



# Calculus 1 Workbook Solutions

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Implicit differentiation

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MATH

## IMPLICIT DIFFERENTIATION

- 1. Use implicit differentiation to find  $dy/dx$  at (3,4).

$$4x^3 - 3xy^2 + y^3 = 28$$

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$12x^2 - 3y^2 - 6xy \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$$

$$(3y^2 - 6xy) \frac{dy}{dx} = 3y^2 - 12x^2$$

$$\frac{dy}{dx} = \frac{3y^2 - 12x^2}{3y^2 - 6xy}$$

$$\frac{dy}{dx} = \frac{y^2 - 4x^2}{y^2 - 2xy}$$

Evaluate  $dy/dx$  at (3,4).

$$\frac{dy}{dx}(3,4) = \frac{(4)^2 - 4(3)^2}{(4)^2 - 2(3)(4)} = \frac{16 - 36}{16 - 24} = \frac{5}{2}$$

- 2. Use implicit differentiation to find  $dy/dx$ .



$$5x^3 + xy^2 = 4x^3y^3$$

*Solution:*

Rearrange the function. We'll do this to get all the terms that include  $y$  on one side of the equation, which will make it easier to solve for  $dy/dx$  later on.

$$5x^3 + xy^2 = 4x^3y^3$$

$$xy^2 - 4x^3y^3 = -5x^3$$

Use implicit differentiation to take the derivative of both sides.

$$y^2 + 2xy \frac{dy}{dx} - 12x^2y^3 - 12x^3y^2 \frac{dy}{dx} = -15x^2$$

$$2xy \frac{dy}{dx} - 12x^3y^2 \frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$(2xy - 12x^3y^2) \frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$\frac{dy}{dx} = \frac{12x^2y^3 - 15x^2 - y^2}{2xy - 12x^3y^2}$$

■ 3. Use implicit differentiation to find  $dy/dx$ .

$$3x^2 = (3xy - 1)^2$$



*Solution:*

Rearrange the function. We'll do this to get all the terms that include  $y$  on one side of the equation, which will make it easier to solve for  $dy/dx$  later on.

$$3x^2 = (3xy - 1)^2$$

$$3x^2 = 9x^2y^2 - 6xy + 1$$

Use implicit differentiation to take the derivative of both sides.

$$6x = 18xy^2 + 18x^2y \frac{dy}{dx} - 6y - 6x \frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = 18x^2y \frac{dy}{dx} - 6x \frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = (18x^2y - 6x) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x - 18xy^2 + 6y}{18x^2y - 6x}$$

$$\frac{dy}{dx} = \frac{x - 3xy^2 + y}{3x^2y - x}$$

■ 4. Use implicit differentiation to find  $dy/dx$ .

$$\sin(2x + 5y) = \cos^2 x + \cos^2 y$$



*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$\cos(2x + 5y) \left( 2 + 5 \frac{dy}{dx} \right) = 2 \cos x (-\sin x) + 2 \cos y (-\sin y) \frac{dy}{dx}$$

$$2 \cos(2x + 5y) + 5 \cos(2x + 5y) \frac{dy}{dx} = -2 \sin x \cos x - 2 \sin y \cos y \frac{dy}{dx}$$

$$2 \sin y \cos y \frac{dy}{dx} + 5 \cos(2x + 5y) \frac{dy}{dx} = -2 \sin x \cos x - 2 \cos(2x + 5y)$$

$$(2 \sin y \cos y + 5 \cos(2x + 5y)) \frac{dy}{dx} = -2 \sin x \cos x - 2 \cos(2x + 5y)$$

$$\frac{dy}{dx} = \frac{-2 \sin x \cos x - 2 \cos(2x + 5y)}{2 \sin y \cos y + 5 \cos(2x + 5y)}$$

$$\frac{dy}{dx} = - \frac{2 \sin x \cos x + 2 \cos(2x + 5y)}{2 \sin y \cos y + 5 \cos(2x + 5y)}$$

$$\frac{dy}{dx} = - \frac{\sin(2x) + 2 \cos(2x + 5y)}{\sin(2y) + 5 \cos(2x + 5y)}$$

■ 5. Use implicit differentiation to find  $dy/dx$ .

$$e^{2xy} = 3x^3 - \ln(xy^2)$$

*Solution:*



Rearrange the equation. We'll do this to get all the terms that include  $y$  on one side of the equation, which will make it easier to solve for  $dy/dx$  later on.

