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Time: 3:30
Difficulty: 3.5

Simplify Logarithms

a) $\log(x) + \log(y) - \log z$

$\Rightarrow \log\left(\frac{xy}{z}\right)$

b) $2\log(x) + 1$

$\Rightarrow \log(x^2) + \log(e)$

$\Rightarrow \log(x^2 e)$

c) $\log(x) - 2$

$\Rightarrow \log(x) - \log(e^2)$

$\Rightarrow \log\left(\frac{x}{e^2}\right)$

Sequences

a) $U_n = 5 + 3n$

$n = 1$	$n = 2$	$n = 3$
$\Rightarrow U_1 = 8$	$\Rightarrow U_2 = 11$	$\Rightarrow U_3 = 14$

This is an arithmetic sequence.

b) $U_n = 3^n$

$n = 1$	$n = 2$	$n = 3$
$\Rightarrow U_1 = 3$	$U_2 = 9$	$U_3 = 27$

Ratio:

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2}$$

$$= \frac{\cancel{9}\,3}{\cancel{3}} = \frac{\cancel{27}\,3}{\cancel{9}}$$

$$= 3 = 3$$

This is a geometric sequence.

c) $U_n = n \times 3^n$

$n = 1$	$n = 2$	$n = 3$
$U_1 = 1 \times 3^1$	$U_2 = 2 \times 3^2$	$U_3 = 3 \times 3^3$
$U_1 = 3$	$U_2 = 18$	$U_3 = 81$

This sequence is neither an arithmetic progression nor a geometric one.

Find the limit

a) $U_n = 1 + \frac{1}{2}n$

Sol:

In this sequence, the term $\left(\frac{1}{2}n = \frac{n}{2}\right)$ increases without bounds as $n \to \infty$, and as $n \to -\infty$, the term decrease ~~without bounds~~. As such, this sequence does not have a finite limit which we can call the general limit. ~~However,~~

b) $U_n = \left(\frac{1}{2}\right)^n$

Sol;

For this sequence, as n approaches +ve infinity, the sequence converges to 0; thus the limit ~~in this~~ from the right hand side tends to be 0. From the left hand side, i.e., as n approaches -ve infinity, the sequence diverges to infinity. As such, this sequence does not have a general limit.

c) $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4}$

when $x = -3.99$	when $x = -4.01$
$\frac{(-3.99)^2+5(-3.99)+4}{(-3.99)^2+3(-3.99)-4}$	$\frac{(-4.01)^2+5(-4.01)+4}{(-4.01)^2+3(-4.01)-4}$
$\Rightarrow 0.599 \approx 0.6$	$\Rightarrow 0.6007 \approx 0.6$

This seq. does have a limit as x approaches -4 which is $\simeq 0.6$

Determine convergence or divergence

a) $a_n = \frac{3+5n^2}{n+n^2}$

div both num & den by n^2

$$a_n = \frac{3/n^2 + 5}{1/n + 1}$$

As $n \longrightarrow +\infty$, both $3/n^2$ & $1/n$ tend to be 0,

$$a_n = \frac{0+5}{0+1} = 5$$

The same is true in case where $n \longrightarrow -\infty$. As such, this seq. converges and the limit is 5.

b) $a_n = \frac{(-1)^{n-1}\, n}{n^2+1}$

when $n=1$	when $n=2$
$a_n = \frac{(-1)^{1-1}(1)}{(1)^2+1}$	$a_n = \frac{(-1)^{2-1}(2)}{(2)^2+1}$
$a_n = \frac{1}{2}$	$a_n = \frac{-2}{5}$

when $n = -1$

$$a_n = \frac{(-1)^{-1-1}(-1)}{(-1)^2+1} = \frac{1/(-1)^2 \cdot (-1)}{1+1} = \frac{-1}{2}$$

As n approaches both +ve & -ve infinity, the sequence converges to 0, thus the limit, too, is zero.

Find more limits.

$\lim_{x \to a} f(x) = -3$, $\lim_{x \to a} g(x) = 0$, $\lim_{x \to a} h(x) = 8$

a) $\lim_{x \to a} [f(x) + h(x)]$

Sol;

$$\Rightarrow \lim_{x \to a} [f(x) + h(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} h(x)$$

$$= -3 + 8$$

$$= 5$$

The limit exists at 5 as $x \to a$.

b) $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{-3}{0} = \text{undefined}$$

The limit does not exist as $x \to a$ as we get an undefined term & it violates the rule.

