Technische Universität Hamburg

INSTITUT FÜR NACHRICHTENTECHNIK
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Exam WS 2017/2018

Information Theory and Coding

Name:	Student ID:	
	Points	From
Task 1		
Task 2		
Total points		
Grade		

- The following aids are allowed in this exam:
  - 2 DIN A4 sheets, **handwritten** on both sides (4 pages in total)
  - Calculator (non-programmable, not graphical, not capable of communication)
  - Pens
- Other aids are not allowed.
- Please use a separate solution sheet for each task.
- Write your name and matriculation number on each solution sheet.
- An arrow next to a question means that this part of the task can be solved independently of the rest of the task.
- For calculations the approach as well as the steps must be specified.
- Please do not write with pencils and do not use a red pen.
- The duration of the exam is 90 minutes.
- The exam consists of 6 pages (including this cover page).
- Switch off your cell phones!

Task 1: Ternary Huffman Code

The symbols of a source are described by the random variable X with realizations $x_i \in \{a, b, c, d, e\}$.
The realizations x_i occur according to the following probabilities:

x_i	$p_X(x_i)$
a	0.25
b	0.15
c	0.2
d	0.15
e	0.25

Now, the source symbols shall be encoded by a ternary Huffman code, i.e. the code symbols of the Huffman code take values in the set $c_i \in \{0, 1, 2\}$.

a) Construct a ternary Huffman-Code for encoding of the realizations x_i. Make sure that all steps in the construction of the Huffman code are clearly given. Represent the mapping of realizations x_i to codewords $\mathbf{c}$ of the Huffman code in a table. Answer in complete sentences!

b) Sketch the tree representation of your Huffman code from a).

c) Determine the average codeword length of the Huffman code from a).

d) Is the code prefix-free? Give reasons and answer in complete sentences!

Task 2: Polar Codes

Polar Codes are currently discussed for many applications such as 5G cellular systems as they have been shown to be capacity-achieving in binary memoryless channels. The idea of Polar Codes is that the encoder in combination with a successive cancellation decoder transforms the channel into a set of equivalent channels for the information bits, where a fraction of those channels is error-free (capacity $C_i = 1$), whereas the remaining channels are totally noisy (capacity $C_i = 0$). This effect of transforming the physical channel into perfect and noisy (good and bad) channels is called polarization. Information is transmitted only on the perfect channels. We will investigate the principle of Polar Codes in this task.

We consider Polar Codes in a Binary Erasure Channel (BEC). The capacity of a BEC with erasure probability p_e is given by

$$C_{\text{BEC}} = 1 - p_e$$

The basic building block of a Polar Code is depicted in the following figure:

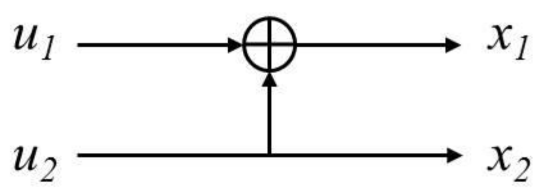

Figure 1: Basic building block of a Polar Code encoder.

Two information bits u_1 and u_2 are mapped to two code bits x_1 and x_2. The code bits x_i are then transmitted through a Binary Erasure Channel (BEC). The received bits at the output of the BEC are denoted y_i.

The information bits are uniformly distributed, i.e. $P(u_i = 0) = P(u_i = 1) = 1/2$.

a) Sketch the state transition diagram of a Binary Erasure Channel. Label the figure completely.

b) State the definition of a memoryless channel and show, that the Binary Erasure Channel is memoryless. Answer in complete sentences!

c) Derive the channel capacity $C_{2\mathrm{BEC}}$ of two parallel Binary Erasure Channels with the same erasure probability p_e.

d) Derive the equations for the code bits x_1 and x_2 depending on u_1 and u_2.

e) Sketch a Tanner graph element for the information bit u_1, i.e. a Tanner graph which contains u_1,x_1 and x_2 as variable nodes. Which type of code is represented by this Tanner graph element?

f) Sketch a Tanner graph element for the information bit u_2. Assume that the bit u_1 is perfectly known due to prior decoding. Which type of code is represented by this Tanner graph element ?

The basic decoder of a Polar Code is a successive cancellation decoder: First, the information bit u_1 is decoded. Then, the information bit u_2 is decoded under the assumption, that the decision on u_1 was correct.
Thus, for each information bit an equivalent channel with a corresponding capacity can be derived.
The channel capacity C_1 for an equivalent channel for the information bit u_1 is given as

$$C_1 = (1 - p_e)^2.$$

The channel capacity C_2 for an equivalent channel for the information bit u_2 is given as

$$C_2 = 1 - p_e^2.$$

g) Show that $C_2 > C_1$. Give an intuitive explanation for this relation. Answer in complete sentences!

h) Derive the sum capacity $C_1 + C_2$ based on the given channel capacities for the equivalent channels. Compare the sum capacity to your result from c).

i) Interprete your results from g) and h) in view of the target of *polarization* as explained in the introduction to the task. Answer in complete sentences!

⇨ j) Derive a generator matrix $\mathbf{G}_2$ for the basic encoder building block in Figure 1.

⇨ k) Derive the code rate R_2 of the basic building block in Figure 1. Can error correction be achieved with this code? Give reasons and answer in complete sentences!

A powerful Polar Code with strong polarization effect can be obtained by combining a large number of copies of the basic building block from Figure 1. The procedure is illustrated for a Polar Code of codeword length $N = 4$ in the following figure:

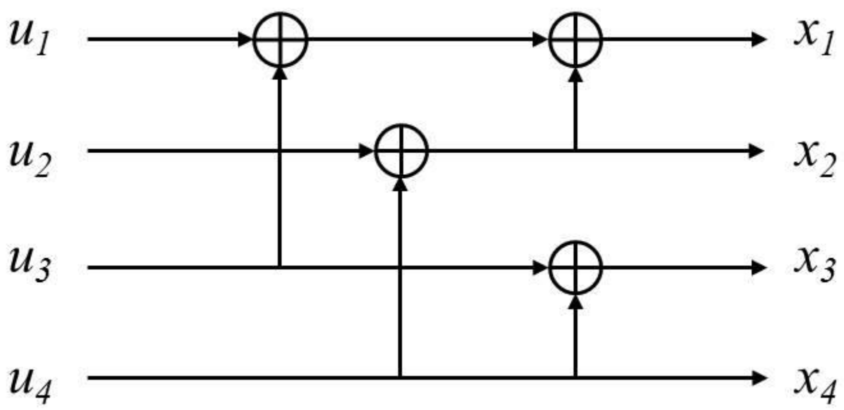

Figure 2: Encoder of Polar Code with codeword length $N = 4$.

⇨ l) Derive a generator matrix $\mathbf{G}_4$ for the encoder in Figure 2.

m) Write the generator matrix $\mathbf{G}_4$ depending on the generator matrix $\mathbf{G}_2$ from j).

In order to obtain an error-correcting code with code rate R<1, some of the bits u_i are not used for transmission of information. Instead, they are replaced by fixed values, e.g. by 0. We call the fixed bits *frozen bits.* The polarization effect is exploited for choosing the positions of the frozen bits: Information is transmitted on the good effective channels, while the frozen bits are put to the bad (noisy) effective channels.

⇨ n) How many bits u_i need to be frozen in the encoder from Figure 2 in order to obtain a code rate of $R = 1/2$?