$$e^{2xy} = 3x^3 - \ln(xy^2)$$

$$e^{2xy} + \ln(xy^2) = 3x^3$$

Use implicit differentiation to take the derivative of both sides.

$$e^{2xy} \left( 2y + 2x \frac{dy}{dx} \right) + \frac{1}{xy^2} \left( y^2 + 2xy \frac{dy}{dx} \right) = 9x^2$$

$$2ye^{2xy} + 2xe^{2xy} \frac{dy}{dx} + \frac{y^2}{xy^2} + \frac{2xy}{xy^2} \left( \frac{dy}{dx} \right) = 9x^2$$

$$2ye^{2xy} + 2xe^{2xy} \frac{dy}{dx} + \frac{1}{x} + \frac{2}{y} \left( \frac{dy}{dx} \right) = 9x^2$$

$$2xe^{2xy} \frac{dy}{dx} + \frac{2}{y} \left( \frac{dy}{dx} \right) = 9x^2 - 2ye^{2xy} - \frac{1}{x}$$

$$\left( 2xe^{2xy} + \frac{2}{y} \right) \frac{dy}{dx} = 9x^2 - 2ye^{2xy} - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{9x^2 - 2ye^{2xy} - \frac{1}{x}}{2xe^{2xy} + \frac{2}{y}}$$

■ 6. Use implicit differentiation to find  $dy/dx$  at  $(0,2)$ .



$$\frac{2x - y^3}{y + x^2} = 5x - 4$$

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$\frac{\left(2 - 3y^2 \frac{dy}{dx}\right)(y + x^2) - (2x - y^3)\left(\frac{dy}{dx} + 2x\right)}{(y + x^2)^2} = 5$$

$$\frac{2y + 2x^2 - 3y^3 \frac{dy}{dx} - 3x^2 y^2 \frac{dy}{dx} - \left(2x \frac{dy}{dx} + 4x^2 - y^3 \frac{dy}{dx} - 2xy^3\right)}{(y + x^2)^2} = 5$$

$$\frac{2y + 2x^2 - 3y^3 \frac{dy}{dx} - 3x^2 y^2 \frac{dy}{dx} - 2x \frac{dy}{dx} - 4x^2 + y^3 \frac{dy}{dx} + 2xy^3}{(y + x^2)^2} = 5$$

$$2y + 2x^2 - 3y^3 \frac{dy}{dx} - 3x^2 y^2 \frac{dy}{dx} - 2x \frac{dy}{dx} - 4x^2 + y^3 \frac{dy}{dx} + 2xy^3 = 5(y + x^2)^2$$

$$-3y^3 \frac{dy}{dx} - 3x^2 y^2 \frac{dy}{dx} - 2x \frac{dy}{dx} + y^3 \frac{dy}{dx} = 5(y + x^2)^2 - 2y - 2x^2 + 4x^2 - 2xy^3$$

$$(-3y^3 - 3x^2 y^2 - 2x + y^3) \frac{dy}{dx} = 5(y + x^2)^2 - 2y + 2x^2 - 2xy^3$$

$$(-2y^3 - 3x^2 y^2 - 2x) \frac{dy}{dx} = 5(y + x^2)^2 - 2y + 2x^2 - 2xy^3$$

$$\frac{dy}{dx} = \frac{5(y + x^2)^2 - 2y + 2x^2 - 2xy^3}{-2y^3 - 3x^2 y^2 - 2x}$$



$$\frac{dy}{dx} = - \frac{5(y + x^2)^2 - 2y + 2x^2 - 2xy^3}{2y^3 + 3x^2y^2 + 2x}$$

Evaluate  $dy/dx$  at  $(0,2)$ .

$$\frac{dy}{dx}(0,2) = - \frac{5(2 + 0^2)^2 - 2(2) + 2(0)^2 - 2(0)(2)^3}{2(2)^3 + 3(0)^2(2)^2 + 2(0)}$$

$$\frac{dy}{dx}(0,2) = - \frac{5(2)^2 - 2(2)}{2(8)}$$

$$\frac{dy}{dx}(0,2) = - \frac{20 - 4}{16}$$

$$\frac{dy}{dx}(0,2) = - \frac{16}{16}$$

$$\frac{dy}{dx}(0,2) = - 1$$





## EQUATION OF THE TANGENT LINE WITH IMPLICIT DIFFERENTIATION

- 1. Use implicit differentiation to find the equation of the tangent line to  $5y^2 = 2x^3 - 5y + 6$  at  $(3,3)$ .

