1) a) $$H(X) = \sum_x p_X(x) \log_2 \frac{1}{P_X(x)} \quad 1$$

$$= \frac{1}{2}\log_2 2 + \frac{1}{4}\cdot\log_2 4 + 2\cdot\frac{1}{8}\cdot\log_2 8$$

$$= \frac{1}{2} + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 3 \quad 1$$

$$= \frac{7}{4} \text{ bit} \quad 1$$

3

b)

	$P_X(x)$	
a	1/2	
b	1/4	
c	1/8	0
d	1/8	1

1

	$P(x)$	
a	1/2	
b	1/4	0
c, d	1/4	1

1

a	0
b, c, d	1

1

x	c
a	0
b	1 0
c	1 1 0
d	1 1 1

2

5

c) $$E\{l\} = P_X(a)\cdot 1 + p_X(b)\cdot 2$$

$$+ P_X(c)\cdot 3 + p_X(d)\cdot 3 \quad 1$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{8}\cdot 3 \quad 1$$

3

$$= \frac{4 + 2\cdot 2 + 3 + 3}{8} = \frac{14}{8} = \frac{7}{4} \text{ bit/symbol} \quad 1$$

$$= 1{,}75 \text{ bit/symbol}$$

2

1d)

	$P(x)$	
d	$1/2$	
c	$1/4$	
b	$1/8$	0
a	$1/8$	1

1

	$P(x)$	
d	$1/2$	
c	$1/4$	0
b, a	$1/4$	1

1

d	0
c, b, a	1

X	$\underline{w}$
a	1 1 1
b	1 1 0
c	1 0
d	0

2

4

e) $E\{l\} = p_X(a)\cdot 3 + p_X(b)\cdot 3 + p_X(c)\cdot 2 + p_X(d)\cdot 1$ 2

$= \frac{1}{2}\cdot 3 + \frac{1}{4}\cdot 3 + \frac{1}{8}\cdot 2 + \frac{1}{8}\cdot 1$ 6

$= \frac{12+6+2+1}{8} = \frac{21}{8}$ bit/symbol $= 2{,}625 > 1{,}75 \frac{\text{bit}}{\text{symbol}}$ 1 1 1

The average codeword length is larger than for the Huffman Code from b)/c). 1

f)

$$D(p\|q) = \sum_x p_X(x) \log_2 \frac{p_X(x)}{q_X(x)} \quad 1$$

$$= p_X(a) \log_2 \frac{p_X(a)}{q_X(a)} + p_X(b) \log_2 \frac{p_X(b)}{q_X(b)}$$

$$+ p_X(c) \log_2 \frac{p_X(c)}{q_X(c)} + p_X(d) \log_2 \frac{p_X(d)}{q_X(d)} \quad 1$$

$$= \frac{1}{2} \log_2 \frac{\frac{1}{2}}{\frac{1}{8}} + \frac{1}{4} \log_2 \frac{\frac{1}{4}}{\frac{1}{8}}$$

$$+ \frac{1}{8} \log_2 \frac{\frac{1}{8}}{\frac{1}{4}} + \frac{1}{8} \log_2 \frac{\frac{1}{8}}{\frac{1}{2}} \quad 1$$

$$= \frac{1}{2} \log_2 \frac{8}{2} + \frac{1}{4} \log_2 \frac{8}{4} + \frac{1}{8} \log_2 \frac{4}{8} + \frac{1}{8} \log_2 \frac{2}{8}$$

$$= \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 1 - \frac{1}{8} \cdot 1 - \frac{1}{8} \cdot 2$$

4

$$= \frac{8 + 2 - 1 - 2}{8} = \frac{7}{8} \text{ bit} \quad 1$$

g) The average codeword length of the Huffman code from d), which was constructed based on the wrong probability distribution $q_X(x)$, is given by 8

$$E\{l\} = H(X) + D(p\|q) = \frac{7}{4} + \frac{7}{8} = \frac{21}{8} \frac{\text{bit}}{\text{symbol}}.$$

The relative entropy $D(p\|q)$ is the penalty in terms of achievable average codeword length when using the ~~wrong~~ wrong probability distribution $q_X(x)$ instead of the true probability distribution $p_X(x)$ for Huffman code construction.

h) $D(p \| q) = \sum_x p_X(x) \log_2 \frac{p_X(x)}{q_X(x)}$ 1

$-D(p \| q) = \sum_x p_X(x) \log_2 \frac{q_X(x)}{p_X(x)}$ 2

$= \cancel{\sum_x p_X(x) \log_2 \frac{q_X(x)}{p_X(x)} + \sum_x p_X(x) \log_2 q_X(x)}$

$= E_X\left\{ \log_2 \frac{q_X(x)}{p_X(x)} \right\}$ 1 $\cancel{+ E_X\{\log_2 q_X(x)\}}$

Jensen's inequality

$\leq \log_2 \left(E_X\left\{ \frac{q_X(x)}{p_X(x)} \right\} \right)$ 3

$= \log_2 \left(\sum_x p_X(x) \frac{q_X(x)}{p_X(x)} \right)$ 1

$= \log_2 \underbrace{\sum_x q_X(x)}_{1}$ 1

$= \log_2 1 = 0$ 1

(12)

$\Rightarrow -D(p \| q) \leq 0$ 1

$\Rightarrow D(p \| q) \geq 0$ 1

5

i) $$D(p\|q) = \sum_x p_X(x) \underbrace{\log_2 \frac{p_X(x)}{q_X(x)}}_{= 0 \text{ for } p_X(x) = q_X(x)} \quad 1$$

1 1

6

$$\Rightarrow D(p\|q) = 0 \text{ if } p_X(x) = q_X(x) \ \forall x$$

1 2

j) $$D(p\|q) = \sum_x p_X(x) \log_2 \frac{p_X(x)}{q_X(x)}$$

$$\underset{\text{in general}}{\overset{1}{\neq}} \sum_x q_X(x) \log_2 \frac{q_X(x)}{p_X(x)} = D(q\|p)$$

1 1

4

$\Rightarrow$ The relative entropy is not symmetric and, therefore, is actually not a true distance.

