LECTURE NOTES 2-5: CONTINUITY (DAY 2)

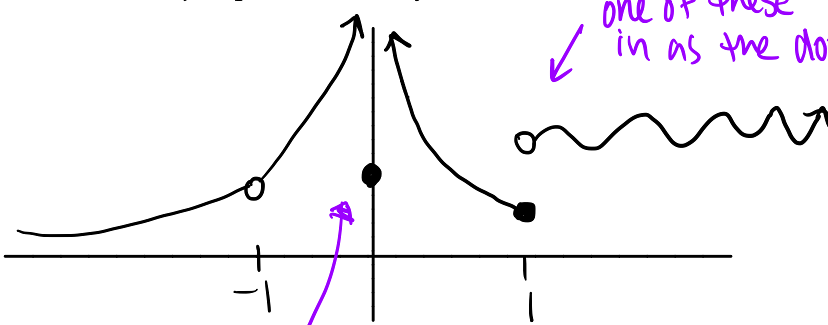
REVIEW: A function $f(x)$ is continuous at the number $x = a$ if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

check: ① $f(a)$ must be defined ② $\lim_{x \to a} f(x)$ must exist ③ $\lim_{x \to a} f(x) = f(a)$

Sketch a function with domain $(\infty, -1) \cup (-1, \infty)$ that has a removable discontinuity at $x = -1$, an infinite discontinuity at $x = 0$, and a jump discontinuity $x = 1$.

one of these must be filled in as the domain includes $x=1$



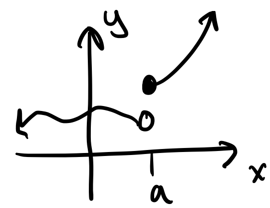
need this as $x=0$ is in domain

GOALS: In this lesson, we will practice using the definition of continuity, define right- and left-continuity, and learn (& apply) several very powerful theorems concerning continuous functions.

DEFINITION: A function $f(x)$ is **continuous from the right** at the number $x = a$ if

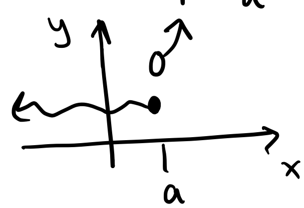
$$\boxed{\lim_{x \to a^+} f(x) = f(a)},$$

picture



A function $f(x)$ is **continuous from the left** at the number $x = a$ if

$$\boxed{\lim_{x \to a^-} f(x) = f(a)},$$



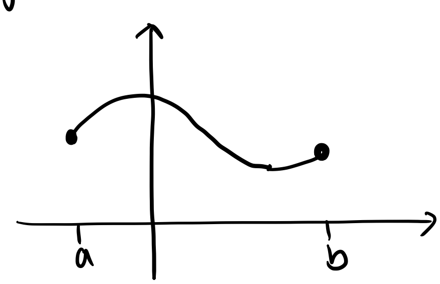
QUESTION: Look at your picture above and determine all the a-values for which your function is continuous from the right and those for which your function is continuous from the left.

- The function is continuous from the left at $x=1$.
- Everywhere the function is continuous (ie. $(-\infty,-1) \cup (-1,0) \cup (0,1) \cup (1,\infty)$) the function is continuous from the right & left.

QUESTION: Why would we want one-sided continuity?

So we can say $f(x)$ is continuous on an interval $[a, b]$.

The function $f(x)$ is continuous on $[a, b]$ if it is continuous on (a,b) and left continuous at $x=b$, right continuous at $x=a$.



QUESTION: Assume $f(x)$ and $g(x)$ are BOTH continuous at $x = a$, what do you think should be true about the new function $H(x) = f(x) + g(x)$ and how would you JUSTIFY your intuition?

It seems like $H(x)$ should be continuous.

Why: $\lim_{x\to a} H(x) = \lim_{x\to a} (f(x)+g(x))$ (def. of $H(x)$)

$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ (limit laws)

$= f(a) + g(a)$ (f + g are continuous at $x=a$)

$= H(a)$

Thus $\lim_{x\to a} H(x) = H(a)$. This mean $H(x)$ is continuous at $x=a$.

THEOREMS 4, 5, AND 7 (as numbered in your textbook) all tell us that a large family of familiar functions are continuous. Below we will list this collection. The numbering aligns with the textbook theorem.

4.1 $f(x)+g(x)$

4.2 $f(x)-g(x)$

4.3 $c\,f(x)$

4.4 $f(x)\cdot g(x)$

4.5 $f(x)/g(x)$ provided $g(a)\neq 0$

5a all polynomials are continuous everywhere: $\mathbb{R}$ or $(-\infty,\infty)$

5b All rational functions are continuous where they are defined. watch out for where the denominator $=0$!

7 roots, trig functions, inverse trig functions, exponential and logarithmic functions are continuous where defined

These mean you must find the DOMAIN

PRACTICE PROBLEMS:

1. Determine the intervals over which the function $f(x) = \frac{3e^x+\tan x}{5x}$ is continuous and *justify* your answer using the Theorems above.

Note $\boxed{7}$ tells us $y=3e^x$ and $y=\tan x$ are continuous so long as they are defined. We know $y=3e^x$ has domain $\mathbb{R}$, and $y=\tan x$ is continuous if $x = \frac{\pi}{2}(2k+1)$ [ie $\pm\pi/2, \pm 3\pi/2, ..$]. We also must leave out $x=0$.

So f is continuous on $\boxed{\ldots(-3\pi/2,-\pi/2)\cup(-\pi/2,0)\cup(0,\pi/2)\cup(\pi/2,3\pi/2)\ldots}$.