*Solution:*

Rearrange the function.

$$5y^2 = 2x^3 - 5y + 6$$

$$5y^2 + 5y = 2x^3 + 6$$

Use implicit differentiation to take the derivative of both sides.

$$10y \frac{dy}{dx} + 5 \frac{dy}{dx} = 6x^2$$

$$(10y + 5) \frac{dy}{dx} = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{10y + 5}$$

Evaluate  $dy/dx$  at  $(3,3)$ .

$$\frac{dy}{dx}(3,3) = \frac{6(3)^2}{10(3) + 5} = \frac{54}{35}$$

Then the equation of the tangent line is



$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{54}{35}(x - 3)$$

$$y = \frac{54}{35}(x - 3) + 3$$

■ 2. Use implicit differentiation to find the equation of the tangent line to  $5x^3 = -3xy + 4$  at  $(2, -6)$ .

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$15x^2 = -3y - 3x \frac{dy}{dx}$$

$$15x^2 + 3y = -3x \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{15x^2 + 3y}{-3x}$$

$$\frac{dy}{dx} = -\frac{5x^2 + y}{x}$$

Evaluate  $dy/dx$  at  $(2, -6)$ .

$$\frac{dy}{dx}(2, -6) = -\frac{5(2)^2 + (-6)}{2} = -\frac{20 - 6}{2} = -7$$



Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 6 = -7(x - 2)$$

$$y = -7(x - 2) - 6$$

$$y = -7x + 8$$

■ 3. Use implicit differentiation to find the equation of the tangent line to  $4y^2 + 8 = 3x^2$  at  $(6, -5)$ .

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$8y \frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{8y} = \frac{3x}{4y}$$

Evaluate  $dy/dx$  at  $(6, -5)$ .

$$\frac{dy}{dx}(6, -5) = \frac{3(6)}{4(-5)} = -\frac{18}{20} = -\frac{9}{10}$$

Then the equation of the tangent line is



$$y - y_1 = m(x - x_1)$$

$$y + 5 = -\frac{9}{10}(x - 6)$$

$$y = -\frac{9}{10}(x - 6) - 5$$

$$y = -\frac{9}{10}x + \frac{54}{10} - 5$$

$$y = -\frac{9}{10}x + \frac{27}{5} - \frac{25}{5}$$

$$y = -\frac{9}{10}x + \frac{2}{5}$$

■ 4. Use implicit differentiation to find the equation of the tangent line to  $2x + 3y - 5 = \ln(x^5 + y^5)$  at  $(1,0)$ .

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$2 + 3y' = \frac{1}{x^5 + y^5}(5x^4 + 5y^4y')$$

$$2 + 3y' = \frac{5x^4}{x^5 + y^5} + \frac{5y^4y'}{x^5 + y^5}$$



$$3y' - \frac{5y^4 y'}{x^5 + y^5} = \frac{5x^4}{x^5 + y^5} - 2$$

$$y' \left( 3 - \frac{5y^4}{x^5 + y^5} \right) = \frac{5x^4}{x^5 + y^5} - 2$$

$$y' \left( \frac{3(x^5 + y^5) - 5y^4}{x^5 + y^5} \right) = \frac{5x^4 - 2(x^5 + y^5)}{x^5 + y^5}$$

$$y' = \frac{5x^4 - 2(x^5 + y^5)}{x^5 + y^5} \left( \frac{x^5 + y^5}{3(x^5 + y^5) - 5y^4} \right)$$

$$y' = \frac{5x^4 - 2(x^5 + y^5)}{3(x^5 + y^5) - 5y^4}$$

Evaluate  $y'$  at  $(1,0)$  to find the slope of the tangent line.

$$\frac{dy}{dx}(1,0) = \frac{5(1)^4 - 2(1^5 + 0^5)}{3(1^5 + 0^5) - 5(0)^4} = \frac{5(1) - 2(1 + 0)}{3(1 + 0) - 0} = \frac{5 - 2}{3} = 1$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 0 = 1(x - 1)$$

$$y = x - 1$$

■ 5. Use implicit differentiation to find the equations of the tangent and normal line to  $\cos x = \sin(2y) + 9$  at  $(\pi/2, \pi)$ .



