Distance-based Equilibria in Normal Form Games

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Abstract

We propose a simple uncertainty modification for the agent model in classical game-theoretic setting; at any given strategy profile, the agent can access only to a set of "possible profiles" that are close to the actual profile (rather than observing particular distributions for every other player). We investigate the variations in which the agent determines her strategy using well-known approaches to deal with such uncertainty e.g., considering the worst case, or trying to minimize regret. Any such modification in behavioral model naturally induces a notion of equilibrium; a *distance-based equilibrium*, where all agents prefer to keep their current strategy. We explore the connections among the different variations of the model for normal form games, as well as with other existing solution concepts such as trembling-hand perfection, and demonstrate how solutions apply to several games of interest.

1 Introduction

Decision making under uncertainty is a key issue both in game theory and in artificial intelligence. Whereas models of strict uncertainty, or absence of unique priors were common at the outset of those fields, more recently probabilistic and Bayesian models have become dominant, albeit due to different reasons. In AI, the use of probabilities often leads to better performance in a wide variety of tasks (e.g. Bayesian networks for debugging and information retrieval [Heckerman *et al.*1995], Monte-Carlo methods for robot localization [Thrun *et al.*2001], and many more). In game theory, probabilities are used first and foremost because they allow for clean modeling, and in particular the use of von Neumann-Morgenstern utilities with all the rich theory that they support.

Several solutions concepts suggested in the behavioral game theory literature tackled the problem of imperfect rationality (e.g., Cognitive hierarchy [Camerer *et al.*2004], Quantal response [McKelvey and Palfrey1995], Trembling-hand perfect equilibria [Selten1975] and others), yet many of these still assume that agents optimize or approximate their expected utility over some distribution. Nonetheless, Trembling-hand perfect equilibria which appears to be the foremost central notion of equilibrium refinements, it regained strong attention in machine learning research and AI, in general [Farina *et al.*2018b, Farina *et al.*2018a]. In fact, following a trembling hand perfect strategy can be claimed to be a "super-rational" behavior rather than bounded rational [Aumann1997].

In this work, we suggest a flexible model for an agent inspired by early work in AI on uncertainty and reasoning, and some recent works on specific games (such as voting [Conitzer *et al.*2011, Meir *et al.*2014, Lev *et al.*2019] and routing [Meir and Parkes2015]), and look at the very foundations; normal form games, from the lenses of this model. Our model is *distance-based*; that is, at any given action profile, a set of "possible profiles" (that are close to the actual profile) is constructed w.r.t. a metric, without assigning them any probabilities. Then, the agent determines her action using one of the many available methods for decision making under strict uncertainty, such as considering the worst case [Wald1939], or trying to minimize regret [Savage1951, Hyafil and Boutilier2004], etc. The intuition of such setting is due to imprecision caused by limitations in observations (of the agent). Note that such model can be considered as a bounded rationality model, as it limits the agent's reasoning about the other agents' strategies.

It is flexible in the sense that potentially different distance metrics and decision rules can be plugged into the model, to fit specific games or types of behavior. Once we fix our behavioral model though, it naturally induces a notion of *equilibrium*, which is an action profile where no agent is inclined to change her action.

Some other related works (and the references therein) that are worth mentioning: [Marinacci2000] makes use of Choquet expected utility model based on non-additive probabilities [Gilboa and Schmeidler1989], their pessimistic/optimistic choices shares similar intuition with our worst-case/best-case responses. [Aghassi and Bertsimas2006] uses robust optimization (hence worst-case behaviour) to model uncertainty in payoffs. None of those works uses distance as basic machinery, and different technical subletlies and challanges apply.

Our contribution and the organization of the paper is as follows: After giving basics and the existing equilibrium notions in the next section, we explore the implications among the different variations of the model, as well as with other central solution concepts such as the aforementioned trembling-hand perfection (Section 2-3), and demonstrate how these solution concepts apply to several common games of interest (Section 4). Further, we provide with existence results for our notions of distance-based equilibria (Section 5). To underline its usefulness, we introduce a class of games such that these notions potentially guarantee better outcomes (Section 6). And related to this, as last, we provide with a result which gives a price of anarchy bound in terms of our notion (Section 7). Conclusion and future work closes the paper (Section 8).

2 Preliminaries and Notation

We define n-player *normal-form game* $\mathcal{G} = (N, A, u)$, where N is a finite set of n players, indexed by i; $A = A_1 \times \ldots \times A_n$, where A_i is a finite set of *actions* (or *pure strategies*) available to player i. Each vector $a = (a_1, \ldots, a_n) \in A$ is called an action profile; $u = (u_1, \ldots, u_n)$ where $u_i : A \to \mathbb{R}$ is a real-valued *utility function* (or *payoff function*) for player i. A mixed strategy π_i for player i is a probability distribution over the set of available actions A_i for player i. Further, we denote the set of *mixed-strategies* for player i by Π_i, which is the set of all probability distributions over the set A_i of actions for player i. The set of mixed-strategy profiles is simply the Cartesian product of the individual mixed-strategy sets i.e., $\Pi = \Pi_1 \times \ldots \times \Pi_n$. We denote a (mixed-strategy) profile by $\pi \in \Pi$. Further, for a player i, we denote the probability that an action a_i is played under mixed strategy π_i, by $\pi_i(a_i)$. The *support* of a mixed strategy π_i for a player i is the set of pure strategies $\{a_i | \pi_i(a_i) > 0\}$. For a player i, a mixed-strategy π_i is *totally* (or *completely*) *mixed* if its support subsumes A_i. A strategy profile π is totally mixed if its every component is totally mixed.

