Your Name

Solutions

Your Signature

Instructor Name

End Time

Desk Number

- The total time allowed for this exam is 90 minutes.
- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question** where appropriate.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

This exam is printed double-sided.

There are problems on both sides of the page!

If you need more space, you may use extra sheets of paper. If you use extra pages:

- Put your name on each extra sheet
- Label your work with the problem you're working on
- Write on the exam problem that there is additional work at the end
- Turn in your additional pages at the end of your exam.

$\boxed{1}$ (10 points) Consider the function $g(x) = \dfrac{4}{x} + x$. $= 4x^{-1} + x$

(a) Find the critical number(s) of $g(x)$.

$$g'(x) = 4(-x^{-2}) + 1 = \frac{-4}{x^2} + 1$$

$$g'(x) = 0 \Rightarrow \frac{-4}{x^2} + 1 = 0 \Rightarrow \frac{-4}{x^2} = -1 \Rightarrow x^2 = 4 \Rightarrow$$

$$x = 2 \text{ or } x = -2$$

$$g'(x) \text{ DNE} \Rightarrow x = 0.$$

$$\boxed{\text{Critical \#s are } x = -2,\ x = 0,\ x = 2.}$$

(b) Find the absolute maximum and absolute minimum values of $g(x)$ on the interval $[1/2, 3]$.

Extreme value theorem says abs max/min occur either at endpoints or at critical points in the interval.

| type | $x$ | $f(x)$ |
|---|---|---|
| end | $1/2$ | $8\frac{1}{2}$ ← ABS MAX |
| crit. | 2 | 4 ← ABS MIN |
| end | 3 | $4\frac{1}{2}$ |

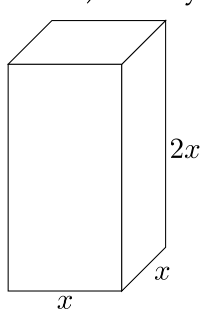
$$f(1/2) = \frac{4}{1/2} + 1/2 = 8 + 1/2 = 8\tfrac{1}{2}$$

$$f(2) = \frac{4}{2} + 2 = 4$$

$$f(3) = \frac{4}{3} + 3 = \frac{13}{2} = 4\tfrac{1}{2}$$

$$\boxed{\text{The abs. max is } y = 8\tfrac{1}{2} \text{ and abs min is } y = 4}$$

$\boxed{2}$ (10 points) A box has a square base and a height that is twice as large as the length of the base. If the length of the base is measured to be 4 cm with an error of $\pm 1$ mm ( = 1/10 cm), what is the (absolute) error in the volume of the box? (That is, how much "extra" or "missing" volume is there?) Show your work.




The intent of this problem was to use linearization/differentials to estimate the error in volume, but we forgot to say "estimate". So we are accepting either a direct computation or an estimation. :-)

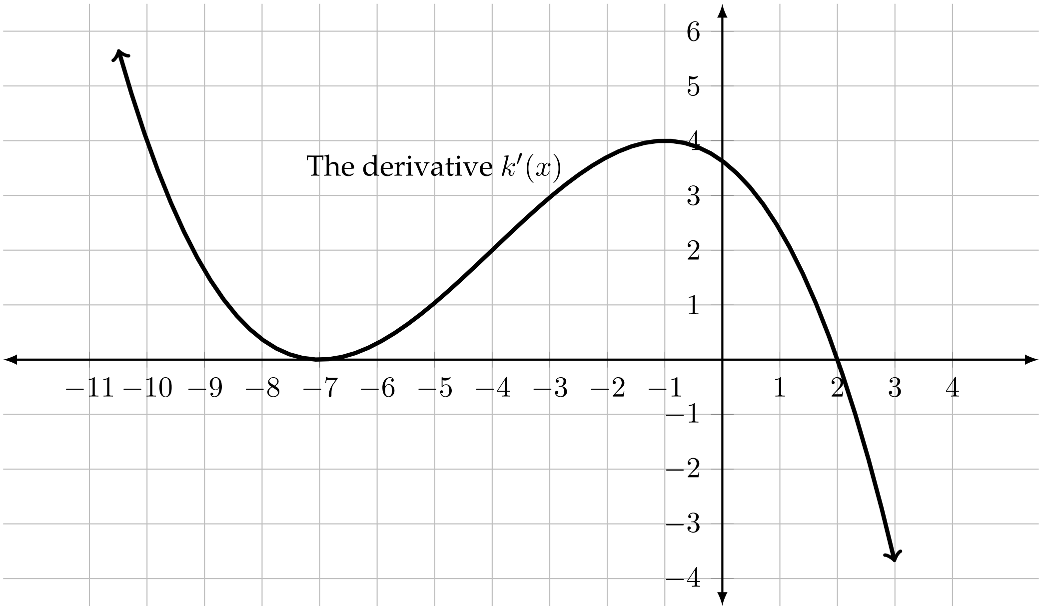
Know $V(x) = 2x^3$, so $\dfrac{\Delta V}{\Delta x} \approx \dfrac{dV}{dx} = 6x^2$. If

