TUHH
Hamburg University of Technology

INSTITUT FÜR NACHRICHTENTECHNIK

Exam SS 2018

Information Theory and Coding

Name: Sample solution	Student ID:	
	Points	From
Task 1		
Task 2		
Total points		
Grade		

- The following aids are allowed in this exam:
  - 2 DIN A4 sheets, **handwritten** on both sides (4 pages in total)
  - Calculator (non-programmable, not graphical, not capable of communication)
  - Pens
- Other aids are not allowed.
- Please use a separate solution sheet for each task.
- Write your name and matriculation number on each solution sheet.
- An arrow next to a question means that this part of the task can be solved independently of the rest of the task.
- For calculations the approach as well as the steps must be specified.
- Please do not write with pencils and do not use a red pen.
- The duration of the exam is 90 minutes.
- The exam consists of 9 pages (including this cover page).
- Switch off your cell phones!

Task 1: Tail-Biting Convolutional Codes

Due to their simple encoding and efficient decoding combined with very good error correction characteristics, convolutional codes are part of many applications and standards. Convolutional codes are very efficient for large codeword length N. In recent years, research focused on the analysis and design of error correction schemes for very short code words. In this context, the so-called tail-biting convolutional codes were developed. This extension of convolutional codes will be examined in this task.

In the first subtasks we consider a convolutional code with generator polynomials $[515, 677]_8$ given in octal representation. The number of information bits is set to $K = 4096$ and the codeword length equals $N = 8192$ ignoring the termination sequence.

a) Determine the code rate R of the given convolutional code without termination.

From the tuple (8192, 4096) given in the task description, the code rate R can be found as $R = \frac{4096}{8192} = \frac{1}{2}$.

b) How is the encoding of convolutional codes done? Answer in complete sentences.

The encoding of convolutional codes is done in a continues manner using a shift register. The information bits are fed into the shift register successively. Due to the memory elements in this shift register, previous information bits influence the resulting code bits.

c) Determine the memory M and the constraint length of the code. Give a brief explanation what the constraint length describes. Answer in complete sentences!

Converting the generator polynomials from octal into binary representation, i.e.

$$G_1 = 515_8 = 101\,001\,101$$
$$G_2 = 677_8 = 110\,111\,111$$

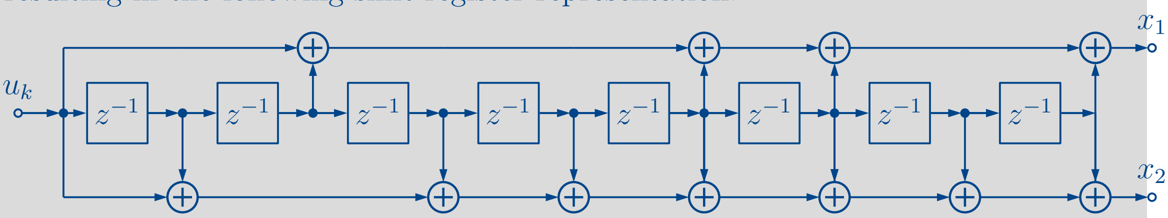
shows that the number of memory elements equals $M = 8$. The constraint length is defined as $M + 1$, i.e., in this case the constraint length is 9. The constraint length indicates that each code symbol depends on $M + 1$ information symbols.

d) Sketch the shift register representation of the given code! Is the given encoder a recursive or non-recursive convolutional encoder? Answer in complete sentences.

For the given code the generator polynomials are

$$G_1 = 515_8 = 1 + 0 \cdot D + 1 \cdot D^2 + 0 \cdot D^3 + 0 \cdot D^4 + 1 \cdot D^5 + 1 \cdot D^6 + 0 \cdot D^7 + 1 \cdot D^8$$
$$G_2 = 677_8 = 1 + 1 \cdot D + 0 \cdot D^2 + 1 \cdot D^3 + 1 \cdot D^4 + 1 \cdot D^5 + 1 \cdot D^6 + 1 \cdot D^7 + 1 \cdot D^8$$

resulting in the following shift register representation:



Since there is no feedback loop, the encoder is non-recursive.

e) How is a non-recursive convolutional code terminated? Why is a termination sequence required? Answer in complete sentences!

Without termination, the last M info symbols of a block have less protection since they have impact on a lower number of code bits. E.g. in a rate 1/2 convolutional code, the last info symbol has impact only on the last 2 code symbols. In contrast, an info symbol in the middle of the block has impact on $2(M+1)$ code symbols. This clearly enables better error correction. In order to provide at least the same error protection to the last symbols of an info block, the code is usually terminated, i.e. M known symbols are appended to the info block such that the encoder shift register is forced to a predetermined state. Usually, this is the all-zero state. For non-recursive convolutional codes, termination in the zero state is achieved by appending M zero symbols to the info block.

f) Determine the effective code rate R_{eff}, considering the termination sequence. Compute the rate reduction R_{eff}/R due to termination compared to the rate from a) in percent.

