Intro Video: Section 3.6
Logarithmic Differentiation

Math F251X: Calculus I

What is the derivative of $f(x) = \ln(x)$? $g(x) = \log_b(x)$?

Use implicit differentiation!

$y = \ln(x) \Rightarrow x = e^y \Rightarrow$

$\frac{d}{dx}(x) = \frac{d}{dx}(e^y) \Rightarrow$

$1 = e^y \frac{dy}{dx} \Rightarrow$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

$y = \log_b(x) = \frac{\ln(x)}{\ln(b)}$ ← fact!

$\frac{dy}{dx} = \frac{1}{x\ln(b)}$

Example: If $y = \ln(x^2 - 5x + 6)$ what is $\frac{dy}{dx}$?

$y' = \frac{1}{x^2 - 5x + 6}(2x - 5)$.

If $y = \ln(f(x))$ then

$y' = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}$.

Logarithmic differentiation:

Idea: $y =$ complicated function

We can take natural log of both sides to make complicated stuff simpler.

Example: $\boxed{y = \sqrt{\dfrac{x-1}{x^4+1}}} = \left(\dfrac{x-1}{x^4+1}\right)^{1/2}$

$$\boxed{\ln(y)} = \ln\left[\left(\frac{x-1}{x^4+1}\right)^{1/2}\right] = \frac{1}{2}\ln\left[\frac{x-1}{x^4+1}\right] = \boxed{\frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x^4+1)}$$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{4x^3}{x^4+1}\right)$$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$$

$$\boxed{y'} = \left(\frac{1}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{4x^3}{x^4+1}\right)\right) y = \boxed{\left(\frac{1}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{4x^3}{x^4+1}\right)\right)\left(\frac{x-1}{x^4+1}\right)^{1/2}}$$

Example: $\boxed{y = \cos(x)\, e^{7x^3} \sqrt{\tan(x) - x^2}}$

$$\ln(y) = \ln\left(\cos(x)\, e^{7x^3} \left(\tan(x) - x^2\right)^{1/2} \right)$$

$$= \ln(\cos(x)) + \ln\left(e^{7x^3}\right) + \frac{1}{2} \ln\left(\tan(x) - x^2\right)$$

$$= \ln(\cos(x)) + 7x^3 + \frac{1}{2} \ln\left(\tan(x) - x^2\right).$$

$$\frac{y'}{y} = -\frac{\sin(x)}{\cos(x)} + 21x^2 + \frac{1}{2} \frac{(\sec(x))^2 - 2x}{\tan(x) - x^2}$$

$$y' = \left(-\frac{\sin(x)}{\cos(x)} + 21x^2 + \frac{1}{2}\left(\frac{(\sec(x))^2 - 2x}{\tan(x) - x^2} \right) \right) y$$

$$y' = \left(-\frac{\sin(x)}{\cos(x)} + 21x^2 + \frac{1}{2}\left(\frac{(\sec(x))^2 - 2x}{\tan(x) - x^2} \right) \right) \left(\cos(x)\, e^{7x^3} \sqrt{\tan(x) - x^2} \right)$$

Logarithmic differentiation is mandatory to differentiate

$$y = f(x)^{g(x)} \quad \text{(in general)}.$$

Example: $y = \tan(x)^{1/x}$.

$$\ln(y) = \ln\left(\tan(x)^{1/x}\right) = \frac{1}{x}\ln(\tan(x))$$

$$\frac{y'}{y} = \frac{1}{x}\frac{d}{dx}\left(\ln(\tan(x))\right) + \ln(\tan(x)) \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{y'}{y} = \frac{1}{x}\left[\frac{(\sec(x))^2}{\tan(x)}\right] + \ln(\tan(x))\left(-x^{-2}\right)$$

$$\frac{dy}{dx} = \left(\frac{1}{x}\left(\frac{(\sec(x))^2}{\tan(x)}\right) - \frac{\ln(\tan(x))}{x^2}\right)\underbrace{\tan(x)^{1/x}}_{y}$$