

Spring 2025

Math F113X

Exam 1

Name: Solutions

Section: ☐ 10:30 am (Leah Berman)
☐ 2:15pm (Jill Faudree)

Rules:

- Partial credit will be awarded, but you must show your work.
- You may have 1/2 of a standard page of paper ($8.5'' \times 5.5''$ or $11'' \times 4.25''$) of notes, both sides.
- Calculators are allowed.
- Place a box around your FINAL ANSWER to each question where appropriate.
- Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	20	
2	12	
3	20	
4	12	
5	18	
6	18	
Extra Credit	(6)	
Total	100	

1. (20 points)

A certain borough in Alaska has switched to using **Instant Runoff Voting (Ranked Choice Voting)** to determine the winner of its mayoral races.

In a recent municipal election, the preference schedule for the race was as follows:

	47	24	15	10	14	10
1st choice	Sampson	Hopkins	Kassel	Sampson	Kassel	Ward
2nd choice	Hopkins	Kassel	Ward	Hopkins	Sampson	Kassel
3rd choice	Ward	Sampson	Sampson	Kassel	Ward	Sampson
4th choice	Kassel	Ward	Hopkins	Ward	Hopkins	Hopkins

a. How many voters voted in the election? 120

$$47 + 24 + 15 + 10 + 14 + 10 = 120$$

b. How many voters are needed to have a majority of the votes? 61

$$120/2 = 60. \text{ But } 60 = 50\% \text{ exactly which is not more than } 50\%$$

c. Was there a winner after round 1 (that is, before anyone was eliminated)? Why or why not? Explain your answer.

No. No one had a majority after round 1.

$$S = 47 + 10 = 57$$

$$K = 15 + 14 = 29$$

$$H = 24$$

$$W = 10$$

d. Was anyone eliminated in round 1? Explain your answer.

Ward had the fewest votes and was eliminated.

e. Determine the winner of the election. Show your work clearly, in a way that someone else can follow. If you require multiple rounds, show the computations clearly, and clearly state which candidate is eliminated.

Round 2

$$\begin{array}{cccccc} 47 & 24 & 15 & 10 & 14 & 10 \\ \hline S & H & K & S & K & K \end{array}$$

$$S = 47 + 10 = 57$$

$$H = 24$$

$$K = 15 + 14 + 10 = 39$$

H is eliminated

The winner of the election was Kassel after 3 rounds.

Round 3

$$\begin{array}{cccccc} 47 & 24 & 15 & 10 & 14 & 10 \\ \hline S & K & K & S & K & K \end{array}$$

$$S = 47 + 10 = 57$$

$$K = 24 + 15 + 14 + 10 = 63$$

↑ wins!

2. (12 points)

ASUAF is holding elections to decide who will represent UAF on the Board of Regents. There are four candidates (labeled A, B, C, and D for convenience). The preference schedule is in the table below.

number of voters	8	15	20	10	9	16
1st choice	A	B	C	D	A	B
2nd choice	C	D	A	C	C	D
3rd choice	D	C	D	A	B	A
4th choice	B	A	B	B	D	C

- a. Find the winner under the plurality method. Show the calculations that give your answer.

$$A = 8 + 9 = 17$$

B wins with 31 votes.

$$B = 15 + 16 = 31$$

$$C = 20$$

$$D = 10$$

- b. Determine who would win if the only candidates were A and B. (That is, determine the winner in a head-to-head comparison of A and B.) Show the calculations that give your answer.

$$\begin{array}{cccccc} 8 & 15 & 20 & 10 & 9 & 16 \\ \hline A & B & A & A & A & B \end{array}$$

← Yellow highlights

$$A = 8 + 20 + 10 + 9 = 47 \quad \leftarrow \underline{A \text{ wins}}$$

$$B = 15 + 16 = 31$$

- c. Based only on your calculation in part (b), is it possible for A to be the Condorcet Winner? Justify your answer.

Yes, because the Condorcet winner wins all their head-to-head matches, and A won this head-to-head match, at least.

