Name: ____

This page contains information and techniques you will need for Sections 4.5 and 4.6.

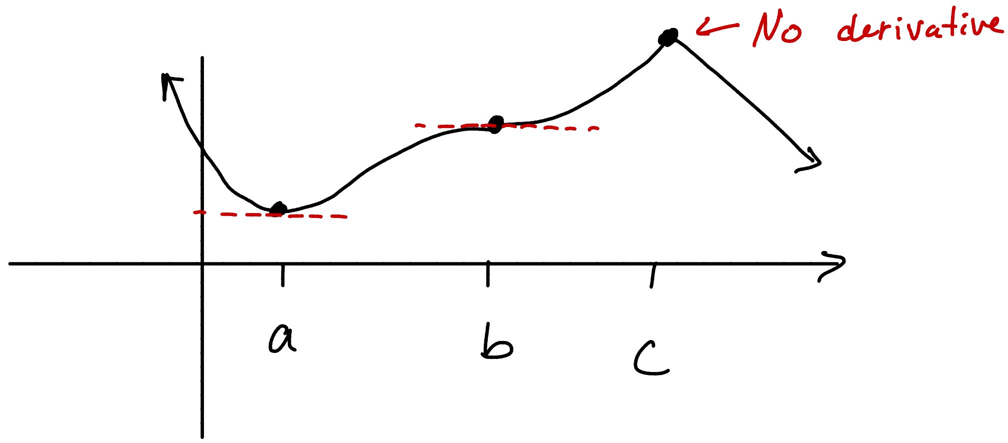
1. What is a critical point? How do you find them?

   An x-value in the domain of $f(x)$ where $f'(x)=0$ OR f' is undefined.

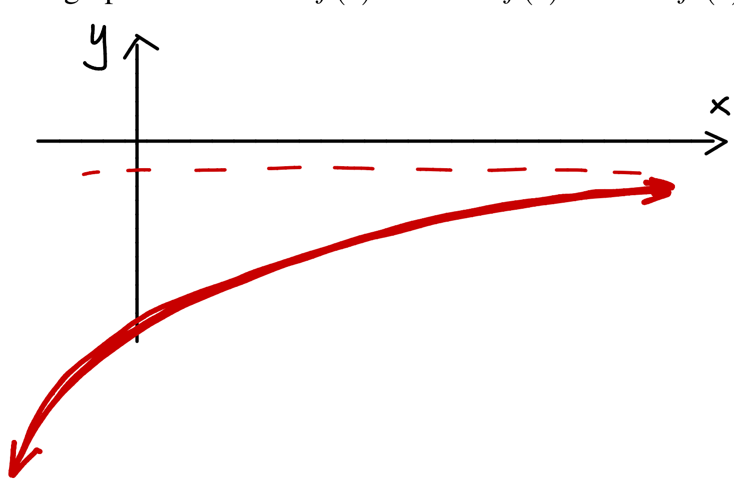
2. What is the importance of critical points? What are they good for?

   High & low points of a function will be found at critical points

3. Draw a graph of a function $f(x)$ with three critical points (at $x=a$, $x=b$ and $x=c$ such that
   (i) $f'(a)=f'(b)=0$ and $f'(c)$ is undefined,
   and
   (i) f has a local minimum at $x=a$, a local maximum at $x=c$ and neither at $x=b$.



4. Draw a graph of a function $f(x)$ such that $f(x)<0$ and $f'(x)>0$.



5. For each function below, find (a) its domain and (b) all its critical points.

(a) $f(x) = x^3 - 2x^2$

domain: $(-\infty,\infty)$

$f'(x) = 3x^2 - 4x = x(3x-4)$

f' never undefined.

$f' = 0$ if $x=0$ or $3x-4=0$

answer: crit. pts: $x=0$ or $x=4/3$

(b) $f(x) = x^{1/3}$

domain: $(-\infty,\infty)$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

f' never zero.

f' undefined when $x=0$.

answer: crit pts: $x=0$

(c) $f(x) = \arctan(x)$

domain: $(-\infty, \infty)$

$f'(x) = \frac{1}{x^2+1}$

f' is never zero (ie $1 \neq 0$)

f' is never undefined (ie $x^2+1 \neq 0$)

answer: crit pts: none

(d) $f(x) = \frac{x^2}{x^2-4}$ (Note: $f'(x) = \frac{-8x}{x^2-4}$.)

domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

$f'=0$ when $x=0$

f' undefined when $x=\pm 2$

Answer: crit. pts $x=-2,0,2$

(e) $f(x) = e^{-x^2}$

domain: $(-\infty,\infty)$

$f'(x) = -2x\,e^{-x^2}$

$= \frac{-2x}{e^{x^2}}$

$f'=0$ when $x=0$

f' never undefined (b/c $e^{x^2} \neq 0$)

answer: crit. pts $x=0$

(f) $f(x) = \sqrt{x^2-4} = (x^2-4)^{1/2}$

domain: $(-\infty,-2]\cup[2,\infty)$

$f'(x) = \frac{1}{2}(x^2-4)^{-1/2}(2x)$

$= \frac{x}{\sqrt{x^2-4}}$

$f'=0$ when $x=0$

f' undefined at $x=\pm 2$

answer: crit. pts $x=-2,0,2$

6. For $f(x) = x^{1/3}$, you should have gotten $x = 0$ as the one and only critical point. Explain (in whatever method works for you) **HOW** you will determine the **sign** of $f'(x)$? Repeat the process for $f''(x)$.