$x = 4$ and $\Delta x = \pm\frac{1}{10}$, $\Delta V \approx 6(4)^2\left(\pm\frac{1}{10}\right) = \pm\dfrac{6\cdot4\cdot4}{10} = \pm\dfrac{6\cdot4\cdot2}{5} = \pm\dfrac{48}{5}$

$= \pm\dfrac{96}{10} = \pm 9.6$. That is $\boxed{\Delta V \approx \pm\frac{48}{5} = 9.6}$

If you tediously computed the exact #s, $V(4+\frac{1}{10}) = 2(4.1)^3 = 137.842$ and $V(4) = 2(4)^3 = 128$ for an error of $9.842$, and $V(4-\frac{1}{10}) = 118.638 \Rightarrow$ error is $-9.362$.

$\boxed{3}$ (14 points) The following graph shows the $\boxed{\text{DERIVATIVE}}$ $k'$ of some function $k$.




**The following questions are about the function $k(x)$, not the graphed $k'(x)$.**

(a) Critical points of $k(x)$: $x = -7$ & $x = 2$

(b) On what intervals is $k$ increasing or decreasing?

Increasing: $(-\infty, 2)$

Decreasing: $(2, \infty)$

(c) At what values of $x$ does $k$ have a local maximum or minimum? If none, say so.

Local Maxima: $x =$ 2 Local Minima: $x =$ none

(d) On what intervals is $k$ concave up or concave down? Use interval notation.

Concave up: $(-7, -1)$ Concave down: $(-\infty, -7) \cup (-1, \infty)$

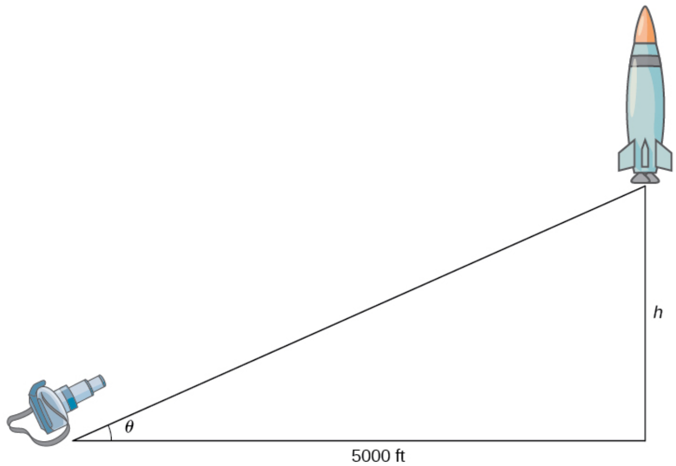
(e) At what values of $x$ does $k$ have inflection points? If none, say so.

Inflection points: $x =$ $-7, -1$

**4** (14 points) A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically according to the position equation

$$h(t) = 50t^2,$$

where **h** is measured in feet and is **t** measured in seconds (*see picture below*). The camera is 5000 feet from the launch pad.




(a) Find the height and velocity [i.e., change in height] of the shuttle 10 seconds after lift-off.

