Math 252 Final Exam
12/08/2020

Name: SOLUTIONS

Instructor: Rhodes

with comments by Bueler for Spring 2022 Final

On all problems, show enough work to indicate how you arrived at your answers.

1. (6 pts. – 3 pts. each) A parametric curve is given by

$$x(t) = t^2 - 2t + 3, \quad y(t) = t^3 - 4t,$$

for $-\infty < t < \infty$.

(a) Find the slope of the tangent line to the curve at the point where $t = 0$.

$$m = \frac{y'(t)}{x'(t)} = \frac{3t^2-4}{2t-2} \underset{t=0}{=} 2$$

(b) Give the x- and y- coordinates of all points on the curve where the tangent line is vertical.

$x'(t) = 0$

$2t-2=0$

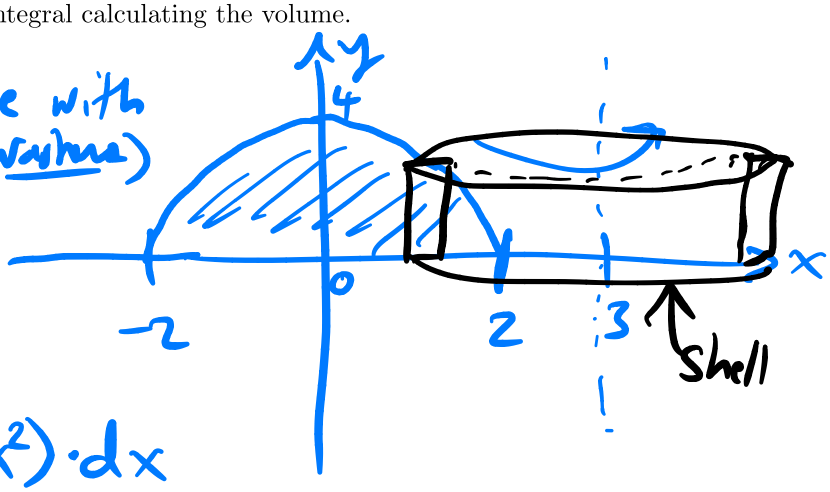
$t=1$ $\rightarrow$ $(x,y) = (2,-3)$

2. (8 pts.) The region bounded by the graphs of $y = 4 - x^2$ and $y = 0$ is rotated about a vertical axis at $x = 3$. Give, but **do not evaluate**, an integral calculating the volume.

Shells (can also be done with washers)

$$V = \int_{-2}^{2} 2\pi(3-x)\cdot y \cdot dx$$

$$= \int_{-2}^{2} 2\pi(3-x)\cdot(4-x^2)\cdot dx$$



this problem is around $x=3$ as an axis. The Spring 2022 Final will only involve $x=0$ and $y=0$ as the axes

3. (25 pts. – 5 pts. each) Compute the following, showing your work.

(a) $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\,dx$

$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{(4\sec^2\theta)^{3/2}}$$

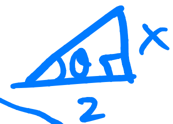
$dx = 2\sec\theta\tan\theta\,d\theta$

$x = 2\tan\theta$

$x^2+4 = 4(1+\tan^2\theta)$

$= 4\sec^2\theta$

$$= \frac{2}{8}\int \frac{\sec\theta\tan\theta}{\sec^3\theta}\,d\theta = \frac{1}{4}\int \frac{\tan\theta}{\sec^2\theta}\,d\theta$$



$$= \frac{1}{4}\int \frac{\sin\theta\cos^2\theta}{\cos\theta}\,d\theta = \frac{1}{4}\int u\,du = \frac{1}{8}u^2 + C$$

$$= \frac{1}{8}\sin^2\theta + C$$

$$= \frac{1}{8}\left(\frac{x}{\sqrt{x^2+2^2}}\right)^2 + C$$

$$= \frac{1}{8}\frac{x^2}{x^2+4} + C$$

(b) $\displaystyle\int_0^{\pi/4} \tan^4 x \sec^4 x\,dx$

$$= \int_0^{\pi/4} \tan^4 x\,(\underbrace{\sec^2 x}_{=1+\tan^2 x})\sec^2 x\,dx$$

$$= \int_0^{\pi/4} \tan^4 x\,(1+\tan^2 x)\sec^2 x\,dx$$

$u = \tan x$

$du = \sec^2 x\,dx$

$$= \int_0^1 u^4(1+u^2)\,du = \int_0^1 u^4 + u^6\,du = \frac{1}{5}u^5 + \frac{1}{7}u^7\Big]_0^1 = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}$$

