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Exam SS 2019

Information Theory and Coding

Name: Sample solution	Student ID:	
	Points	From
Task 1		
Task 2		
Total points		
Grade		

- The following aids are allowed in this exam:
  - 2 DIN A4 sheets, **handwritten** on both sides (4 pages in total)
  - Calculator (non-programmable, not graphical, not capable of communication)
  - Pens
- Other aids are not allowed.
- Please use a separate solution sheet for each task.
- Write your name and matriculation number on each solution sheet.
- An arrow next to a question means that this part of the task can be solved independently of the rest of the task.
- For calculations the approach as well as the steps must be specified.
- Please do not write with pencils and do not use a red pen.
- The duration of the exam is 90 minutes.
- The exam consists of 11 pages (including this cover page).
- Switch off your cell phones!

Task 1: Channel Capacity of Bandpass and Baseband AWGN Channels

In the following problems, we consider the channel capacities of bandpass and baseband AWGN channels. The two channels deal with real valued signals. In both cases the energy per symbol E_s is the same.
Figure 1a depicts the noise power spectral density of a bandpass AWGN channel with bandwidth B and center frequency f_c. Figure 1b depicts the noise power spectral density of a baseband AWGN channel with bandwidth B:

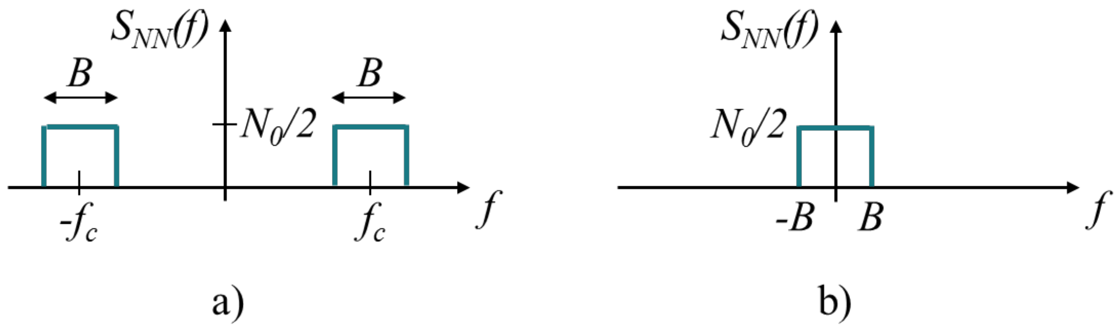


Figure 1: Power spectral densities of AWGN channels.

a) Give the equation for the channel capacity C^* in bit/s of the bandpass channel in Figure 1a. Express the capacity in terms of E_s and N_0.
Hint: For the bandpass channel the bandwidth is related to the symbol spacing through $T_s = \frac{1}{B}$.

From the lecture we know that the channel capacity is given by

$$C^* = B \log_2 \left(1 + \frac{P_X}{P_N}\right) \text{ in } \frac{\text{bit}}{\text{s}} \tag{1}$$

Considering the bandpass channel of figure 1a, the noise power is computed as $P_N = \frac{N_0}{2} 2B$. Further, in a *bandpass channel* the bandwidth B is connected to the symbol spacing T_s through $B = \frac{1}{T_s}$. Defining E_s as the energy used for each symbol, the power of the signal yields $P_X = \frac{E_s}{T_s}$.

Inserting the definitions into the capacity formula leads to

$$\begin{aligned} C^* &= B \log_2 \left(1 + \frac{P_X}{N_0 B}\right) \\ &= B \log_2 \left(1 + \frac{E_s/T_s}{N_0/T_s}\right) \\ &= B \log_2 \left(1 + \frac{E_s}{N_0}\right) \text{ in } \frac{\text{bit}}{\text{s}} \end{aligned} \tag{2}$$

b) Derive the normalized channel capacity C in bit/channel use of the bandpass channel in Figure 1a, starting from your result in a). Express the capacity in terms of E_s and N_0.

The normalized capacity C states the limit of maximum error-free transmitted information for one channel use. A channel is used each time we transmit a symbol. The average amount of information contained in each symbol is computed through normalizing by the symbol spacing $T_s = \frac{1}{B}$. Thus,

$$\begin{aligned} C &= C^* T_s \\ &= \log_2\left(1 + \frac{E_s}{N_0}\right) \text{ in } \frac{\text{bit}}{\text{channel use}} \text{ or } \frac{\text{bit}}{\text{s Hz}} \end{aligned} \tag{3}$$

c) Give the equation for the channel capacity C^* in bit/s of the baseband channel in Figure 1b. Express the capacity in terms of E_s and N_0.
Hint: For the baseband channel of figure 1b the bandwidth is related to the symbol spacing through $T_s = \frac{1}{2B}$.

