Your Name

Solutions

Your Signature

Instructor Name

End Time

Desk Number

- The total time allowed for this exam is 60 minutes.
- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question** where appropriate.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

This exam is printed double-sided.

There are problems on both sides of the page!

If you need more space, you may use extra sheets of paper. If you use extra pages:

- Put your name on each extra sheet
- Label your work with the problem you're working on
- Write on the exam problem that there is additional work at the end
- Turn in your additional pages at the end of your exam.

$\boxed{1}$ (14 points) Evaluate the following limits. Justify your answers with words and/or any relevant algebra. Be sure to use proper notation, as points will be deducted for not doing so.

(a) $\displaystyle\lim_{x\to -5} \frac{x^2-25}{x^2+2x-15}$

$$= \lim_{x \to -5} \frac{(x-5)(x+5)}{(x+5)(x-3)}$$

$$= \lim_{x \to -5} \frac{x-5}{x-3}$$

$$= \frac{-10}{-8} = \frac{5}{4}$$

(b) $\displaystyle\lim_{t\to -3} \frac{6+4t}{t^2+1}$

$$= \frac{6+4(-3)}{(-3)^2+1}$$

$$= \frac{6-12}{9+1}$$

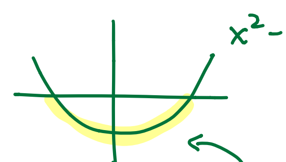
$$= \frac{-6}{10} = \frac{-3}{5}$$

(c) (i) $\displaystyle\lim_{x\to -1^-} \sqrt{x^2-1}$

$$= \sqrt{(-1)^2-1}$$

$$= \sqrt{0}$$

$$= 0$$



(ii) Why do we not evaluate $\displaystyle\lim_{x\to -1^+} \sqrt{x^2-1}$? Explain using a sentence.

The function $f(x) = \sqrt{x^2-1} = \sqrt{(x-1)(x+1)}$ is not defined on the interval $(-1, 1)$

$\boxed{2}$ (10 points) Evaluate the following limits. Justify your answers with words and/ or any relevant algebra. Be sure to use proper notation, as points will be deducted for not doing so.

(a) $\displaystyle\lim_{x\to 1} \frac{x-1}{\sqrt{2x-1}-1}$

$$= \lim_{x\to 1} \frac{x-1}{(\sqrt{2x-1}-1)} \cdot \frac{(\sqrt{2x-1}+1)}{(\sqrt{2x-1}+1)}$$

$$= \lim_{x\to 1} \frac{(x-1)(\sqrt{2x-1}+1)}{2x-1-1}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(\sqrt{2x-1}+1)}{2\cancel{(x-1)}}$$

$$= \lim_{x\to 1} \frac{\sqrt{2x-1}+1}{2} = 1$$

(b) $\displaystyle\lim_{x\to\infty} \sin\left(\frac{\pi x^2-3x}{6x^2-4x+7}\right)$

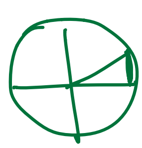
Note $\sin(x)$ is a continuous function!

$$= \sin\left(\lim_{x\to\infty}\left(\frac{\pi x^2-3x}{6x^2-4x+7}\right)\frac{1/x^2}{1/x^2}\right)$$

