

SOLUTIONS

Name: _____

Midterm Exam 2

No book, electronics, calculator, or internet access. Only “Summary of Convergence Tests” notes allowed. 100 points possible. 70 minutes.

1. (5 pts) Verify that $y = e^{2x^2}$ is a solution to the differential equation $y' - 4xy = 0$.

$$y' = e^{2x^2} \cdot 4x$$

$$y' - 4xy = e^{2x^2} \cdot 4x - 4x \cdot e^{2x^2} = 0 \checkmark$$

2. Compute and simplify the improper integrals, or show they diverge. Use correct limit notation.

$$(a) (5 \text{ pts}) \quad \int_0^1 \frac{dx}{x^3} = \lim_{t \rightarrow 0^+} \int_t^1 x^{-3} dx = \lim_{t \rightarrow \infty} \left[\frac{-x^{-2}}{2} \right]_t^1$$

$$= \lim_{t \rightarrow 0^+} \left[-\frac{1}{2} + \frac{1}{2t^2} \right] = +\infty \quad \text{so } \text{diverges}$$

$$(b) (5 \text{ pts}) \quad \int_1^\infty 2xe^{-x^2} dx = \lim_{t \rightarrow \infty} \int_1^t 2x e^{-x^2} dx \quad \begin{matrix} u = x^2 \\ du = 2x dx \end{matrix}$$

$$= \lim_{t \rightarrow \infty} \int_1^{t^2} e^{-u} du = \lim_{t \rightarrow \infty} [-e^{-u}]_{t^2}^1$$

$$= \lim_{t \rightarrow \infty} [-e^{-t^2} + e^{-1}] = 0 + e^{-1} = \textcircled{C}$$

3. Do the following series converge or diverge? Show your work, including naming any test you use.

(a) (5 pts) $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

$$\frac{n+1}{n^2} \geq \frac{n}{n^2} = \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

so

diverges

by comparison test

(b) (5 pts) $\sum_{n=1}^{\infty} \frac{2n+1}{5n-1}$

$$\lim_{n \rightarrow \infty} \frac{2n+1}{5n-1} \stackrel{L'H}{=} \frac{2}{5} \neq 0$$

diverges

by divergence test

(c) (5 pts) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n}}$

$$b_n = \frac{1}{\sqrt{2n}} \cdot \lim_{n \rightarrow \infty} b_n = 0$$

b_n decreases

so

Converges

by AST

$$(d) \ (5 \ pts) \quad \sum_{n=0}^{\infty} \frac{2^n n!}{(n+2)!} = \sum_{n=0}^{\infty} \frac{2^n n!}{(n+2)(n+1)n!} = \sum_{n=0}^{\infty} \frac{2^n}{(n+2)(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{2^n}{(n+2)(n+1)} \stackrel{L'H}{\rightarrow} \lim_{n \rightarrow \infty} \frac{(2n)^2 2^n}{2} = +\infty$$

diverges

by divergence test

$$(e) \ (5 \ pts) \quad \sum_{n=0}^{\infty} \left(\frac{n+3}{2n-1} \right)^n$$

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n+3}{2n-1} \right)^n} = \lim_{n \rightarrow \infty} \frac{n+3}{2n-1}$$

$$\stackrel{L'H}{=} \frac{1}{2} < 1$$

Converges

by root test

4. (5 pts) Does the following series converge or diverge? Show your work, including naming any test you use. (Hint. Previous problem? Or another test?)

$$\sum_{n=1}^{\infty} \frac{2n}{e^{(n^2)}}$$

integral test using Z(b) :

$$\int_1^{\infty} 2x e^{-x^2} dx = \frac{1}{e} \text{ converges}$$

so series converges

5. (5 pts) Compute and simplify the value of the infinite series $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{n+1}$.

geometric series

$$= \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \dots$$

$$\underline{\text{so}} \quad a = \left(\frac{2}{5}\right)^2 = \frac{4}{25}, \quad r = \frac{2}{5} < 1 \quad (\text{converges})$$

$$\underline{\text{So}}: \quad \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{n+1} = \frac{\frac{4}{25}}{1 - \frac{2}{5}} = \frac{\frac{4}{25}}{\frac{3}{5}} = \frac{4}{25} \cdot \frac{5}{3}$$

$$= \frac{4}{15}$$

6. Consider the infinite series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots$

(a) (5 pts) Write the series using sigma (\sum) notation.

