Indicate **units** in your answer. Justify that the solution you found really is a solution.

Area $= xy$ and Area $= 1200 \Rightarrow xy = 1200 \Rightarrow y = \frac{1200}{x}$

So $C(x) = 8x + \frac{6(1200)}{x}$

$= 8x + \frac{7200}{x}$ ← Domain is $(0, \infty)$ since we can't have negative side length

$$\begin{array}{r} 1200 \\ \times\ 6 \\ \hline 7200 \end{array}$$

$C'(x) = 8 - \frac{7200}{x^2}$

$C'(x) = 0 \Rightarrow 8 = \frac{7200}{x^2} \Rightarrow x^2 = \frac{7200}{8} = 900 = 3^2 \cdot 10^2$

$\Rightarrow x = 30$

Is this a max/min? $C''(x) = -7200(-2)x^{-3}$

$= \frac{2 \cdot 7200}{x^3} > 0$ when $x > 0$

So function is CU $\Rightarrow \cup$ so $x = 30$ is a local min

If $x = 30$ $y = \frac{1200}{30} = \frac{120}{3} = 40$

and $C(30) = 8(30) + \frac{7200}{30} = 240 + 240 = 480$

$$\begin{array}{r} 240 \\ 3\overline{)720} \\ 6 \\ \hline 12 \end{array}$$

Dimensions: $x =$ 30 $y =$ 40

Total minimum cost: \$480

3 (12 points) Evaluate the following integrals. For full credit, include a constant of integration whenever one would be justified, and show your work clearly.

(a) (4 points) $\displaystyle\int \left(x^3 + \pi e^x - \frac{\sec^2(x)}{4}\right)\, dx$

$$= \frac{x^4}{4} + \pi e^x - \frac{1}{4}\tan(x) + C$$

(b) (4 points) $\displaystyle\int_0^1 \left(\frac{2}{t+1} + \frac{1}{3(1+t^2)}\right) dt$. Simplify your answer.

$$= \int_0^1 \frac{2}{t+1}\, dt + \frac{1}{3}\int_0^1 \frac{1}{1+t^2}\, dt$$

$u = t+1 \Rightarrow du = dt$

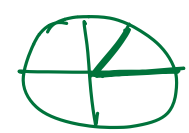
$$= \int_{t=0}^{t=} \frac{2}{u}\, du + \frac{1}{3}\int_0^1 \frac{1}{1+t^2}\, dt$$

$$= 2\ln(u)\Big|_1^2 + \frac{1}{3}\arctan(t)\Big|_0^1$$

$$= 2\ln(2) - 2\ln(1) + \frac{1}{3}\arctan(1) - \frac{1}{3}\arctan 0$$

$$= 2\ln(2) - 0 + \frac{1}{3}\cdot\frac{\pi}{4} - 0$$

$$= 2\ln(2) + \frac{\pi}{12}$$



(c) (4 points) $\displaystyle\int 10\sin(2y)\cos(2y)\sqrt{\cos^2(2y)+5}\, dy$

Let $u = (\cos(2y))^2 + 5$. Then $\dfrac{du}{dy} = 2\cos(2y)(\sin(2y)(2) \Rightarrow$