$$= \sin\left(\lim_{x\to\infty}\frac{\pi-3/x}{6-4/x+7/x^2}\right)$$

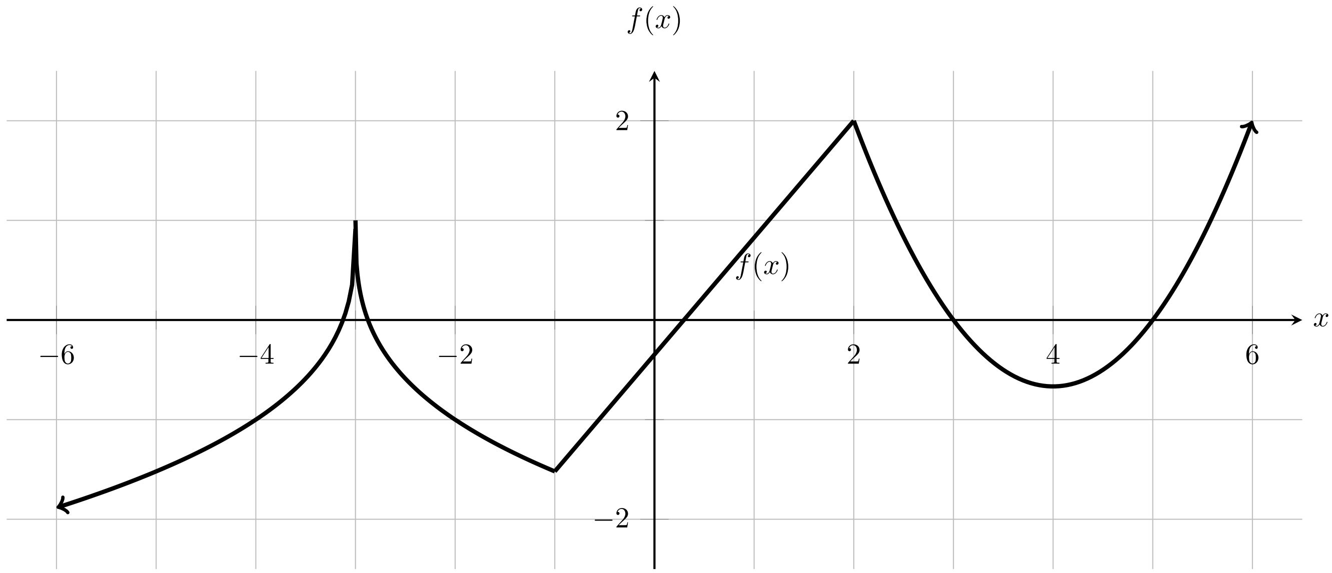
$$= \sin(\pi/6)$$

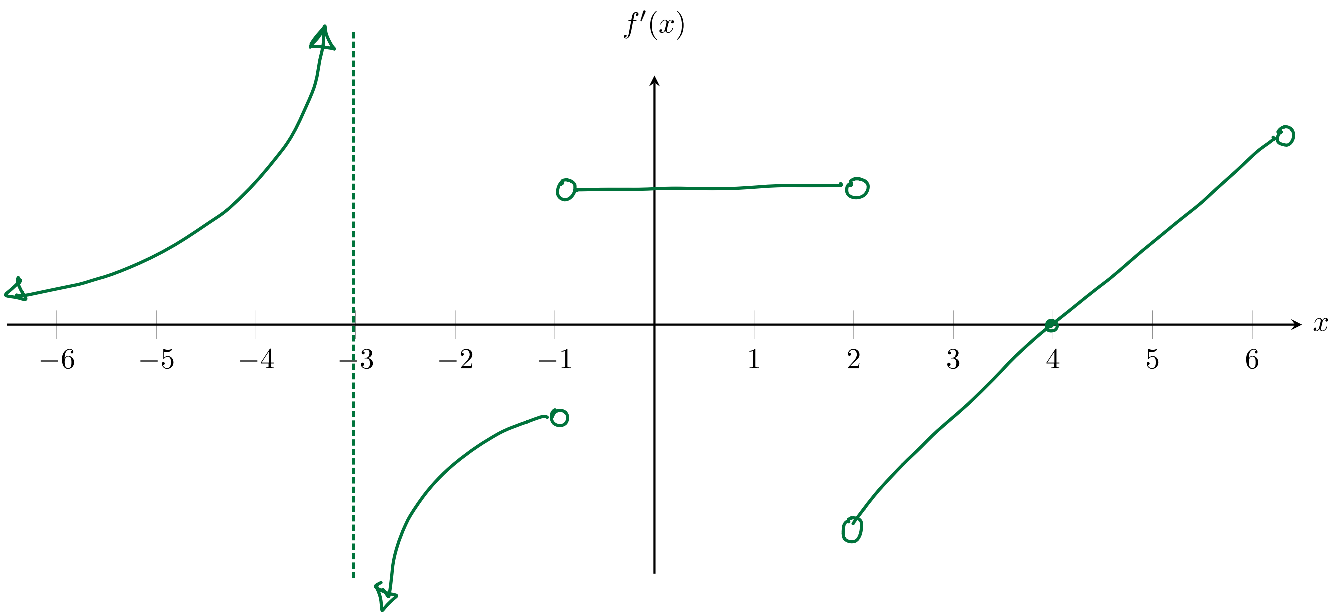
$$= 1/2$$



$\boxed{3}$ (14 points)

(a) The graph of $f(x)$ is shown on the top set of axes. Sketch the graph of $f'(x)$ on the second set of axes.





