Calculus 1 Workbook Solutions

Exponential growth and decay

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HALF LIFE

■ 1. Find the half-life of Tritium if its decay constant is 0.0562.

Solution:

Since we're calculating half-life, the exponential decay formula $y = Ce^{-kt}$ can be simplified to

$$\frac{1}{2} = e^{-kt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.0562t}$$

$$\ln\frac{1}{2} = \ln e^{-0.0562t}$$

$$-\ln 2 = -0.0562t$$

$$t = \frac{-\ln 2}{-0.0562} = \frac{\ln 2}{0.0562} \approx 12.33 \text{ years}$$

■ 2. Find the half-life of Cobalt-60 if its decay constant is 0.1315.

Solution:

Since we're calculating half-life, the exponential decay formula $y = Ce^{-kt}$ can be simplified to

$$\frac{1}{2} = e^{-kt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.1315t}$$

$$\ln\frac{1}{2} = \ln e^{-0.1315t}$$

$$-\ln 2 = -0.1315t$$

$$t = \frac{-\ln 2}{-0.1315} = \frac{\ln 2}{0.1315} \approx 5.27 \text{ years}$$

■ 3. Find the half-life of Berkelium-97 if its decay constant is 0.000503.

Solution:

Since we're calculating half-life, the exponential decay formula $y = Ce^{-kt}$ can be simplified to

$$\frac{1}{2} = e^{-kt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.000503t}$$

$$\ln\frac{1}{2} = \ln e^{-0.000503t}$$

$$-\ln 2 = -0.000503t$$

$$t = \frac{-\ln 2}{-0.000503} = \frac{\ln 2}{0.000503} \approx 1{,}378 \text{ years}$$

■ 4. Radium-224 has a half life of 3.66 days. If 3.25 g of Radium-224 remains after 9 days, what was the original mass of Radium-224?

Solution:

Substitute $t = 3.66$ into the half life formula,

$$\frac{1}{2} = e^{-kt}$$

$$\frac{1}{2} = e^{-3.66k}$$

then apply the natural logarithm to both sides in order to solve for k.

$$\ln\frac{1}{2} = \ln(e^{-3.66k})$$

$$\ln \frac{1}{2} = -3.66k$$

$$k = -\frac{1}{3.66} \ln \frac{1}{2}$$

With a value for k and $t = 9$ days and $y = 3.25$, we can now solve for the original amount of the substance, C.

$$y = Ce^{-kt}$$

$$3.25 = Ce^{-\left(-\frac{1}{3.66} \ln \frac{1}{2}\right)9}$$

$$3.25 = Ce^{\frac{9}{3.66} \ln \frac{1}{2}}$$

Solve for C.

$$C = \frac{3.25}{e^{\frac{9}{3.66} \ln \frac{1}{2}}}$$

$$C = \frac{3.25}{e^{\ln \left(\frac{1}{2}\right)^{\frac{9}{3.66}}}}$$

$$C = \frac{3.25}{\left(\frac{1}{2}\right)^{\frac{9}{3.66}}}$$

$C = 17.87$ g

The original amount of Radium-224 was 17.87 g

■ 5. The half-life of Potassium-40 is 1.25 billion years. A scientist analyzes a rock that contains only 9.5% of the Potassium-40 it contained originally when the rock was formed. How old is the rock?

Solution:

We don't know the original mass of the Potassium-40, but regardless of the size of the mass, we can say that the starting amount was 100% of the mass.

Substitute $t = 1.25$ billion years into the half life equation,

$$\frac{1}{2} = e^{-kt}$$

$$\frac{1}{2} = e^{-1.25k}$$

then apply the natural logarithm to both sides in order to solve for k.

$$\ln\frac{1}{2} = \ln(e^{-1.25k})$$

$$\ln\frac{1}{2} = -1.25k$$

$$k = -\frac{1}{1.25}\ln\frac{1}{2}$$

With a value for k, we can now solve for the number of years it'll take for the substance to decay from 100% to 9.5% of its original mass. We'll substitute $C = 1$ and $y = 0.095$, along with the value we've just found for the decay constant k.

$$y = Ce^{-kt}$$

$$0.095 = 1e^{-\left(-\frac{1}{1.25}\ln\frac{1}{2}\right)t}$$

$$0.095 = e^{\left(\frac{1}{1.25}\ln\frac{1}{2}\right)t}$$

Apply the natural logarithm to both sides in order to solve for t.

$$\ln 0.095 = \ln\left(e^{\left(\frac{1}{1.25}\ln\frac{1}{2}\right)t}\right)$$

$$\ln 0.095 = \left(\frac{1}{1.25}\ln\frac{1}{2}\right)t$$

$$1.25\ln 0.095 = \left(\ln\frac{1}{2}\right)t$$

$$t = \frac{1.25\ln 0.095}{\ln\frac{1}{2}}$$

$t \approx 4.24$ billion years

It would take about 4.24 billion years for the Potassium-40 to decay to $9.5\,\%$ of its original amount, so the scientist concludes that the rock is 4.24 billion years old.

