Exam WS 2018/2019

Information Theory and Coding

Name: Sample solution	Student ID:	
	Points	From
Task 1		
Task 2		
Total points		
Grade		

- The following aids are allowed in this exam:
  - 2 DIN A4 sheets, **handwritten** on both sides (4 pages in total)
  - Calculator (non-programmable, not graphical, not capable of communication)
  - Pens
- Other aids are not allowed.
- Please use a separate solution sheet for each task.
- Write your name and matriculation number on each solution sheet.
- An arrow next to a question means that this part of the task can be solved independently of the rest of the task.
- For calculations the approach as well as the steps must be specified.
- Please do not write with pencils and do not use a red pen.
- The duration of the exam is 90 minutes.
- The exam consists of 7 pages (including this cover page).
- Switch off your cell phones!

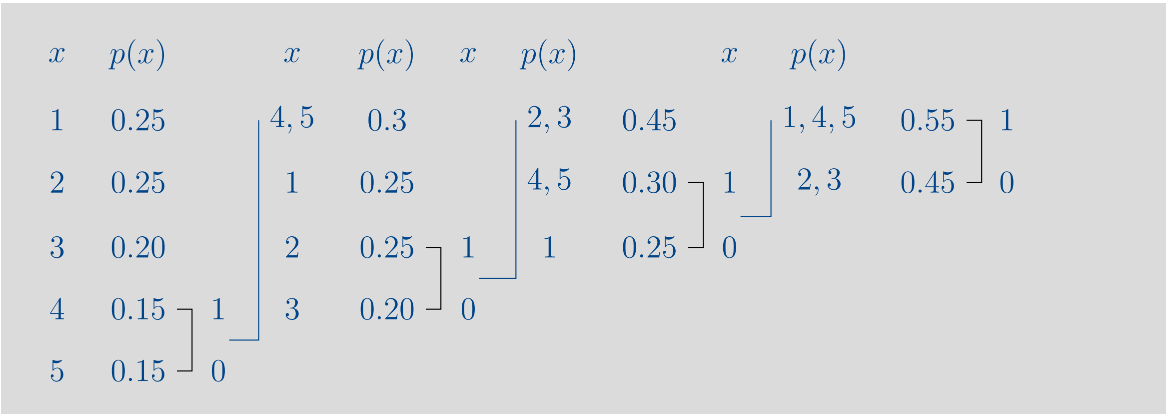
Task 1: Slice Codes

We consider source coding for a source which emits the symbols $x \in \{1, 2, 3, 4, 5\}$ according to the following probabilities:

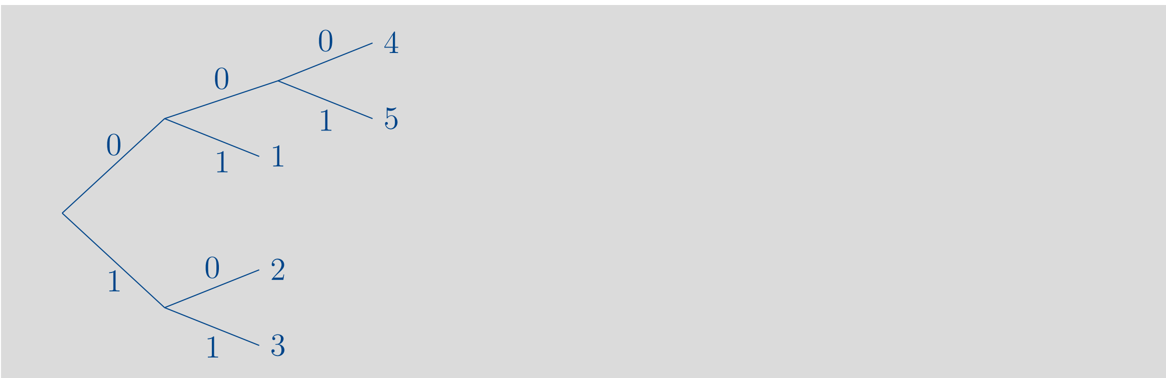
x	$p(x)$
1	0.25
2	0.25
3	0.2
4	0.15
5	0.15

It can be observed from the table above, that the symbols are ordered according to ascending probability.

a) Construct a binary Huffman-Code for encoding of the realizations x. Make sure that all steps in the construction of the Huffman code are clearly given.



b) Sketch the tree representation of your Huffman code from a).



