

November 16, 2007
Faudree

NAME: Solutions

There are 8 questions on this exam for a total of 100 points. You may not use calculators, books, or notes. You must show your work to receive full credit. There is an extra credit problem on the last page.

You have one hour to complete the exam.

problem	points
1	
2	
3	
4	
5	
6	
7	
8	
total	

1. (10 points)

- (a) Find the critical numbers of $f(x) = x^3 e^x$.

$$f'(x) = 3x^2 e^x + x^3 e^x = x^2 e^x (3+x).$$

$f' = 0$ when $x=0$ or $x=-3$

f' never undefined.

answer: critical numbers are $x=0$ and $x=-3$

- (b) Apply the Second Derivative Test to the critical numbers of $f(x)$ and describe what this test tells you about the behavior of $f(x)$ at these points?

Since $f'(x) = e^x (3x^2 + x^3)$,

$$\begin{aligned} f''(x) &= e^x (3x^2 + x^3) + e^x (6x + 3x^2) \\ &= e^x (3x^2 + x^3 + 6x + 3x^2) \\ &= e^x (x^3 + 6x^2 + 6x). \end{aligned}$$

Now $f''(0)=0$. So 2nd derivative test is inconclusive.

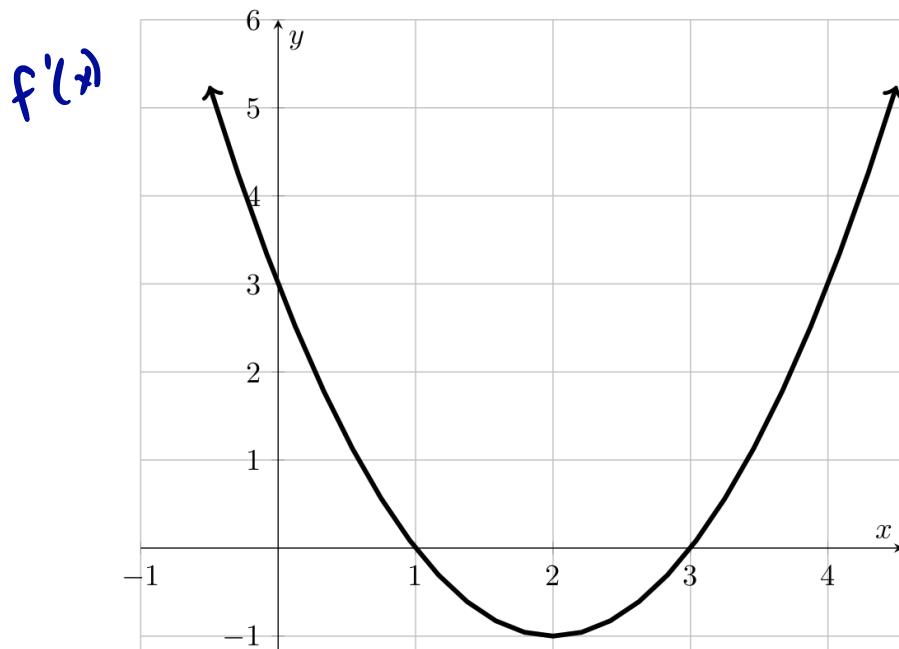
$$f''(-3) = e^{-3} (-27 + 54 - 18) > 0$$

Since $f''(-3) > 0$, f is concave up and f has a local minimum at $x=-3$.

arithmetic
aside:

$$\begin{array}{r} 27 \\ 18 \\ \hline 45 \end{array}$$

2. (10 points) Use the graph of $f'(x)$ (the derivative), sketched below to answer the following questions. If there is not enough information to answer the question, answer "don't know."



- (a) On what interval(s) is $f(x)$ increasing? $(-\infty, 1) \cup (3, \infty)$ (where $f' > 0$)
- (b) Does $f(x)$ have a relative maximum? at $x=1$ (yes) (where f' changes from + to -) we don't really know.
- (c) Does $f(x)$ have an absolute maximum? $(2, \infty)$ where f' is increasing.
- (d) On what interval(s) is $f(x)$ concave up? we can't know this.
- (e) On what interval(s) is $f(x)$ positive?

