

Name: Solutions**Rules:**

You have 90 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten 3×5 notecard.

Calculators are not allowed.

Place a box around your **FINAL ANSWER** to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	16	
3	8	
4	8	
5	14	
6	12	
7	15	
8	15	
Extra Credit	5	
Total	100	

1. (12 points)

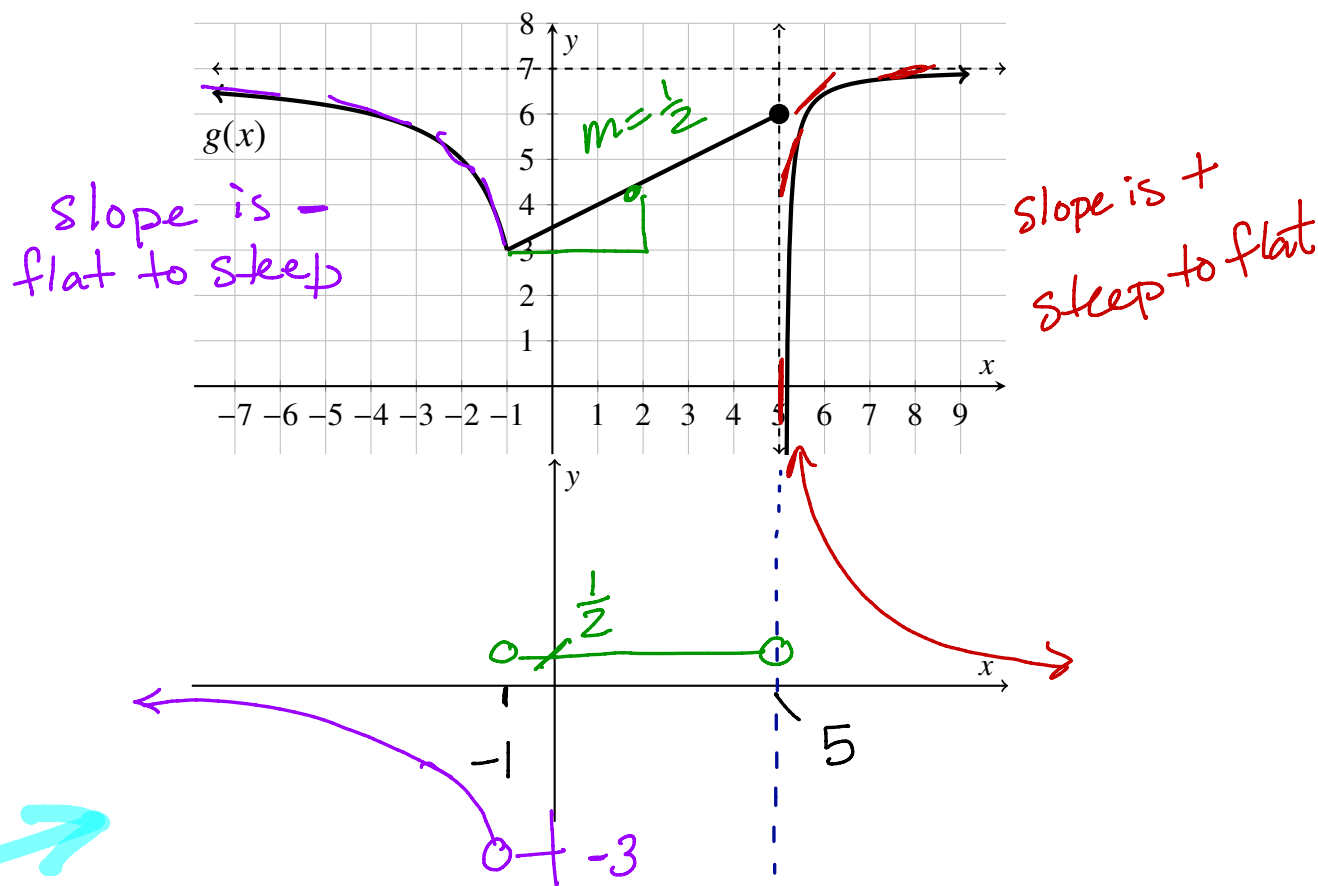
(a) State the definition of $f'(x)$, the derivative of the function $f(x)$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

(b) Find the derivative of $f(x) = \sqrt{5-x}$ using the limit definition of the derivative. No credit will be awarded for using other methods.

