10. Transformation of GPS results

10.1 Introduction

The reference frame of GPS is the World Geodetic System 1984 (WGS-84), cf. Decker (1986). When using GPS, the coordinates of terrestrial sites for example are obtained in the same reference frame. However, the surveyor is not, usually, interested in computing the coordinates of the terrestrial points in a global frame. Rather, the results are preferred in a local coordinate frame either as geodetic (i.e., ellipsoidal) coordinates, as plane coordinates, or as vectors combined with other terrestrial data. Since the WGS-84 is a geocentric system and the local system usually is not, certain transformations are required. The subsequent sections deal with the transformations most frequently used.

10.2 Coordinate transformations

10.2.1 Cartesian coordinates and ellipsoidal coordinates

Denoting the Cartesian (rectangular) coordinates of a point in space by X, Y, Z and assuming an ellipsoid of revolution with the same origin as the Cartesian coordinate system, the point can also be expressed by the ellipsoidal coordinates φ, λ, h, see Fig. 10.1. The relation between the Cartesian coordinates and the ellipsoidal coordinates, given in Eq. (3.6), is:

$$\begin{aligned} X &= (N + h)\cos\varphi\cos\lambda \\ Y &= (N + h)\cos\varphi\sin\lambda \\ Z &= \left(\frac{b^2}{a^2}N + h\right)\sin\varphi\,, \end{aligned} \tag{10.1}$$

with N the radius of curvature in prime vertical

$$N = \frac{a^2}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}\,, \tag{10.2}$$

and a, b are the semiaxes of the reference ellipsoid. Recall that the Cartesian coordinates related to WGS-84 are also denoted ECEF coordinates and that

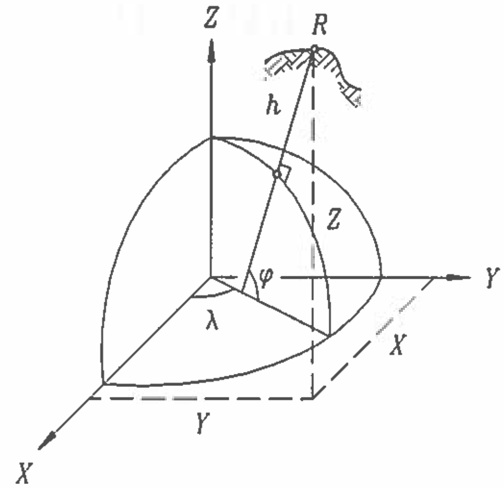

Fig. 10.1. Cartesian coordinates X, Y, Z and ellipsoidal coordinates φ, λ, h

the origins of the ECEF coordinate system and of the WGS-84 ellipsoid of revolution are identical (i.e., geocentric).

The formulas (10.1) transform ellipsoidal coordinates φ, λ, h into Cartesian coordinates X, Y, Z. For GPS applications, the inverse transformation is more important since the Cartesian coordinates are given and the ellipsoidal coordinates sought. Thus, the task is now to compute the ellipsoidal coordinates φ, λ, h from the Cartesian coordinates X, Y, Z. Usually, this problem is solved iteratively although a solution in closed form is possible. From X and Y the radius of a parallel,

$$p = \sqrt{X^2 + Y^2} = (N + h)\cos\varphi\,, \tag{10.3}$$

can be computed. This equation is rearranged as

$$h = \frac{p}{\cos\varphi} - N \tag{10.4}$$

so that the ellipsoidal height appears explicitly. Introducing by

$$e^2 = \frac{a^2 - b^2}{a^2} \tag{10.5}$$

the first numerical eccentricity, it follows $b^2/a^2 = 1 - e^2$ which can be substituted into the equation for Z in (10.1). The result

$$Z = (N + h - e^2 N)\sin\varphi \tag{10.6}$$

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can be written as

$$Z = (N + h)\left(1 - e^2 \frac{N}{N + h}\right) \sin\varphi \tag{10.7}$$

equivalently. Dividing this expression by Eq. (10.3) gives

$$\frac{Z}{p} = \left(1 - e^2 \frac{N}{N + h}\right) \tan\varphi \tag{10.8}$$

which yields

$$\tan\varphi = \frac{Z}{p}\left(1 - e^2 \frac{N}{N + h}\right)^{-1}. \tag{10.9}$$

For the longitude λ the equation

$$\tan\lambda = \frac{Y}{X} \tag{10.10}$$

is obtained from Eq. (10.1) by dividing the first and the second equation.

The longitude can be directly computed from Eq. (10.10). The height h and the latitude φ are determined by Eqs. (10.4) and (10.9). The problem with (10.4) is that it depends on the (unknown) latitude. Equation (10.9) is even worse because the desired latitude is implicitly contained in the right-hand side in N. Based on these three equations, a solution can be found iteratively by the following steps:

1. Compute $p = \sqrt{X^2 + Y^2}$.
2. Compute an approximate value $\varphi_{(0)}$ from
   $$\tan\varphi_{(0)} = \frac{Z}{p}(1 - e^2)^{-1}.$$
3. Compute an approximate value $N_{(0)}$ from
   $$N_{(0)} = \frac{a^2}{\sqrt{a^2\cos^2\varphi_{(0)} + b^2\sin^2\varphi_{(0)}}}.$$
4. Compute the ellipsoidal height by
   $$h = \frac{p}{\cos\varphi_{(0)}} - N_{(0)}.$$
5. Compute an improved value for the latitude by
   $$\tan\varphi = \frac{Z}{p}\left(1 - e^2 \frac{N_{(0)}}{N_{(0)} + h}\right)^{-1}.$$

6. Check for another iteration step: if $\varphi = \varphi_{(0)}$ then go to the next step otherwise set $\varphi_{(0)} = \varphi$ and continue with step 3.

7. Compute the longitude λ from
$$\tan \lambda = \frac{Y}{X}.$$

The formulas in closed form for the transformation of X, Y, Z into φ, λ, h are

$$\begin{aligned}\varphi &= \arctan \frac{Z + e'^2\, b \sin^3\theta}{p - e^2\, a \cos^3\theta}\\ \lambda &= \arctan \frac{Y}{X}\\ h &= \frac{p}{\cos\varphi} - N\end{aligned} \tag{10.11}$$

where

$$\theta = \arctan \frac{Z\, a}{p\, b} \tag{10.12}$$

is an auxiliary quantity and

$$e'^2 = \frac{a^2 - b^2}{b^2} \tag{10.13}$$

is the second numerical eccentricity. Actually, there is no reason why these formulas are less popular than the iterative procedure. Either method works equally well and can be easily programmed.

10.2.2 Ellipsoidal coordinates and plane coordinates

In contrast to the previous section, only points on the ellipsoid are considered. Thus, ellipsoidal latitute φ and longitude λ are of interest here. The objective is mapping a point φ, λ on the ellipsoid into a point x, y on a plane.

There are many kinds of map projections, some being more popular than others. In principle,

$$\begin{aligned}x &= x(\varphi, \lambda;\, a, b)\\ y &= y(\varphi, \lambda;\, a, b)\end{aligned} \tag{10.14}$$

is the general formulation of the desired map projection. Geodetic applications require conformal projections be used. Conformality means that an angle on the ellipsoid is preserved after mapping it into the plane. More