3. (10 points) Identify any local and absolute extreme values of the function $f(x) = \frac{1}{4}x + \frac{1}{x}$ on the interval $[1, 8]$, if any exist. You must show your work.

local minimum(s): 1

local maximum(s): 1.125, 2.125

absolute minimum(s): 1

absolute maximum(s): 2.125

$$f(x) = \frac{1}{4}x + x^{-1}$$

$$f'(x) = \frac{1}{4} - 1x^{-2} = \frac{1}{4} - \frac{1}{x^2}$$

f' is undefined when $x=0$.

$$f' = 0 \text{ when } \frac{1}{4} - \frac{1}{x^2} = 0$$

$$\text{so } \frac{1}{4} = \frac{1}{x^2} \text{ or } x = \pm 2$$

But only $x=2$ falls into $[1, 8]$.

I only need to check the critical point and the end points.

x	1	8	2
$f(x)$	1.25	2.125	1

\nwarrow max value \nearrow min value

$$f(1) = \frac{1}{4} + \frac{1}{1} = 1.25$$

$$f(8) = \frac{8}{4} + \frac{1}{8} = 2 + \frac{1}{8}$$

$$f(2) = \frac{2}{4} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

4. (10 points each) Evaluate the limits below. Give the most complete answer possible.
Show your work.

$$(a) \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

\uparrow form $\frac{\infty}{\infty}$

$$(b) \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^{3x} = e^6$$

\nwarrow form 1^∞ .

Let $y = (1 + 2x^{-1})^{3x}$

so $\ln y = 3x \cdot \ln(1 + 2x^{-1})$

Now $\lim_{x \rightarrow \infty} 3x \ln(1 + 2x^{-1}) = \lim_{x \rightarrow \infty} \frac{3 \cdot \ln(1 + 2x^{-1})}{x^{-1}}$ $\stackrel{\text{H}}{=} \lim_{x \rightarrow \infty} \frac{3 \cdot \frac{1}{1+2x^{-1}} \cdot -2x^{-2}}{-1x^{-2}}$
 \uparrow form $\infty \cdot 0$ \nwarrow form $\frac{0}{0}$

$$= \lim_{x \rightarrow \infty} \frac{6}{1 + 2x^{-1}} = 6$$

5. (15 points) Let $f(x) = \frac{5x^2 - 1}{\sqrt{x^4 - 1}}$.

(a) Find the domain of $f(x)$.

We require $x^4 - 1 > 0$.

So $x^4 > 1$.

So $x > 1$ or $x < -1$.

Answer: $(-\infty, -1) \cup (1, \infty)$

(b) Find any x -intercepts for $f(x)$.

Set $y=0$. Solve for x .

$f(x)=0$ forces $5x^2 - 1 = 0$.

$$x^2 = \frac{1}{5}$$

$$x = \pm \frac{1}{\sqrt{5}}$$

(c) Find any y -intercepts for $f(x)$.

Set $x=0$. Solve for y .

But $x=0$ is not in the domain of $f(x)$. Thus, $f(x)$ has no y -intercepts.

Answer: f has x -intercepts at $x = -\frac{1}{\sqrt{5}}$ and $x = \frac{1}{\sqrt{5}}$

Correction! These values do NOT lie in the domain of $f(x)$. Thus $f(x)$ has no x -intercepts.

(d) Find any horizontal asymptotes and show your answer is correct.

Check

$$\lim_{x \rightarrow \infty} \frac{5x^2 - 1}{\sqrt{x^4 - 1}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{5 - \frac{1}{x^2}}{\sqrt{1 - \frac{1}{x^4}}} = 5$$

$$\lim_{x \rightarrow -\infty} \frac{5x^2 - 1}{\sqrt{x^4 - 1}} = 5 \text{ (by symmetry)}$$

Answer:
 $y=5$ is the horizontal asymptote.