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{5-(x+h)} - \sqrt{5-x}}{h} \\
 &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{5-(x+h)} - \sqrt{5-x}}{(h)} \right) \left(\frac{\sqrt{5-(x+h)} + \sqrt{5-x}}{\sqrt{5-(x+h)} + \sqrt{5-x}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{5-x-h-(5-x)}{h(\sqrt{5-(x+h)} + \sqrt{5-x})} = \lim_{h \rightarrow 0} \frac{-h}{h(\sqrt{5-x-h} + \sqrt{5-x})} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{5-x-h} + \sqrt{5-x}} = \boxed{\frac{-1}{2\sqrt{5-x}}}
 \end{aligned}$$

2. (16 points) Use the graph of $g(x)$, in the figure below, to answer the questions (a)-(g). The dashed lines in the figure represent asymptotes of the graph of $g(x)$.



(a) $\lim_{x \rightarrow 5^-} g(x) = 6$

(c) $g(5) = 6$

(b) $\lim_{x \rightarrow 5^+} g(x) = -\infty$

(d) $\lim_{x \rightarrow -1} g(x) = 3$

- (e) Sketch the graph of $g'(x)$ on the axes below the graph $g(x)$. Label important points on the x and y axes.

- (f) For what x -values does the function $g(x)$ fail to be continuous?

$$x=5$$

- (g) For what x -values does the **derivative** of $g(x)$, $g'(x)$, fail to be continuous?

$$x=-1, x=5$$

3. (8 points) Does the function $K(x) = \frac{13}{x-6}$ have any vertical asymptotes? **Justify your answer.** Your justification will require both a limit calculation and a sentence of explanation.

Answer : vertical asymptote at $x=6$

Justification : $\lim_{x \rightarrow 6^+} \frac{1}{x-6} = \infty$ since as $x \rightarrow 6^+$
 $x-6 \rightarrow 0^+$ and $13 > 0$.

4. (8 points) Find any x -values where the graph of $f(x) = \frac{1}{9}x + x^{-1}$ has a horizontal tangent line or explain why none exist.

$$f'(x) = \frac{1}{9} - x^{-2}$$

horizontal tangent means $f'(x) = 0$.

$$\text{So set } \frac{1}{9} - x^{-2} = 0$$

$$\frac{1}{9} = \frac{1}{x^2}$$

$$\boxed{\text{So } x = \pm 3}$$

5. (14 points) For 12 minutes of an experiment, the temperature T in degrees Celsius of a pan of water is modeled by the equation

$$T(x) = 11 - 4x + x^2$$

where x is measured in minutes.

- (a) Calculate $T(1)$ and interpret this value in the context of the problem. Your answer should be a sentence and it should include units.

$$T(1) = 11 - 4 + 1 = 8^\circ\text{C}$$

After one minute of the experiment, the temperature of the water is 8°C .

- (b) Find the average rate of change of the temperature between $x = 1$ and $x = 4$. Your answer should be a sentence and it should include units.

$$T(4) = 11 - 16 + 16 = 11, \text{ avg. rate of change} = \frac{T(4) - T(1)}{4 - 1} = \frac{11 - 8}{3} = \frac{3}{3} = 1^\circ\text{C/min}$$

The average rate of change of temperature between the first and fourth minute was 1°C/min .

- (c) Find $T'(1)$ and include units with your answer.

$$T'(x) = -4 + 2x, \quad T'(1) = -4 + 2 \cdot 1 = -2^\circ\text{C/min}$$

- (d) Explain in simple terms what your calculation in part (c) indicates in the context of the problem. Your answer should be a sentence and it should include units.

At the instant 1 minute has passed, the temperature of the water is decreasing at a rate of 2°C per minute.

- (e) Write an equation of the line tangent to the graph of $T(x)$ when $x = 1$.

$$m = -2$$

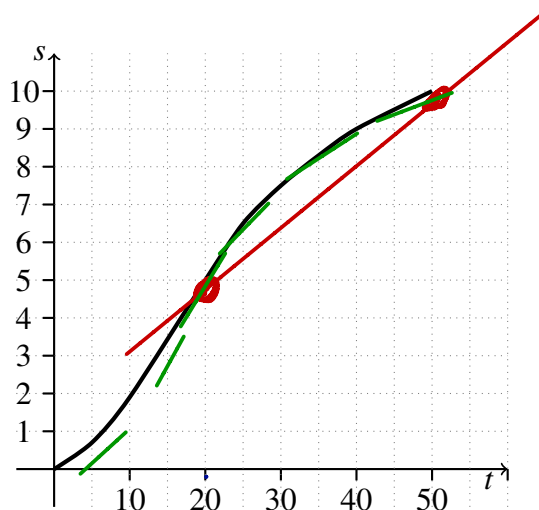
$$\text{point: } (1, 8)$$

$$\text{line: } y - 8 = -2(x - 1) \quad \text{or}$$

$$y = 8 - 2(x - 1) \quad \text{or}$$

$$y = 10 - 2x$$

6. (12 points) The graph in the figure below models the position, $s(t)$, of a runner along a 10 km trail where s is measured in kilometers and t is measured in minutes.