(c) $\displaystyle\int \frac{1}{x^2-2x}\,dx$

$$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} \iff 1 = A(x-2) + Bx$$

$\iff A + B = 0$

$-2A = 1$

$$= \int \frac{-1/2}{x} + \frac{1/2}{x-2}\,dx$$

$$= -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$$

(d) $\displaystyle\int_0^1 \frac{1}{x^{3/2}}\,dx$ $= \lim\limits_{t\to 0^+} \int_t^1 x^{-3/2}\,dx = \lim\limits_{t\to 0^+} \left[-2x^{-1/2}\right]_t^1$

$$= \lim_{t\to 0^+}\left[-2 + 2t^{-1/2}\right] = -2 + \infty = +\infty$$

diverges

(e) $\displaystyle\int x \ln x\,dx$ $= (\ln x)\left(\tfrac{1}{2}x^2\right) - \int \tfrac{1}{2}x^2 \cdot \tfrac{1}{x}\,dx$ $\left\{\begin{array}{ll} u = \ln x & v = \tfrac{1}{2}x^2 \\ du = \tfrac{1}{x}dx & dv = x\,dx \end{array}\right.$

$$= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\int x\,dx$$

$$= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C$$

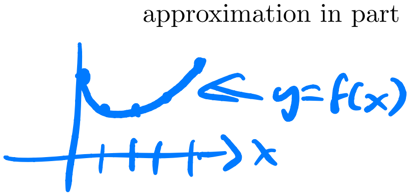
4. (6 pts. – 2 pts. each) A function's values are given in a table:

	x_1	x_2	x_3	x_4	x_5
x	0	0.25	0.5	0.75	1
$f(x)$	0.14	0.31	0.75	1.31	2

(a) Give a trapezoid sum approximating $\int_0^1 f(x)\,dx$. (Do not simplify.)

$$T_4 = \frac{\Delta x}{2}\left(f(x_1) + 2f(x_1) + 2f(x_2) + 2f(x_3) + x_4\right) = \frac{0.25}{2}\left(0.14 + 2(0.31) + 2(0.75) + 2(1.31) + 2\right)$$

(b) If you know the graph of the function $f(x)$ is concave up, what does that tell you about your approximation in part (a)? Explain briefly, using a drawing.



$$T_4 \geq \int_0^1 f(x)\,dx$$

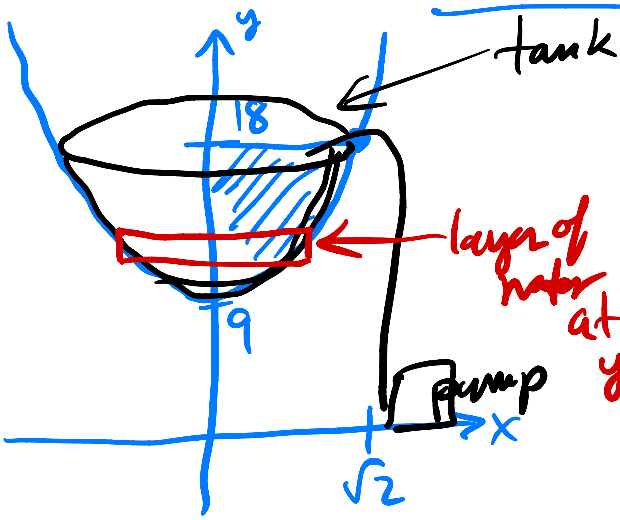
(c) ~~Use Simpson's rule to approximate $\int_0^1 f(x)\,dx$. (Do not simplify.)~~

I won't ask this

5. (10 pts.) An elevated water tank is shaped like a paraboloid, obtained by rotating about the y-axis the region bounded by that axis, the curve $y = 9 + x^2$ for $x \geq 0$, and the line $y = 18$. Both x and y are measured in feet, and the ground is at $y = 0$.

Give an integral for computing the work that would be done in pumping enough water from ground level to fill the tank. **Do not evaluate** the integral. (The weight density of water is 62.4 lbs/ft^3.)

$= \rho g$



$$W = \int_9^{18} \underbrace{(y-0)}_{\text{distance}} \cdot \underbrace{62.4\, \overbrace{\pi x^2\, dy}^{\text{volume}}}_{\text{force}}$$

$$= \int_9^{18} y\, 62.4\pi (y-9)\, dy$$

$$= 62.4\pi \int_9^{18} y(y-9)\, dy$$

$y = 9 + x^2$

$x^2 = y - 9$

6. (6 pts. – 2 pts. each) The subparts of this problem can be done most easily using each one to answer the next, but you may also do them independently. No work needs to be shown.