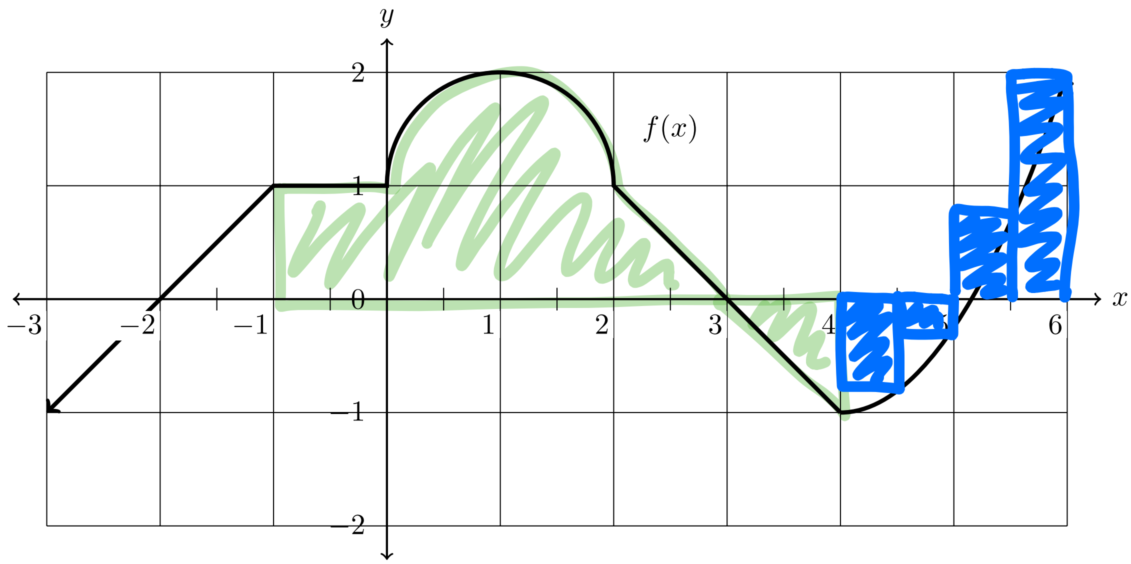
$\dfrac{du}{4\cos(2y)\sin(2y)} = dy$. So

$$\int 10\sin(2y)\cos(2y)\sqrt{u}\cdot\frac{du}{4\cos(2y)\sin(2y)} = \frac{5}{2}\int \sqrt{u}\, du$$

$$= \frac{5}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{5}{3}\left((\cos(2y))^2 + 5\right)^{3/2} + C$$

$\boxed{4}$ (14 points) The function $f(x)$ has been graphed below. The curve for $0 < x < 2$ is an upper half circle. Define a new function $g(x)$, as

$$g(x) = \int_{-1}^{x} f(s)\, ds.$$



Use the graph above to answer the questions below.
Note: Pay attention to whether question concerns the function f, f', g or g'. Also pay attention to the scale and the labels on the axes.

(a) (2 points) Where is $f(x)$ NOT differentiable? If nowhere, say so. $x =$ $-1, 0, 2, 4$

(b) (2 points) $f(1) =$ 2

(c) (2 points) $g(4) =$ $3 + \pi/2$

(d) (2 points) $g(-2) =$ $-1/2$ $= \int_{-1}^{-2} f(s)\,ds = -\int_{-2}^{-1} f(s)\,ds = -(1/2)$

(e) (2 points) $f'(3) =$ -1

(f) (2 points) $g'(1) =$ 2

(g) (2 points) Clearly draw on the graph an approximation of $\int_{4}^{6} f(s)\, ds$ using four RIGHT-HAND RECTANGLES. Shade in your rectangles. You just have to draw and shade the rectangles; you do not have to compute anything in this part.

Drawn in blue.

5 (24 points) Consider the following function:

$$h(x) = \frac{\ln(x)}{x^2} \qquad \left(\text{and notice } h'(x) = \frac{1 - 2\ln(x)}{x^3} \quad \text{and} \quad h''(x) = \frac{6\ln(x) - 5}{x^4}\right)$$

(a) (1 points) What is the domain of h?

$x > 0$, that is, $(0, \infty)$

(b) (1 points) Compute $h(1)$.

$h(1) = \frac{\ln(1)}{1^2} = \frac{0}{1} = 0.$

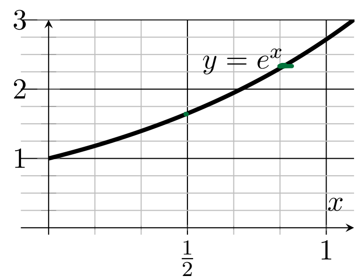
(c) (3 points) Find all critical numbers of h, if any.

$h'(x) = \frac{1 - 2\ln(x)}{x^3}$ Critical Point: $h'(x) = 0 \Rightarrow 1 - 2\ln(x) = 0 \Rightarrow \ln(x) = 1/2 \Rightarrow \boxed{x = e^{1/2}}$

Undefined at $x = 0$ (not a critical point)

(d) (4 points) Determine the intervals on which h is increasing or decreasing. If none, say so. Show your work.