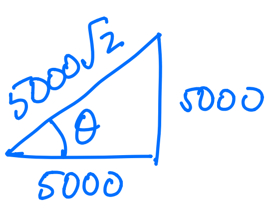
$$h(10) = 50(10)^2 = 50 \cdot 100 = 5000$$

$$\frac{dh}{dt} = 100t \quad \text{so} \quad \left.\frac{dh}{dt}\right|_{t=10} = 100 \cdot 10 = 1000$$

(b) Find the rate of change in the angle of elevation of the camera ($\theta$) at 10 seconds after lift-off. [Include **units** in your answer]

Want $\frac{d\theta}{dt}$ when $h=10$. Know $\frac{h}{5000} = \tan\theta \Rightarrow \underline{h = 5000\tan\theta}$

So $\frac{dh}{dt} = 5000(\sec\theta)^2 \cdot \frac{d\theta}{dt}$. Observe when $h=10$,




$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}} = \sqrt{2}$, so

$$\frac{dh}{dt} = 5000(\sec(\theta))^2 \frac{d\theta}{dt} \Rightarrow 1000 = 5000(\sqrt{2})^2 \frac{d\theta}{dt}$$

$$\Rightarrow \frac{1}{5} = 2\frac{d\theta}{dt} \Rightarrow \boxed{\frac{d\theta}{dt} = \frac{1}{10} \text{ radians/second}}$$

5 (12 points) For each limit:

(i) Write the **form** of the limit AND state whether the **form** is **indeterminate** (include the type).

(ii) Find the limit. If you use a L'Hôpital Rule, indicate it by a symbol (such as **L'H** or **H**) over the equal sign.

(a) $\lim\limits_{x\to 0} \dfrac{\sin(2x)+7x^2}{x(x+1)} = \dfrac{\sin(2x)+7x^2}{x^2+x}$ Type: $0/0$ $\dfrac{\sin(0)+0}{0(1)}$

$\overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{\cos(2x)(2)+14x}{2x+1}$

$= \dfrac{\cos(0)\cdot 2 + 14(0)}{2(0)+1}$

$= \boxed{2}$

(b) $\lim\limits_{x\to 0} \dfrac{2\cos(\pi x) - 1 + x^2}{2e^{4x}}$ Type: not indeterminate!

$= \dfrac{2\cos(\pi\cdot 0) - 1 + 0^2}{2\cdot e^{4(0)}}$

$= \dfrac{2-1+0}{2\cdot e^0} = \boxed{\dfrac{1}{2}}$

(c) $\lim\limits_{t\to\infty} t\ln\left(1+\dfrac{3}{t}\right)$ Type: $\infty\cdot 0$

$\lim\limits_{t\to\infty} \ln\left(1+\frac{3}{t}\right) = \ln\left(\lim\limits_{t\to\infty} 1+\frac{3}{t}\right)$
$= \ln(1) = 0$

$= \lim\limits_{t\to\infty} \dfrac{\ln\left(1+\frac{3}{t}\right)}{1/t}$ type $\frac{0}{0}$

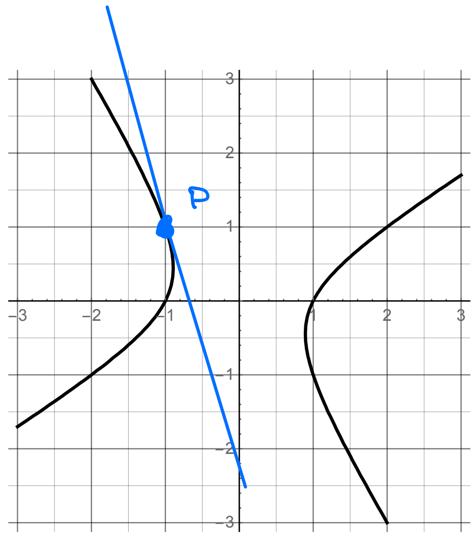
$\overset{L'H}{=} \lim\limits_{t\to\infty} \dfrac{\frac{1}{1+3/t}\left(3(-1)t^{-2}\right)}{-t^{-2}}$

$= \lim\limits_{t\to\infty} \dfrac{3}{1+3/t}$

$= \boxed{3}$

**6** (10 points) Consider the implicitly defined curve given by

$$x^2 - y^2 = 1 + xy.$$




(a) Show that the point $P = (-1, 1)$ is on the curve. Then **draw and label** the point $P$ in the figure.

