Intro video: Section 2.3 part 1
Limit laws and calculating limits algebraically

Math F251X: Calculus I

Remember: we say $\lim_{x\to a} f(x) = L$ if as $x \to a$, $f(x) \to L$.

Suppose c is a constant and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

EXAMPLES

$\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = 5$

LAWS OF LIMITS

formal notation	sentences	example
$\lim_{x\to a}\left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$	limit of a sum is the sum of the limits	$\lim_{x\to 2} f(x)+g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = 3+5 = 8$
$\lim_{x\to a}\left(cf(x)\right) = c\lim_{x\to a} f(x)$	you can pull constants through limits	$\lim_{x\to 2} 7f(x) = 7\lim_{x\to 2} f(x) = 7(5) = 35$
$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$	The limit of a product is the product of the limits	$\lim_{x\to 2} f(x)g(x) = \left(\lim_{x\to 2} f(x)\right)\left(\lim_{x\to 2} g(x)\right) = 3\cdot 5 = 15$
$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x) \neq 0$	limit of a quotient is the quotient of the limits as long as limit in denom. $\neq 0$	$\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)} = \frac{3}{5}$

Example: Suppose $\boxed{\lim_{x\to 3} f(x) = 8}$ and $\boxed{\lim_{x\to 3} g(x) = 4.}$

a) What is $\lim_{x\to 3}\left(3f(x) - g(x)\right)$?

$$\lim_{x\to 3}\left(3f(x)-g(x)\right) = \lim_{x\to 3}\left(3f(x) + (-g(x))\right) = \lim_{x\to 3} 3f(x) + \lim_{x\to 3}(-g(x))$$

$$= 3\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)$$

$$= 3(8) - 4 = 20.$$

b) What is $\lim_{x\to 3}\dfrac{g(x)}{7f(x)}$?

$$\lim_{x\to 3}\frac{g(x)}{7f(x)} = \frac{\lim_{x\to 3} g(x)}{\lim_{x\to 3}\left(7f(x)\right)} = \frac{\lim_{x\to 3} g(x)}{7\lim_{x\to 3} f(x)} = \frac{4}{7(8)} = \frac{1}{14}$$

More limit laws

- Suppose n is a positive integer
  c is a constant

$\boxed{\text{LAWS OF LIMITS}}$

$$\lim_{x \to a} x = a$$

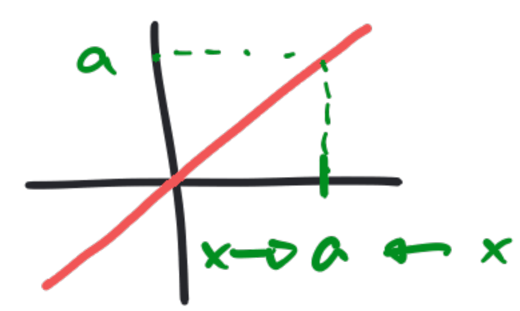
$$\lim_{x \to a} c = c$$

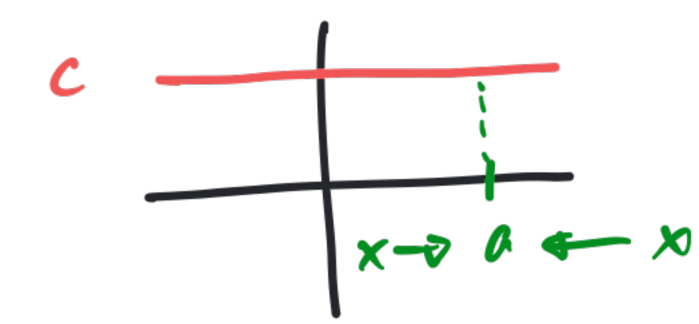
$$(f(x))^n = \underbrace{f(x) \cdot f(x) \cdot \ldots \cdot f(x)}_{n}$$

$$\lim_{x \to a} \left((f(x))^n \right) = \left(\lim_{x \to a} f(x) \right)^n$$

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

Example $\lim_{x \to 2} f(x) = 3$





$$\lim_{x \to 2} (f(x))^5 = \left(\lim_{x \to 2} f(x) \right)^5$$
$$= 3^5 = 243$$

$$\lim_{x \to 2} \sqrt{f(x)} = \sqrt{\lim_{x \to 2} f(x)}$$
$$= \sqrt{3}$$

Example What is $\lim_{x \to 2} 3x^2 - 5x + 7$?

$$\lim_{x \to 2} 3x^2 - 5x + 7 = \lim_{x \to 2} 3x^2 - \lim_{x \to 2} 5x + \lim_{x \to 2} 7$$

$$= 3 \lim_{x \to 2} x^2 - 5 \lim_{x \to 2} x + \lim_{x \to 2} 7$$

$$= 3 \left(\lim_{x \to 2} x \right)^2 - 5 \lim_{x \to 2} x + \lim_{x \to 2} 7$$

$$= 3(2)^2 - 5(2) + 7$$

$$= 3 \cdot 4 - 10 + 7$$

$$= 12 - 10 + 7 = 2 + 7 = 9$$

Theorem: If $f(x)$ is a polynomial, then $\lim_{x \to a} f(x) = f(a)$

Example: What is $\lim_{x \to 5} 3x^2 - 4x^3$?

$$\lim_{x \to a} 3x^2 - 4x^3 = 3(5)^2 - 4(5^3) = 5^2(3 - 4 \cdot 5) = 25(-17) = -425$$

Theorem: If $h(x) = \frac{f(x)}{g(x)}$ where f and g are polynomials, and a is in the domain of $h(x)$, then $\lim_{x \to a} h(x) = \frac{f(a)}{g(a)}$.