For the given code with 4096 information bits and 8192 code bits the effective rate R_{eff} can be found as

$$R_{eff} = \frac{4096}{2 \cdot 4096 + 2 \cdot 8} = 0.4990, \tag{1}$$

where $2 \cdot M = 16$ addtional bits need to be considered due to the termination.

Using $R_{eff} = 0.4990$ and $R = 0.5$ the rate reduction can be found as $R_{eff}/R = 0.998$, i.e., 0.2 %.

From now on we consider a short convolutional code with the same generator polynomials $[515, 677]$ given in octal representation but $K = 64$ and $N = 128$.

g) Determine the effective code rate R_{eff}, considering an appropriate termination sequence. Compute the rate reduction R_{eff}/R due to termination for the short code.

From the tuple (128, 64) given in the task description, the code rate R can be found as $R = \frac{64}{128} = \frac{1}{2}$.

For the given code with 64 information bits and 128 code bits the effective rate R_{eff} can be found as

$$R_{eff} = \frac{64}{2 \cdot 64 + 2 \cdot 8} = 0.44, \tag{2}$$

where $2 \cdot M = 16$ addtional bits need to be considered due to the termination.

Using $R_{eff} = 0.44$ and $R = 0.5$ the rate reduction can be found as $R_{eff}/R = 0.88$, i.e., 12 %.

h) Compare the rate reduction encountered in f) to the rate loss determined in g). Which problem of short convolutional codes, decoded using the Viterbi algorithm with termination do you identify?

The reduction for the short code is 12 % whereas the rate reduction due to termination for the long code was negligible with 0.2%. For short codes the required termination sequence to ensure a similar protection of all information bits affects significantly the code rate.

Tail-biting convolutional codes differ from conventional convolutional codes mainly in the initialization of the shift registers and the decoding. To avoid the need of an additional termination sequence the memory elements of the shift register are initialized with the last information bits.

i) How does this different initialization effect the first and last state of the correct path through the trellis? Give reasons and answer in complete sentences!

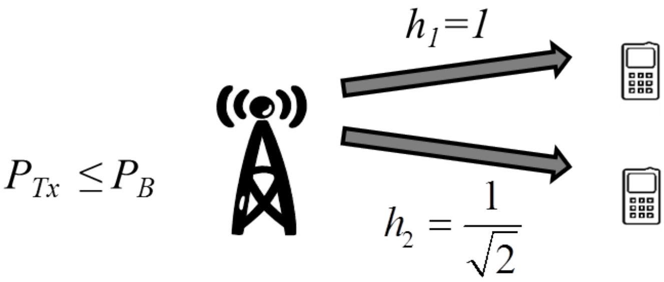
For non-tailbiting codes, the initial state of the path is known to the decoder and equals the all-zeros state. Thus in the first trellis segment not all transitions are possible. This also holds for the last state which is typically forced to the all-zero state due to termination. Thus, both ends of the path through the trellis are fixed and known to the decoder. In contrast, for tail-biting convolutional codes, the initial state depends on the last information bits, since they are used to initialize the memory. This state is typically unknown to the decoder. However, the last and first state of the path have to be identical due to the chosen initialization approach. This knowledge can be exploited in the decoder.

j) Devise a modified version of the Viterbi algorithm suitable for tail-biting codes. Answer in complete sentences!

In contrast to classical convolutional codes, the first and last state are not fixed to the all-zeros state but to an arbitrary state. The only constraint is that the initial and final state are the same. The Viterbi algorithm has to be adopted accordingly. It is thus possible to run several Viterbi decoders in parallel to check all possible initial states and choose the most likely path. An alternative approach is to perform several iterations instead of only one run, where each iteration is initialized with the most likely final state from the previous iteration.

Task 2: Capacity Region of Downlink TDMA and FDMA in Wireless Communications

Consider the following downlink transmission scenario in a wireless system:



A base station serves two users with independent data. The available bandwidth is $B = 10MHz$ and the maximum transmit power of the base station is $P_B = 10W$. Each user faces an AWGN channel with noise power spectral density $N_0 = 10^{-20}W/Hz$ in the complex equivalent baseband. The channel coefficients h_k of the two users k, $k \in \{1, 2\}$ are given by

$$h_1 = 1$$
$$h_2 = \frac{1}{\sqrt{2}}.$$

The received power at user k is given by $|h_k|^2 P_{\mathbf{Tx}}$, where $P_{\mathbf{Tx}}$ is the transmit power of the base station. The achievable rates with different transmission strategies shall be analyzed in the following problems.

a) Determine the single user capacity C_1 for user 1, i.e. the capacity which is achieved if only user 1 is served.

