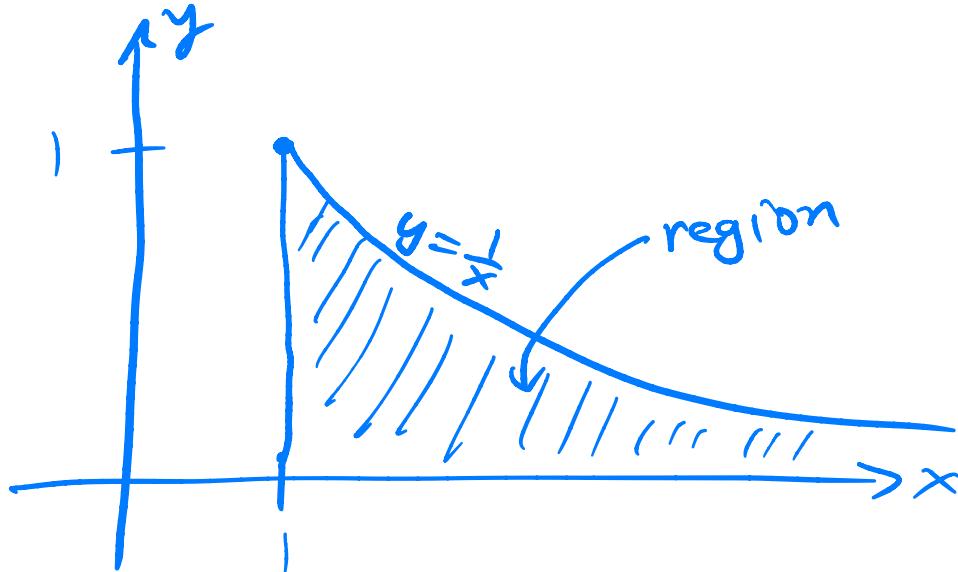


Name: SOLUTIONS

Final Exam

No book, electronics, calculator, or internet access. 125 points possible.
 125 minutes maximum. Allowed notes: 1/2 sheet of letter paper (i.e. half of 8.5×11 sheet), with anything written on both sides.

1. (a) (5 pts) Sketch the region bounded by $y = \frac{1}{x}$ and the x -axis, on the interval $1 \leq x < +\infty$.



- (b) (8 pts) Compute the volume of the solid of revolution found by rotating the region in (a) around the x -axis, or show that it is infinite.

$$\begin{aligned}
 V &= \int_1^\infty \pi y(x)^2 dx = \pi \int_1^\infty \frac{dx}{x^2} \\
 &= \pi \lim_{t \rightarrow \infty} \int_1^t x^{-2} dx = \pi \lim_{t \rightarrow \infty} \left[-x^{-1} \right]_1^t \\
 &= \pi \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right) = \pi(0+1) = \cancel{\pi} \\
 &\quad (\text{converges})
 \end{aligned}$$

2. (7 pts) Set up, but **do not evaluate**, an integral for the arclength of the parametric curve defined by $x(t) = e^t$, $y(t) = \ln t$ on the interval $1 \leq t \leq 10$.

$$L = \int_1^{10} \sqrt{(e^t)^2 + \left(\frac{1}{t}\right)^2} dt$$

$$\left(\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = \frac{1}{t} \right)$$

3. (8 pts) Compute the indefinite integral:

$$\int x 5^{x^2} dx = \int 5^u \frac{du}{2} = \frac{1}{2} \frac{5^u}{\ln 5} + C$$

\uparrow
 $\boxed{u = x^2}$
 $du = 2x dx$

$$= \boxed{\frac{5^{x^2}}{2 \ln 5} + C}$$

4. Evaluate the integrals:

$$(a) (6 \text{ pts}) \int_0^3 x e^{-x} dx = \left[x(-e^{-x}) \right]_0^3 - \int_0^3 (-e^{-x}) dx$$

↑

by parts: $u = x$ $v = -e^{-x}$
 $du = dx$ $dv = e^{-x} dx$

$$= -\left[xe^{-x} \right]_0^3 + \int_0^3 e^{-x} dx = -(3e^{-3} - 0) - [e^{-x}]_0^3$$

$$= -3e^{-3} - e^{-3} + e^{-0} = \boxed{1 - 4e^{-3}}$$

$$(b) (6 \text{ pts}) \int \frac{dx}{x(1+x)} = \int \frac{1}{x} - \frac{1}{1+x} dx$$

$$= \ln|x| - \ln|1+x| + C$$

$$= \ln \left| \frac{x}{1+x} \right| + C$$

partial fractions:

$$\frac{1}{x(1+x)} = \frac{A}{x} + \frac{B}{1+x}$$

$$0x + 1 = A(1+x) + Bx \\ = (A+B)x + A$$

$$A=1, B=-1$$

5. Evaluate the indefinite integrals:

$$(a) (6 \text{ pts}) \quad \int \cos^2 \theta \sin^2 \theta d\theta = \frac{1}{4} \int (1 + \cos 2\theta)(1 - \cos 2\theta) d\theta$$

$$= \frac{1}{4} \int 1 - \cos^2(2\theta) d\theta = \frac{1}{4} \int 1 - \frac{1}{2}(1 + \cos(4\theta)) d\theta$$

$$= \frac{1}{8} \int 1 - \cos(4\theta) d\theta = \frac{1}{8} \left[\theta - \frac{\sin(4\theta)}{4} \right] + C$$

$$= \frac{\theta}{8} - \frac{\sin(4\theta)}{32} + C$$

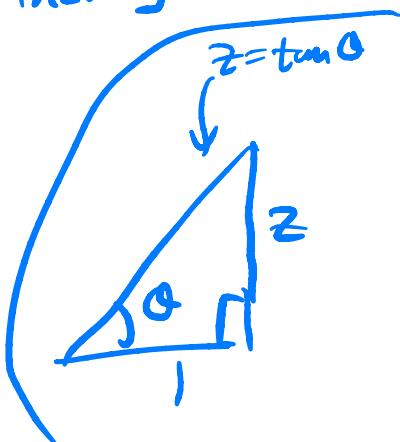
$$(b) (6 \text{ pts}) \quad \int \frac{dz}{\sqrt{1+z^2}} = \int \frac{\sec^2 \theta d\theta}{\sqrt{1+\tan^2 \theta}} = \int \frac{\sec^2 \theta}{\sec \theta} d\theta$$

$$\begin{bmatrix} z = \tan \theta \\ dz = \sec^2 \theta \end{bmatrix}$$

$$= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$

[↑] [from memory of "trick"]

$$= \ln |\sqrt{1+z^2} + z| + C$$



6. Determine whether the following series converge or diverge. Explain your reasoning and identify any test used.

(a) (6 pts) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

$b_n = \frac{1}{\sqrt{n+3}} > 0$, b_n decreasing,

$$\lim_{n \rightarrow \infty} b_n = 0$$

Converges

by A.S.T.

(b) (6 pts) $\sum_{n=1}^{\infty} \frac{n}{10 + \sqrt{n}}$

Divergence test:

$$\lim_{n \rightarrow \infty} \frac{n}{10 + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{n \cdot \frac{1}{\sqrt{n}}}{(10 + \sqrt{n}) \cdot \frac{1}{\sqrt{n}}} =$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\frac{10}{\sqrt{n}} + 1} = \frac{\infty}{0+1} = +\infty$$

\therefore Diverges

or limit comparison: compare to $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\lim_{n \rightarrow \infty} \frac{\frac{n}{10 + \sqrt{n}}}{\frac{n}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{n(\sqrt{n})}{(10 + \sqrt{n})n}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{3/2}}{n^{3/2} + 10n} = 1 \neq 0, \infty$$

p-series with
 $p = -\frac{1}{2} \leq 1$ diverges

7. (a) (8 pts) Find the Maclaurin series by any convenient method:

$$f(x) = x^2 e^{-x^2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{so} \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\text{so} \quad f(x) = x^2 e^{-x^2} = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!} \right)$$

- (b) (8 pts) Use the result in (a) to compute the following antiderivative **as a power series**:

$$\int x^2 e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n+2} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+3}}{2n+3} + C$$

$$(f(0) = x^2 e^{-x^2} \Big|_{x=0} = 0 \quad \text{so} \quad C = 0.)$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)} x^{2n+3} \right)$$

8. (a) (6 pts) Does the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge or diverge? Explain your reasoning and identify any test used.

ratio test: $\rho = \lim_{n \rightarrow \infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}}$

$$= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n+1}{n} \stackrel{\text{L'H}}{=} \frac{1}{2} < 1 \quad \therefore \text{converges}$$