The baseband channel capacity C can be computed using the same formula as in task 1 a):

$$C^* = B \log_2\left(1 + \frac{P_X}{P_N}\right) \text{ in } \frac{\text{bit}}{\text{s}} \tag{4}$$

According to figure 1b, the spectrum of the baseband channel has a width of $2B$. Thus, a real valued signal can exploit this spectral width and we can transmit symbols at the Nyquist rate with symbol spacing $T_s = \frac{1}{2B}$. The noise power is computed as $P_N = \frac{N_0}{2} 2B$. Defining E_s as the energy used for each symbol, the power of the signal yields $P_X = \frac{E_s}{T_s}$.
Inserting the definitions into the capacity formula leads to

$$\begin{aligned} C^* &= B \log_2\left(1 + \frac{P_X}{N_0 B}\right) \\ &= B \log_2\left(1 + \frac{E_s/T_s}{N_0/(2T_s)}\right) \\ &= B \log_2\left(1 + \frac{2E_s}{N_0}\right) \text{ in } \frac{\text{bit}}{\text{s}} \end{aligned} \tag{5}$$

Note that we fixed the energy per symbol to the same value E_s as for the bandpass channel capacity derived in task 1a. As a result the power used in the baseband channel $P_{X,c)} = E_s/T_{s,c)} = E_s 2B$ is by a factor of 2 larger than the power used in the bandpass channel $P_{X,a)} = E_s/T_{s,a)} = E_s B$.

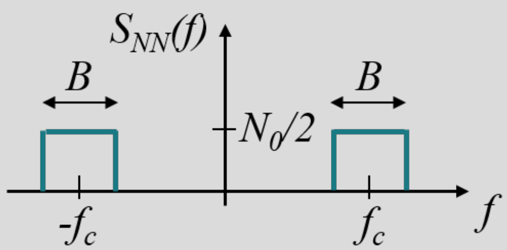
d) Derive the normalized channel capacity C in bit/channel use of the baseband channel in Figure 1b, starting from your result in c). Express the capacity in terms of E_s and N_0.

Like in task 1b) the normalized capacity is computed by normalizing with the symbol spacing $T_s = 1/(2B)$ leading to

$$\begin{aligned} C &= C^* T_s \\ &= \frac{1}{2}\log_2\left(1 + \frac{2E_s}{N_0}\right) \text{ in } \frac{\text{bit}}{\text{channel use}} \text{ or } \frac{\text{bit}}{\text{s Hz}} \end{aligned} \tag{6}$$

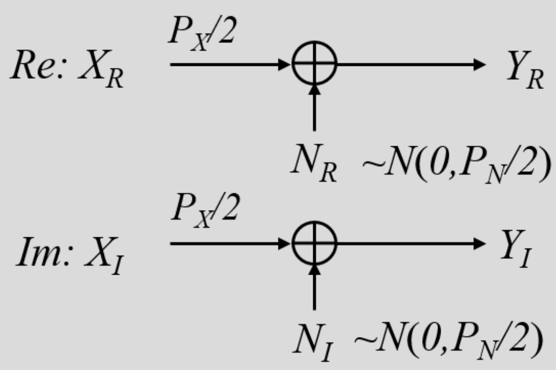
e) Sketch the noise power spectral density of the **equivalent** baseband AWGN channel corresponding to the **bandpass** channel in Figure 1a. Label the axes completely.

The bandpass noise power spectral density $S_{NN}(f)$ can be represented by the power spectral density $S_{N_{BB}N_{BB}}(f)$ of the equivalent baseband model:



f) Give the equation for the channel capacity C in bit/channel use of the **equivalent** baseband channel according to e). Compare your solution to the solutions from b) and d). Explain differences.

A equivalent baseband channel is a complex AWGN channel. Real and imaginary part are real AWGN channels with independent noise of same power $P_{N,R} = P_{N,I} = \frac{N_0}{2}B$. According to the waterfilling principle, the signal power P_X is split uniformly among the two constituend channels.



The capacity is the sum capacity of those two channels.

$$
\begin{aligned}
C^* = C_I^* + C_Q^* = 2 \cdot C_I^* &= 2\frac{B}{2}\log_2\left(1 + \frac{P_X/2}{\frac{N_0}{2}B}\right) \\
&= B\log_2\left(1 + \frac{P_X}{N_0 B}\right) \\
&= B\log_2\left(1 + \frac{E_s}{N_0}\right)
\end{aligned}
\tag{7}
$$

The normalized capacity yields

$$
C = C^* T_s = \log_2\left(1 + \frac{E_s}{N_0}\right) \tag{8}
$$

The solution corresponds to b) as the equivalent baseband channel represents a bandpass AWGN channel according to figure 1a).

The baseband channel of d) has lower normalized capacity, since only one complex dimension is available rather than two (R and I).