*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$-\sin x = \cos(2y)(2y')$$

$$-\sin x = 2y' \cos(2y)$$

$$y' = -\frac{\sin x}{2 \cos(2y)}$$

Evaluate  $y'$  at  $(\pi/2, \pi)$  to find the slope of the tangent line.

$$y' \left( \frac{\pi}{2}, \pi \right) = -\frac{\sin \frac{\pi}{2}}{2 \cos(2\pi)} = -\frac{1}{2(1)} = -\frac{1}{2}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - \pi = -\frac{1}{2} \left( x - \frac{\pi}{2} \right)$$

$$y - \pi = -\frac{1}{2}x + \frac{\pi}{4}$$

$$y = -\frac{1}{2}x + \frac{5\pi}{4}$$

Since the normal line is perpendicular to the tangent line at the point of tangency, the slope of the normal line is 2 (the negative reciprocal of  $-1/2$ ), so the equation of the normal line is



$$y - \pi = 2 \left( x - \frac{\pi}{2} \right)$$

$$y - \pi = 2x - \pi$$

$$y = 2x$$

■ 6. Use implicit differentiation to find the equation of the tangent line to  $4x^2 - xy + y^2 = 6$  at the points in the second and third quadrant when  $x = -1$ .

*Solution:*

First we need to find points on the curve in second and third quadrant at  $x = -1$ .

$$4(-1)^2 - (-1)y + y^2 = 6$$

$$4 + y + y^2 = 6$$

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$$y = -2 \text{ and } y = 1$$

Therefore, we get two points  $(-1, -2)$  and  $(-1, 1)$ .

Now use implicit differentiation to take the derivative of both sides of  $4x^2 - xy + y^2 = 6$ .



$$8x - y - xy' + 2yy' = 0$$

$$2yy' - xy' = y - 8x$$

$$y'(2y - x) = y - 8x$$

$$y' = \frac{y - 8x}{2y - x}$$

Evaluate  $y'$  at  $(-1, -2)$ .

$$y'(-1, -2) = \frac{-2 - 8(-1)}{2(-2) - (-1)} = \frac{-2 + 8}{-4 + 1} = \frac{6}{-3} = -2$$

Then the equation of the tangent line at  $(-1, -2)$  is

$$y - y_1 = m(x - x_1)$$

$$y + 2 = -2(x + 1)$$

$$y + 2 = -2x - 2$$

$$y = -2x - 4$$

Evaluate  $y'$  at  $(-1, 1)$ .

$$y'(-1, 1) = \frac{1 - 8(-1)}{2(1) - (-1)} = \frac{1 + 8}{2 + 1} = \frac{9}{3} = 3$$

Then the equation of the tangent line at  $(-1, 1)$  is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 3(x + 1)$$





$$y - 1 = 3x + 3$$

$$y = 3x + 4$$



## HIGHER-ORDER DERIVATIVES

- 1. Find the second and third derivatives of the function at  $x = -1$ .

$$y = 2x^5 - 3x^4 + x^3 + x^2 - 7$$

*Solution:*

Find the first derivative.

$$y = 2x^5 - 3x^4 + x^3 + x^2 - 7$$

$$y' = 10x^4 - 12x^3 + 3x^2 + 2x$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (10x^4 - 12x^3 + 3x^2 + 2x)'$$

$$y'' = 40x^3 - 36x^2 + 6x + 2$$

Evaluate the second derivative at  $x = -1$ .

$$y''(-1) = 40(-1)^3 - 36(-1)^2 + 6(-1) + 2$$

$$y''(-1) = -40 - 36 - 6 + 2$$

$$y''(-1) = -80$$

Find the third derivative by taking the derivative of the second derivative.



$$y''' = (y'')' = (40x^3 - 36x^2 + 6x + 2)'$$

$$y''' = 120x^2 - 72x + 6$$

Evaluate the third derivative at  $x = -1$ .

$$y'''(-1) = 120(-1)^2 - 72(-1) + 6$$

$$y'''(-1) = 120 + 72 + 6$$

$$y'''(-1) = 198$$

■ 2. Find the second derivative of the function  $y = -3x^{\frac{2}{3}} + x^{-\frac{1}{2}}$ .