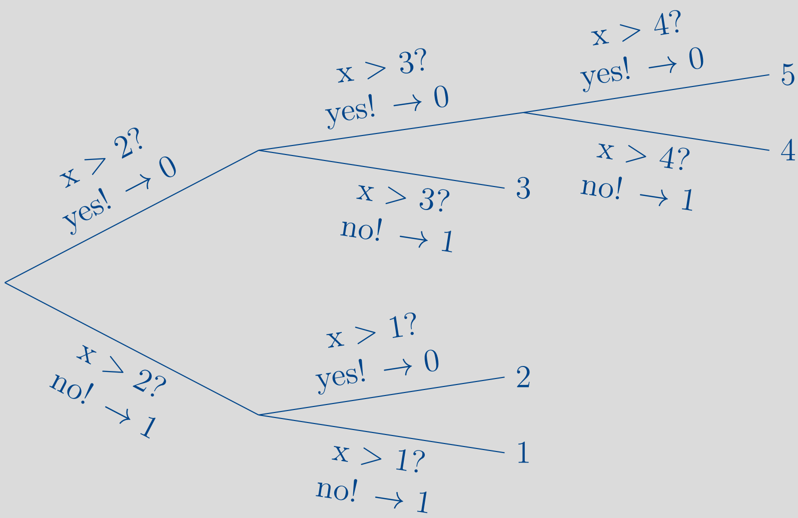
c) Determine the average codeword length of the Huffman code from a).

The average codeword length is computed as follows:

$$\begin{aligned} \mathbb{E}\{L\} &= 3 \cdot (0.15 + 0.15) + 2 \cdot (0.25 + 0.25 + 0.2) && (1) \\ &= 3 \cdot 0.3 + 2 \cdot 0.7 && (2) \\ &= 2.3 \text{bits / symbol} && (3) \end{aligned}$$

A slice code uses the same code tree structure as a Huffman code but maps the code bits in a different way: Starting from the root of the tree, at each level a question of the following form is asked: "Is $x > a$?" for some $a \in \{1, 2, 3, 4, 5\}$. The answer "yes" is encoded by a code bit "0", the answer "no" is encoded by a code bit "1".

d) Determine the mapping of source symbols x to codewords c for a slice code. Use the tree structure of the Huffman code from a). Make sure that each step in your derivation can be clearly understood. Represent the mapping of the slice code both in a table and in a code tree.



x	$p(x)$	c
1	0.25	11
2	0.25	10
3	0.2	01
4	0.15	001
5	0.15	000

.

e) Determine the average codeword length of the slice code from d) and compare to the codeword length of the Huffman code from a)-c).

The average codeword length is computed as follows:

$$\mathbb{E}\{L\} = 3 \cdot (0.15 + 0.15) + 2 \cdot (0.25 + 0.25 + 0.2) \tag{4}$$
$$= 3 \cdot 0.3 + 2 \cdot 0.7 \tag{5}$$
$$= 2.3\text{bits / symbol} \tag{6}$$

Comparing with the averaging code word length of the Huffman code, we find that both the average code word length of the slice code and the Huffman code are equal.

f) Give a lower bound and an upper bound on the average codeword length $E\{L\}$ of the slice code.

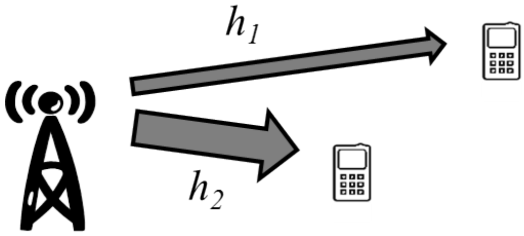
The average code word length is bounded by

$$H(X) \leq \mathbb{E}\{L\} \leq H(X) + 1, \tag{7}$$

i.e. similar as for the Huffman code.