6. 25 grams of a substance decayed to 13.25 grams in 13 seconds. Determine the half-life of a substance.

Solution:

Substitute $y = 13.25$, $C = 25$, and $t = 13$ into the exponential decay formula,

$$y = Ce^{-kt}$$

$$13.25 = 25e^{-13k}$$

$$0.53 = e^{-13k}$$

then apply the natural logarithm to both sides in order to solve for k.

$$\ln 0.53 = -13k$$

$$k = -\frac{\ln 0.53}{13}$$

$$k \approx 0.0488$$

Now that we have the decay constant, we can substitute it into the half-life formula to find t.

$$\frac{1}{2} = e^{-kt}$$

$$\ln \frac{1}{2} = \ln e^{-0.0488t}$$

$$-\ln 2 = -0.0488t$$

$$t = \frac{-\ln 2}{-0.0488} = \frac{\ln 2}{0.0488} \approx 14.2038 \text{ seconds}$$

NEWTON'S LAW OF COOLING

1. A cup of coffee is 195° F when it's brewed. Room temperature is 74° F. If the coffee is 180° F after 5 minutes, to the nearest degree, how hot is the coffee after 25 minutes?

Solution:

Use the information given and the temperature after 5 minutes to solve for k in the Newton's Law of Cooling formula.

$$T(5) - T_a = (T_0 - T_a)e^{-kt}$$

$$180 - 74 = (195 - 74)e^{-5k}$$

$$106 = 121e^{-5k}$$

$$\frac{106}{121} = e^{-5k}$$

$$\ln\frac{106}{121} = \ln e^{-5k}$$

$$\ln\frac{106}{121} = -5k$$

$$k = -\frac{1}{5}\ln\frac{106}{121} \approx 0.02647$$

Then use k to solve for $T(25)$.

$$T(25) - 74 = (195 - 74)e^{-0.02647(25)}$$

$$T(25) - 74 = 121e^{-0.66175}$$

$$T(25) - 74 = 62.42966$$

$$T(25) = 136.42966$$

The coffee is approximately 136° F after 25 minutes.

■ 2. A boiled egg that's 99° C is placed in a pan of water that's 24° C. If the egg is 62° C after 5 minutes, how much longer, to the nearest minute, will it take the egg to reach 32° C.

Solution:

Use the information given and the temperature after 5 minutes to solve for k in the Newton's Law of Cooling formula.

$$T(5) - T_a = (T_0 - T_a)e^{-kt}$$

$$62 - 24 = (99 - 24)e^{-5k}$$

$$38 = 75e^{-5k}$$

$$\frac{38}{75} = e^{-5k}$$

$$\ln\frac{38}{75} = \ln e^{-5k}$$

$$\ln \frac{38}{75} = -5k$$

$$k = -\frac{1}{5} \ln \frac{38}{75} \approx 0.13598$$

Then use k to solve for t.

$$32 - 24 = (62 - 24)e^{-0.13598t}$$

$$8 = 38e^{-0.13598t}$$

$$\frac{8}{38} = e^{-0.13598t}$$

$$\ln \frac{8}{38} = \ln e^{-0.13598t}$$

$$\ln \frac{8}{38} = -0.13598t$$

$$t = \frac{1}{-0.13598} \ln \frac{8}{38} \approx 11.4586$$

The egg will be 32° C after about 11 and a half more minutes.

■ 3. Suppose a cup of soup cooled from 200° F to 161° F in 10 minutes in a room whose temperature is 68° F. How much longer will it take for the soup to cool to 105° F?