2. Evaluate $\lim_{x\to\pi/4} \frac{3e^x + \tan x}{5x}$ and *justify* your strategy.

$$\lim_{x\to\pi/4}\left(\frac{3e^x+\tan x}{5x}\right) = \frac{3e^{\pi/4}+\tan(\pi/4)}{(5\pi/4)}$$

$$= \boxed{\frac{4}{5\pi}\left(3e^{\pi/4}+1\right)}$$

as $f(x)$ is continuous at $x=\pi/4$, we can plug in because of the direct substitution property!

THEOREMS 8 AND 9 (as numbered in your textbook) tell us that continuity is preserved by function *composition* **provided** the resulting function is defined.

EXAMPLE: Determine all x-values for which the function $f(x) = \ln\left(\frac{1}{x} - 1\right)$ is continuous.

f(x) is defined if: $x \neq 0$ and $1/x - 1 > 0$

so $1/x > 1 \Rightarrow 1 > x$

$\Rightarrow x < 1$ (leave out 0)

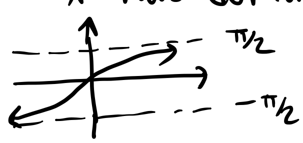
$\boxed{\text{so } f \text{ is continuous on } (-\infty, 0) \cup (0, 1)}$

PRACTICE PROBLEMS:

1. Determine the domain of the function $g(r) = \tan^{-1}(1 + e^{-r^2})$ and explain why $g(r)$ is continuous at every number in its domain.

• note $y = \tan^{-1} x$ has domain $\mathbb{R}$

recall:



• so $1 + e^{-r^2}$ can be any number

• Also $1 + e^{-r^2}$ is always defined.

Domain of $g(r)$ is thus $\mathbb{R}$, and $g(r)$ is continuous for all real numbers because we have a composition of continuous functions.

2. Use continuity to evaluate $\lim_{x \to 4} 3^{\sqrt{x^2 - 2x - 4}}$.

$$\lim_{x \to 4} 3^{\sqrt{x^2-2x-4}} = 3^{\sqrt{4^2 - 2(4) - 4}}$$
$$= 3^{\sqrt{16 - 8 - 4}}$$
$$= 3^{\sqrt{4}}$$
$$= 3^2 = \boxed{9}$$

← when you have a continuous function $f(x)$ & you are finding $\lim_{x \to a} f(x)$ you can just input a!

3. Let $f(x) = 1/x$ and $g(x) = 1/x^2$. (a) Find $(f \circ g)(x)$. (b) Explain why $f \circ g$ is not continuous everywhere.

$$(f \circ g)(x) = f(g(x))$$
$$= f(1/x^2)$$
$$= 1/(1/x^2)$$
$$= x^2, \ x \neq 0$$

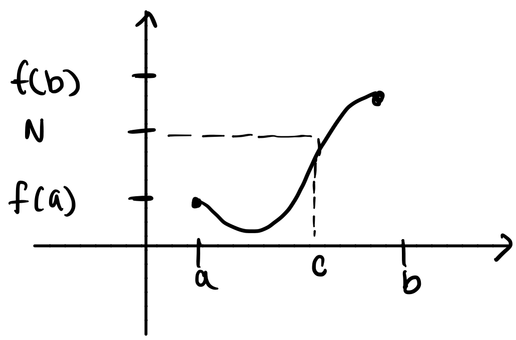
but $x \neq 0$ because $g(x)$ is not defined at $x = 0$.

Thus g is continuous on $(-\infty, 0) \cup (0, \infty)$.

$\boxed{\text{THE INTERMEDIATE VALUE THEOREM:}}$ Suppose $f(x)$ is a function such that

- $f(x)$ is continuous on $[a, b]$,
- $f(a) \neq f(b)$, and
- N is a number between $f(a)$ and $f(b)$,

then, $\longrightarrow$ there exists some number c (maybe more than one!) in the open interval (a,b) so that f(c) = N.



Note: to use this theorem you must show (check!) that the three bullet points after the "if" hold.

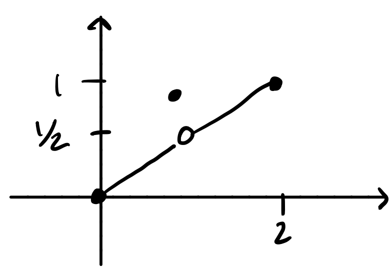
$\boxed{\text{PRACTICE PROBLEMS:}}$

1. Use the Intermediate Value Theorem to show that the equation $x^4 + x - 3 = 0$ must have a root in the interval $(1, 2)$.

showing our equation has a root is the same as showing f(x)=0.

Let $f(x) = x^4 + x - 3$. Note $f(x)$ is continuous on $[1,2]$.
Also $f(1) = 1 + 1 - 3 = -1 < 0$ and $f(2) = 16 + 2 - 3 = 15 > 0$.
Taking $N = 0$ we know that there is a c in $(1,2)$ so that $f(c) = 0$. Thus the equation has a solution in $(1,2)$.

2. Give an example of a function $f(x)$ that is defined for every number in the interval $[0, 2]$ such that $f(0) = 0$, $f(2) = 1$ but there does not exist a single x-value in the interval $(0, 2)$ such that $f(x) = 1/2$.



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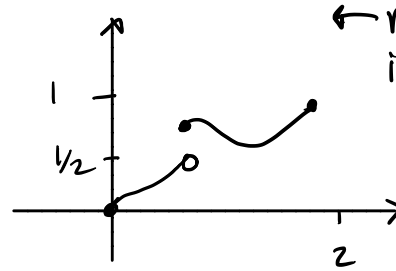

← note, f(x) is not continuous!