Also, the limit doesnot exist even if we approximate $g(x)$ as 0.001 or -0.001 as we get different results.

c) $\lim\limits_{x \to a} \dfrac{2f(x)}{h(x) - f(x)}$

$$= \frac{2\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} (h(x) - f(x))} = \frac{2\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} h(x) - \lim\limits_{x \to a} f(x)}$$

$$= \frac{2(-3)}{8-(-3)} = \frac{-6}{11}$$

The limit exists at $\frac{-6}{11}$

Check for discontinuities.

a) $f(x) = \dfrac{9x^3 - x}{(x-1)(x+1)}$

This function is discontinuous when $x = 1$ or $x = -1$ as it leads to an Undefined values.

b) $f(x) = e^{-x^2}$

This function is continuous as x approaches or moves away from 0 in either direction.

Find Finite Limits

a) $\lim\limits_{x \to 1} = \left[\dfrac{x^4 - 1}{x - 1}\right]$

Sol:

$$\lim_{x \to 1} \left[\frac{(x^2+1)(x^2-1)}{x-1}\right]$$

$$\Rightarrow \lim_{x \to 1} \left[\frac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{x-1}}\right]$$

$$\lim_{x \to 1} (x^2+1)(x+1) = (1+1)(1+1) = 4$$

b) $\lim_{x \to -4} \left[\frac{x^2+5x+4}{x^2+3x-4} \right]$

Sol:

$\lim_{x \to -4} \left[\frac{x^2+4x+x+4}{x^2+4x-x-4} \right]$

$\Rightarrow \lim_{x \to -4} \left[\frac{\cancel{(x+4)}(x+1)}{\cancel{(x+4)}(x-1)} \right]$

$\Rightarrow \lim_{x \to -4} \left[\frac{x+1}{x-1} \right] = \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}$

Find Infinite Limits

a) $\lim_{x \to \infty} \left[\frac{9x^2}{x^2+3} \right]$

Applying L'Hopital's Rule ~~Rules~~.

$\lim_{x \to \infty} \left[\frac{9x^2}{x^2+3} \right] = \lim_{x \to \infty} \frac{\overset{9}{\cancel{18}}\cancel{x}}{\cancel{2}\cancel{x}} = 9$

b) $\lim_{x \to \infty} \left[\frac{3^x}{x^3} \right]$

Applying L'Hopital's Rule

$\lim_{x \to \infty} \left[\frac{3^x}{x^3} \right] = \lim_{x \to \infty} \frac{3^x \log(3)}{3x^2}$

Applying L'Hopital's law again,

$$\lim_{x \to \infty} \frac{3^x \log(3)}{3x^2} = \frac{3^x (\log(3))^2}{6x}$$

Applying L'Hopital's law again,

$$\lim_{x \to \infty} \frac{3^x (\log(3))^2}{6x} = \frac{3^x (\log(3))^3}{6}$$

As $x \to \infty$, so does 3^x or 3^∞. As such the limit of $\frac{3^x}{x^3}$ as $x \to \infty$ is ∞.

Assessing Continuity & Differentiability

a) $f(x) = \begin{cases} +x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

Sol:

For $x \geq 0$ and $x < 0$:

$f(0) = 0^2 = 0$

$f(0) = -0^2 = 0$

Continuous

Differentiating both x^2 & $-x^2$

$2x$ & $-2x$

$2(0)$ & $-2(0)$

$0 = 0$

Differentiable

b) $f(x) = \begin{cases} x^3, & x \leq 1 \\ x, & x > 1 \end{cases}$

For $x \leq 1$ & $x > 1$;

$f(1) = x^3 = (1)^3 = 1$
$f(1) = x = 1$

Continuous

Differentiating both x^3 & x

$3x^2$ & 1
$3(1)^2$ & 1
$3 \neq 1$

Not differentiable

Possible Derivatives

A) Not possible - it is of a straight line
B) Not possible - it is of a straight line
C) C seems to be right as the line seems like that of a parabola & aligns with the zero
D) Seems wrong as the derivative seems both beneath & to the right of where the original graph of f(x)'s vertex is.