Solution:

Use the information given and the temperature after 10 minutes to solve for k in the Newton's Law of Cooling formula.

$$T(10) - T_a = (T_0 - T_a)e^{-kt}$$

$$161 - 68 = (200 - 68)e^{-10k}$$

$$93 = 132e^{-10k}$$

$$\frac{93}{132} = e^{-10k}$$

$$\ln\frac{93}{132} = \ln e^{-10k}$$

$$\ln\frac{93}{132} = -10k$$

$$k = -\frac{1}{10}\ln\frac{93}{132} \approx 0.03502$$

Then use k to solve for t.

$$105 - 68 = (161 - 68)e^{-0.03502t}$$

$$37 = 93e^{-0.03502t}$$

$$\frac{37}{93} = e^{-0.03502t}$$

$$\ln\frac{37}{93} = \ln e^{-0.03502t}$$

$$\ln\frac{37}{93} = -0.03502t$$

$$t = \frac{1}{-0.03502} \ln \frac{37}{93} \approx 26.3187$$

The egg will be 105° F after about 26 more minutes.

■ 4. A thermometer is measuring 18° C indoors. The thermometer is moved outdoors where the temperature is -5° C, and after 2 minutes the thermometer reads 11° C. How many more minutes will it take for the thermometer to read 0° C?

Solution:

Substitute what we've been given into the Newton's Law of Cooling formula, then solve for k.

$$T(2) - T_a = (T_0 - T_a)e^{-kt}$$

$$11 - (-5) = (18 - (-5))e^{-2k}$$

$$16 = 23e^{-2k}$$

$$\frac{16}{23} = e^{-2k}$$

$$\ln \frac{16}{23} = -2k$$

$$k = -\frac{1}{2} \ln \frac{16}{23} \approx 0.18145$$

Now we'll determine how much longer it'll take for the thermometer to read 0° C. T_0 is now equal to $11^\circ C$.

$$T - T_a = (T_0 - T_a)e^{-kt}$$

$$0 - (-5) = (11 - (-5))e^{-0.18145t}$$

$$5 = 16e^{-0.18145t}$$

$$\frac{5}{16} = e^{-0.18145t}$$

$$\ln\frac{5}{16} = -0.18145t$$

$$t = \frac{1}{-0.18145}\ln\frac{5}{16} \approx 6.41031$$

The thermometer will read 0° C after about 6 more minutes.

■ 5. A cake baking inside an oven currently has a temperature of 220° C. Find the decay constant if the cake's temperature is 168° C 5 minutes after it's removed from the oven, given that the room temperature is 23° C.

Solution:

Substitute what we've been given into the Newton's Law of Cooling formula, then solve for k.

$$T(5) - T_a = (T_0 - T_a)e^{-kt}$$

$$168 - 23 = (220 - 23)e^{-5k}$$

$$145 = 197e^{-5k}$$

$$\frac{145}{197} = e^{-5k}$$

$$\ln \frac{145}{197} = -5k$$

$$k = -\frac{1}{5} \ln \frac{145}{197} \approx 0.06129$$

■ 6. Using the decay constant we calculated in the previous problem, determine the number of minutes that will pass before the cake's temperature will be 50° C.

Solution:

With T_0 now equal to 168° C, we'll calculate time in minutes.

$$T - T_a = (T_0 - T_a)e^{-kt}$$

$$50 - 23 = (168 - 23)e^{-0.06129t}$$

$$27 = 145e^{-0.06129t}$$

$$\frac{27}{145} = e^{-0.06129t}$$

$$\ln \frac{27}{145} = -0.06129t$$

$$t = \frac{1}{-0.06129} \ln \frac{27}{145} \approx 27.4253$$

The cake's temperature will be 50° C after about 27 more minutes.

SALES DECLINE

1. Suppose a pizza company stops a special sale for their three-topping pizza. They will resume the sale if sales drop to 70% of the current sales level. If sales decline to 90% during the first week, when should the company expect to start the special sale again?

Solution:

Use the exponential function $F = Pe^{-rt}$. Plug in what we know.

$$F = Pe^{-rt}$$

$$90 = 100e^{-r(1)}$$

$$\frac{90}{100} = e^{-r}$$

$$\ln \frac{90}{100} = \ln e^{-r}$$

$$\ln 90 - \ln 100 = -r$$

$$r = \ln 100 - \ln 90 \approx 0.10536$$

Find t using a sales level of 70% and $r = 0.10536$.

$$70 = 100e^{-0.10536t}$$

$$\frac{70}{100} = e^{-0.10536t}$$

$$\ln \frac{70}{100} = \ln e^{-0.10536t}$$

$$\ln 70 - \ln 100 = -0.10536t$$

$$t = \frac{\ln 70 - \ln 100}{-0.10536} \approx 3.385$$

Since time t is in weeks, this means the company should expect to start the sale again in about 3 and a half weeks.

