Exam WS 2016/2017

Information Theory and Coding

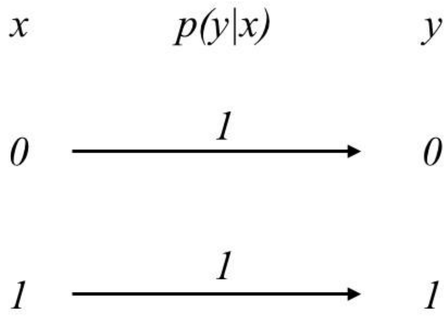
Name: ………………………………………………..	Immatriculation number: …………………………..	
	Points	of
Problem 1		
Problem 2		
Problem 3		
Total points		
Grade		

- The following aids are allowed in this exam:
    - 2 Din A4 sheets, handwritten on both sides (4 pages in total)
    - calculator (non-programmable)
    - Pens
- Other aids are not allowed.
- Please use a separate solution sheet for each task.
- Write your name and matriculation number on each solution sheet.
- An arrow next to a question means that this part of the task can be solved. independently of the previous questions.
- Please do not write with pencils and do not use a red pen.
- The duration of the exam is 90 minutes.
- The exam consists of **4** pages (including this cover page).
- Switch off your cell phones!

Good luck!

Problem 1: Capacity of the Noiseless Binary Channel

The *noiseless binary channel* is given by the following state transition diagram:



a) Determine the channel capacity by intuition. Explain your solution clearly and answer in complete sentences.

b) Derive the channel capacity formally. Make sure that all steps in your derivation are clearly given.

c) Determine the capacity achieving probability distribution *p(x)* of the channel input symbols.

Problem 2: Maximum Entropy

In this problem, we will prove, that the entropy of a discrete random variable X with N different realizations x and probability distribution $p_X(x)$ is upper bounded by

$$H(X) \leq \log_2 N.$$

We will make use of the *relative entropy* (also called the *Kullback Leibler distance*)

$$0 \leq D(p \| q) = \sum_x p_X(x) \log_2 \frac{p_X(x)}{q_X(x)},$$

which is a measure for the distance of two probability distributions $p_X(x)$ and $q_X(x)$. The relative entropy $D(p||q)$ is always non-negative.

a) Determine the probability distribution $q_X(x)$ of a uniformly distributed random variable X, which can take N different realizations x.

b) Determine the entropy $H_u(X)$ of a uniformly distributed random variable X, which can take N different realizations x.

c) Determine the relative entropy $D(p||q)$ between a probability distribution $p_X(x)$ and a uniform probability distribution $q_X(x)$ depending on the entropy $H(X)$ of a random variable with probability distribution $p_X(x)$.

d) Use your result from c) in order to prove that $H(X) \leq \log_2 N$.

e) Which probability distribution delivers the maximum possible entropy of a random variable X with N different realizations ?

Problem 3: Random Codes

When Shannon published his famous paper in 1948, channel codes did not exist. Shannon's theorems have been proven based on the idea of random codes. In this problem, we will address binary random codes, which map information words of length K bits to codeword of length N bits. A random code is constructed by randomly choosing the codewords among all possible binary sequences of length N.

a) Construct a random code of rate $R=1/2$ with codeword length $N=6$. State all the codewords and the mapping of information words to codewords in a table.

b) How many different binary sequences of length N exist ?

c) How many different binary sequences of length N with Hamming weight w exist ?

d) How many codewords exist for a binary $(N, K)_2$ code ?

e) Assume, that a random code has been constructed. We then randomly choose a binary sequence of length N. Determine the probability, that this randomly chosen sequence is a codeword.

f) Determine the expected number $E\{A_w\}$ of codewords with Hamming weight w for a random $(N, K)_2$ code.

g) Is a random code in general a linear code ? Give reasons !

h) Which minimum Hamming distance $d_{\min}$ can be guaranteed by the random code construction ?

i) Determine the minimum Hamming weight $w_{\min}$ of your code from a).

j) Determine the minimum Hamming distance $d_{\min}$ of your code from a).