- (a) Draw the secant line between the points when $t = 20$ to $t = 50$ and find the slope of this line.

$$m_{\text{sec}} = \frac{10-5}{50-20} = \frac{5}{30} = \frac{1}{15}$$

- (b) What does the slope from part (a) represent in the context of the problem? Include units with your answer.

In the last half of the run (from 5 km to 10 km) the average velocity (or average speed) of the runner was $\frac{1}{15}$ km/min.

- (c) Describe what $s'(t)$, the derivative of $s(t)$, represents in the context of the problem and describe how it behaves as t increases.

$s'(t)$ is the instantaneous velocity (or speed) of the runner. The velocity of the runner increases at first and then decreases to the end of the run.

- (d) Estimate when the runner was running the fastest and explain how you drew this conclusion.

I estimate that the runner was the fastest around 20 minutes into the run. This is where the tangent to the graph appears to be the steepest.

7. (15 points) Evaluate the following limits. Show your work to earn full credit. Be careful to use proper notation.

(a) $\lim_{x \rightarrow 2} \frac{x^2 - x - 2}{2x^2 - x - 6} = \lim_{x \rightarrow 2} \frac{(x-2)(x+1)}{(x-2)(2x+3)} = \lim_{x \rightarrow 2} \frac{x+1}{2x+3} = \frac{2+1}{4+3} = \frac{3}{7}$

plug in $\frac{4-2-2}{8-2-6} = \frac{0}{0}$
 try algebra factor

(b) $\lim_{x \rightarrow 4^-} \frac{1+x}{\sqrt{x}+1} = \frac{1+4}{\sqrt{4}+1} = \frac{5}{3}$

plug in

(c) $\lim_{h \rightarrow 0} \frac{\frac{5}{x+h} - \frac{5}{x}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{5}{x+h} - \frac{5}{x} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{5x - 5(x+h)}{(x+h)(x)} \right)$

plug in $\frac{0}{0}$

Do algebra.
 Common denominator

$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-5h}{(x+h)(x)} \right) = \lim_{h \rightarrow 0} \frac{-5}{(x+h)(x)}$

$= \frac{-5}{(x+0)(x)} = \frac{-5}{x^2} = -5x^{-2}$

8. (15 points) Find the derivative of each function below. You do not need to simplify your answer. Be careful to appropriately parenthesize your answer.

(a) $f(x) = 6x^{1/2} - 8x^2 + \pi^{1/2}$

$$\begin{aligned} f'(x) &= 6 \cdot \frac{1}{2} x^{-1/2} - 8 \cdot 2 \cdot x^1 + 0 \\ &= 3x^{-1/2} - 16x \end{aligned}$$

(b) $g(x) = x^4 \sin(x)$

$$\begin{aligned} g'(x) &= (4x^3)(\sin(x)) + (x^4)(\cos(x)) \\ &= 4x^3 \sin(x) + x^4 \cos(x) \end{aligned}$$

(c) $H(q) = \frac{q+3}{q^3+4q-2}$

$$H'(q) = \frac{(q^3+4q-2)(1) - (q+3)(3q^2+4)}{(q^3+4q-2)^2}$$

Extra Credit: (5 points) Use a theorem we have learned about in class to prove that there exists a solution to the equation $2^x = -2x$. A complete answer requires some calculations **and** complete sentences justifying your conclusion.

If $2^x = -2x$ then $2^x + 2x = 0$.

Pick $f(x) = 2^x + 2x$. This function is continuous

so the Intermediate Value Theorem applies.

Observe $f(0) = 2^0 + 2 \cdot 0 = 1 + 0 = 1$ and $f(-1) = 2^{-1} + 2(-1) = -1.5$.

Since $f < 0$ when $x = -1$ and $f > 0$ when $x = 0$, the Intermediate value theorem implies that $f(x) = 0$ for some x between $x = -1$ and $x = 0$.⁸