Observe $(-1)^2 - 1^2 = 0$ and $1 + (-1)(1) = 1 - 1 = 0$. Thus the point $(-1, 1)$ satisfies the equation $x^2 - y^2 = 1 + xy$.

(b) Compute $y'$ at $P$.

$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(1 + xy) \Rightarrow 2x - 2y\,y' = 0 + xy' + y$

So at $x = -1$, $y = 1$, we have

$2(-1) - 2(1)y' = (-1)y' + (1) \Rightarrow -2 - 2y' = -y' + 1 \Rightarrow$

$-2y' + y' = 1 + 2 \Rightarrow \boxed{y' = -3}$ ← plausible from diagram!

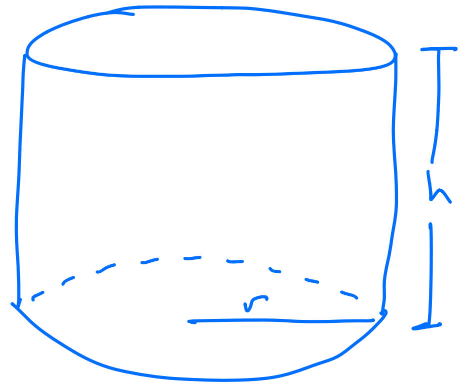
(c) Find the equation of the tangent line at $P$. Then **draw** this tangent line in the figure.

Tangent line:

$$y = -3(x - (-1)) + 1 \Rightarrow y = -3(x + 1) + 1$$

7 (14 points) Suppose an open cup in the shape of a cylinder is to be made with surface area 48 in$^2$. What dimensions (radius and height) will maximize the volume of the cup?

[surface area = $\pi r^2 + 2\pi r h$ and volume = $\pi r^2 h$, where $r$ is the radius of the cup and $h$ is the height.]




$V = \pi r^2 h$

Know $48 = \pi r^2 + 2\pi r h \Rightarrow$

$h = \dfrac{48 - \pi r^2}{2\pi r} = \dfrac{24}{\pi r} - \dfrac{r}{2}$

So $V(r) = \pi r^2\left(\dfrac{24}{\pi r} - \dfrac{r}{2}\right) = 24r - \dfrac{\pi r^3}{2}$

and $V'(r) = 24 - \dfrac{\pi}{2}(3r^2) = 24 - \dfrac{3\pi r^2}{2}$

$V'(r)$ is never undefined, so only critical point is

$V'(r) = 0 \Rightarrow 24 = \dfrac{3}{2}\pi r^2 \Rightarrow r^2 = \dfrac{48}{3\pi} = \dfrac{16}{\pi}$

So $r = \dfrac{4}{\sqrt{\pi}}$ or $r = \dfrac{-4}{\sqrt{\pi}}$ but only the positive answer makes sense.

Height? $h = \dfrac{24}{\pi\left(\frac{4}{\sqrt{\pi}}\right)} - \dfrac{\frac{4}{\sqrt{\pi}}}{2} = \dfrac{6\sqrt{\pi}}{\pi} - \dfrac{2}{\sqrt{\pi}} = \dfrac{4}{\sqrt{\pi}}$

$\boxed{8}$ (16 points) We want to sketch a graph of a function $f(x)$ with certain specified properties.

(a) Fill in the following tables. (You can use words or pictures.)

| function information | what you conclude about the behavior of $f$ |
|---|---|
| Domain of $f$ is $(-\infty, \infty)$ | f is continuous* |
| $\lim_{x\to-\infty} f(x) = -2$ | $y=-2$ is a horiz. asymp. as $x \to -\infty$ |
| $\lim_{x\to\infty} f(x) = 5$ | $y=5$ is a horiz. asymp. as $x\to\infty$ |
| $f(0) = 10$ | fn passes through $(0, 10)$ |

* or, f is defined for all real #s, but that's just restating what "domain" means...

| $x$ | $x<0$ | $0$ | $x>0$ |
|---|---|---|---|
| sign/value of $f'(x)$ | $+$ | $0$ | $-$ |
| Behavior of $f(x)$ | ↗ | ↑ | ↘ |

max (optional)

