Worksheet §2.7 and 2.8

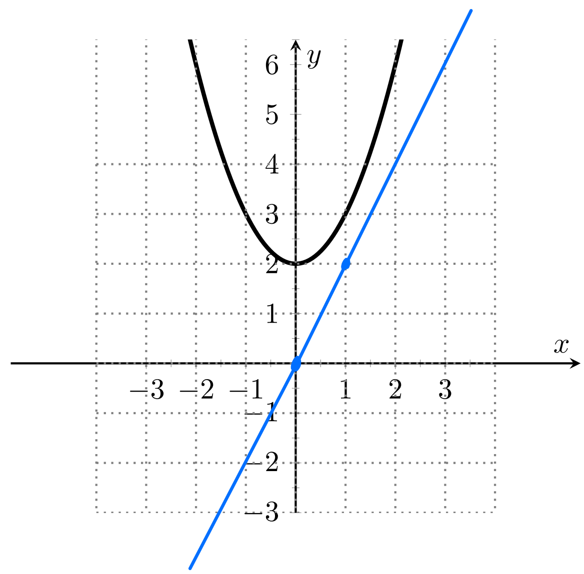
The function

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is called the **derivative of** f. The value of f' at x can be interpreted geometrically as the SLOPE of the tangent line to f at the point $(x, f(x))$. *Note: f' is called the derivative because it has been derived from f using the limit operation defined above. The domain of f' is the set of all x such that this limit exists and may be smaller than the domain of f.*

1. Let $f(x) = x^2+2$, shown below. Use the definition of the derivative as a function to compute $f'(x)$. Then graph $f'(x)$ on the same axes.

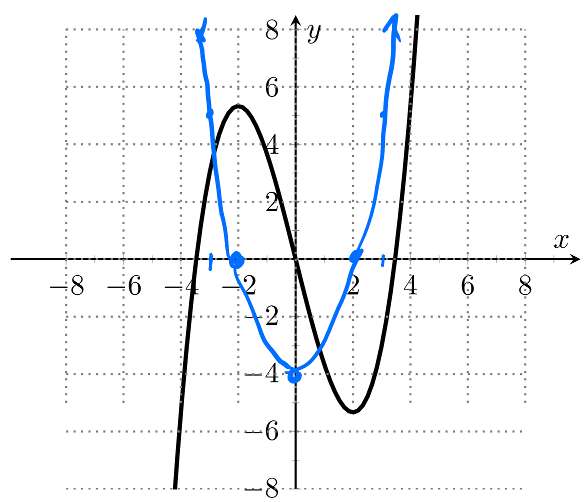
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\left((x+h)^2+2\right)-\left(x^2+2\right)\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\cancel{x^2}+2xh+h^2+2-\cancel{x^2}-2\right)$$
$$= \lim_{h\to 0}\frac{1}{\cancel{h}}\left(2x\cancel{h}+h^{\cancel{2}}\right)$$
$$= \lim_{h\to 0} 2x+h = 2x$$



2. Let $f(x) = \frac{1}{3}x^3 - 4x$.

   (a) Use the definition of the derivative (as a function) to find a formula for $f'(x)$. You may find it helpful to use the fact that $(a+b)^3 = a^3+3ab^2+3ab^2+b^3$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{3}(x+h)^3-4(x+h)-\left[\frac{1}{3}x^3-4x\right]\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{3}(x^3+3x^2h+3xh^2+h^3-4x-4h)-\frac{1}{3}x^3+4x\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\cancel{\frac{1}{3}x^3}+x^2h+xh^2+\frac{1}{3}h^3-\cancel{4x}-4h-\cancel{\frac{1}{3}x^3}+\cancel{4x}\right)$$
$$= \lim_{h\to 0}\frac{1}{\cancel{h}}\left(x^2\cancel{h}+xh^{\cancel{2}}+\frac{1}{3}h^{\cancel{3}\,2}-4\cancel{h}\right)$$
$$= \lim_{h\to 0} x^2+xh+\frac{h^2}{3}-4 = \boxed{x^2-4}$$



   (b) Factor the formula and use the factorization to plot the graph of $f'(x)$ on the same axes that show $f(x)$.

$$f'(x) = x^2-4 = (x-2)(x+2)$$

$$f'(-3) = 5$$

   (c) What do you notice about the relationship between ~~$f(x)$ and $f'(x)$~~? ~~Explain why this makes sense by thinking about the slopes of tangent lines to $f(x)$.~~ the zeroes of $f'(x)$ and the tangent lines of $f(x)$?

The TL are horizontal where the derivatives are 0.

3. Consider the function

$= -\frac{x^2}{8} + \frac{x}{2} + 4$

$$f(x) = \left| \frac{x^2}{8} - \frac{x}{2} - 4 \right| = \begin{cases} \frac{x^2}{8} - \frac{x}{2} - 4 & \text{if} \quad x \le -4 \text{ or } x \ge 8 \\ -(\frac{x^2}{8} - \frac{x}{2} - 4) & \text{if} \quad -4 < x < 8 \end{cases}.$$

(a) The graph of $f(x)$ is given on the top set of axes shown below. By thinking about slopes of tangent lines, sketch a graph of the derivative on the second set of axes.

When I ask you to sketch, I am interested in the qualitative behavior of the derivative: Where does it cross the x-axis? Is it positive or negative? Is it a lot positive or a little positive? Are the slopes growing steeper or getting less steep? (This is why the y-axis is unmarked on the answer graph.)

(b) Use the definition of the derivative to determine $f'(x)$ algebraically, for two cases: (i) $x < -4$ or $x > 8$; (ii) $-4 < x < 8$. Explain why your algebraic calculations match your sketch.