*Solution:*

Find the first derivative.

$$y = -3x^{\frac{2}{3}} + x^{-\frac{1}{2}}$$

$$y' = -3 \left( \frac{2}{3} x^{-\frac{1}{3}} \right) - \frac{1}{2} x^{-\frac{3}{2}}$$

$$y' = -2x^{-\frac{1}{3}} - \frac{1}{2} x^{-\frac{3}{2}}$$

Now we can find the second derivative by taking the derivative of the first derivative.



$$y'' = (y')' = \left( -2x^{-\frac{1}{3}} - \frac{1}{2}x^{-\frac{3}{2}} \right)'$$

$$y'' = -2 \left( -\frac{1}{3} \right) x^{-\frac{4}{3}} - \frac{1}{2} \left( -\frac{3}{2} \right) x^{-\frac{5}{2}}$$

$$y'' = \frac{2}{3}x^{-\frac{4}{3}} + \frac{3}{4}x^{-\frac{5}{2}}$$

■ 3. Find the second derivative of the function.

$$y = -3x^7 \sin x$$

*Solution:*

Find the first derivative.

$$y = -3x^7 \sin x$$

$$y' = -21x^6 \sin x + (-3x^7 \cos x)$$

$$y' = -21x^6 \sin x - 3x^7 \cos x$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (-21x^6 \sin x - 3x^7 \cos x)'$$

$$y'' = -126x^5 \sin x + (-21x^6 \cos x) - (21x^6 \cos x + 3x^7(-\sin x))$$



$$y'' = -126x^5 \sin x - 21x^6 \cos x - (21x^6 \cos x - 3x^7 \sin x)$$

$$y'' = -126x^5 \sin x - 21x^6 \cos x - 21x^6 \cos x + 3x^7 \sin x$$

$$y'' = -126x^5 \sin x - 42x^6 \cos x + 3x^7 \sin x$$

$$y'' = 3x^7 \sin x - 42x^6 \cos x - 126x^5 \sin x$$

$$y'' = 3x^5(x^2 \sin x - 14x \cos x - 42 \sin x)$$

■ 4. Find the second and the third derivatives of the function.

$$y = \ln(x^5 \sqrt{x})$$

*Solution:*

Find the first derivative.

$$y = \ln(x^5 \sqrt{x})$$

$$\frac{dy}{dx} = \frac{1}{x^5 \sqrt{x}} \left( 5x^4 \sqrt{x} + x^5 \frac{1}{2\sqrt{x}} \right)$$

$$\frac{dy}{dx} = \frac{5x^4 \sqrt{x}}{x^5 \sqrt{x}} + \frac{\frac{x^5}{2\sqrt{x}}}{x^5 \sqrt{x}}$$

$$\frac{dy}{dx} = \frac{5}{x} + \frac{1}{2x}$$



$$\frac{dy}{dx} = \frac{10}{2x} + \frac{1}{2x}$$

$$\frac{dy}{dx} = \frac{11}{2x}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{11}{2x} \right)$$

$$\frac{d^2y}{dx^2} = \frac{11}{2}(-1)x^{-2}$$

$$\frac{d^2y}{dx^2} = -\frac{11}{2x^2}$$

Find the third derivative by taking the derivative of the second derivative.

$$\frac{d}{dx} \left( \frac{d^2y}{dx^2} \right) = \frac{d}{dx} \left( -\frac{11}{2x^2} \right)$$

$$\frac{d^3y}{dx^3} = -\frac{11}{2}(-2)x^{-3}$$

$$\frac{d^3y}{dx^3} = 11x^{-3}$$

$$\frac{d^3y}{dx^3} = \frac{11}{x^3}$$



■ 5. Find the second derivative of the function.

$$y = \frac{2x}{\sin(x^2)}$$

*Solution:*

Find the first derivative.

$$y = \frac{2x}{\sin(x^2)}$$

$$y' = \frac{2 \sin(x^2) - 2x \cos(x^2)(2x)}{\sin^2(x^2)}$$

$$y' = \frac{2 \sin(x^2) - 4x^2 \cos(x^2)}{\sin^2(x^2)}$$

$$y' = \frac{2 \sin(x^2)}{\sin^2(x^2)} - \frac{4x^2 \cos(x^2)}{\sin^2(x^2)}$$

$$y' = \frac{2}{\sin(x^2)} - \frac{4x^2 \cot(x^2)}{\sin(x^2)}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = \left( \frac{2}{\sin(x^2)} - \frac{4x^2 \cot(x^2)}{\sin(x^2)} \right)'$$