(b) What is the domain of $f'(x)$? Write your answer using interval notation.

Domain: $(-\infty, -3) \cup (-3, -1) \cup (-1, 2) \cup (2, \infty)$

$\boxed{4}$ (10 points) In the first few years after a coal mine's operation, the total deposit of coal (in millions of tons) t years after opening is approximately

$$C(t) = 400 - \frac{t^{3/2}}{2}.$$

(a) Find the average rate of change of the amount of coal in the deposit from the opening of the mine to year 4. Include correct units in your answer.

average rate of change $= \dfrac{C(4) - C(0)}{4 - 0} = \dfrac{\left(400 - \frac{4^{3/2}}{2}\right) - \left(400 - \frac{0^{3/2}}{2}\right)}{4}$

$= \frac{1}{8}\left(200 - 8 - 200 + 0\right)$

$= -1$ millions of tons / year

(b) It is a fact that $C'(t) = -\frac{3}{4}\sqrt{t}$. Compute $C'(4)$ and indicate what this quantity tells us about the mine. Write your answer in a sentence. Again, include correct units in your description.

$C'(4) = -\frac{3}{4}\sqrt{4} = \frac{-3}{4} \cdot 2 = -3/2$

At the end of the 4^{th} year, coal production is decreasing by 3/2 million tons/year.

$\boxed{5}$ (14 points) Let $g(x) = \frac{3x^2+6x}{x^2-4}$. $= \frac{3x^2+6x}{(x-2)(x+2)} = \frac{3x(x+2)}{(x-2)(x+2)}$

(a) What is the domain of g? Write your answer using interval notation.

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(b) Use limits to determine all **vertical** asymptotes of $g(x)$. Show your work clearly and justify your conclusion using limits. Write the equations of the vertical asymptote(s) in the space provided; if none exist write DNE.

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} \frac{3x(x+2)}{(x-2)(x+2)} = \lim_{x \to -2^-} \frac{3x}{x-2} = \frac{-6}{-4}$$

$\to$ That is, $x = -2$ is a hole!

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} \frac{3x^2+6x \xrightarrow{} 12+12=24}{x^2-4 \xrightarrow{} 0^+} = \infty$$

Equation(s) of vertical asymptote(s): $x = 2$ is a vertical asymptote

(c) Use limits to determine all **horizontal** asymptotes of $g(x)$. Show your work clearly and justify your conclusion using limits. Write the equations of the horizontal asymptote(s) in the space provided; if none exist write DNE.

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{3x^2+6x}{x^2-4} = \lim_{x \to \infty} \frac{3+6/x}{1-4/x^2} = 3$$

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{3x^2+6x}{x^2-4} = \lim_{x \to \infty} \frac{3(-x)^2+6(-x)}{(-x)^2-4}$$

$$= \lim_{x \to \infty} \frac{3x^2-6x}{x^2-4} = \lim_{x \to \infty} \frac{3-6/x}{1-4/x} = 3$$

Equation(s) of horizontal asymptote(s): $y = 3$ is a HA, in both directions

6 (14 points) Let k be the piecewise defined function below.

$$k(x) = \begin{cases} ax + x^3 & x < 1 \\ \ln(x) - 1 & 1 \le x \le e \\ \dfrac{1}{x-1} & x > e \end{cases}$$

(a) Determine $\lim_{x \to e} k(x)$ or explain why it doesn't exist. (As usual, e is Euler's Constant, $e \approx 2.71828$.)

$\lim_{x \to e^-} k(x) = \lim_{x \to e^-} \ln(x) - 1 = \ln(e) - 1 = 0$

$\lim_{x \to e^+} k(x) = \lim_{x \to e^+} \frac{1}{x-1} = \frac{1}{e-1} \neq 0 \quad \left(\frac{1}{e-1} \approx \frac{1}{1.718}\right)$

Since $\lim_{x \to e^-} k(x) \neq \lim_{x \to e^+} k(x)$, the limit does not exist

(b) Determine a value for a such that the function $k(x)$ is continuous at $x = 1$, and write your answer in the space below. Show that your choice for a is correct using the *definition of continuity at a point*. (A correct answer will involve writing and computing an appropriate limit or limits.)