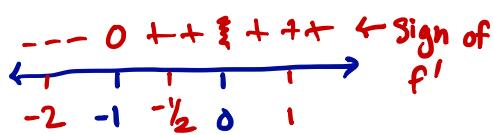
6. (15 points) Given $f(x) = 4x^{1/3} + x^{4/3}$. Answer the questions below about the graph of $f(x)$.

NOTE: $f'(x) = \frac{4(x+1)}{3x^{2/3}}$ and $f''(x) = \frac{4(x-2)}{9x^{5/3}}$

- (a) Determine the intervals where f is increasing and where f is decreasing.

$f' = 0$ when $x = -1$;

f' undefined when $x = 0$



(Note $x^{2/3} \geq 0$ always)

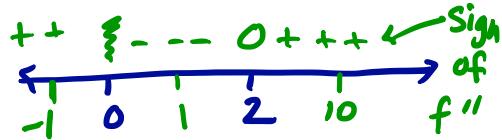
Answer:

f is increasing on $(-1, \infty)$
and decreasing $(-\infty, -1)$

- (b) Determine the intervals where f is concave up and where f is concave down.

$f'' = 0$ when $x = 2$

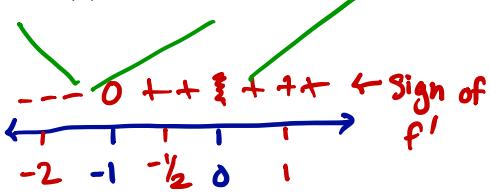
f'' undefined at $x = 0$



Answer:

f cc up on $(-\infty, 0) \cup (2, \infty)$
and cc down on $(0, 2)$

- (c) Identify any local maximums or minimums.



↑ local min here!

So $f(-1) = -3$ is a local minimum

- (d) Identify any inflection points.