- d. Determine the point value candidate C would receive if the election were held using the Borda Count Method.

$$\begin{aligned} C \text{ gets } & 3(8) + 2(15) + 4(20) + 3(10) + 3(9) + 1(16) \\ & = 207 \text{ points} \end{aligned}$$

(see cyan highlights)

3. (20 points)

Consider the weighted voting system $[q : 10, 10, 5, 5, 5, 5, 1]$

1. What is the smallest value q can take? Justify your answer with a calculation.

By definition, q must be more than half the total votes.

$$10 + 10 + 5 + 5 + 5 + 5 + 1 = 20 + 20 + 1 = 41$$

So $q > \frac{41}{2} = 20.5$. So the smallest q can be is 21.

2. Explain why there is no choice of q for which this voting system can have a dictator.

We need $q > 21$, but all of the weights are less than 21. So no one can be a dictator.

3. Suppose q is 36. So, the weighted voting system is $[36 : 10, 10, 5, 5, 5, 5, 1]$.



- (a) Identify any players with veto power or state that none exist. Justify your answer.

Observe: $5 + 5 + 5 + 5 + 1 = 21$ and $10 + 21 = 31 < 36$.

So in fact, any winning coalition must include $P_1 \neq P_2$.

But $10 + 10 + \underbrace{5 + 5 + 5 + 5 + 1}_{\text{any of } P_3 \text{ thru } P_7} = 36$

and $10 + 10 + 5 + 5 + 5 + 5 = 40 > 36$

So we don't need a particular choice among $P_3 \dots P_7$

$P_1 \neq P_2$
have veto power

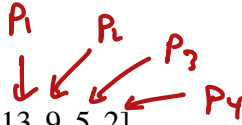
- (b) Identify any dummies or state that none exist. Justify your answer.

There are no dummies:

in $\{P_1, P_2, P_3, P_4, P_5, P_7\}$ $10 + 10 + 5 + 5 + 5 + 1 = 36$ and each player is critical

in $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ $10 + 10 + 5 + 5 + 5 + 5 = 40$ and each player is critical.

4. (12 points)

Consider the weighted voting system $[17 : 13, 9, 5, 2]$ 

- a. Determine all winning coalitions with one or two players. List of them in the space below (b).
- b. All winning coalitions using 3 or 4 players are listed below. Underline the players that are critical in each coalition (both the 2 player and the 3 or more players coalitions). Then find the Banzhaf power distribution for this system.

winning coalitions with
1 or 2 players

P_1, P_2
 P_1, P_3

winning coalitions with
3 or more players P_1, P_2, P_3 P_1, P_2, P_4 P_1, P_3, P_4 P_1, P_2, P_3, P_4

	#	BPI %
P_1	6	$6/10 = 60\%$
P_2	2	$2/10 = 20\%$
P_3	2	$2/10 = 20\%$
P_4	0	$0/10 = 0\%$
total	10	

- c. Based on your calculations in part (b), does this system contain any dummy players? Justify your conclusion.

Player 4 is a dummy because they have no power /
they are not critical in any winning coalition

Arithmetic to check winning coalitions:

1) No dictators = no single player winning coalitions.

$$13 + 9 = 22 > 17 \Rightarrow P_1, P_2 \text{ winning}$$

$$13 + 5 = 18 > 17 \Rightarrow P_1, P_3 \text{ winning}$$

$$9 + 5 + 2 = 16 \Rightarrow \text{can swap out any of } P_2, P_3, P_4 \text{ in } P_1, P_2, P_3, P_4$$

5. (18 points)

Amanda and Bernadette pooled their money to buy a fancy box of handmade valentine's day heart-shaped truffles. The candy company makes boxes of truffles that contain three flavors of filling: caramel, raspberry, and hazelnut. Each box of candy contains 12 truffles, 3 each of caramel and raspberry and 6 of hazelnut. (See box right.) The box of truffles costs \$24.

c	c	c	h	h	h
r	r	r	h	h	h

a. What is the dollar value of a fair share? $\underline{24/2 = \$12}$

b. Amanda will not eat caramel, and she likes raspberry twice as much as hazelnut.

(i) What is the value of **each** truffle to her?

Let h = value of a hazelnut truffle and r = value of rasp. Then $r = 2h$
 so $6h + 3(2h) = 24 \Rightarrow 12h = 24 \Rightarrow h = 2$ and

single caramel: $\$0$ single raspberry: $\$4$ single hazelnut: $\$2$

(ii) Which of the following collections of truffles are a fair share for Amanda (if any)? Circle the answer, and write in the total value for Amanda.