Hint:



x	1	$e^{1/2}$	2
$h'(x)$	$+$	0	$-$
$h(x)$	$\nearrow$	MAX	$\searrow$

Note $e^{1/2} \approx 1.2$

$h'(1) = \frac{1 - 2(0)}{1} = 1 > 0$

$h'(2) = \frac{1 - 2\ln(2)}{1} < 0$

Increasing: $(0, e^{1/2})$

Decreasing: $(e^{1/2}, \infty)$

(e) (4 points) Determine the intervals where $h(x)$ is concave up or concave down. If none, say so. Show your work.

$h''(x) = 0 \Rightarrow \frac{6\ln(x) - 5}{x^4} = 0 \Rightarrow \ln(x) = 5/6 \Rightarrow x = e^{5/6}$

Note $e^{5/6} \approx 2.3$

x	1	$e^{5/6}$	100
$h''(x)$	$-$	0	$+$
$h(x)$	$\cap$	INF	$\cup$

$h''(1) = \frac{6(0) - 5}{1} < 0$

$h''(100) = \frac{6\ln(8) - 5}{1} > 0$

Concave up: $(e^{5/6}, \infty)$

Concave down: $(0, e^{5/6})$

continued on next page...

Recall that $h(x) = \dfrac{\ln(x)}{x^2}$. ...continued from previous page

(f) (4 points) Compute the following. If you use L'Hospital's rule, indicate where you are doing so with **H** or **L'H**.

$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \dfrac{\ln(x)}{x^2} =$ type $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \dfrac{1/x}{2x} = \lim_{x\to\infty} \dfrac{1}{2x^2} = 0$

$\lim_{x\to 0+} h(x) = \lim_{x\to 0^+} \dfrac{\ln(x)}{x^2} =$ type $\frac{-\infty}{0}$ (not indeterminate)

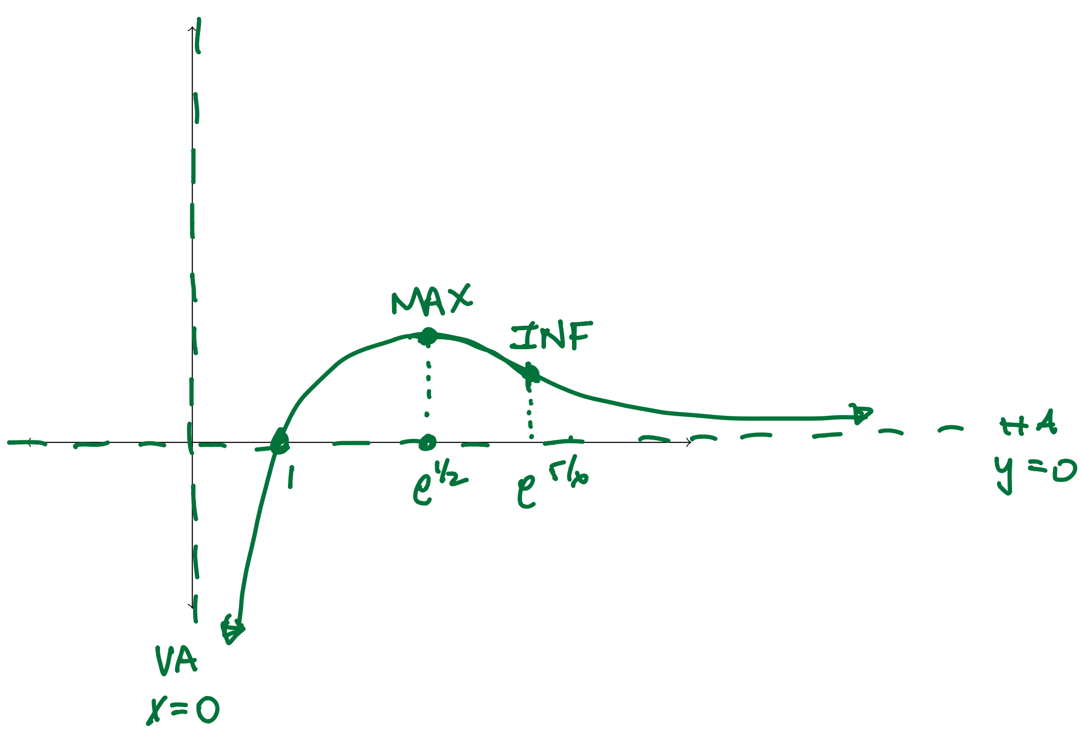
$= \lim_{x\to 0^+} \underbrace{\ln(x)}_{\to -\infty} \cdot \underbrace{\frac{1}{x^2}}_{\to \infty} = -\infty$