For simplicity, we overload the function symbol u_i to define the (expected) utility u_i of a strategy profile π for player i in a normal form game as $u_i(\pi) = \sum_{a \in A} u_i(a) \prod_{j \in N} \pi_j(a_j)$.[1] A (mixed) strategy π_i is a *best response* to π_{-i} if $u_i(\pi_i, \pi_{-i}) \geq u_i(\pi'_i, \pi_{-i})$ for every $\pi'_i \in \Pi_i$. A (mixed) strategy profile is a *Nash equilibrium* (MN) if for every player $i \in N$, π_i is a best-response to π_{-i}. A pure strategy Nash equilibrium (PN) is a MN where every player's strategy has a support of cardinality 1. A *totally mixed Nash equilibrium* is denoted by TMN. Given a sequence of profiles, Given a profile π, *Social Welfare* $SW(\pi) = \sum_{i \in N} u_i(\pi)$. And finally, *Price of Anarchy* PoA for a game is defined as the ratio of the maximum social welfare to the minimum social welfare in an equilibrium.

2.1 Equilibria with Mistakes and Imprecision

We mention definitions of several well-known equilibrium concepts involved with *slight mistakes* or *imprecision* of agents, from the literature. These are *Trembling-Hand Perfect Equilibrium* (T) of [Selten1975], its stronger version, *Truly Perfect Equilibrium* (TP) [Kohlberg1981], and *Robust equilibrium* (R) [Messner and Polborn2005]. These concepts are of particular importance since we shall reveal their connections to the distance-based equilibrium concepts, introduced later in the next section.

Definition 1 (Trembling-Hand Perfect equilibrium)**. [Selten1975]** *Given a finite game $\mathcal{G}$, a mixed strategy profile π is* trembling-hand perfect equilibrium *if there is a sequence $\{\pi^k\}_{k=0}^{\infty}$ of totally mixed strategy profiles which converges to π such that for each agent $i \in N$, π_i is a best response to π^k_{-i} for all k.*

Selten's notion of trembling hand perfect equilibrium is based on the notion of best response which is robust against minimal mistakes (hence trembling hand) of opponents, formalized by a sequence of profiles converging to the equilibrium. As it was shown in [Selten1975], every finite game has a T-equilibrium. Note that the notion does not demand a best-response to every such sequence but rather only one. We shall later show that this very notion is entangled to our notions of distance-based equilibria. So is the Truly Trembling Hand Perfect equilibrium, a stronger variant, as we mention next.[2]

Definition 2 (Truly perfect equilibrium)**.**

[1] Note that large $\prod$ in this expression stands for product (instead of Π, the set of mixed profiles).

[2] Another similar variation aims to strengthen Selten's is in [Okada1981].

[Kohlberg1981] *Given a finite game $\mathcal{G}$, a mixed strategy profile π is* truly perfect equilibrium *if for each sequence $\{\pi^k\}_{k=0}^{\infty}$ of totally mixed strategy profiles which converge to π, there is a K such that π_i is a best response to π_{-i}^k for all $k \geq K$.*

It is easy to see that, this notion demands a lot by requiring a best response in it to be a best response in every sequence of profile converging to the equilibrium. There is a cost for this demand, that is, TP does not always exist [Kohlberg1981] (see also [Fudenberg and Tirole1991], Chapter 11).

We also define a stronger variation; namely, *Strict-TP* (s-TP).

Definition 3 (Strict-TP)**.** *A strategy profile is a* Strict-TP *if it is defined as in Def. 2 except that every π_i is strictly better than any other response to π_{-i}^k.*

We also provide a slightly stronger extension of a Selten's original trembling hand equilibrium; namely *strict-T* (s-T):

Definition 4 (Strict-T)**.** *A strategy profile is a* Strict-T *it is defined as in Def. 1 except that for every $i \in N$, there is an infinite subsequence of $\{\pi^k\}_{k=0}^{\infty}$ where π_i is a strict best-response to π_{-i}^k.*

Note that a strict-T has to be a pure Nash equilibrium, since a strict best-response cannot be mixed.

The following notion of robust equilibrium is an adaption from robust political equilibrium [Messner and Polborn2005].[3] Intuitively, an equilibrium is ϵ-Robust, if each player would like to keep her action, even if there is a small chance that other players deviate.

Definition 5 (Robust equilibrium)**. [Messner and Polborn2005]**

- *A mixed profile π is an ϵ-noisy variant of a pure profile a, if for all $j \in N$, $\pi_j(a_j) > 1 - \epsilon$ where $\epsilon > 0$.*
- *Given a pure strategy profile a, player i, and $\epsilon > 0$, action b_i is a ϵ-*Robust response *if b_i is a best response to any ϵ-noisy variant of a_{-i}.*
- *A pure strategy profile a is an ϵ-Robust equilibrium if every a_i is an ϵ-Robust response to a_{-i}.*

Next, we provide a link between those two concepts.

[3] [Messner and Polborn2005] deal with both coalitional stability and noisy actions, assuming that each player "fails" to play with some small probability. We only focus on the latter part. Our definition of Robust Equilibrium is based on their informal description and motivation.

Proposition 1. *If a is an ϵ-Robust equilibrium for some $\epsilon > 0$, then a is a TP.*

Proof. Let a be some ϵ-Robust equilibrium for some $\epsilon > 0$, and consider a sequence $\{\pi^k\}_{k=0}^{\infty}$ converging to a. Thus $\pi_i^k(a_i) \to 1$ for all $i \in N$. In particular, there is some K_i such that for all $k > K_i$, $\pi_i^k(a_i) > 1 - \epsilon$. Let $K := \max_i K_i$. Then for all $k > K$, and for all $j \in N$, we have that $\pi_j^k(a_j) > 1 - \epsilon$, i.e. π^k is an ϵ-noisy variant of a, and thus a_{-i} is a best response to π_{-i}^k. $\square$

3 Distance-based Equilibria

In the following, we define the central notions of the paper.

3.1 Distance-based uncertainty

For every agent $i \in N$, let $r_i \in \mathbb{R}^+$ be the associated *ignorance factor* formalizing the intuition: Greater the ignorance factor, more ignorant/cautious the agent (about the mixed strategies of other agents).