$$y'' = \frac{(0)\sin(x^2) - 2\cos(x^2)(2x)}{\sin^2(x^2)}$$

$$- \frac{(8x \cot(x^2) + 4x^2(-\csc^2(x^2))(2x))\sin(x^2) - 4x^2 \cot(x^2)\cos(x^2)(2x)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x \cos(x^2)}{\sin^2(x^2)} - \frac{(8x \cot(x^2) - 8x^3 \csc^2(x^2))\sin(x^2) - 8x^3 \cot(x^2)\cos(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x \cot(x^2)}{\sin(x^2)} - \frac{8x \sin(x^2)\cot(x^2) - 8x^3 \sin(x^2)\csc^2(x^2) - 8x^3 \cos(x^2)\cot(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x \cot(x^2)}{\sin(x^2)} - \frac{8x \sin(x^2)\cot(x^2)}{\sin^2(x^2)} + \frac{8x^3 \sin(x^2)\csc^2(x^2)}{\sin^2(x^2)} + \frac{8x^3 \cos(x^2)\cot(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x \cot(x^2)}{\sin(x^2)} - \frac{8x \cot(x^2)}{\sin(x^2)} + \frac{8x^3 \csc^2(x^2)}{\sin(x^2)} + \frac{8x^3 \cot^2(x^2)}{\sin(x^2)}$$

$$y'' = -\frac{12x \cot(x^2)}{\sin(x^2)} + \frac{8x^3 \csc^2(x^2)}{\sin(x^2)} + \frac{8x^3 \cot^2(x^2)}{\sin(x^2)}$$

$$y'' = \frac{8x^3 \cot^2(x^2) + 8x^3 \csc^2(x^2) - 12x \cot(x^2)}{\sin(x^2)}$$

■ 6. Find the second derivative of the function at  $x = 0$ .

$$y = \frac{e^x}{4x - 9}$$

*Solution:*





Find the first derivative.

$$y = \frac{e^x}{4x - 9}$$

$$\frac{dy}{dx} = \frac{e^x(4x - 9) - e^x(4)}{(4x - 9)^2}$$

$$\frac{dy}{dx} = \frac{e^x(4x - 9 - 4)}{(4x - 9)^2}$$

$$\frac{dy}{dx} = \frac{e^x(4x - 13)}{(4x - 9)^2}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{e^x(4x - 13)}{(4x - 9)^2} \right)$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x - 13) + e^x(4))(4x - 9)^2 - e^x(4x - 13)2(4x - 9)(4)}{(4x - 9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x - 13) + 4e^x)(4x - 9)^2 - 8e^x(4x - 13)(4x - 9)}{(4x - 9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x - 13) + 4e^x)(4x - 9)^2}{(4x - 9)^4} - \frac{8e^x(4x - 13)(4x - 9)}{(4x - 9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(4x - 13) + 4e^x}{(4x - 9)^2} - \frac{8e^x(4x - 13)}{(4x - 9)^3}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(4x - 13 + 4)}{(4x - 9)^2} - \frac{8e^x(4x - 13)}{(4x - 9)^3}$$



$$\frac{d^2y}{dx^2} = \frac{e^x(4x-9)}{(4x-9)^2} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

$$\frac{d^2y}{dx^2} = \frac{e^x}{4x-9} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

Then evaluate the derivative at  $x = 0$ .

$$\frac{d^2y}{dx^2}(0) = \frac{e^0}{4(0)-9} - \frac{8e^0(4(0)-13)}{(4(0)-9)^3}$$

$$\frac{d^2y}{dx^2}(0) = \frac{1}{-9} - \frac{8(-13)}{(-9)^3}$$

$$\frac{d^2y}{dx^2}(0) = -\frac{1}{9} - \frac{104}{729}$$

$$\frac{d^2y}{dx^2}(0) = -\frac{185}{729}$$



## SECOND DERIVATIVES WITH IMPLICIT DIFFERENTIATION

- 1. Use implicit differentiation to find  $d^2y/dx^2$ .

$$2x^3 = 2y^2 + 4$$

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$6x^2 = 4y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x^2}{4y}$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(6x)(2y) - (3x^2)\left(2 \cdot \frac{dy}{dx}\right)}{(2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - 6x^2 \frac{dy}{dx}}{4y^2}$$



$$\frac{d^2y}{dx^2} = \frac{6xy - 3x^2 \frac{dy}{dx}}{2y^2}$$

Substitute the first derivative for  $dy/dx$  and then simplify.