Example: Compute $\lim_{x \to -1} \frac{x^2+1}{2x-4}$.

$x=-1$ is in the domain, so $\lim_{x \to -1} \frac{x^2+1}{2x-4} = \frac{(-1)^2+1}{2(-1)-4} = \frac{2}{-6} = \frac{-1}{3}$

$\lim_{x \to 2^+} \frac{x^2+1}{2x-4}$ ← 2 is not in the domain of the function!

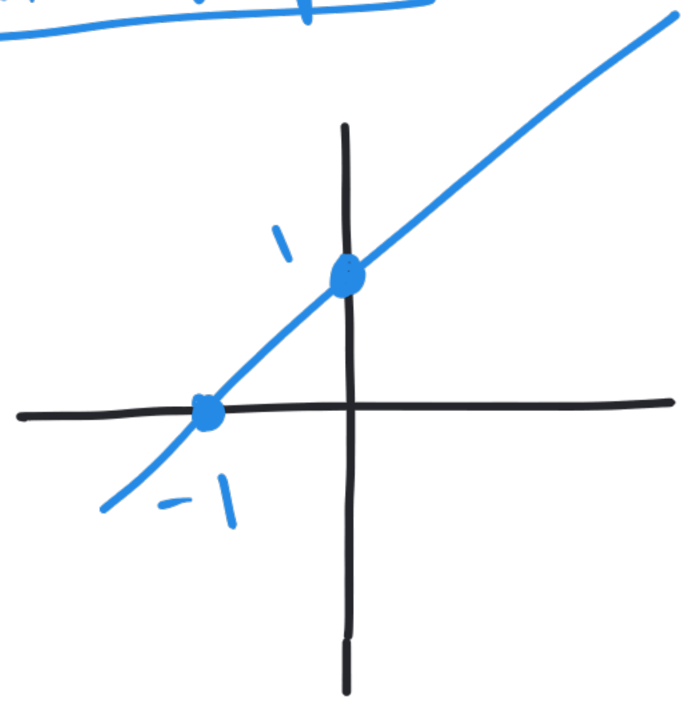
* Cannot direct substitute!

As $x \to 2^+$, $x^2+1 \to 5$ (we can direct substitute here...)

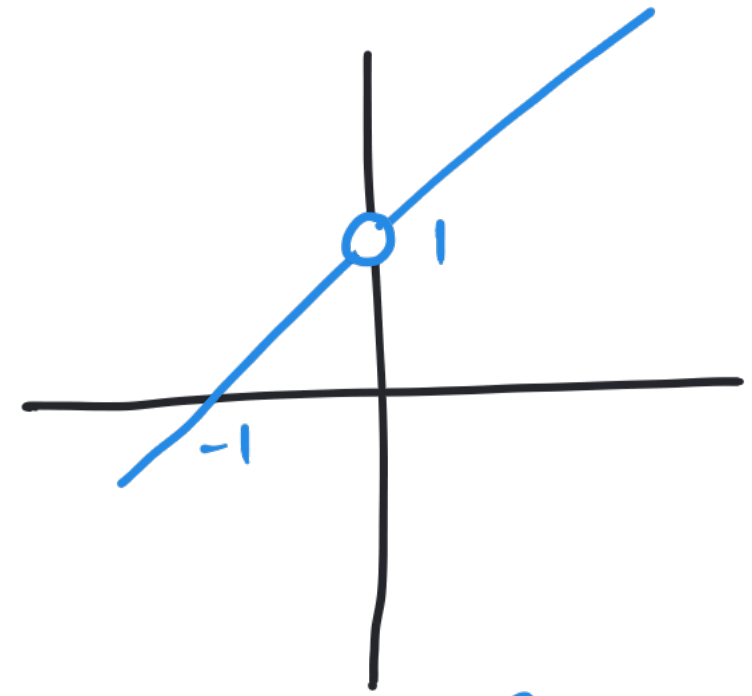
But $2x-4 \to 0^+$, so

$$\lim_{x \to 2^+} \frac{x^2+1}{2x-4} = \infty$$

Example



$$f(x) = x+1$$



$$g(x) = \frac{x^2+x}{x} = \frac{x(x+1)}{x}$$

$$\lim_{x \to 0} g(x) = 1.$$

Theorem: If $f(x) = g(x)$ everywhere except at $x = a$, then as long as the limits exist, then

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$$

Example: compute $\lim_{x \to 1} \frac{x^2+x-2}{x-1}$

Direct sub.?

$\frac{1^2+1-2}{1-1} = \frac{0}{0}$

Bad news! Cannot direct sub.

$$\lim_{x \to 1} \frac{x^2+x-2}{x-1} = \lim_{x \to 1} \frac{(x+2)(x-1)}{x-1}$$

$$= \lim_{x \to 1} x+2$$

$$= 1+2$$ ← direct substitution, so no more limits!

$$= 3.$$

Example

What is $\lim_{h \to 0} \frac{(2+h)^2-2^2}{h}$?

$$\lim_{h \to 0} \frac{(2+h)^2-2^2}{h} = \lim_{h \to 0} \frac{1}{h}\left(\cancel{2^2}+4h+h^2-\cancel{2^2}\right)$$

$$= \lim_{h \to 0} \frac{4h+h^2}{h} = \lim_{h \to 0} \frac{h(4+h)}{h}$$

$$= \lim_{h \to 0} 4+h = 4+0 = 4$$