$$(0, f(0)) = (0, 0)$$

$$(2, f(2)) = (2, 6\sqrt[3]{2})$$

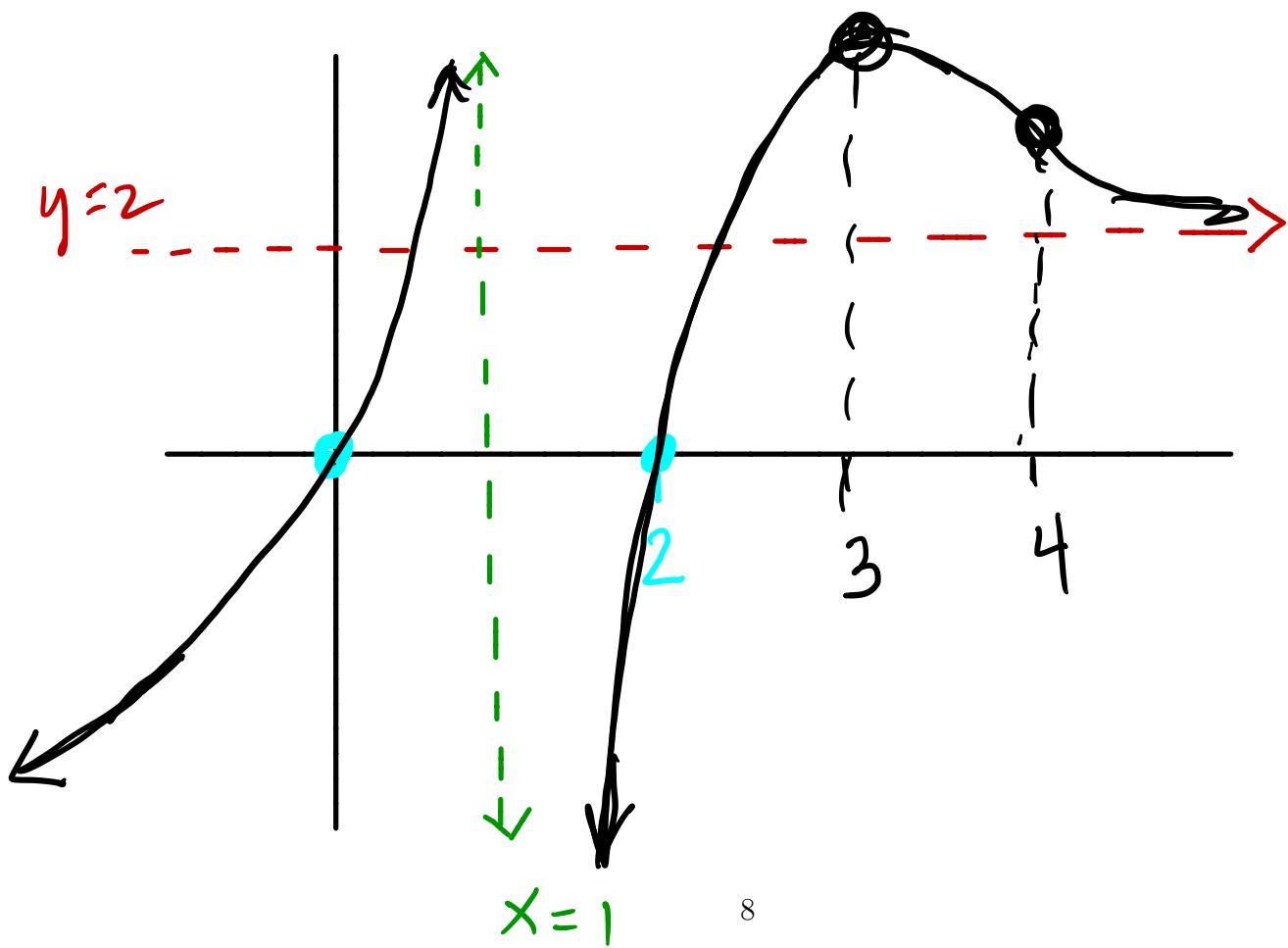
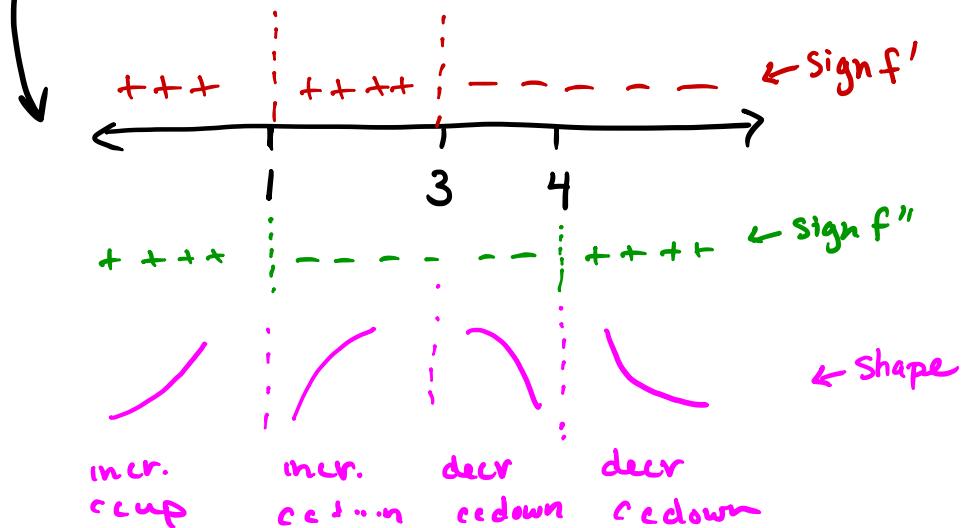
$$\begin{aligned} f(2) &= 4 \cdot 2^{\frac{1}{3}} + 2^{\frac{4}{3}} \\ &= 4 \cdot 2^{\frac{1}{3}} + 2 \cdot 2^{\frac{1}{3}} = 6 \cdot 2^{\frac{1}{3}} \end{aligned}$$

7. (10 points) Clearly and neatly sketch ONE graph containing ALL the properties below.