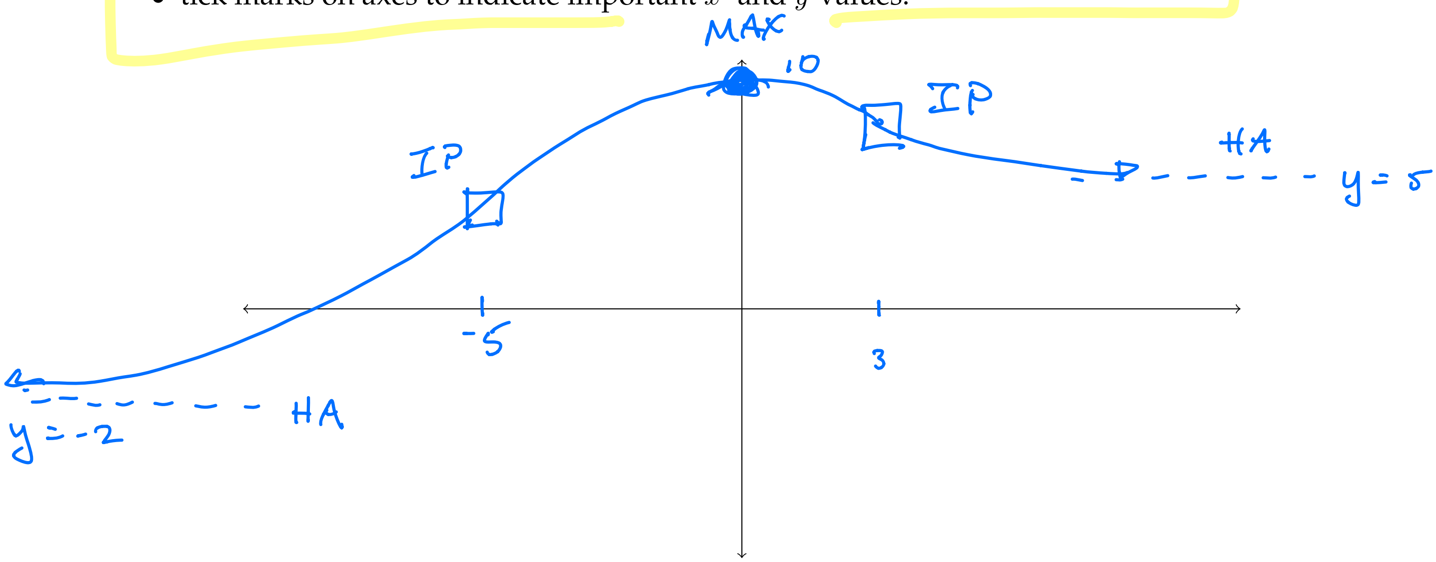
| $x$ | $x<-5$ | $-5$ | $-5<x<3$ | $3$ | $x>3$ |
|---|---|---|---|---|---|
| sign of $f''(x)$ | $+$ | $0$ | $-$ | $0$ | $+$ |
| Behavior of $f(x)$ | ∪ | IP | ∩ | IP | ∪ |

optional

(b) Sketch the graph of $f$ that has all of the properties listed in the tables (does not need to be drawn to scale). Label/draw $\boxed{\textbf{on the graph}}$ the following:

- a point at any local maxima/minima,
- a box at any inflection points,
- a dashed line for any horizontal/vertical asymptotes along with equation,
- tick marks on axes to indicate important $x$- and $y$-values.

← Required!



$\boxed{\text{Extra Credit}}$ (5 points)

Use the Mean Value Theorem to prove that $a - b \leq \sin b - \sin a \leq b - a$ given the interval $[a, b]$.

Let $f(x) = \sin(x)$. Since $\sin(x)$ is continuous on $[a, b]$ and d'ble on $(a, b)$ we know, by the MVT, that there exists some $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

That is,

$$\cos(c) = \frac{\sin(b) - \sin(a)}{b - a}.$$

Since $-1 \leq \cos(x) \leq 1$, it follows that

$$-1 \leq \frac{\sin(b) - \sin(a)}{b - a} \leq +1 \implies$$

$$a - b \leq \sin(b) - \sin(a) \leq b - a$$

which is what we were asked to show!