According to Shannon the channel capacity for an AWGN is given by

$$C = B \cdot \log_2\left(1 + \frac{P_R}{P_N}\right). \qquad (3)$$

At first we compute the noise power as

$$\begin{aligned} P_N &= \frac{N_0}{2} 2B \\ &= 10^{-20}\ \text{W/Hz} \cdot 10 \cdot 10^6\ \text{Hz} \\ &= 10^{-13}\ \text{W}. \end{aligned}$$

Thus, the resulting capacity C_1 equals

$$\begin{aligned} C_1 &= B \cdot \log_2\left(1 + \frac{|h_1|^2 P_{\text{Tx}}}{P_N}\right) \\ &= 10 \cdot 10^6 \text{ Hz} \cdot \log_2\left(1 + \frac{1 \cdot 10W}{10^{-13}\text{ W}}\right) \\ &= 465.07\,\text{Mbits / s}. \end{aligned}$$

b) Determine the single user capacity C_2 for user 2, i.e. the capacity which is achieved if only user 2 is served.

According to Shannon the channel capacity for an AWGN is given by

$$C = B \cdot \log_2\left(1 + \frac{P_R}{P_N}\right). \tag{4}$$

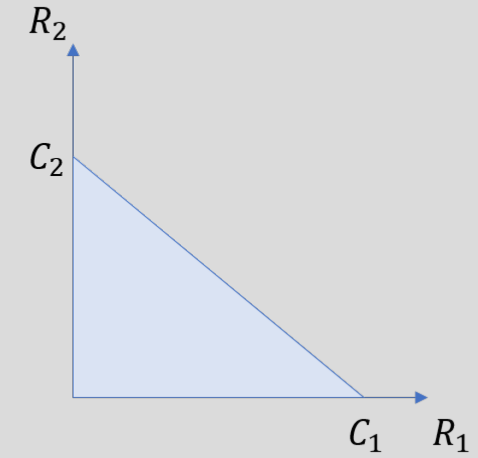
At first we compute the noise power as

$$\begin{aligned} P_N &= \frac{N_0}{2} 2B \\ &= 10^{-20}\text{ W/Hz} \cdot 10 \cdot 10^6\text{ Hz} \\ &= 10^{-13}\text{ W}. \end{aligned}$$

Thus, the resulting capacity C_2 equals

$$\begin{aligned} C_2 &= B \cdot \log_2\left(1 + \frac{|h_2|^2 P_{\text{Tx}}}{P_N}\right) \\ &= 10 \cdot 10^6 \text{ Hz} \cdot \log_2\left(1 + \frac{\frac{1}{2} \cdot 10W}{10^{-13}\text{ W}}\right) \\ &= 455.07\,\text{Mbits / s}. \end{aligned}$$

c) Sketch the achievable rate region for time division multiple access (TDMA) with power constraint.



⇨ d) Determine the achievable rate pair $R_{1,TDMA}, R_{2,TDMA}$ for TDMA, if the channel is allocated to both users for the same fraction of the total transmission time.

Since the channel is allocated to both users for the same fraction, each user transmits half of the time, i.e., $\tau = 0.5$. The achievable rate is then

$$R_{1,TDMA} = \tau \cdot C_2 = 0.5 \cdot C_1 = 232.53 \text{ Mbits / s}$$
$$R_{2,TDMA} = \tau \cdot C_2 = 0.5 \cdot C_2 = 227.53 \text{ Mbits / s}$$

⇨ e) Determine the achievable rate pair $R_{1,FDMA}, R_{2,FDMA}$ for FDMA, if the same fraction of the channel bandwidth is allocated to the two users.

Since both users allocate half of the available bandwidth, $B_k = 0.5B, k \in \{1, 2\}$. Similarly, the power assigned to a user needs to be adapted to meet the sum-power constraint of the base station. Consequently, $P_K = 0.5P_{\mathbf{Tx}}, k \in \{1, 2\}$. Due to the reduced bandwidth allocated for user k the noise power changes accordingly, i.e. $P_{N,k} = \frac{N_0}{2} \cdot 0.5 \cdot 2B = 0.5P_N$. The resulting achievable rate pair equals

$$\begin{aligned} R_{k,FDMA} &= B_k \log_2 \left(1 + \frac{|h_k|^2 P_K}{P_{N,k}}\right) \\ &= 0.5B \log_2 \left(1 + \frac{|h_k|^2 \cdot 0.5P_{\mathbf{Tx}}}{0.5P_N}\right) \\ &= 0.5C_k \end{aligned}$$

Hence,

$$R_{1,FDMA} = 0.5 \cdot C_1 = 232.53 \text{ Mbits / s}$$
$$R_{2,FDMA} = 0.5 \cdot C_2 = 227.53 \text{ Mbits / s}$$

f) Determine the achievable sum rate for the TDMA and FDMA schemes in d) and e).

The achievable sum rate is

$$R_{sum} = R_1 + R_2 = 0.5C_1 + 0.5C_2$$

for both TDMA with $\tau = 0.5$ and FDMA with $\beta = 0.5$.

g) Which TDMA and FDMA transmission strategy maximizes the sum rate?

Since $|h_1| > |h_2$, the sum rate is maximized, if only user 1 is served.

h) Determine the maximum sum rate according to your solution from g).

The maximum sum rate is achieved, if only the user with the highest rate is served, i.e.,

$$R_{sum,max} = \max\{R_1, R_2\} = C_1$$