For $k(x)$ to be continuous at $x=1$, we need

$\lim_{x \to 1^-} k(x) = \lim_{x \to 1^+} k(x)$. Note

$\lim_{x \to 1^-} k(x) = \lim_{x \to 1^-} ax + x^3 = a+1$

and $\lim_{x \to 1^+} k(x) = \lim_{x \to 1^+} \ln(x) - 1 = \ln(1) - 1 = -1$.

So we need $a + 1 = -1 \Rightarrow a = -2$.

$a =$ -2

$\boxed{7}$ (14 points) Consider the function

$$f(x) = 2x + \frac{1}{x}.$$

(a) Using the $\boxed{\textbf{definition of the derivative}}$, find $f'(a)$. Show all your steps using correct notation. No credit will be given if a different method is used. [It is recommended you start by writing the definition of the derivative.]

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left(2(a+h) + \frac{1}{a+h}\right) - \left(2a + \frac{1}{a}\right)}{h}$$

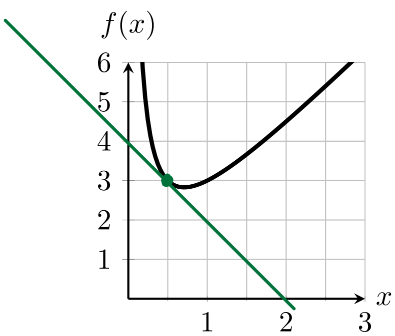
$$= \lim_{h \to 0} \frac{1}{h}\left[(2a + 2h - 2a) + \frac{1}{a+h} - \frac{1}{a}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2h + \frac{a - a - h}{a(a+h)}\right] = \lim_{h \to 0} \frac{1}{h}\left[2h + \frac{-h}{a(a+h)}\right]$$

$$= \lim_{h \to 0} 2 - \frac{1}{a(a+h)}$$

$$= 2 - \frac{1}{a^2}$$

(b) It is a fact that for this function, $f'(\frac{1}{2}) = -2$. Use this fact to write the equation of the tangent line to the curve at the point with $x = \frac{1}{2}$, and sketch the tangent line on the graph.



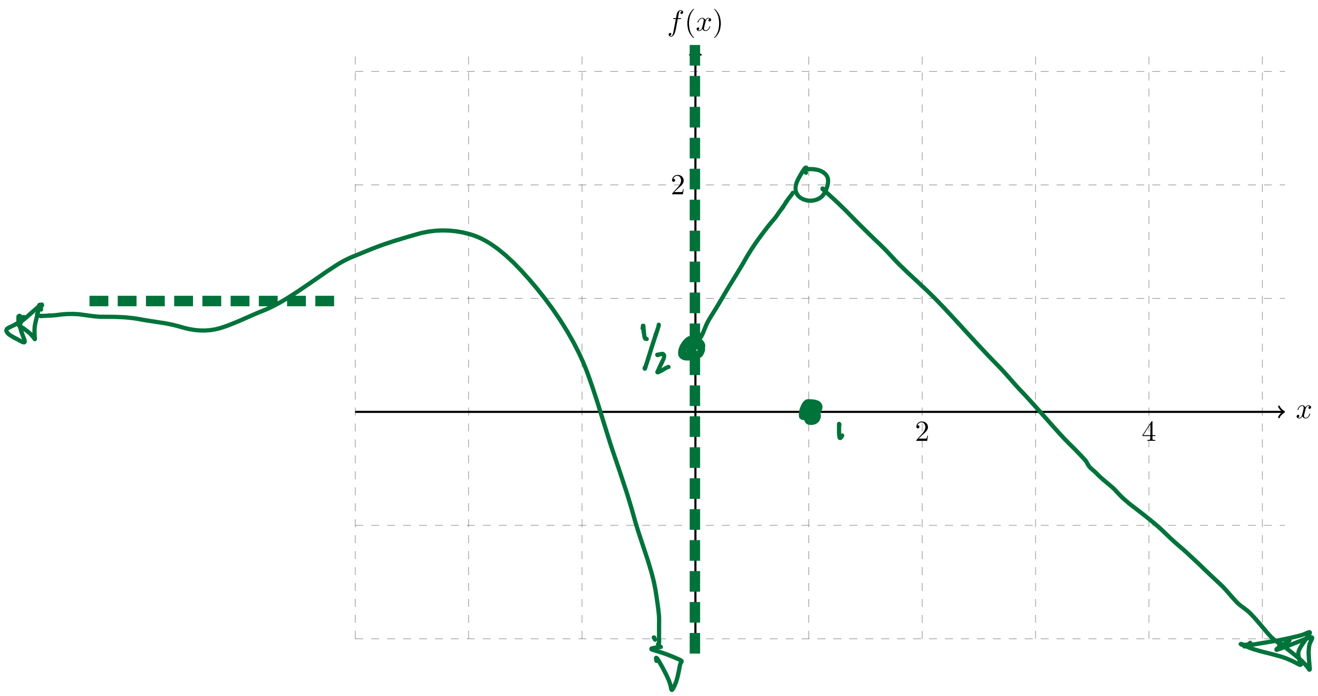
$$f(1/2) = 2(1/2) + \frac{1}{1/2}$$
$$= 1 + 2 = 3$$