(g) (2 points) Find all asymptotes of f, both vertical and horizontal, if they exist, and use limits to justify how you know that they are asymptotes.

There is a horizontal asymptote at $y=0$ since $\lim_{x\to\infty} h(x) = 0$

There is a VA at $x=0$ since $\lim_{x\to 0^+} h(x) = -\infty$.

(h) (5 points) Sketch the graph on the axes. **Label** with "Max" or "Min" any **points on the graph** that are local extrema, and label with "Inf" any **points on the graph** that are inflection points. Draw any asymptotes with dashed lines, and label them with their equations. Clearly mark on the x-axis the x-values of local extrema or inflection points, as well as any intercepts.



$\boxed{6}$ (6 points) The velocity of a particle in m/sec is given by $v(t) = t + e^{-2t}$.

(a) (2 points) Find a function that gives the acceleration of the particle at time t (in other words, determine $a(t)$).

$$a(t) = v'(t) = 1 + e^{-2t}(-2)$$

(b) (4 points) Find a function that gives the position of the particle at time t (in other words, find $s(t)$) if $s(0) = 2$.

$$s(t) = \int v(t)\,dt = \int t + e^{-2t}\,dt = \frac{t^2}{2} + \frac{e^{-2t}}{-2} + c$$

$u = -2t$
$du = -2\,dt$

$$s(0) = 2 \Rightarrow \frac{0^2}{2} + \frac{e^{-2(0)}}{-2} + c = 2$$

$$\Rightarrow -\frac{1}{2} + c = 2$$

$$\Rightarrow c = 2 + 1/2 = 5/2$$

So $s(t) = \dfrac{t^2}{2} - \dfrac{1}{2}e^{-2t} + \dfrac{5}{2}$.

7 (10 points) A very simple model of a particular epidemic moving through a finite population is given by the following equation:

$$N(t) = \frac{1000}{1 + 199e^{-t/4}} = 1000(1+199e^{-t/4})^{-1}$$

where time t is measured in days and $N(t)$ measures the total number of people who have been infected at time t. (Note that this model assumes that once people have been infected, that they stay infected – no one recovers, and no one leaves the population.)

(a) (2 points) Find and interpret $N(0)$.

$$N(0) = \frac{1000}{1+199} = \frac{100\cancel{0}}{20\cancel{0}} = 5$$

At time t=0, 5 people were infected.

(b) (2 points) Compute $\lim_{t\to\infty} N(t)$. Write a sentence explaining what this limit determines.

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{1000}{1+\cancel{999e^{-t/4}} \to 0} = 1000$$

After a long period of time, the # of people infected approaches 1000.

(c) (4 points) Determine $N'(t)$. (You do not have to simplify.)

$$N'(t) = -1(1000(1+199e^{-t/4}))^{-2}(199e^{-t/4}(-1/4)) \quad \leftarrow \text{chain rule}$$

or

$$N'(t) = \frac{(1+199e^{-t/4})(0) - (1000)(199e^{-t/4})(-1/4)}{(1+199e^{-t/4})^2} \quad \leftarrow \text{quotient rule}$$

(d) (1 points) It is computed that $N'(10) \approx 13.5$. What does this mean? Write your answer in a sentence.

After 10 days, the rate of new infections is about 13.5 new cases/day

(e) (1 points) If $N(10) \approx 58$ and $N'(10) \approx 13.5$, approximately how many people would you expect to be infected on day 11?