$$\frac{d^2y}{dx^2} = \frac{6xy - 3x^2 \left( \frac{3x^2}{2y} \right)}{2y^2}$$

$$\frac{d^2y}{dx^2} = \frac{6xy - \frac{9x^4}{2y}}{2y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

Multiply through the numerator and denominator by  $y$  to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = \frac{12xy^2 - 9x^4}{4y^3}$$

■ 2. Use implicit differentiation to find  $d^2y/dx^2$ .

$$4x^2 = 2y^3 + 4y - 2$$



*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$8x = 6y^2 \frac{dy}{dx} + 4 \frac{dy}{dx}$$

$$8x = (6y^2 + 4) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{8x}{6y^2 + 4}$$

$$\frac{dy}{dx} = \frac{4x}{3y^2 + 2}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(4)(3y^2 + 2) - (4x) \left( 6y \cdot \frac{dy}{dx} \right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy \frac{dy}{dx}}{(3y^2 + 2)^2}$$

Substitute the first derivative for  $dy/dx$  and then simplify.

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy \left( \frac{4x}{3y^2 + 2} \right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - \frac{96x^2y}{3y^2 + 2}}{(3y^2 + 2)^2}$$



Multiply through the numerator and denominator by  $3y^2 + 2$  to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = \frac{12y^2(3y^2 + 2) + 8(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{(12y^2 + 8)(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{4(3y^2 + 2)(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{4(3y^2 + 2)^2 - 96x^2y}{(3y^2 + 2)^3}$$

■ 3. Use implicit differentiation to find  $d^2y/dx^2$  at  $(0,3)$ .

$$3x^2 + 3y^2 = 27$$

*Solution:*

Rewrite the equation.

$$3x^2 + 3y^2 = 27$$

$$x^2 + y^2 = 9$$

Use implicit differentiation to take the derivative of both sides.



$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = -\frac{(1)(y) - (x)(1)\left(\frac{dy}{dx}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x \frac{dy}{dx}}{y^2}$$

Substitute the first derivative for  $dy/dx$  and then simplify.

$$\frac{d^2y}{dx^2} = -\frac{y - x \left(-\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y + \frac{x^2}{y}}{y^2}$$

Multiply through the numerator and denominator by  $y$  to get rid of the fraction in the numerator.



$$\frac{d^2y}{dx^2} = -\frac{y^2 + x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-x^2 - y^2}{y^3}$$

Evaluate the second derivative at (0,3).

$$\frac{d^2y}{dx^2}(0,3) = \frac{-0^2 - 3^2}{3^3} = \frac{-9}{27} = -\frac{1}{3}$$

■ 4. Use implicit differentiation to find  $d^2y/dx^2$  at (2,1).

$$e^{x-2y} = 2x - y$$

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$e^{x-2y} \left( 1 - 2\frac{dy}{dx} \right) = 2 - \frac{dy}{dx}$$

$$e^{x-2y} - 2e^{x-2y} \frac{dy}{dx} = 2 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - 2e^{x-2y} \frac{dy}{dx} = 2 - e^{x-2y}$$

$$\frac{dy}{dx}(1 - 2e^{x-2y}) = 2 - e^{x-2y}$$





$$\frac{dy}{dx} = \frac{2 - e^{x-2y}}{1 - 2e^{x-2y}}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{-e^{x-2y} \left(1 - 2\frac{dy}{dx}\right)(1 - 2e^{x-2y}) - (2 - e^{x-2y})(-2e^{x-2y}) \left(1 - 2\frac{dy}{dx}\right)}{(1 - 2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(1 - 2\frac{dy}{dx}\right) e^{x-2y} [-(1 - 2e^{x-2y}) - (2 - e^{x-2y})(-2)]}{(1 - 2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(1 - 2\frac{dy}{dx}\right) e^{x-2y} (-1 + 2e^{x-2y} + 4 - 2e^{x-2y})}{(1 - 2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(1 - 2\frac{dy}{dx}\right) e^{x-2y} (3)}{(1 - 2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{3e^{x-2y} \left(1 - 2\frac{dy}{dx}\right)}{(1 - 2e^{x-2y})^2}$$

Substitute the first derivative for  $dy/dx$  and then simplify.