Recall that a *Maclauren series* is just a Taylor series at $a = 0$.

(a) Give the Maclauren series for the function $f(x) = \dfrac{1}{1-x}$.

either form

$$f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

(b) Give the Maclauren series for the function $f(x) = \dfrac{1}{1+x^2}$.

$$f(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-x^2)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

integrate $\int_0^x \ldots dt$

(c) Give the Maclauren series for the function $f(x) = \arctan x$.

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Show pattern or

7. (9 pts. – 3 pts. each) The time, in minutes, a customer must wait to check out at a certain business is modeled as a random variable with probability density

$$f(t) = \begin{cases} ce^{-t/7} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

for some constant c.

(a) What must the value of c be?

I won't ask probability questions

(b) What is the probability a customer will wait less than 3 minutes? (Evaluate any integrals in your answer.)

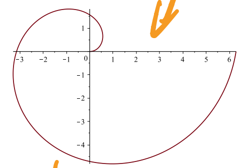
(c) What is the average time a customer must wait to check out? (Evaluate any integrals in your answer.)

I expect you to know this!

I don't remember the order of quadrants

I would ask you to sketch this graph

8. (5 pts.) The graph of the polar curve $r = \theta$ for $0 \le \theta \le 2\pi$ is shown.



(a) (3 pts.) ~~Using $A = \int_a^b \frac{1}{2} r^2\, d\theta$~~, fully compute the area in the ~~fourth quadrant~~ that is inside the graph.

$x>0, y<0$ quadrant

$$A = \frac{1}{2}\int_{3\pi/2}^{2\pi} \theta^2\, d\theta = \frac{\theta^3}{6}\Big]_{3\pi/2}^{2\pi} = \frac{(2\pi)^3}{6} - \frac{(3\pi/2)^3}{6} = \frac{\pi^3}{6}\left(8 - \frac{27}{8}\right)$$

(b) (2 pts.) Give parametric formulas for x, y tracing the curve.

$x = \theta\cos\theta$

$y = \theta\sin\theta$

9. (9 pts.) A function f is defined by a power series as

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(3^n)\sqrt{n+1}}(x-5)^n$$

(a) (6 pts.) Find the interval of convergence of the series.

root test: $\rho = \lim_{n\to\infty} \sqrt[n]{\frac{|x-5|^n}{3^n\sqrt{n+1}}} = \lim_{n\to\infty} \frac{|x-5|}{3\sqrt{\sqrt[n]{n+1}}} = \frac{|x-5|}{3}$

$\rho < 1$: $|x-5| < 3 \iff 2 < x < 8$

$\underline{x=8}$: $\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{3^n}}{\cancel{3^n}\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges AST

$I = (2, 8]$

$\underline{x=2}$: $\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges (p-series $p = \frac{1}{2}$)

(b) (3 pts.) Give numerical values of $f(5)$, $f'(5)$, and $f''(5)$

$$f(x) = 1 - \frac{x-5}{3\sqrt{2}} + \frac{(x-5)^2}{3^2\sqrt{3}} - \cdots$$

$\therefore f(5) = 1$, $f'(5) = -\frac{1}{3\sqrt{2}}$, $f''(5) = \frac{2}{3^2\sqrt{2}}$

10. (16 pts. – 4 pts. each) Determine whether the following series converge or diverge. State what test you use, and show enough work to make clear that you have applied the test correctly.

(a) $\sum_{n=0}^{\infty} \frac{\sqrt{n}+1}{3n^2+1}$

limit comp. test to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $p=3/2$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}+1}{3n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{(\sqrt{n}+1)\, n^{3/2}}{3n^2+1} = \lim_{n\to\infty} \frac{n^2+n^{3/2}}{3n^2+1}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{\sqrt{n}}}{3+\frac{1}{n^2}} = \frac{1}{3} \neq 0, \infty \quad \therefore \text{Converge}$$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{n^2}$

diverges: divergence test

$$\lim_{n\to\infty} \frac{(-1)^n e^n}{n^2} \neq 0$$

L'H

(c) $\sum_{n=1}^{\infty} \frac{7^n}{(n+1)!}$

ratio test: $\rho = \lim_{n\to\infty} \frac{\frac{7^{n+1}}{(n+2)!}}{\frac{7^n}{n!}} = \lim_{n\to\infty} \frac{7}{(n+2)(n+1)}$

$= 0 < 1$ ✓

converges

(d) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+2}}$

Converges

A.S.T.

$b_n = \frac{1}{\sqrt{n+2}} \geq 0$

$\lim_{n\to\infty} b_n = 0$

b_n decreasing