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

(b) (5 pts) Compute and simplify S_3 , the partial sum of the first three terms.

$$S_3 = 1 - \frac{1}{4} + \frac{1}{9} = \frac{36 - 9 + 4}{36} = \frac{31}{36}$$

(c) (5 pts) Does this series converge absolutely, conditionally, or neither (diverge)? Show your work, identify any test(s) used, and circle one answer.

$$\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad p=2 \text{ series converges}$$

CONVERGES ABSOLUTELY

CONVERGES CONDITIONALLY

DIVERGES

7. Use the well known geometric series $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ to find power series representations for the following functions. Show your work.

(a) (5 pts) $\frac{1}{1+x^3}$

$$r = -x^3$$



$$\frac{1}{1+x^3} =$$

$$\sum_{n=0}^{\infty} (-x^3)^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^{3n}}$$

(b) (7 pts) $\ln(1+x)$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = \int \frac{1}{1+x} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}$$

$$x=0: 0 = \ln 1 = C + 0 \therefore C=0$$

$$\ln(1+x) =$$

$$\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}}$$

8. (7 pts) If $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$, find a power series representation for $f'(5x)$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{\sqrt{n}} = \sum_{n=1}^{\infty} \sqrt{n} x^{n-1}$$

any of these

$$f'(5x) = \sum_{n=1}^{\infty} \sqrt{n} (5x)^{n-1} = \sum_{m=0}^{\infty} \sqrt{m+1} (5x)^m$$

$$= \sum_{n=0}^{\infty} \sqrt{n+1} 5^n x^n$$

9. Find the interval of convergence of the following power series.

(a) (8 pts) $\sum_{n=1}^{\infty} \frac{2^n x^n}{n!}$

ratio test

$$\rho = \lim_{n \rightarrow \infty} \frac{\frac{2^{n+1} |x|^{n+1}}{(n+1)!}}{\frac{2^n |x|^n}{n!}} = \lim_{n \rightarrow \infty} \frac{2^{n+1} |x|^{n+1}}{(n+1)! 2^n |x|^n}$$

$$= \lim_{n \rightarrow \infty} \frac{2 |x|}{(n+1)} = 0 < 1$$

$\therefore \boxed{I = (-\infty, \infty)}$ is int. of conv.

(b) (8 pts) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n 3^n}$

$$\rho = \lim_{n \rightarrow \infty} \frac{\frac{|x-1|^{n+1}}{(n+1) 3^{n+1}}}{\frac{|x-1|^n}{n 3^n}} = \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1}}{(n+1) 3^n |x-1|^n}$$

$$= \frac{|x-1|}{3} \lim_{n \rightarrow \infty} \frac{n}{n+1} \stackrel{L'H}{=} \frac{|x-1|}{3} \cdot 1 = \frac{|x-1|}{3} < 1$$

$$\Leftrightarrow -3 < x-1 < 3 \Leftrightarrow -2 < x < 4$$

$x = -2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge AST

$x = 4$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges harmonic

$\boxed{I = [-2, 4]}$

{ is int. of conv.

Extra Credit. (3 pts) The function $f(x) = \arctan(x)$ can be represented by the power series

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Suppose I choose $x = 1/\sqrt{3}$ and compute the partial sum $S_{20} = \sum_{n=0}^{20} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1}$ as an approximation to $\arctan(1/\sqrt{3}) = \frac{\pi}{6}$.

How accurate is this approximation? Use a known fact about remainders of alternating series.

$$\begin{aligned} \frac{\pi}{6} = \arctan\left(\frac{1}{\sqrt{3}}\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1} = S \\ |R_{20}| = |S - S_{20}| &\leq b_{21} = \frac{(1/\sqrt{3})^{43}}{43} = \frac{1}{43 \cdot 3^{43/2}} \\ &= \boxed{\frac{1}{43 \cdot 3^{21.5}}} \\ &\text{Very small!} \end{aligned}$$

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