Jill's method

"tornado" b/c f' is undefined

$+++$ ⚡ $+++$ ← Sign of f'

$-1 \quad 0 \quad 1$ ← testpoints

$f'(-1) = \dfrac{1}{3(-1)^{2/3}} > 0$

$f'(1) = \dfrac{1}{3(1)^{2/3}} > 0$

Answer: $f' > 0$ for all $x \neq 0$.

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

$= \dfrac{-2}{9x^{5/3}}$

$++++$ ⚡ $----$ ← Sign f''

$-1 \quad 0 \quad 1$

$\dfrac{-2}{9(-1)^{5/3}} = \dfrac{-}{-} = +$

$\dfrac{-}{++} = -$

Answer: $f'' > 0$ on $(0, \infty)$

$f'' < 0$ on $(-\infty, 0)$

7. Assume the expression below is the second derivative of some function. How will you determine where y'' is positive or negative? Are some parts of the expression that are more important than other parts? What would have happened in the term in the denominator was raised to the **fourth** power instead of the third power?

$$y'' = \frac{8(3x^2+4)}{(x^2-4)^3}$$

Numerator is always positive.

numerator is zero when $x = \pm 2$

I only look at (x^2-4) term!

$+++$ ⚡ $- - -$ ⚡ $+++$ ← sign y''

$-3 \quad -2 \quad 0 \quad 2 \quad 3$ ← sample points

$9-4 \quad 0-4 \quad 9-4$

$+ \quad - \quad +$

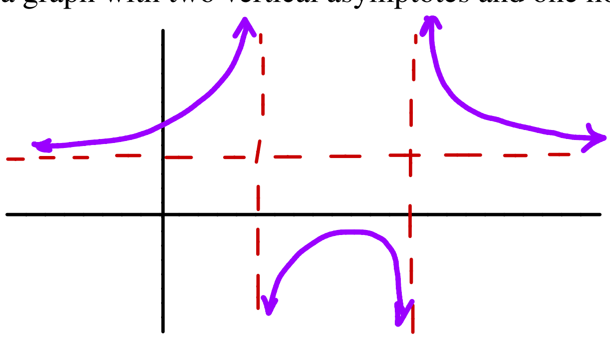
Answer:

$y'' > 0$ on $(-\infty, -2)$

$\cup (2, \infty)$

$y'' < 0$ on $(-2, 2)$

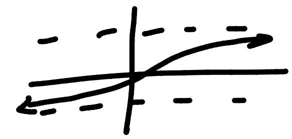
8. Sketch a graph with two vertical asymptotes and one horizontal asymptote.



9. Write a formula for a function $f(x)$ such that $f(x)$ has asymptotes $x = 1$, $x = 4$ and $y = 2$.

$$f(x) = 2 + \frac{1}{(x-1)(x-4)}$$

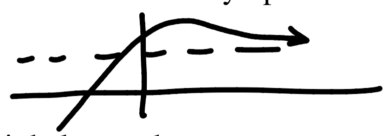
10. Can a function have more than one **horizontal** asymptote? Explain.



Yes. Consider $f(x) = \arctan(x)$

11. Can a function with a horizontal asymptote ever cross that asymptote? Explain.

Sure.



12. Evaluate each limit below and **justify** your answer. Your justification can be in words, using a graph, and/or numerical (or all of the above).

(a) $\lim_{x\to 2^+} \frac{5}{x-2} = +\infty$

pick $x = 2.01$; So $x-2 \to 0^+$

(b) $\lim_{x\to 2^-} \frac{5}{x-2} = -\infty$

pick $x = 1.99$; So $x-2 \to 0^-$

(c) $\lim_{x\to 2} \frac{5}{x-2} = $ DNE

left + right go different ways

(d) $\lim_{x\to \infty} \frac{5}{x-2} = 0$

as $x \to \infty$, $x-2 \to \infty$; So $\frac{1}{\text{big\#}} \to 0$

(e) $\lim_{x\to -\infty} \frac{5}{x-2} = 0$

as $x \to -\infty$, $x-2 \to -\infty$; So $\frac{1}{-(\text{big\#})} \to 0$

(f) $\lim_{x\to \infty} \left(8 + \frac{5}{x-2}\right) = \lim_{x\to\infty} (8) + \lim_{x\to\infty} \frac{5}{x-2} = 8 + 0 = 8$

(g) $\lim_{x\to \infty} \left(x + \frac{5}{x-2}\right) = \infty$

as $x \to \infty$, $\frac{5}{x-2} \to 0$; So $x + \frac{5}{x-2} \to \infty + 0 = \infty$.