Given a mixed strategy profile $\pi = \langle \pi_i, \pi_{-i} \rangle$, $\mathcal{B}_i(\pi, r_i) := \{\pi'_{-i} \mid d(\pi_{-i}, \pi'_{-i}) \leq r_i\}$ is the set of *possible response profiles* of others that the agent i is considering, equipped with a metric d which is assumed to have the axioms of non-negativity i.e., $d(x, y) \geq 0$; identity of indiscernibles i.e., $d(x, y) = 0 \iff x = y$; symmetry i.e., $d(x, y) = d(y, x)$; and triangle inequality i.e., $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \Pi$. Although, our metric-dependent results comply with any compact space with a metric which has the same axioms, in the examples throughout the paper we work with Euclidean metric for convenience (due to its wide-spread use). Often, we will use the shorthand notation $\mathcal{B}_i(\pi)$ when r_i is clear from the context.

Another way to put it, $\mathcal{B}_i(\pi)$ is the set that captures i's *belief* or *subjective uncertainty* about other agents' strategies in a strategy profile π, hence we will call it often *belief set*. Yet another description is that instead of writing a belief as a distribution over profiles, the belief of agent i is written as a *point estimate* π_{-i} plus an uncertainty parameter r_i (which together induce a set $\mathcal{B}_i$). The use of a ball to capture uncertainty is both motivated by its formal simplicity, and the degree of freedom it provides which is set aside from any obvious domain specific/context-dependent constraints.

Notice that as r_i approaches to 0, i becomes almost sure about other agents' strategies, and thus $\mathcal{B}_i$ becomes $\{\pi_{-i}\}$. Noteworthy is that for two-player games r reduces to the distance between two probability distributions. For more than two players, any distance on probabilities induces a natural metric d on uncorrelated profiles where for each $j \neq i$ we consider all π'_j close to π_j. [4]

3.2 Local responses

Let Π_i denote the set of all strategies available to i. First, we introduce some notions of best response.

Definition 6. π'_i locally dominates π_i *in the set* $\Pi^*_{-i} \subseteq \Pi_{-i}$ *if: (a) for all* $\pi_{-i} \in \Pi^*_{-i}$, $u_i(\pi'_i, \pi_{-i}) \geq u_i(\pi_i, \pi_{-i})$*; and (b) there exists* $\pi'_{-i} \in \Pi^*_{-i}$ *such that* $u_i(\pi'_i, \pi'_{-i}) > u_i(\pi_i, \pi'_{-i})$. π'_i strictly locally dominates π_i *in* Π^*_{-i} *if (a) holds with strict inequality.*

Note that when $\Pi^*_{-i} = \Pi_{-i}$, local dominance and strict local dominance boil down to weak and strict strategic dominance, respectively [Shoham and Leyton-Brown2008].

Given a mixed strategy profile π, for each $i \in N$ with r_i, a strategy π_i is a distance based

- **(W)orst-case** best response (or *maximin*) if $\pi_i = \arg\max_{\pi'_i \in \Pi_i} \{\min(u_i(\pi'_i, \pi_{-i})) \mid \pi_{-i} \in \mathcal{B}_i(\pi)\}$.
- **(B)est-case** best response (or *maximax*) if $\pi_i = \arg\max_{\pi'_i \in \Pi_i} \{\max(u_i(\pi'_i, \pi_{-i})) \mid \pi_{-i} \in \mathcal{B}_i(\pi)\}$.
- **(WR) Worst-Case Regret** best response if $\pi_i = \arg\min_{\pi'_i \in \Pi_i} \max\{\mathtt{reg}_i(\pi'_i, \pi_{-i}) \mid \pi_{-i} \in \mathcal{B}_i(\pi)\}$ where $\mathtt{reg}_i(\pi_i, \pi_{-i}) = \max_{\pi'_i \in \Pi_i}(u_i(\pi'_i, \pi_{-i})) - u_i(\pi_i, \pi_{-i})$.
- **(U)ndominated** best response if there is no π'_i that locally dominates π_i in the set $\mathcal{B}_i(\pi)$.
- **(D)ominant** best response if π_i locally dominates all π'_i in the set $\mathcal{B}_i(\pi)$.
- **(SD) Strictly dominant** best response if π_i strictly locally dominates all π'_i in the set $\mathcal{B}_i(\pi)$.

By the following proposition, we characterize the relations between these notions. We make no assumption on the metric since it only uses single sets of possible strategy profiles (of opponents), a.k.a. belief sets. In other words, these relations are independent from the choice of the metric.

Theorem 1. *Given any ignorance factor* r*, the following statements hold:*

(a) If π_i *is a* SD_r *best response, then* π_i *is a* D_r *best response.*

(b) If π_i *is a* D_r *best response, then* π_i *is a* W_r *and* B_r *best response.*

(c) If π_i *is a* D_r *best response, then* π_i *is a* WR_r *best response.*

(d) If π_i *is a unique* W_r, B_r *or* WR_r *best response, then* π_i *is a* U_r *best response.*

Proof. (a) Easily follows by definitions.

(b) Consider any action $\pi'_i \neq \pi_i$. Since $u_i(\pi_i, \pi_{-i}) \geq u_i(\pi'_i, \pi_{-i})$ for any state $\pi_{-i} \in \mathcal{B}_i(\pi)$, this holds in particular for the states with maximal utility and minimal utility.

(c) If π_i is a D_r best response, then $\mathtt{reg}_i(\pi_i, \pi_{-i}) = \max_{\pi'_i \in \Pi_i}(u_i(\pi'_i, \pi_{-i}) - u_i(\pi_i, \pi_{-i})) \leq \max_{\pi'_i \in \Pi_i}(u_i(\pi_i, \pi_{-i}) - u_i(\pi_i, \pi_{-i})) = 0$ for all $\pi_{-i} \in \mathcal{B}_i(\pi)$. The regret of any other action π'_i can only be higher, thus π'_i is a WR_r response.

(d) Suppose that π_i is a B_r best response. If π_i is not a U_r response, then there is an action $\pi'_i \neq \pi_i$ that locally dominates π_i. In particular, $u_i(\pi'_i, \pi^*_{-i}) \geq u_i(\pi_i, \pi^*_{-i})$ in the best state π^*_{-i}, which means that π'_i is also a B_r best response. Note that uniqueness is a necessary condition, otherwise, we can consider two actions π_i, π'_i that have the same utility in the best case, but one of them dominates the other. The proof for W_r and WR_r is similar. $\square$

The following result also holds for any metric since it only uses containment.