Task 2: Protograph-Based Raptor-Like (PBRL) LDPC Codes for 5G Wireless Communications Systems.

For 5G wireless communications systems, so-called protograph-based Raptor-like (PBRL) LDPC codes have been standardized. A PBRL code is defined via a small protograph as depicted in the following Figure 2:

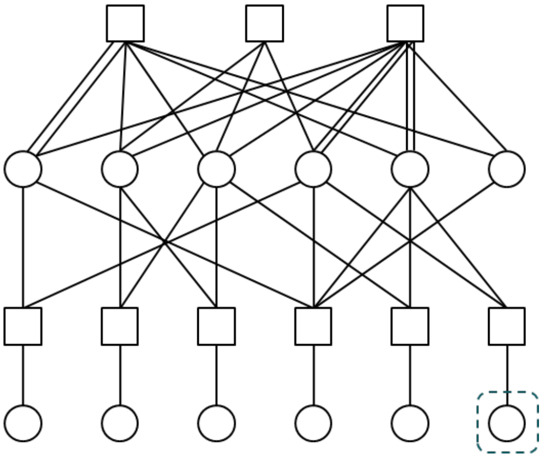


Figure 2: Protograph of a PBRL LDPC code.

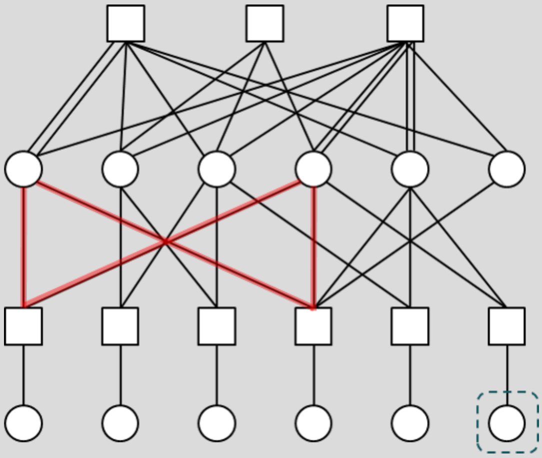
Bit nodes (variable nodes) are represented by circles, check nodes are represented by squares. Note that the nodes are a bit shuffled compared to the conventional drawing of a Tanner graph where usually all bit nodes are aligned in a row and all check nodes are aligned in another row. Also note that the protograph contains double edges. The Tanner graph of a larger code is obtained by making multiple copies of the protograph and reconnecting the nodes without changing the degree distribution.

a) Determine the parity check matrix $\mathbf{H}^T$ of the protograph.
Hint: Represent a double edge in the protograph by a "2" in the parity check matrix.

$$\mathbf{H}^T = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

b) Determine the girth of the protograph.
Hint: A double edge can be treated like a single edge.

The girth of the graph is the length of the shortest cycle. Here the girth is 4.



c) Determine the bit node degree distribution $\mathbf{d}_b$ of the protograph.

$$\mathbf{d}_b = \begin{bmatrix} \frac{6}{12} & 0 & \frac{1}{12} & 0 & \frac{2}{12} & \frac{3}{12} \end{bmatrix}^T \tag{10}$$

d) Determine the check node degree distribution $\mathbf{d}_c$ of the protograph.

$$\mathbf{d}_c = \begin{bmatrix} 0 & 0 & \frac{6}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} \end{bmatrix}^T \tag{11}$$

e) Does the protograph represent a regular or an irregular LDPC code? Give reasons.

The code is irregular since among the check nodes the node degree is not constant which holds also for the variable nodes.

f) Determine the code rate R of the code which is represented by protograph.

The code rate can be compute using the degree distribution according to

$$
\begin{aligned}
R &= 1 - \frac{\sum_k k d_{b,k}}{\sum_l l d_{c,l}} \\
&= 1 - \frac{1\frac{6}{12} + 3\frac{1}{12} + 5\frac{2}{12} + 6\frac{3}{12}}{3\frac{6}{9} + 5\frac{1}{9} + 6\frac{1}{9} + 8\frac{1}{9}} \\
&= 1 - \frac{\frac{37}{12}}{\frac{37}{9}} \\
&= \frac{1}{4}
\end{aligned}
\tag{12}
$$

Alternatively the code rate is given by

$$
R = \frac{N - \text{rank}(\mathbf{H}^T)}{N} = \frac{N - (N - K)}{N} = \frac{1}{4} \tag{13}
$$

g) Assume that a longer code is obtained by copying the protograph three times. Determine the number K_c of information bits which is mapped to one codeword.

The code rate of the original protograph is

$$
R = \frac{K}{N} = \frac{1}{4} \tag{14}
$$

Thus, the number of information bits is $K = R \cdot N = 1/4 \cdot 12$.