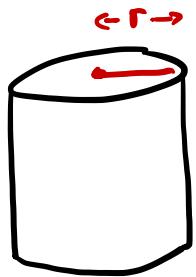
(a) $\lim_{x \rightarrow \infty} f(x) = 2$ and $\lim_{x \rightarrow 1^-} f(x) = \infty$ $\rightarrow y=2$ HA $x=1$ V.A.

(b) x -intercepts at 0 and 2

- (c) $f' > 0$ on $(-\infty, 1) \cup (1, 3)$ and $f' < 0$ on $(3, \infty)$
 (d) $f'' > 0$ on $(-\infty, 1) \cup (4, \infty)$ and $f'' < 0$ on $(1, 4)$



8. (10 points) A cylindrical tank is to be constructed. The cost of construction per square unit of surface area is four times as much for the top and bottom of the tank as for the sides. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. You must show your work. Make sure to clearly define your variables, identify the domain of your function, show (using calculus) that your answer is correct, and completely answer the question.



goal: minimize cost.

Note: Volume is fixed, say V .

$$\text{So } V = \pi r^2 h \Rightarrow h = \frac{1}{\pi} V r^{-2}$$

$$\text{Cost: } C = 4 \cdot 2 \cdot \pi r^2 + 2\pi r h = 8\pi r^2 + 2\pi r h$$

↑ ↑ ↑
 area top sides
 bottom, too

4x cost of sides

Plug in to get a function of 1 variable!

$$C(r) = 8\pi r^2 + 2\pi r \left(\frac{1}{\pi} V r^{-2} \right) = 8\pi r^2 + 2V r^{-1} \text{ with domain } (0, \infty).$$

$$C'(r) = 16\pi r - 2V r^{-2} = 0 \quad \leftarrow \text{find critical #}'s.$$

$$\text{So } 16\pi r = \frac{2V}{r^2} \text{ or } r^3 = \frac{2V}{16\pi}$$

$$\text{So } r = \sqrt[3]{\frac{V}{8\pi}} = \frac{1}{2} \sqrt[3]{\frac{V}{\pi}}$$

(--- 0 + + + + \rightarrow
 0 0.1 $\frac{1}{2} \sqrt[3]{\frac{V}{\pi}}$ 100V

Sign of C'
 ↗ local min here.

Still got to find $[h]$:

$$\begin{aligned} h &= \frac{V}{\pi} \left(\frac{V}{8\pi} \right)^{-2/3} = \frac{V}{\pi} \cdot \frac{4\pi^{2/3}}{V^{2/3}} \\ &= 4 \left(\frac{V}{\pi} \right)^{1/3} \end{aligned}$$

Answer: dimensions

$$r = \frac{1}{2} \sqrt[3]{\frac{V}{\pi}}, h = 4 \left(\frac{V}{\pi} \right)^{1/3}$$

An absolute minimum occurs here because $r = \frac{1}{2} \sqrt[3]{\frac{V}{\pi}}$ is the unique critical # on $(0, \infty)$.

EXTRA CREDIT

A marathoner ran the 26.2 mile New York City Marathon in 2.2 hours. Use the Mean Value Theorem to prove that at least twice, the marathoner was running precisely 11 mph. You MUST use the Theorem explicitly to receive credit. That is, you must carefully identify your $f(x)$, make sure the MVT applies, etc. Hint: You will need another important theorem too...

Mean Value Theorem: Let $f(x)$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) , then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Observe that the marathoner's average velocity was:

$$\frac{26.2 \text{ mi}}{2.2 \text{ hr}} > 11 \text{ mi/hr}$$

If $s(t)$ is the runner's position from the start at time $t=0$ and we assume the runner's position is smooth + continuous on $[0, 2.2]$, then there exists a c in $[0, 2.2]$ such that $s'(c) = \frac{26.2}{2.2} > 11$.

If we further assume (quite reasonably) that $s'(t)$ is continuous, then the Extreme Value theorem implies the runner must achieve every velocity from 0 (when starting) to $s'(c)$. Since 11 mph lies between this velocity, too, must be achieved. \square .