Proposition 2. *If* $r'_i < r_i$ *then* SD_{r_i} *response implies* $SD_{r'_i}$ *response.*

Proof. Since $\mathcal{B}(\pi, r'_i) \subseteq \mathcal{B}(\pi, r_i)$, condition (a) of Def. 6 must hold in all states. $\square$

This does not hold in any of the other variations. To see this, consider W-equilibrium, since for any response π_i, $\min\{u_i(\pi_i, \pi_{-i}) \mid \pi_{-i} \in \mathcal{B}_i(\pi, r)\} \leq \min\{u_i(\pi_i, \pi_{-i} \mid \mathcal{B}_i(\pi, r^*)\})\}$ whenever $\mathcal{B}_i(\pi, r^*) \subseteq \mathcal{B}_i(\pi, r)$. The other variations are similar.

3.3 Equilibrium

Next, $\mathbf{r} := (r_1, \ldots, r_n)$ is the *ignorance vector* which stores ignorance factor for each agent $i \in N$. Assume that $\star \in \{W, B, WR, U, D, SD\}$. Then, π is called a **distance-based** $\star_{\mathbf{r}}$**-equilibrium** if for every agent i, whose belief set $\mathcal{B}_i(\pi)$ is defined w.r.t. r_i where $r_i = \mathbf{r}_i$,

[4] This can further be extended to correlated profiles where π_j are independent given a signal σ.

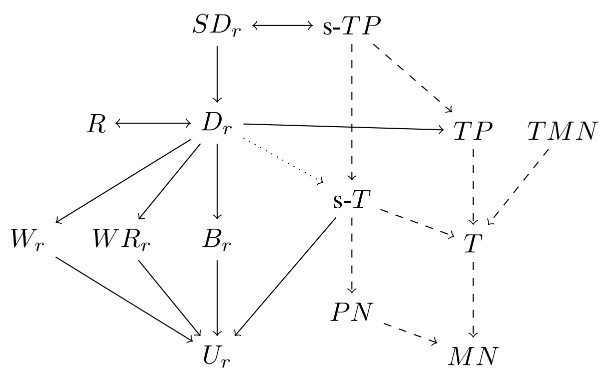

Figure 1: Entailment of equilibrium concepts; An arrow to $\star_r$ means that there is some $r > 0$ for which the entailment holds. An arrow from $\star_r$ means the entailment holds for any $r > 0$ (except for the dotted arrow which is slightly weaker, see Prop. 4). Dashed arrows mark entailments that are known or obvious.

π_i is a $\star$-best response. When all the agents have the same r, we will use r instead of $\mathbf{r}$ as a subscript, or totally omit it whenever it is clear from the context. As a note of related work, [Meir *et al.*2014] employs U_r while [Meir and Parkes2015] employs W_r and WR_r.

Observation 1. *For all definitions above, if $r_i = 0$ for all $i \in N$, then $\star$-best response is any best-response, and a $\star$-equilibrium is any Nash equilibrium.*

The statements below follow immediately from the relations between corresponding responses (i.e., Proposition 1).

Corollary 1. *Given any r, the following statements hold:*

(a) If π is a SD_r-equilibrium, then π is a D_r-equilibrium.

(b) If π is a D_r-equilibrium, then π is a W_r, B_r and WR_r-equilibrium.

(c) If π is a W_r, B_r or WR_r-equilibrium, then π is a U_r-equilibrium.

A brief summary of our results is illustrated in Figure 1.

4 Locally Best Response and Trembling Hand

Definitions 1-4 capture stability of equilibrium in a rigorous mathematical way, but require reasoning about sequences of profiles that do not seem have a clear cognitive interpretation.

In this section we aim to get a better understanding of these concepts and of our distance-based equilibrium concepts, by exploring the connections between them. Moreover, the proposed distance-based best-response (and hence equilibrium) do have cognitive interpretation which is the observational limitation that an agent has. One can give plenty of real-world examples: the necessary amount of difference between two colors or two sounds to be able to distinguish between them (e.g., two close tones of red in a dark room, or moaning of a cat at night vs. human baby cry).

We first argue that robustness (under the appropriate Def. 1-4) implies stability under uncertainty (under Def. 5) when the ignorance factors of all agents are sufficiently small.

Observation 2. *Given a profile π and an agent $i \in N$, for any mixed strategy π_i, $\max_{\pi_{-i} \in \mathcal{B}_i(\pi)} u_i(\pi_i, \pi_{-i}) = \min_{\pi_{-i} \in \mathcal{B}_i(\pi)} u_i(\pi_i, \pi_{-i}) = u_i(\pi_i, \pi_{-i})$ if $r_i = 0$.*

Having Observation 2 in mind, realize that the classical notion of MN implies our notions of W, B, WR and U for $r = 0$. The following result provides a partial picture.

Proposition 3. *Given a game $\mathcal{G}$, if π is a strict trembling-hand perfect equilibrium, then there is an $\epsilon > 0$ such that if $r_i < \epsilon$ for every $i \in N$, then π is a $U_{\mathbf{r}}$-equilibrium.*

Proof. Assume a game $\mathcal{G}$ and a trembling hand perfect equilibrium π in $\mathcal{G}$. By Definition 1, there is a sequence $\{\pi^k\}_{k=0}^{\infty}$ of totally mixed strategies which converges to π. Take an arbitrary $\epsilon > 0$. By convergence there is a K such that $d(\pi^k, \pi) < \epsilon$ whenever $k \geq K$. Moreover, it follows that for every $i \in N$, $d(\pi^k_{-i}, \pi_{-i}) < \epsilon$ as well for $k \geq K$. For each agent $i \in N$, let $r_i = d(\pi^K_{-i}, \pi_{-i})$ i.e., $\mathbf{r} = (r_1, \ldots, r_n)$. Now, for any agent i, we know that $\pi^k_{-i} \in \mathcal{B}_i(\pi)$ whenever $k \geq K$, and also since π is strict-T, π_i is a best response to π^k_{-i}. This shows that π_i is U_{r_i}-response and that π is a $U_{\mathbf{r}}$ equilibrium. □

In the other direction, it seems that a D_r-equilibrium (w.r.t. any $r > 0$) must be a strict trembling hand equilibrium.