■ 2. Suppose a donut store experiments with raising the price of a dozen donuts to see if sales are affected. They'll resume the sale if sales drop to 80% of the current sales level. If sales decline to 90% after two weeks, when should the store change back to the original price?

Solution:

Use the exponential function $F = Pe^{-rt}$. Plug in what we know.

$$F = Pe^{-rt}$$

$$90 = 100e^{-r(2)}$$

$$\frac{90}{100} = e^{-2r}$$

$$\ln \frac{90}{100} = \ln e^{-2r}$$

$$\ln 90 - \ln 100 = -2r$$

$$r = \frac{\ln 90 - \ln 100}{-2} \approx 0.05268$$

Find t using a sales level of 80% and $r = 0.05268$.

$$80 = 100e^{-0.05268t}$$

$$\frac{80}{100} = e^{-0.05268t}$$

$$\ln \frac{80}{100} = \ln e^{-0.05268t}$$

$$\ln 80 - \ln 100 = -0.05268t$$

$$t = \frac{\ln 80 - \ln 100}{-0.05268} \approx 4.2358$$

Since time t is in weeks, this means the store should expect to change back to the original price in about 4 and a quarter weeks.

■ 3. Suppose a flower shop decides to stop ordering roses in the winter time to see if sales are affected. They will resume the sale if sales drop to 90% of the current sales level. If sales decline to 96% after three weeks, when should the shop begin ordering roses again?

Solution:

Use the exponential function $F = Pe^{-rt}$. Plug in what we know.

$$F = Pe^{-rt}$$

$$96 = 100e^{-r(3)}$$

$$\frac{96}{100} = e^{-3r}$$

$$\ln \frac{96}{100} = \ln e^{-3r}$$

$$\ln 96 - \ln 100 = -3r$$

$$r = \frac{\ln 96 - \ln 100}{-3} \approx 0.01361$$

Find t using a sales level of 90% and $r = 0.01361$.

$$90 = 100e^{-0.01361t}$$

$$\frac{90}{100} = e^{-0.01361t}$$

$$\ln \frac{90}{100} = \ln e^{-0.01361t}$$

$$\ln 90 - \ln 100 = -0.01361t$$

$$t = \frac{\ln 90 - \ln 100}{-0.01361} \approx 7.7414$$

Since time t is in weeks, this means the store should begin ordering roses again in about 7 and three-quarter weeks.

■ 4. Mark has been selling lemonade for the last 5 years. Five years ago, he sold $3{,}850$ glasses of lemonade, but this year he's only sold $2{,}985$. Assuming that sales have declined exponentially, what's been the annual rate of decline?

Solution:

Both the sales decline and time have units in years, so with matching units we can plug directly into the sales decline formula to find the rate of decline.

$$F = Pe^{-rt}$$

$$2{,}985 = 3{,}850e^{-5r}$$

$$0.77532 = e^{-5r}$$

Apply the natural log to both sides.

$$\ln(0.77532) = \ln(e^{-5r})$$

$$\ln(0.77532) = -5r$$

$$r = \frac{\ln(0.77532)}{-5}$$

$$r = 0.0509 = 5.09\,\%$$

Over the last 5 years, sales declined by $5.09\,\%$ per year.

■ 5. A bakery sold $5{,}465$ croissants 3 years ago. If the sales declined at a rate of $1.5\,\%$ per month, how many croissants were sold last year?

Solution:

Since the rate of decline and time have different units, let's convert the monthly rate to an annual rate by multiplying the monthly rate by 12.

$$r = 1.5\% \cdot 12 = 18\,\%$$

Substitute what we know into the sales decline formula.

$$F = Pe^{-rt}$$

$$F = 5{,}465e^{-0.18(3)}$$

$$F \approx 3{,}185$$

About $3{,}185$ croissants were sold last year.

■ 6. Suppose a convenience store decides to stop their sale on ice cream in the summer time to see if sales are affected. They will resume the sale if sales drop to $87\,\%$ of their current level. If sales fall to $87\,\%$ of their current level after two weeks, what was the monthly rate of decline?

Solution:

Since we have to find the monthly rate of decline, we need to convert the given time from weeks to months. Two weeks is half a month.

$$t = 2/4 = 0.5$$

Substitute what we know into the sales decline formula.

$$F = Pe^{-rt}$$

$$87 = 100e^{-0.5r}$$

$$0.87 = e^{-0.5r}$$

Apply the natural log to both sides.

$$\ln(0.87) = \ln(e^{-0.5r})$$

$$\ln(0.87) = -0.5r$$

$$r = \frac{\ln(0.87)}{-0.5}$$

$$r \approx 0.2785$$

The rate of decline in sales was $27.85\,\%$.