Equation of Tangent Line: $y = -2(x - 1/2) + 3$

$f'(1/2)$ $x = 1/2$ $f(1/2)$

8 (10 points) Sketch the graph of a function f that satisfies all of the given conditions. Indicate any asymptotes using dashed lines.

$\lim_{x\to-\infty} f(x) = 1$	$\lim_{x\to 0^-} f(x) = -\infty$	$\lim_{x\to 1} f(x) = 2$	$\lim_{x\to\infty} f(x) = -\infty$
	$f(0) = 1/2$	$f(1) = 0$	
	$\lim_{x\to 0^+} f(x) = 1/2$		



(There are lots of valid ways to sketch this graph.)

$\boxed{\text{Extra Credit}}$ (5 points) You may choose only ONE of the following two problems. Clearly mark which one you want graded.

$\boxed{\text{EC I}}$ Grade This One □

Show that $\lim_{x \to 0} x^2(1 + \sin(1/x)) = 0$. You must clearly explain your work and cite any relevant theorems for full credit.

Observe that since $-1 \leq \sin(\text{blah}) \leq 1$,

$$-1 \leq \sin(1/x) \leq 1 \implies$$

$$0 \leq 1 + \sin(1/x) \leq 2 \implies$$

$$0 \leq x^2(1+\sin(1/x)) \leq 2 \cdot x^2$$

So $\lim_{x \to 0} 0 \leq \lim_{x\to 0} x^2(1+\sin(1/x)) \leq \lim_{x \to 0} 2x^2 = 0$

Therefore by the squeeze theorem, $\lim_{x\to 0} x^2(1+\sin(1/x)) = 0$.

$\boxed{\text{EC II}}$ Grade This One □

Evaluate $\lim_{x \to -\infty} \arctan\left(x^2 - \frac{2x^3}{3\sqrt{1+x^4}}\right)$. Your answer must be preceded by relevant steps and correct notation.

$$= \arctan\left(\lim_{x\to\infty} x^2 - \frac{2x^3}{3\sqrt{1+x^4}}\right)$$

Observe $\arctan(x)$ is continuous, so we can pull it through the limit.

$$= \arctan\left(\lim_{x\to\infty} \frac{3x^2\sqrt{1+x^4} - 2x^3}{3\sqrt{1+x^4}}\right)$$

$$= \arctan\left(\lim_{x\to\infty} \frac{x^2(3\sqrt{1+x^4} - 2x)}{3\sqrt{1+x^4}} \cdot \frac{1/x^2}{1/x^2}\right)$$

Note $1/x^2 = \frac{1}{\sqrt{x^4}}$

$$= \arctan\left(\lim_{x\to\infty} \frac{3\sqrt{x^4+1} - \cancel{2/x}^{\to 0}}{3\sqrt{\cancel{1/x^4}^{\to 0}+1}}\right)$$

$$= \arctan\left(\frac{\left[\lim_{x\to\infty} (3\sqrt{x^4+1})\right]^{\to \infty} - 0}{3}\right) = \pi/2$$