Proposition 4. *If there is an r^*, such that π is D_r - equilibrium for all $r \in (0, r^*)$, then π is strict-T.*

Proof. Consider an arbitrary sequence of $r^k < r^*$ that converges to 0. Since π is a D_{r^k}-equilibrium, then for every player $i \in N$, π_i is a best response to every $\pi_{-i} \in \mathcal{B}(\pi, r^k)$, and a strict best response to at least one profile $\pi^{k,i}_{-i}$. Moreover, $\pi^{k,i}_{-i}$ is w.l.o.g. totally mixed (we can mix it with a low probability for any other profile such that π_i remains a best-response). Let $\pi_i^{r_k}$ be a mixed strategy that selects π_i with probability $1 - r_k$.

We therefore get n sequences of totally-mixed profiles converging to π, where in each sequence

$((\pi^{k,i}, \pi_i^{r_k}))_{k=0}^{\infty}$, π_i is a strict best-response to the entire sequence.

Let $\pi^k = (\pi^{k,i}, \pi_i)$ for all k such that $k \mod n = (i-1)$ (that is, we interleave subsequences). Note that $(\pi^k)_{k=0}^{\infty}$ converges to π, and for every agent i there is a subsequence for which π_i is a strict best-response. Thus π is a strict trembling hand equilibrium (strict-T). $\square$

Proposition 5. *For any r, if π is D_r, then π is TP.*

Proof. Fix any r. It follows from the fact that whatever sequence we choose in $\mathcal{B}_i(\pi_{-i}, r)$, π_i will be a best response to it by definition, hence satisfying the condition of TP. $\square$

Proposition 6. *If π is a TP, then there is an r such that π is a U_r.*

Proof. Assume that π is a TP, then for each sequence $\{\pi^k\}_{k=0}^{\infty}$ of totally mixed strategy profiles which converge to π, there is a K such that π_i is a best response to π_{-i}^k for all $k \geq K$. All possible sequences form a ball for each player, and we take the least largest K of those (all) possible sequences; call it $K^{\star}$. Then, for all $i \in N$, we define $r_i = d(\pi_{-i}, \pi_{-i}^{K^{\star}})$. Obviously, π_i is a locally dominates all the other responses to every $\pi'_{-i} \in \mathcal{B}_i(\pi, r_i)$ for every player i, which implies that π is a U_r. $\square$

The following result provides a link between strict-TP and SD.

Proposition 7. *π is a strict-TP if and only if there is an $\epsilon > 0$ such that if $r_i < \epsilon$ for all $i \in N$ then π is SD_r-equilibrium.*

Proof. Assume that π is not a SD_r-equilibrium for any $r > 0$. We will construct a sequence of states $\{\pi^k\}_{k=0}^{\infty}$ that converges to π, but such that for every k there is some agent i for which p_i is not a strict best response to π_{-i}^k. Let $r_k = \frac{1}{k}$. Since π is not a SD_r-equilibrium, there is some $i \in N$, a profile $\pi_{-i}^k \in \mathcal{B}_i(\pi, r_k)$, and an action a'_i such that $u_i(\pi_{-i}^k, \pi_i) \leq u_i(\pi_{-i}^k, a'_i)$. That is, π_i is not a strict best response to π_{-i}^k. We set $\pi^k = (\pi_{-i}^k, \pi_i)$. By construction, $d(\pi^k, \pi) \leq \frac{1}{k}$ and thus $\{\pi^k\}_{k=0}^{\infty}$ converges to π. $\square$

Proposition 8. *Given a game $\mathcal{G}$, if π is an R, then there is an $r > 0$ such that π is a W_r and B_r.*

Proof. We give a proof for B_r (W_r is similar). Assume that π is an R, then by definition 5, there is an ϵ such that for every $\epsilon^* \leq \epsilon$, π^* such that $\pi^* \to \pi$ is an equilibrium. Let π' be a profile such that $d(\pi', \pi) = 0$. By robustness, π' is an equilibrium. Now, for every player $i \in N$, observe that $d(\pi', \pi) > d(\langle \pi_i, \pi'_{-i} \rangle, \pi)$. Then, $\langle \pi_i, \pi'_{-i} \rangle \in \mathcal{B}_i(\pi', \epsilon)$. Hence, π_i is a best response to any sequence of $\{\langle \pi_i, \pi'_{-i} \rangle\}$ converging to π. Therefore, π_i must be a B_r best response for $r = d(\langle \pi_i, \pi'_{-i} \rangle, \pi)$. $\square$

Let $p \geq 1$ denote the p-norm [Rudin1976]. The following result suggests that our D_r definition naturally generalizes Robust equilibrium.

Proposition 9. *Let a be a pure profile in D_r, then a is $\frac{r}{n^{1/p}}$-Robust. Moreover, any ϵ-Robust equilibrium is in D_{ϵ}.*

Proof. Assume that $a \in D_r$ for some r, and consider some ϵ-noisy variant of a. $d(a_{-i}, \pi_{-i}) = \left(\sum_{j \neq i}(1 - \pi(a_j))^p\right)^{1/p} \leq (n\epsilon^p)^{1/p}$, thus $\pi_{-i} \in \mathcal{B}_i(a, r)$ for $r = (n\epsilon^p)^{1/p} = n^{1/p}\epsilon$. This means that a is ϵ-robust for $\epsilon = \frac{r}{n^{1/p}}$. In the other direction, suppose that a is ϵ-Robust and consider some π. If $\pi(a_j) > 1 - \epsilon$, then $d(\pi, a) \geq ((\pi(a_j) - 1)^p)^{1/p} > \epsilon$, which means that all vectors in $\mathcal{B}_i(a, r)$ are (at most) r-noisy variants of a. Thus for any $r \leq \epsilon$, a_i is a best-response to all of $\mathcal{B}_i(a, r)$, and is therefore in D_{ϵ}. $\square$

The following result follows immediately from the Proposition 8 and Proposition 1.