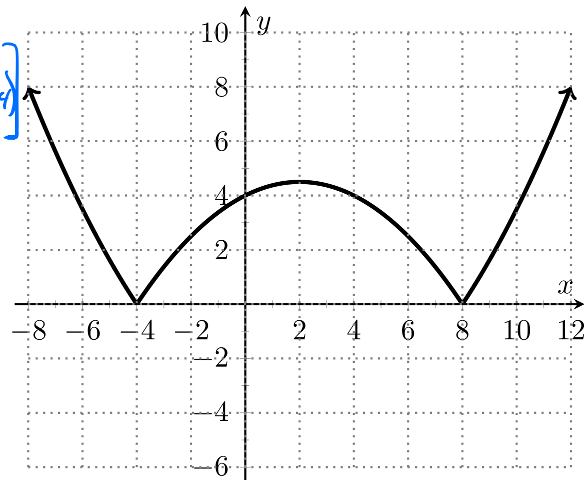
(i) $\boxed{x < -4 \text{ or } x > 8}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{8}(x+h)^2 - \frac{1}{2}(x+h) - 4 - \left(\frac{x^2}{8} - \frac{x}{2} - 4\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{8}(x^2 + 2xh + h^2) - \frac{x}{2} - \frac{h}{2} - 4 - \frac{x^2}{8} + \frac{x}{2} + 4\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cancel{\frac{x^2}{8}} + \frac{xh}{4} + \frac{h^2}{8} - \cancel{\frac{x}{2}} - \frac{h}{2} - \cancel{4} - \cancel{\frac{x^2}{8}} + \cancel{\frac{x}{2}} + \cancel{4}\right]$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{x\cancel{h}}{4} + \frac{h^{\cancel{2}}}{8} - \frac{\cancel{h}}{2}\right] = \lim_{h \to 0} \frac{x}{4} + \frac{h}{8} - \frac{1}{2} = \boxed{\frac{x}{4} - \frac{1}{2}}$$



(ii) $-4 \le x < 8$

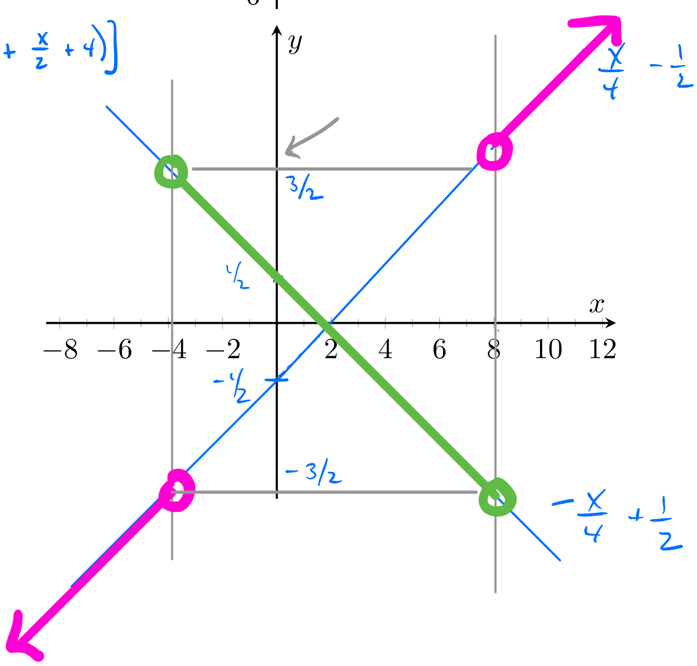
$$\lim_{h \to 0} \frac{1}{h}\left(f(x+h) - f(x)\right) = \lim_{h \to 0} \frac{1}{h}\left(-\frac{1}{8}(x+h)^2 + \frac{(x+h)}{2} + 4 - \left(-\frac{x^2}{8} + \frac{x}{2} + 4\right)\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(-\frac{1}{8}(x^2 + 2xh + h^2) + \frac{x}{2} + \frac{h}{2} + 4 + \frac{x^2}{8} - \frac{x}{2} - 4\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(-\cancel{\frac{x^2}{8}} - \frac{xh}{4} - \frac{h^2}{8} + \cancel{\frac{x}{2}} + \frac{h}{2} + \cancel{4} + \cancel{\frac{x^2}{8}} - \cancel{\frac{x}{2}} - \cancel{4}\right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(-\frac{x\cancel{h}}{4} - \frac{h^{\cancel{2}}}{8} + \frac{\cancel{h}}{2}\right) = \lim_{h \to 0} -\frac{x}{4} - \frac{h}{8} + \frac{1}{2}$$

$$= \boxed{-\frac{x}{4} + \frac{1}{2}}$$



(c) Using your formula from (a), compute

- $\lim_{x \to -4^-} f'(x) =$ $\lim_{x \to -4^-} \frac{x}{4} - \frac{1}{2} = -\frac{3}{2}$
- $\lim_{x \to -4^+} f'(x) =$ $\lim_{x \to -4^+} -\frac{x}{4} + \frac{1}{2} = 3/2$
- $\lim_{x \to 8^-} f'(x) =$ $\lim_{x \to 8^-} -\frac{x}{4} + \frac{1}{2} = -\frac{8}{4} + 1/2 = -3/2$
- $\lim_{x \to 8^+} f'(x) =$ $\lim_{x \to 8^+} \frac{x}{4} - \frac{1}{2} = \frac{8}{4} - 1/2 = 3/2$

Using the language of calculus, what can you say about $f'(x)$ at $x = -4$ and $x = 8$? Why does this make sense geometrically? (Does it match your picture?)

It is not continuous!