Copying the graph three time triples the number of information bits to $K_c = 9$.

A different code rate can be obtained by puncturing the degree-one bit nodes at the bottom of Figure 2. In the remaining problems, we consider a code represented by the protograph (no further copies) where the right most degree-one bit node (which is marked in Figure 2 by the dashed box) is punctured.

h) What does puncturing mean? Answer in complete sentences.

An encoder delivers a codeword of length N bits. Then a certain number of codebits may be dropped, i.e. not transmitted.

This process of dropping codebits is called puncturing.

i) Determine the code rate R_p of the punctured code. Derive the code rate R_p from the code rate R of the mother code (i.e. the non-punctured code).

The code rate before puncturing is $R = \frac{K}{N}$. Dropping n bits reduces the number of code bits which need to be transmitted to $N_p = N - n$. Then the punctured code rate is given by

$$R_p = \frac{K}{N_p} = \frac{K}{N-n} = \frac{K}{N}\frac{N}{N-n} = R\frac{N}{N-n} \tag{15}$$

For the given protograph $N = 12$. We set $n = 1$ since only the degree one bit-node is punctured. Thus, the code rate including puncturing yields

$$R_p = \frac{1}{4}\frac{12}{12-1} = \frac{3}{11} \tag{16}$$

j) What is the effect of puncturing on the parity check represented by the check node to which the punctured bit node is connected? Answer in complete sentences.

The bit node x_{11}, where puncturing is applied, is only connected to one check node. Two other bit nodes x_3 and x_4 are also connected to this check node. The corresponding parity check equation yields

$$x_3 \bigoplus x_4 \bigoplus x_{11} = 0 \tag{17}$$

From the principle of soft message passing decoding we know that *extrinsic* information is send from the check node to connected bit nodes. E.g. we can infer information about x_3 through the parity check equation from the bit nodes x_4 and x_{11}, according to

$$x_3 = x_4 \bigoplus x_{11} \tag{18}$$

The reliablity about the estimate for x_3 from that equation, can be computed as a log-likelihood ratio:

$$L(x_3) = L(x_4) \boxplus L(x_{11}) \tag{19}$$

Since no value is received from the channel for x_{11}, the reliability for x_{11} being a 1 or a 0 is the same and the corresponding log likelihood ratio yields $L(x_{11}) = 0$. Thus,

$$L(x_3) = L(x_4) \boxplus L(x_{11}) = 0 \tag{20}$$

Further, x_{11} is only connected to a single checknode and will never receive extrinsic information from other check nodes. As a result the extrinsic information sent towards bit node x_3 is zero in all iterations.

Correspondingly the same as for x_3 does apply for bit node x_4.

Also, since x_{11} is not connected to any other check node, no extrinsic information needs to be computed for x_{11}, except for making the final decision on x_{11}.

Consequently, the check node can be eliminated from the Tanner graph. This corresponds to deleting the last row from $\mathbf{H}^T$. Puncturing essentially removes the last column from $\mathbf{H}^T$.

$$\mathbf{H}^T = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

k) Determine the effective bit node and check node degree distributions d_b and d_c of the punctured protograph.

According to the result of j) due puncturing one checknode can be eliminated from the Tanner graph and effectively the bit node degree distribution from c) changes to

$$\mathbf{d}_b = \begin{bmatrix} \frac{5}{11} & 0 & \frac{1}{11} & 0 & \frac{4}{11} & \frac{1}{11} \end{bmatrix}^T \tag{21}$$

and the check node degree distribution from d) to

$$\mathbf{d}_c = \begin{bmatrix} 0 & 0 & \frac{5}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}^T \tag{22}$$

l) Determine the design code rate resulting from the degree distribution in k). Compare to your solution from i) and explain why the code rates from l) and i) are the same or why they are different.

$$\begin{aligned} R =& 1 - \frac{\sum_k k d_{b,k}}{\sum_l l d_{c,l}} \\ =& 1 - \frac{1\frac{5}{11} + 3\frac{1}{11} + 5\frac{4}{11} + 6\frac{1}{11}}{3\frac{5}{8} + 5\frac{1}{8} + 6\frac{1}{8} + 8\frac{1}{8}} \\ =& 1 - \frac{\frac{34}{11}}{\frac{34}{8}} \\ =& \frac{3}{11} \end{aligned} \tag{23}$$

The code rates from i) and l) are the same. The punctured code is essentially described by the reduced parity check matrix, where the last column and the last row of $\mathbf{H}$ from a) are eliminated. This can also be viewd as deactivating a part of the Tanner graph.

The degree distribution of the reduced parity cgeck matrix determines the code rate of the punctured code.

The deacitvated part of the Tanner graph may only be taken into account by the decoder in the last iteration for the final decision on the punctured bit.