Corollary 2. *Given a game $\mathcal{G}$, if π is an R, then there is an $r > 0$ such that π is a U_r.*

See Figure 1, for a summary of the results obtained in this Section.

5 Existence Results

In Section 4, we had noted that MN implies our notions distance based notions of W, WR, B and U as a special case when $r = 0$. In this section, we deliver (non)-existence results regarding the several variants of distance-based equilibria. As a negative result, we show that D_r which is central in Figure 1 seem to be too strong to exist in general.

Proposition 10. *D_r equilibria in general do not necessarily exist.*

Proof. We give a simple counter example. Assume a game in which all actions have the same payoffs for every player. Obviously any profile is a Nash equilibrium, yet none of them is a D_r equilibrium (including $r = 0$) due to (b) of Definition 6 of local dominance. $\square$

Balancing out this bad news, the remaining distance-based solution concepts entailed by D_r do exist.

Theorem 2. *Every finite normal form game $\mathcal{G}$ has a distance-based equilibrium W_r, B_r and WR_r.*

Proof. We show it for W_r. The other cases are similar. We define the correspondence $\Gamma : \Pi \rightrightarrows \Pi$ such that $\Gamma(\pi) = [\Gamma_i(\pi_{-i})]_{i \in N}$ where $\Gamma_i(\pi_{-i})$ is the set of all W_r responses to $\mathcal{B}_i(\pi, r_i)$. It is enough to show that Kakutani's fixed point theorem [Kakutani1941] applies. Strategy spaces Π_i for each player i are as in standard case (for mixed strategy Nash equilibria), hence convex, compact and non-empty. For $r = 0$ (a zero vector) it boils down to the standard case of existence for mixed-strategies [Nash1951]. Take any $r = (r_1, \ldots, r_n)$ where r has at least one non-zero component i.e., assume for simplicity $r_i > 0$ for an arbitrary $i \in N$.

Next, we show $\Gamma(\pi)$ is non-empty. Recall that the definition of W_r (Definition 6) incorporates the payoff function u_i which is linear and continuous on Π_i, call the overall payoff function $f_i(\pi_i, \pi_{-i}) := \min_{\pi_{-i} \in \mathcal{B}_i(\pi, r)} u_i(\pi_i, \pi_{-i})$ which is continuous and linear as well. It follows by Weierstrass theorem [Rudin1976] that f_i attains an optimal value, hence $\Gamma(\pi)$ is non-empty.

Next, we show that $\Gamma(\pi)$ is a convex-valued correspondence. For all $\pi \in \Pi$, $\Gamma(\pi)$ is convex iff $\Gamma_i(\pi_{-i})$ is convex for all $i \in N$, hence it is enough to show that $\Gamma_i(\pi_{-i})$ is convex. Let $\pi', \pi'' \in \Gamma_i(\pi_{-i})$. For the sake of contradiction, assume that $\Gamma_i(\pi_{-i})$ is not convex. There is a $\lambda \in [0, 1]$ such that $\lambda f_i(\pi'_i, \pi_{-i}) + (1 - \lambda) f_i(\pi''_i, \pi_{-i}) \notin \Gamma_i(\pi_{-i})$. But then, $f_i(\lambda \pi'_i + (1 - \lambda)\pi''_i, \pi_{-i}) = \lambda f(\pi'_i, \pi_{-i}) + (1 - \lambda) f_i(\pi''_i, \pi_{-i})$, which should be in $\Gamma_i(\pi_{-i})$ which is a contradiction. Hence, $\Gamma(\pi)$ is convex.

Last, we show that Γ is upper hemi-continuous (i.e., $\hat{\pi} \in \Gamma(\pi)$ whenever $(\pi^k, \hat{\pi}^k) \to (\pi, \hat{\pi})$ with $\hat{\pi}^k \in \Gamma(\pi^k)$). Assume the contrary, then there exists a sequence $(\pi^k, \hat{\pi}^k) \to (\pi, \hat{\pi})$ with $\hat{\pi}^k \in \Gamma(\pi^k)$ and $\hat{\pi} \notin \Gamma(\pi)$. This means, there is a player $i \in N$ such that $\hat{\pi}_i \notin \Gamma_i(\pi)$. Hence, there is an $\epsilon > 0$ and a π'_i such that $f_i(\pi'_i, \pi_{-i}) > f_i(\hat{\pi}, \pi_{-i}) + 3\epsilon$. By the facts that $(\pi^k, \hat{\pi}^k) \to (\pi, \hat{\pi})$ and f_i is continuous, for large enough k we get $f_i(\pi'_i, \pi^k_{-i}) > f_i(\pi'_i, \pi_{-i} - \epsilon) > f_i(\hat{\pi}_i, \pi_{-i}) + 2\epsilon > f_i(\hat{\pi}^k, \pi^k_{-i})$. Hence, $f_i(\pi'_i, \pi^k_{-i}) > f_i(\hat{\pi}^k, \pi^k_{-i})$ which contradicts $\hat{\pi}^k_i \in \Gamma_i(\pi^k)$. This completes the proof for W_r. We omit the case for B_r since it is similar (i.e., f is modified with the corresponding best response definition, accordingly.)

The case for WR_r is slightly different. In particular, one has to justify that the corresponding payoff function f_i is continuous, and induces the convexity of $\Gamma_i(\pi_{-i})$. Define $f_i(\pi_i, \pi_{-i}) := \max_{\pi_{-i} \in \mathcal{B}_i(\pi)} \mathtt{reg}_i(\pi_i, \pi_{-i})$ where $\mathtt{reg}_i(\pi_i, \pi_{-i}) = \max_{\pi'_i \in \Pi_i}(u_i(\pi'_i, \pi_{-i})) - u_i(\pi_i, \pi_{-i})$. Now, observe that since u_i is continuous and multilinear, the regret expression (which is in the form of substraction) and under $\max$ operation is continuous and multilinear, hence f_i as well, which then enables the similar reasoning (aforementioned in W_r case) about the non-emptiness (i.e., attaining a minimum value by Weierstrass theorem), the upper-hemi continuity, and the convexity of $\Gamma(\pi)$. □

Now the following result is an immediate result from above theorem and Theorem 1.