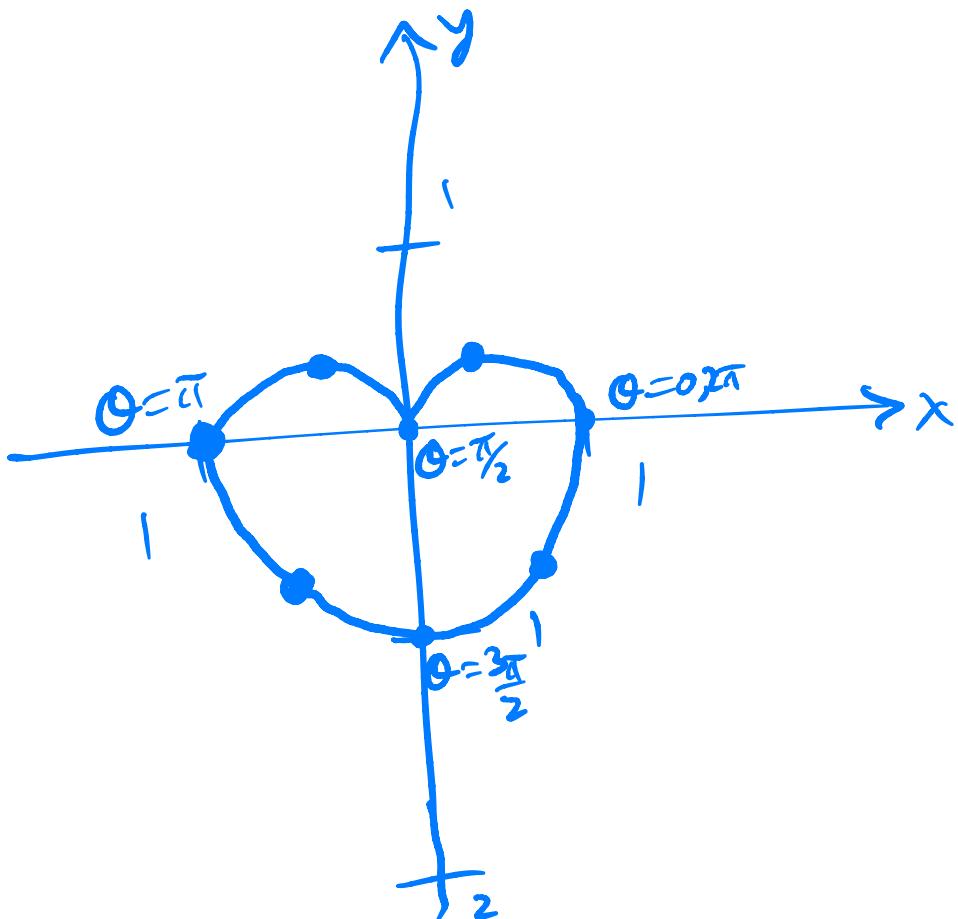
(root test similar)

- (b) (5 pts) Compute and simplify S_3 for the series in part (a).

$$S_3 = \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} = \boxed{\frac{11}{8}}$$

9. (6 pts) Make a careful and reasonably-large sketch of the cardioid $r = 1 - \sin \theta$. (Label the axes and give dimensions/values along the axes.)

r	θ
0	1
$\pi/4$	$1 - \frac{1}{\sqrt{2}} \approx 0.3$
$\pi/2$	0
$3\pi/4$	$1 - \frac{1}{\sqrt{2}} \approx 0.3$
π	1
$5\pi/4$	$1 + \frac{1}{\sqrt{2}} \approx 1.7$
$3\pi/2$	2
$7\pi/4$	$1 + \frac{1}{\sqrt{2}} \approx 1.7$
2π	1



10. Consider the parametric curve $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$.

(a) (6 pts) Find the equation of the tangent line at $t = 1$.

$$m = \left. \frac{dy}{dx} \right|_{t=1} = \left. \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right|_{t=1} = \left. \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} \right|_{t=1} = \frac{2}{0}$$

$$\therefore m = +\infty \quad (\text{vertical})$$

$$x(1) = 1 + \frac{1}{1} = 2$$

$$\therefore \boxed{x = 2}$$

- (b) (6 pts) Compute the second derivative $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{t^2 + 1}{t^2 - 1} \right) = \frac{\frac{2t(t^2 - 1) - (t^2 + 1)(2t)}{(t^2 - 1)^2}}{1 - \frac{1}{t^2}}$$

$$= \frac{2t(t^2 - 1 - t^2 - 1)}{(t^2 - 1)^2 \left(1 - \frac{1}{t^2}\right)} = \frac{2t(-2)}{(t^2 - 1)^2 (t^2 - 1) \frac{1}{t^2}}$$

$$= \boxed{\frac{-4t^3}{(t^2 - 1)^3}}$$

11. (8 pts) Find the radius and interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n^2}$$

root test: $\rho = \lim_{n \rightarrow \infty} \frac{|x+5|}{(\sqrt[n]{n})^2} = \frac{|x+5|}{1} = |x+5| < 1$

$|x+5| < 1 \Leftrightarrow -1 < x+5 < 1 \Leftrightarrow -6 < x < -4$

$x = -6$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges (absolutely) $\rho = 2$