COMPOUNDING INTEREST

1. Suppose you borrow $\$15{,}000$ with a single payment loan, payable in 2 years, with interest growing exponentially at 1.82% per month, compounded continuously. How much will it cost to pay off the loan after 2 years?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of months, we'll convert 2 years into 24 months for time t. The initial investment is $P = 15{,}000$ and the interest rate is $r = 0.0182$.

$$A = Pe^{rt}$$

$$A = 15{,}000e^{0.0182(24)}$$

$$A = \$23{,}216.20$$

2. Your parents deposit $\$5{,}000$ into a college savings account, with interest growing exponentially at 0.875% per quarter, compounded continuously. How much will be in the account after 18 years?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of quarters, we'll convert 18 years into 72 quarters for time t. The initial investment is $P = 5{,}000$ and the interest rate is $r = 0.00875$.

$$A = Pe^{rt}$$

$$A = 5{,}000e^{0.00875(72)}$$

$$A = \$9{,}388.05$$

■ 3. Suppose you win $\$50{,}000$ in a contest and you decide to save it for your retirement. You deposit it into an annuity account that pays $2.4\,\%$ semi-annually, compounded continuously. How much will the account contain after 25 years, when you plan to retire?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of half-years, we'll convert 25 years into 50 half-years for time t. The initial investment is $P = 50{,}000$ and the interest rate is $r = 0.024$.

$$A = Pe^{rt}$$

$$A = 50{,}000e^{0.024(50)}$$

$$A = \$166{,}005.85$$

■ 4. At a 7.5% yearly interest rate compounded semi-annually, how much money would we have to deposit now to have $\$15{,}500$ after 10 years?

Solution:

Using the compound interest formula, plug in what we know. Since the interest is compounded semi-annually, $n = 2$. The interest rate is $r = 0.075$ and we want a balance of $\$15{,}500$ when $t = 10$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$15{,}500 = P\left(1 + \frac{0.075}{2}\right)^{2(10)}$$

$$15{,}500 = P\left(1 + \frac{0.075}{2}\right)^{20}$$

$$15{,}500 = P\left(1 + \frac{0.075}{2}\right)^{20}$$

$$P = \frac{15{,}500}{\left(1 + \frac{0.075}{2}\right)^{20}}$$

$$P \approx \$7{,}422.84$$

5. How many years would it take for $\$25{,}000$ to turn into $\$50{,}000$, at a yearly interest rate of $4.75\,\%$, compounded annually?

Solution:

Using the compound interest formula, plug in what we know. Since the interest is compounded annually, $n = 1$. The initial investment is $P = 25{,}000$, and the interest rate is $r = 0.0475$. We're looking for a final balance of $\$50{,}000$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$50{,}000 = 25{,}000\left(1 + \frac{0.0475}{1}\right)^{1(t)}$$

$$2 = (1 + 0.0475)^t$$

$$2 = 1.0475^t$$

Apply the natural log to both sides.

$$\ln(2) = \ln(1.0475^t)$$

$$\ln(2) = t\ln(1.0475)$$

$$t = \frac{\ln(2)}{\ln(1.0475)}$$

$$t \approx 14.94$$

It would take almost 15 years for the initial amount of $\$25{,}000$ to grow to $\$50{,}000$, at an interest rate of $4.75\,\%$, compounded annually.

■ 6. At a yearly interest rate of $6.5\,\%$ compounded quarterly, how long would it take to triple an initial investment of $\$10{,}000$?

Solution:

Using the compound interest formula, plug in what we know. Since interest is compounded quarterly, $n = 4$. The initial investment is $P = 10{,}000$ and the interest rate is $r = 0.065$. Also, if we want to triple the initial investment, we're looking for a final balance of $\$30{,}000$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$30{,}000 = 10{,}000\left(1 + \frac{0.065}{4}\right)^{4t}$$

$$3 = (1 + 0.01625)^{4t}$$

$$3 = (1.01625)^{4t}$$

Apply the natural log to both sides.

$$\ln(3) = \ln(1.01625^{4t})$$

$$\ln(3) = 4t\ln(1.01625)$$

$$4t = \frac{\ln(3)}{\ln(1.01625)}$$

$$4t \approx 68.15$$

$$t \approx 17.04$$

It will take about 17 years to triple the initial investment from $\$10{,}000$ to $\$30{,}000$, when the $6.5\,\%$ interest rate is compounded quarterly.

W W W . K R I S T A K I N G M A T H . C O M