$$\frac{d^2y}{dx^2} = \frac{3e^{x-2y} \left(1 - 2 \left(\frac{2 - e^{x-2y}}{1 - 2e^{x-2y}}\right)\right)}{(1 - 2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{3e^{x-2y} \left(\frac{1 - 2e^{x-2y} - 2(2 - e^{x-2y})}{1 - 2e^{x-2y}}\right)}{(1 - 2e^{x-2y})^2}$$



$$\frac{d^2y}{dx^2} = \frac{3e^{x-2y} \left( \frac{1 - 2e^{x-2y} - 4 + 2e^{x-2y}}{1 - 2e^{x-2y}} \right)}{(1 - 2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{3e^{x-2y} \left( \frac{-3}{1 - 2e^{x-2y}} \right)}{(1 - 2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{9e^{x-2y}}{1 - 2e^{x-2y}}}{(1 - 2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2} = -\frac{9e^{x-2y}}{(1 - 2e^{x-2y})^3}$$

Evaluate the second derivative at (2,1).

$$\frac{d^2y}{dx^2}(2,1) = -\frac{9e^{2-2(1)}}{(1 - 2e^{2-2(1)})^3} = -\frac{9e^0}{(1 - 2e^0)^3} = -\frac{9}{(1 - 2)^3} = -\frac{9}{-1} = 9$$

■ 5. Use implicit differentiation to find  $y''$ .

$$y \sin x = 7 - 2y^2$$

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$y' \sin x + y \cos x = -4yy'$$

$$y' \sin x + 4yy' = -y \cos x$$



$$y'(\sin x + 4y) = -y \cos x$$

$$y' = -\frac{y \cos x}{\sin x + 4y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$y'' = -\frac{[y' \cos x + y(-\sin x)](\sin x + 4y) - (y \cos x)(\cos x + 4y')}{(\sin x + 4y)^2}$$

$$y'' = -\frac{(y' \cos x - y \sin x)(\sin x + 4y) - (y \cos x)(\cos x + 4y')}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y' \sin x \cos x + 4yy' \cos x - y \sin^2 x - 4y^2 \sin x - y \cos^2 x - 4yy' \cos x}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y' \sin x \cos x - y(\sin^2 x + \cos^2 x) - 4y^2 \sin x}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y' \sin x \cos x - y - 4y^2 \sin x}{(\sin x + 4y)^2}$$

Substitute the first derivative for  $y'$  and then simplify.

$$y'' = -\frac{\left(-\frac{y \cos x}{\sin x + 4y}\right) \sin x \cos x - y - 4y^2 \sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{\left(\frac{y \cos x}{\sin x + 4y}\right) \sin x \cos x + y + 4y^2 \sin x}{(\sin x + 4y)^2}$$



$$y'' = \frac{\left(\frac{y \cos x}{\sin x + 4y}\right) \sin x \cos x}{(\sin x + 4y)^2} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{\frac{y \sin x \cos^2 x}{\sin x + 4y}}{(\sin x + 4y)^2} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{y \sin x \cos^2 x}{(\sin x + 4y)^3} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

■ 6. Use implicit differentiation to find  $y''$  at  $(0,3)$ .

$$e^{2y} - 2x = y^4 - 2$$

*Solution:*

Use implicit differentiation to take the derivative of both sides.

$$e^{2y}(2y') - 2 = 4y^3y'$$

$$2e^{2y}y' - 4y^3y' = 2$$

$$y'(2e^{2y} - 4y^3) = 2$$

$$y' = \frac{2}{2e^{2y} - 4y^3}$$

$$y' = \frac{1}{e^{2y} - 2y^3}$$



Use implicit differentiation again on both sides to find the second derivative.

$$y'' = \frac{(0)(e^{2y} - 2y^3) - (1)(e^{2y}(2y') - 6y^2y')}{(e^{2y} - 2y^3)^2}$$

$$y'' = \frac{-2e^{2y}y' + 6y^2y'}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{2y'(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^2}$$

Substitute the first derivative for  $y'$  and then simplify.

$$y'' = -\frac{2\left(\frac{1}{e^{2y} - 2y^3}\right)(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{\left(\frac{2(e^{2y} - 3y^2)}{e^{2y} - 2y^3}\right)}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{2(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^3}$$

Evaluate the second derivative at (0,3).

$$\frac{d^2y}{dx^2}(0,3) = -\frac{2(e^{2(3)} - 3(3)^2)}{(e^{2(3)} - 2(3)^3)^3} = -\frac{2e^6 - 54}{(e^6 - 54)^3}$$