$x = -4$: $\sum_{n=1}^{\infty} \frac{1^n}{n^2}$ converges $\rho = 2$

$\therefore I = [-6, -4]$

12. (8 pts) Find the general solution to the differential equation $y' = \ln x + \tan x$.

$$\begin{aligned} y(x) &= \int \ln x + \tan x \, dx = \int \ln x \, dx + \int \frac{\sin x}{\cos x} \, dx \\ &= x \ln x - \int x \cdot \frac{1}{x} \, dx - \ln |\cos x| + C \\ &\quad \left[\begin{array}{l} u = \ln x \quad v = x \\ du = \frac{1}{x} dx \quad dv = dx \end{array} \right] \end{aligned}$$

$$= x \ln x - x - \ln |\cos x| + C$$

Extra Credit. (3 pts) First, explain in terms of familiar Maclaurin series, why $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

Second, how accurate is $S_9 = \sum_{n=0}^9 \frac{(-1)^n}{n!}$ as an approximation to $1/e$? Write your bound on the error in the box below.

Note $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Substitute $x = -1$ to

get $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. This series is alternating and converges so $|R_N| \leq b_{N+1}$. Here $N = 9$,

$$\text{so } |R_9| \leq b_{10} = \frac{1}{10!}.$$

You may find the following **trigonometric formulas** useful. However, there are other trig. formulas, not listed here, which you should have in memory, or which can be derived from these.

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

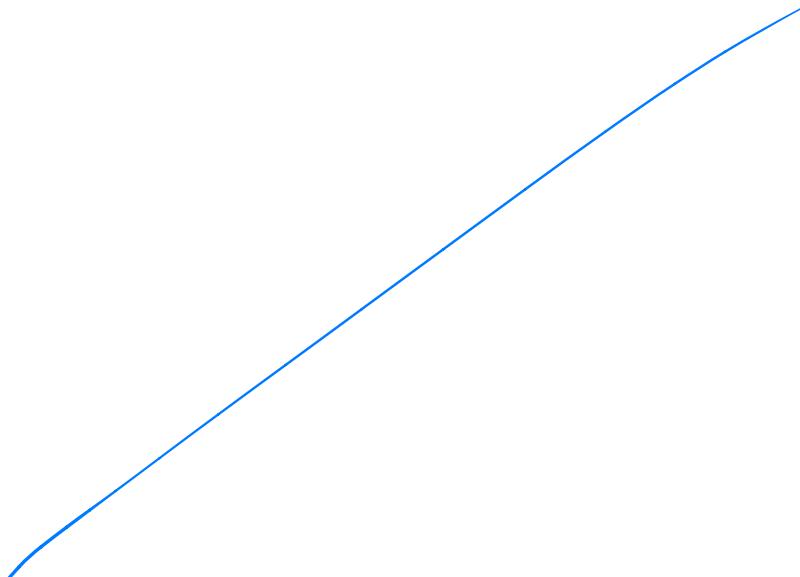
$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\sin(ax) \sin(bx) = \frac{1}{2} \cos((a-b)x) - \frac{1}{2} \cos((a+b)x)$$

$$\sin(ax) \cos(bx) = \frac{1}{2} \sin((a-b)x) + \frac{1}{2} \sin((a+b)x)$$

$$\cos(ax) \cos(bx) = \frac{1}{2} \cos((a-b)x) + \frac{1}{2} \cos((a+b)x)$$

EXTRA SPACE FOR ANSWERS



Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n \rightarrow \infty} a_n$.	If $\lim_{n \rightarrow \infty} a_n = 0$, the test is inconclusive. If $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges.	This test cannot prove convergence of a series.
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $ r < 1$, the series converges to $a/(1-r)$. If $ r \geq 1$, the series diverges.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \dots$, where a is the initial term and r is the ratio.
p-Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges. If $p \leq 1$, the series diverges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p -series. It can sometimes be difficult to find an appropriate series.
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p -series. Often easier to apply than the comparison test.

Series or Test	Conclusions	Comments
	<p>If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.</p> <p>If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.</p>	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx$.	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \rightarrow 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right $.	If $0 \leq \rho < 1$, the series converges absolutely. If $\rho > 1$ or $\rho = \infty$, the series diverges. If $\rho = 1$, the test is inconclusive.	Often used for series involving factorials or exponentials.
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n \rightarrow \infty} \sqrt[n]{ a_n }$.	If $0 \leq \rho < 1$, the series converges absolutely. If $\rho > 1$ or $\rho = \infty$, the series diverges.	Often used for series where $ a_n = b_n^n$.
	If $\rho = 1$, the test is inconclusive.	