Corollary 3. *Every finite normal form game $\mathcal{G}$ has a distance based equilibrium U_r.*

6 Discussion through Examples

To provide a better intuition, we give examples of well-known 2×2 games from the basic game theory literature, and compare the outcomes of standard notions of equilibria against some notions of equilibria that we defined via local responses. We consider both the case of mixed strategies, and pure strategies.

For convenience, the assumed metric is Euclidean; hence, if the opponent plays a mixed strategy $(x, 1 - x)$, the player believes that the strategy is anywhere in the set $\{(y, 1 - y) : \max\{0, x - r\} \leq y \leq \max\{1, x + r\}\}$.

6.1 Trembling-Hand Game

	Left	Right
Up	1, 1	2, 0
Down	0, 2	2, 2

Figure 2: Trembling-Hand Game where both $\langle$Up, Left$\rangle$ and $\langle$Down, Right$\rangle$ are PN, yet only $\langle$Up, Left$\rangle$ is T.

Call the example given in Figure 2 Trembling-Hand Game for demonstration purposes. It seems that the pure strategy Nash equilibrium PN ={(Up, Left), (Down, Right)} while trembling-hand perfect equilibrium T={(Up, Left)} which matches with W.

Assume that $r_1 = r_2 \approx 0.14$. Now consider two mixed Nash strategy equilibria $\pi = \langle (1, 0), (1, 0) \rangle$ and $\pi' = \langle (0, 1), (0, 1) \rangle$ where the former is also a T. See that for $\mathcal{B}_1(\pi)$, every strategy is dominated by π_1 in terms of B-best response and W-best response (and this is the case analogously for the second agent). Moreover, regret increases as agent 1 diverges from $(1, 0)$. Hence $\pi \in B_r \cap W_r$.[5] In the case of π', it is a B_r-response since payoffs are already 2 for both agents. On the other

[5]Indeed, for such value of r, the strategy of the opponent varies only by 0.1. And the worst case is defined by the case that the opponent plays $(1, 0)$.

hand, $\pi' \notin B_r$ since worst case keeps improving for any agent who keeps deviating. Therefore, regret also gets minimized since the best case value is fixed at 2.

6.2 Matching Pennies

In the game of *Matching Pennies*, $PN = \emptyset$ whereas $W = B$ contains all four outcomes.

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Figure 3: Matching Pennies

The set of mixed strategy Nash equilibria is a singleton i.e., $MN = \{\pi\}$ where $\langle(0.5, 0.5), (0.5, 0.5)\rangle$. Take $r \approx 0.14$ as usual for both agents. It is easy to see that $\pi \in W$. Surprisingly enough, however, it seems that $\pi \notin B$ (more generally, it is the case for any value greater than zero). To see that, notice when agent 2 is playing $(0.5, 0.5)$, agent 1 sees it instead as the set of strategies between $(0.4, 0.6)$ and $(0.6, 0.4)$ (i.e., $\mathcal{B}_1(\pi) = [(0.4, 0.6), (0.6, 0.4)]$, and therefore responses with the strategy $(0, 1)$ which has the highest expected payoff (that is 0.2) which is also the strategy that is dominated by any strategy in the worst case.

6.3 Stag Hunt

We continue with the rather well-known coordination game Stag-Hunt illustrated in Figure 4.

	Stag	Hare
Stag	-1, -1	-4, -2
Hare	-2, -4	-3, -3

Figure 4: Stag-Hunt Game

Pure strategy-Nash equilibria are $\langle\text{Stag}, \text{Stag}\rangle$ (i.e., $\langle(1, 0), (1, 0)\rangle$) and $\langle\text{Hare}, \text{Hare}\rangle$ (i.e., $\langle(0, 1), (0, 1)\rangle$). Trembling-hand perfect equilibria are also those equilibria. Mixed Nash equilibria is the one $\langle(0, 5), (0, 5)\rangle$ which is not T, in addition to the ones in PN. Realize that (Stag, Stag) is not stable when it is with a sufficiently large penalty (e.g., $-\infty$). We obtain the same with a finite r.

Now, see that for any given profile π, if $(0.5, 0.5) \notin \mathcal{B}_i(\pi, r)$ and $\liminf \mathcal{B}_i(\pi, r) > 0$, then $W-$ best response (and also the $B-$ best response) is $(1, 0)$ for both players, and exact opposite i.e., for $(0, 1)$ $\limsup \mathcal{B}_i(\pi, r) < 0$. However, if $r > 0$ and is large enough such that $(0.5, 0.5) \in \mathcal{B}_i(\pi, r)$ is an interior point, then the W-best response is $(0, 1)$ whereas B-best response is $(1, 0)$ for both players, yielding a better outcome for each (i.e., -1 for both).

7 A Bliss of Ignorance

Once a new notion of equilibrium is introduced, it is natural to ask whether it leads to any efficient outcome in the game. In this section, we explore this question and provide with a class of normal form games that this is indeed the case. In doing so, we employ the notion of *Price of Anarchy* which is a central to measuring how much a system becomes inefficient due to selfish behaviour [Nisan *et al.*2007] (recall the Section 2 for formal definition). The class of games we introduce is the following.

Definition 7 (Consensus Game)**.** *A normal form game $\mathcal{G} = (N, A, u)$ where $A_i \geq 2$ for every $i \in N$ is called a* consensus game *is $u_i(a') = c'$ and $u(a) = c$ for every $a \in A \setminus \{a'\}$ with $c' > c$.*

Intuitively, a consensus game is a coordination game in which only a single pure strategy profile has a higher payoff i.e., c' for every player compared to all the other pure strategy profiles which has c. Such game model (group) scenarios in which every member player has to agree unilaterally a decision to be taken (e.g., World Trade Organization).

One can observe that any given consensus game has at least two pure strategy equilibria a' and a such that only one of them has a more desirable outcome i.e., $SW(a') = n \cdot c'$, and $SW(a) = n \cdot c$. The following result guarantees that once the rational agents are endowed with an ignorance factor $r > 0$, then the undesirable outcome is not a prediction anymore (since it is not an equilibrium anymore).