Task 2: Capacity Region of Superposition Coding in a Broadcast Channel

Consider a wireless broadcast channel, where a base station serves two users:



The available bandwidth is denoted B and the maximum transmit power of the base station is P_B. The received signal at user k is given by

$$y_k = h_k x + n_k, \tag{8}$$

where h_k is the channel coefficient of user k, x is the transmit signal and n_k is additive white Gaussian noise with the same power P_N at both users.
The base station applies superposition coding strategy such that the transmit signal is a superposition of the signals s_k intended for the two users $k \in \{1, 2\}$:

$$x = s_1 + s_2, \tag{9}$$

where s_k is contained in x with power P_k and

$$P_1 + P_2 \leq P_B.$$

Under the assumption that

$$|h_1|^2 < |h_2|^2,$$

the achievable rates for the superposition coding strategy have been shown to be given by the following expressions:

$$R_1 = B \log_2 \left(1 + \frac{|h_1|^2 P_1}{|h_1|^2 P_2 + P_N}\right), \tag{10}$$

$$R_2 = B \log_2 \left(1 + \frac{|h_2|^2 P_2}{P_N}\right). \tag{11}$$

a) Which power allocation P_1, P_2 in equation (9) corresponds to a single user transmission?

$P_1 \leq P_B, P_2 = 0 \Rightarrow$ only user 1 is served
$P_2 \leq P_B, P_1 = 0 \Rightarrow$ only user 2 is served

b) Give an interpretation of equations (10) and (11): Which transmission and detection strategies can achieve the rates according to (10) and (11)? Answer in complete sentences!

User 2 transmits at its single user capacity. User 1 transmits at a rate which is supported, if the signal of user 2 occurs as additional AWGN.

User 2 experiences the better channel quality. Hence, user 2 can detect for user 1 and subtract it from its received signal. Then, it can detect its own signal as if the signal for user 1 was not present. User 1 experiences the worse channel. It cannot detect the signal for user 2 as the rate of user 2 exceeds the channel capacity of user 1, i.e. $R_2 > C_1$. Hence, user 1 detects its signal while treating the signal for user 2 as additional additive white Gaussian noise (AWGN) with power $|h_1|^2 P_2$. The signal for user 2 can be regarded as independent white Gaussian noise, since the optimum transmit signal in an AWGN channel is Gaussian distributed and consequently, both s_1 and s_2 are Gaussian distributed.

c) Determine an equation for the achievable rate region of superposition coding.

$$C_{SC} = \bigcup_{P_1, P_2 : P_1 + P_2 \leq P_B} \left[R_1 = B \log_2 \left(1 + \frac{|h_1|^2 P_1}{|h_1|^2 P_2 + P_N} \right), R_2 = B \log_2 \left(1 + \frac{|h_2|^2 P_2}{P_N} \right) \right]$$

d) Is superposition coding an optimum transmission scheme in the sense that it can achieve the maximum possible rate region? Give clear reasons and answer in complete sentences!

The rate region in c) is equal to the capacity region of the broadcast channel. (It is the union of the dual MAC capacity regions for all power allocations P_1, P_2). Therefore, superposition coding is an optimum transmission strategy.

e) Which transmission strategy maximizes the sum rate of superposition coding in case of unequal channel quality, i.e. $|h_1|^2 \neq |h_2|^2$?

In case of $|h_1|^2 \neq |h_2|^2$, the sum rate is maximilized, if all resources are allocated to the user with the better channel quality. I.e., if $|h_k| > |h_l|$, only user k should be served with $P_k = P_B, P_l = 0$. User k can then transmit at its single user capacity.

f) Determine an equation for the maximum sum rate according to your solution from e).

$$\max\{R_1 + R_2\} = \max\left\{B\log_2 \frac{|h_1|^2 P_B}{P_N}, B\log_2 \frac{|h_2|^2 P_B}{P_N}\right\}$$

g) Under which condition for $|h_1|^2$ and $|h_2|^2$ does superposition coding provide a larger achievable rate region than time division multiple access (TDMA) ?

For $|h_1| = |h_2|$, the broadcast channel capacity region is equal to the achievable rate region of TDMA. Hence, superposition coding provides a larger achievable rate region than TDMA only if $|h_1| \neq |h_2|$.