About 58 + 13.5 = 71.5 total infections on day 11

$\boxed{8}$ (12 points) Consider the function $f(x) = \sqrt{x}$.

(a) (7 points) Use the **definition of the derivative** to determine $f'(x)$. Clearly show all the steps of your work. An answer with no work (and no limits) will not receive credit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

(b) (3 points) Write the equation of the tangent line to the function $f(x)$ at the point $(9, f(9))$.

$f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$ and $f(9) = \sqrt{9} = 3$

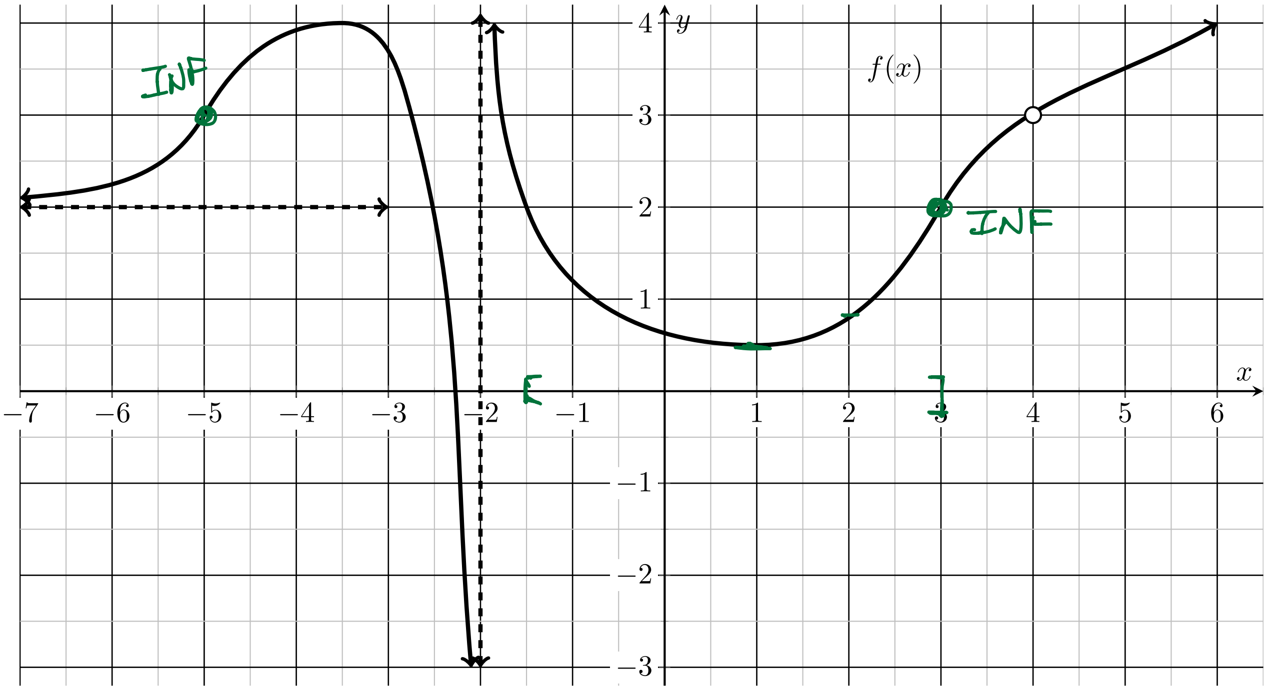
Tangent line: $\boxed{L(x) = y = \frac{1}{6}(x-9) + 3}$

(c) (2 points) Use the tangent line (linearization) to approximate $\sqrt{9.1}$.

$$\sqrt{9.1} \approx L(9.1) = \frac{1}{6}\left(9 + \frac{1}{10} - 9\right) + 3 = \boxed{3 + \frac{1}{60}}$$

9 (12 points)

Answer the following questions about the function $f(x)$ whose graph is shown below. Assume the graph continues in both directions as indicated outside the bounds of the graph. If there is no answer, write "DNE" or "none".