Proposition 11. *Every consensus game has a unique D_r equilibrium where $r_i > 0$ for all $i \in N$. Moreover, PoA is 1.*

Proof. Observe that profile a' has the best possible payoff for every single agent, hence it is an equilibrium. Moreover, due to the linearity of utilities (i.e., $\pi_i(a')c' + (1 - \pi_i(a'))c > \pi'_i(a')c' + 1 - \pi'_i(a')c$ whenever $\pi_i(a') > \pi'_i(a')$, pure strategy $\pi_i(a') = 1$ locally dominates every other strategy (i.e., $\alpha = 1$) for any $\mathcal{B}_i(\pi'', r)$ with $r > 0$, hence it is unique (since, by definition 6 there cannot be two distinct best response which can locally dominate each other). As it is best possible outcome PoA becomes 1. □

Exploring such scenarios and extending them to more general class of games is a part of future work. Yet still to develop a general understanding, it is of paramount importance to look at PoA from the lenses of distance-based uncertainty. In this regard, we deliver our final technical result. In particular, we provide a bound (in terms of r) on the gain/loss of social welfare in an equilibrium when uncertainty grows.

The smoothness framework provides a convenient tool to bound the PoA in games [Roughgarden2009]: If there are $\lambda, \mu > 0$ s.t. for any two pure profiles a, a' we have

$$\sum_{i\in N} u_i(a'_i, a_{-i}) \geq \lambda \sum_{i\in N} u_i(a') - \mu \sum_{i\in N} u_i(a),$$

then for any pure/mixed/correlated/coarse-correlated equilibrium π^* and any profile $\vec{a}$:

$$\frac{SW(\pi^*)}{SW(a')} \geq \frac{\lambda}{1+\mu}.$$

The proof is trivial for pure equilibria. Now, the question we ask is "can we extend this result to $\star_r$ equilibria (perhaps with a relaxed bound)"?

For a game $\mathcal{G}$, let

$$\delta_G(r) = \max\{\max\left\{\frac{u_i(a_i, \pi_{-i})}{u_i(a_i, \pi'_{-i})}, \frac{u_i(a_i, \pi'_{-i})}{u_i(a_i, \pi_{-i})}\right\} : i \in N, a_i \in A_i, \pi'_{-i} \in B_i(\pi_{-i}, r)\}, \quad (1)$$

i.e., the maximal utility ratio of an agent within a sphere of radius r.

Theorem 3. *If there are $\lambda, \mu > 0$ s.t. for any two pure profiles a, a' we have*

$$\sum_{i\in N} u_i(a'_i, a_{-i}) \geq \lambda \sum_{i\in N} u_i(a') - \mu \sum_{i\in N} u_i(a),$$

then for any $\star_r$-pure equilibrium a^* (for any $\star \in \{U, D, W, B\}$) and any profile a':

$$\frac{SW(a^*)}{SW(a')} \geq \frac{\lambda}{\delta_G(r)^2 + \mu}.$$

Note that when $r = 0$, we have $\delta_G(r) = 1$ and the bound boils down to the familiar smoothness bound.

Proof. We prove for U_r-equilibrium. Since a^*_i is undominated, there is some $\pi'_{-i} \in B_i(a^*_{-i}, r)$ s.t. $u_i(a^*_i, \pi'_{-i}) \geq u_i(a'_i, \pi'_{-i})$. We denote $\delta = \delta_G(r)$.

$$\begin{aligned} u_i(a^*) = u_i(a^*_i, a^*_{-i}) &\geq \frac{1}{\delta} u_i(a^*_i, \pi'_{-i}) \\ &\geq \frac{1}{\delta} u_i(a'_i, \pi'_{-i}) \geq \frac{1}{\delta^2} u_i(a'_i, a^*_{-i}). \end{aligned}$$

Thus

$$\begin{aligned} SW(a^*) = \sum_{i\in N} u_i(a^*) &\geq \sum_{i\in N} \left(\frac{1}{\delta^2} u_i(a'_i, a^*_{-i})\right) \\ &\geq \frac{1}{\delta^2}\lambda \sum_{i\in N} u_i(a') - \frac{1}{\delta^2}\mu \sum_{i\in N} u_i(a^*) \\ &= \frac{\lambda}{\delta^2} SW(a') - \frac{\mu}{\delta^2} SW(a^*). \end{aligned}$$

Rearranging,

$$\left(1 + \frac{\mu}{\delta^2}\right) SW(a^*) \geq \frac{\lambda}{\delta^2} SW(a'),$$

and thus

$$SW(a^*) \geq \frac{\lambda}{\delta^2(1 + \frac{\mu}{\delta^2})} SW(a') = \frac{\lambda}{\delta^2 + \mu} SW(a'),$$

as required. □

8 Conclusion and Future Avenues

We have introduced a distribution-free agent model based on strict uncertainty, and studied consequent equilibria notions under different best-response behaviours. In the context of normal-form games, we explored the links between the notions we defined and a handful of existing well-known solution concepts which model mistakes and imprecision such as Trembling Hand Perfect equilibrium (variants) and robust equilibrium. For instance, it is shown that our notion is naturally generalizes Robust equilibrium. It seems that strict equilibrium notion D_r does not exist in general while all other entailed distance-based notions exist. Complementing those existence results with complexity results is an interesting line of future work.

We looked for a possible scenario in which such solution concepts could potentially be useful, and introduced a coordination game in which ignorance was indeed helpful for the players to avoid a worst-outcome. Investigating more general game classes that distance-based uncertainty solutions give rise to nice outcome guarantees deserves a further study on its own, and is our high priority for future research. As a more general outlook, we showed how to bound the loss of social welfare in any equilibrium (PoA) as uncertainty grows in terms of ignorance factor r. It would be nice to obtain finer bounds for games with different local-best responses.

Moreover, studying these notions on certain classes of games e.g., repeated games, as well as extending to extensive form games in general is our future research agenda.

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