2

k) $$I(X,Y) = \sum_x \sum_y p_{X,Y}(x,y) \log_2 \frac{p_{X,Y}(x,y)}{p_X(x)\, p_Y(y)}$$

2

3

$$= D\left(p_{X,Y}(x,y) \,\|\, p_X(x)\, p_Y(y)\right)$$

1

$\sum$ ~~58~~ 46

overhead: ~~46~~ 6 – 8

min: 22–23

1 2 a) $N = 7$ 1

b) $K = 4$

1 $R = \frac{K}{N} = \frac{4}{7}$ 1

1 c) $d_{min} = 3$ 1 from

from: $(N, K, d_{min})_2 = (7, 4, 3)_2$

d) Number of parity checks is number of columns

2 in $\underset{\sim}{H} \Rightarrow$ 3 parity checks 2

e)

$$\underset{\sim}{H} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$2 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad 1 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 2 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{\sim}{\bar{H}}$$

cont e)

$$\underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{P}^T \\ \underset{\sim}{I} \end{bmatrix} \quad 1$$

10

$$\underset{\sim}{G} = \begin{bmatrix} \underset{\sim}{I} \\ \underset{\sim}{P} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad 1$$

2

f) Number of codewords = number of info words

5 2^{K-M} 2 (since 2^M codewords are dropped)

$\Rightarrow$ Info word length: $K-M$ 3

3 g) $R_s = \dfrac{N-M}{K-M}$ 1 1 (for all) 1

h) For linear 1 codes, the min. Hamming distance d_{min} 1 is equal to the min Hamming weight 1 w_{min}.

10 Since only code bits with value 0 are dropped, 1 the minimum Hamming weight and consequently the min. Hamming distance ~~remain the same~~ cannot become smaller 2 of the original code ~~as for the original code.~~ However, codewords with min. Hamming weight could be excluded 1. Therefore, the shortened code will have at least 1 the same 1 min. Hamming distance as the original code, i.e. $d_{min,s} \geq d_{min}$ 1

i) $\underset{\sim}{H}_s$ is obtained by deleting (8)

the first M rows in $\underset{\sim}{H}$, which correspond to the deleted code bits. (3)

$$\underset{\sim}{H} = \begin{array}{c} \\ N \end{array} \overset{N-K}{\begin{bmatrix} \quad \end{bmatrix}}$$

5

$$\underset{\sim}{H}_s = \begin{array}{c} \\ N-M \end{array} \begin{bmatrix} \quad \end{bmatrix}$$

$$N-M-(K-M)$$
$$= N-K$$

$\Rightarrow$ $\underset{\sim}{H}_s$ has dimension $N-M \times N-K$ (2)

j) Choose all code words from ~~table~~ ~~set~~ table which have $M=3$ leading zeros:

0000000
0001111 (2) (4) (1)

5 Delete the three leading zeros of the chosen codewords in order to obtain the codewords of the shortened code:

0000 (2) $\Rightarrow$ Rate $R = \frac{1}{4}$ repetition code.
1111 (1) (4) (1)

8 l

k) Another common method for code shortening is puncturing. 2

The information is encoded using the encoder of the original code. 2 Then, certain codebits are deleted 2 from the codeword (punctured) according to a given puncturing patten.

Puncturing is typically applied for code rate adaptation of convolutional codes.

6

l) $N \to N+1$ 1

(3) $\Rightarrow R = \frac{K}{N+1}$ 1 $= \frac{4}{7+1} = \frac{4}{8} = \frac{1}{2}$ 1

m) $\underset{\sim}{H}_e^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & & & 0 \\ & & \underset{\sim}{H} & & & & & 0 \\ & & & & & & & 0 \end{bmatrix}$ ← 2 ; ↑ 2 ; $\underset{\sim}{H}$ 2

(8) $\Rightarrow \underset{\sim}{H}_e = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ 2

(or with $\underset{\sim}{H}$ according to the solution from e)

n) The min. Hamming weight of the original code is $d_{min} = 3$ i.e. odd. 2

$\Rightarrow$ ~~Adding an~~ Therefore, the additional overall parity check bit ~~to the~~ for the min. weight codeword will be a "1" 2 ~~in order to yield~~

(10) ~~Consequently, the~~ since the mod 2 sum over all code bits needs to be zero for the overall parity check.

Therefore, the min. Hamming weight 1 and, consequently, the min. Hamming distance 1 will increase 2 by 1 compared to the original code.

$\Rightarrow d_{min,e} = d_{min} + 1 = 3 + 1 = 4$ 2

Σ ~~64~~ 49

overhead: ~~25~~ ~~30~~ 15
min: ~~40~~ 8–10

Problem 1: 46

Problem 2: 49

95

necessary for 4,0: 30 - 35

overhead: 23

↳ necessary for 1,0: (possibly more)

66 - 69