Determine the following:

(a) (1 points) Domain of $f(x)$ (in interval notation): $(-\infty,-2)\cup(-2,4)\cup(4,\infty)$

(b) (1 points) $\lim_{x\to\infty} f(x) =$ ∞

(c) (1 points) $\lim_{x\to-\infty} f(x) =$ 2

(d) (1 points) $\lim_{x\to 2} f(x) =$ $\approx 5/6$ or $.8$ ish

Intent was to ask about $\lim_{x\to -2} f(x) =$ DNE

(e) (1 points) $\lim_{x\to 4} f(x) =$ 3

(f) (2 points) The x-value(s) where f has an inflection point: $x =$ $-5, 3$

(g) (2 points) The interval(s) on which $f'(x) \le 0$ $(-3.5,-2)\cup(-2,1)$ $\Rightarrow f$ decreasing

(h) (1 points) The absolute maximum that $f(x)$ attains on the interval $\left[-\frac{3}{2}, 3\right]$ $y = 2$

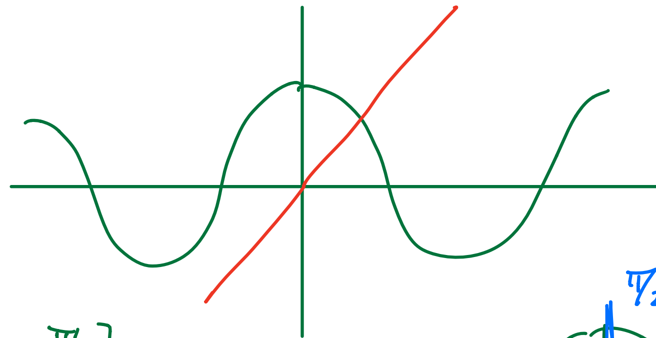
(i) (2 points) The average rate of change of $f(x)$ on the interval $[1,3]$. $3/4$

$$\frac{f(3)-f(1)}{3-1} = \frac{3}{4} \text{ (count squares)}$$

$$= \frac{2-1/2}{3-1} = \frac{3/2}{2} = 3/4$$

Extra Credit (6 points) Consider the equation $\cos(x) = x$.

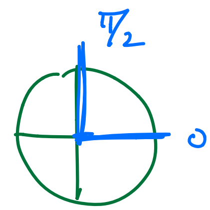
(a) (3 points) Show that a solution of this equation exists. State the names of any theorems or results you are using.



Let $f(x) = \cos(x) - x$.

Observe $f(0) = 1 - 0 = 1 > 0$

and $f(\frac{\pi}{2}) = 0 - \frac{\pi}{2} < 0$.



Since $f(x)$ is continuous on $[0, \pi/2]$

by the intermediate value theorem it follows that

there exists some $c \in (0, \pi/2)$ s.t. $f(c) = 0$. This is a

solution that we are looking for.

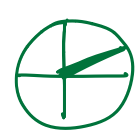
(b) (3 points) Suppose we guess that $x = \pi/6$ is a solution. [It's not, but it's not far off.] Use this to find and simplify a better approximation of the solution to $\cos(x) = x$. State the names of any theorems or results you are using.

We will use newton's method with $x_0 = \pi/6$.

Define $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ where $f(x) = \cos(x) - x$.

Note $f'(x) = -\sin(x) - 1$ so

$$x_1 = x_0 - \frac{\cos(x_0) - x_0}{-\sin(x_0) - 1} = x_0 + \frac{\cos(x_0) - x_0}{\sin(x_0) - 1}$$



$$= \frac{\pi}{6} + \frac{\cos(\pi/6) - \pi/6}{\sin(\pi/6) + 1} = \frac{\pi}{6} + \frac{\sqrt{3}/2 - \pi/6}{1/2 + 1}$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}/2 - \frac{\pi}{6}}{3/2} = \frac{\pi}{6} + \frac{\sqrt{3}}{2} \cdot \frac{2}{3} - \frac{\pi}{6} \cdot \frac{2}{3}$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{3} - \frac{2\pi}{9} = \frac{3\pi + 6\sqrt{3} - 4\pi}{18} = \frac{6\sqrt{3} - \pi}{18}$$