	<table><tr><td>c</td><td>c</td><td>c</td><td>h</td></tr></table>	c	c	c	h	<table><tr><td>c</td><td>c</td><td>c</td><td>h</td></tr><tr><td>r</td><td>r</td><td>h</td><td>h</td></tr></table>	c	c	c	h	r	r	h	h	<table><tr><td>r</td><td>r</td><td>r</td></tr></table>	r	r	r
c	c	c	h															
c	c	c	h															
r	r	h	h															
r	r	r																
Value	\$2	\$14	\$12															
Fair?	yes <u>no</u>	<u>yes</u> no	<u>yes</u> no															

We had intended (b) to mean each raspberry is worth twice each hazelnut. If you viewed it as in aggregate, then total $r = 16$, total $h = 8$ and each $r = \frac{16}{3} = 5.33$, each $h = 1.33$.

c. Bernadette values a single caramel truffle at \$1, a single raspberry truffle at \$5, and a single hazelnut truffle at \$1.

(i) If Bernadette is the divider, show a division of the box of candy that Bernadette might make, and explain why.

$5 + 5 + 1 + 1 = 12$

c	c	c	h	h	h
r	r	r	h	h	h

$1 + 5 + 6 = 12$

Need the total value of each portion to be \$12.

(ii) Which portion of chocolates would Amanda choose? Why? What is the total value to her of that portion?

Amanda would choose the right-hand portion

8	16
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her values

6. (18 points)

Alexis, Jamal, and Kasey buy a small cabin. They decide to divide the time each one occupies the cabin using the lone divider method. One person is chosen to be the divider and divides the year into 3 parts that have equal value to the divider. The table below represents the value of each section in each person's eyes.

	Jan-May	Jun-Sept	Oct-Dec
Alexis	\$25,000	\$25,000	\$25,000
Jamal	\$30,000	\$27,000	\$18,000
Kasey	\$33,000	\$30,000	\$12,000

- a. Identify who the divider was. Alexis
- b. Circle the values in the table that represent a fair share to each person.
- c. What happens in the next step of the lone divider process? Explain.

Distribute Oct-Dec to Alexis.

Then randomly decide how to split Jan-May and June-Sept between Jamal & Kasey - each will be happy with either.

Say: Jan-May June-Sept Oct-Dec
 Jamal Kasey Alexis.

- d. Suppose they instead had submitted the following table of values.

	Jan-May	Jun-Sept	Oct-Dec
Alexis	\$25,000	\$25,000	\$25,000
Jamal	\$5,000	\$70,000	\$5,000
Kasey	\$1,000	\$73,000	\$1,000

What happens in the next step of the lone divider process? Explain what the next step in the process should be and why (but you don't have to actually implement the step).

Distribute some uncontested time (say, oct-dec) to Alexis. Then choose a divider (say, Jamal) and have him re-divide the remaining months into 2 periods of equal value to him, and have Kasey choose which piece she wants. (That is, distribute to Alexis and then do divider-chooser on what remains.) v3

7. (Extra Credit: 6 points)

Andrea and Zeke are dividing up three items: an espresso maker, a cleaning robot, and a microwave. They submit the following sealed bids for the three items.

	Espresso Maker	Cleaning robot	Microwave	
Andrea	\$60	\$120	\$30	total 210
Zeke	\$100	\$50	\$75	225

- a. Determine each person's fair share (in dollars).

Andrea's fair share	Zeke's fair share
105	112.50

- b. Determine which person gets each item.

Espresso Maker	Cleaning Robot	Microwave
Zeke	Andrea	Zeke

- c. Determine the **surplus**. Show your work!

Zeke value of stuff = 175, and $175 - 112.50 = \$62.5$ to the pot
 Andrea value of stuff = 120 and $120 - 105 = \$15$ to the pot

Total surplus is $62.50 + 15 = 77.50$

- d. How many dollars does each person pay or receive in the end? Show your work!

Andrea:

pays in \$15 and receives $77.5 / 2 = 38.75$

So she receives $38.75 - 15 = \$23.75$

Zeke:

pays in \$62.50 and receives \$38.75 in surplus

so he pays $62.50 - 38.75 = \$23.75$

(look, they match! That's a good sign!)