Calculate Derivatives.

a) $f(x) = 4x^3 + 2x^2 + 5x + 11$
$\Rightarrow f'(x) = 12x^2 + 4x + 5$

b) ~~[illegible]~~ $y = \sqrt{30}$
In this case, the derivative is 0 as it's a constant.

c) $h(t) = \log(9t+1)$
$= \frac{1}{9t+1}(9) = \frac{9}{(9t+1)}$

d) $f(x) = \log(x^2 e^x)$
$= \frac{1}{x^2 e^x}(2x e^x + x^2 e^x)$
$= \frac{2x e^x + x^2 e^x}{x^2 e^x}$
$= \frac{\cancel{x}\cancel{e^x}(2+x)}{x^{\cancel{2}}\cancel{e^x}}$
$= \frac{2+x}{x}$
$= \frac{2}{x} + 1$

e) $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y+5y^3)$

$= (y^{-2} - 3y^{-4})(y+5y^3)$

$= (-2y^{-3} + 12y^{-5})(y+5y^3) + \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(15y^2+1)$

$= \left(-\frac{2}{y^3} + \frac{12}{y^5}\right)(y+5y^3) + \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(15y^2+$

f) $h(x) = \frac{x}{\log(x)}$

$= \frac{1(\log(x)) - x(1/x)}{(\log(x))^2}$

$= \frac{\log(x) - 1}{(\log(x))^2}$

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USE THE PRODUCT & QUOTIENT RULE

Q) $f(x) = \dfrac{x^2-2x}{x^4+6}$

$$= \frac{(2x-2)(x^4+6)-(x^2-2x)(4x^3)}{(x^4+6)^2}$$

$$= \frac{2x^5+12x-2x^4-12-4x^5+8x^4}{(x^4+6)^2}$$

$$= \frac{-2x^5+6x^4+12x-12}{x^8+12x^4+36}$$

Using Product Rule.

$$f(x) = (x^2-2x)(x^4+6)^{-1} \qquad (x^4+6)^{-2}$$

$$= (2x-2)(x^4+6)^{-1} + (x^2-2x)(-1)(4x^3)$$

$$= \frac{(2x-2)}{(x^4+6)} + \frac{-4x^5+8x^4}{(x^4+6)^2}$$

$$= \frac{(2x-2)(x^4+6)-4x^5+8x^4}{(x^4+6)^2}$$

$$= \frac{2x^5+12x-2x^4-12-4x^5+8x^4}{(x^4+6)^2}$$

$$= \frac{-2x^5+6x^4+12x-12}{(x^4+6)^2}$$

They are equivalent.

Composite Functions.

a) $g(x) = x^2 + 4$, $h(x) = 5x - 1$

For $g(h(x))$;

$$g(h(x)) = g(5x-1) = (5x-1)^2 + 4$$
$$= 25x^2 - 10x + 5$$

Domain of $g(h(x)) = (-\infty, \infty)$
all real numbers

For $h(g(x)) = h(x^2+4) = 5(x^2+4) - 1$
$$= 5x^2 + 20 - 1$$
$$= 5x^2 + 19$$

Domain of $h(g(x)) = (-\infty, \infty)$
all real numbers

b) $g(x) = x^3$, $h(x) = (x-1)(x+1)$

For $g(h(x))$;

$$g(h(x)) = g((x-1)(x+1)) = ((x-1)(x+1))^3$$
$$= (x^2-1)^3$$
$$= (x^2-1)^2 (x^2-1)$$
$$= x^4 - 2x^2 + 1 \, (x^2-1)$$
$$\cancel{x^5 \; x^4 \; 2x^3 + 2x^2 + \; 1}$$
$$= x^6 - 2x^4 + x^2 - x^4 + 2x^2 - 1$$
$$= x^6 - 3x^4 + 3x^2 - 1$$

Domain of $g(h(x)) = (-\infty, \infty)$
all real numbers

For $h(g(x))$;

$$h(g(x)) = h(x^3) = (x^3 - 1)(x^3 + 1)$$
$$= x^6 - 1$$

Domain of $g(h(x)) = (-\infty, \infty)$
all real numbers

CHAIN RULE

a) $g(x) = x^2 + 4$, $h(x) = 5x - 1$

Sol:

Derivative of $g(h(x))$:

- Directly

$$g(h(x)) = 25x^2 - 10x + 5$$
$$\frac{d}{dx}(g(h(x))) = 50x - 10$$

- Using chain rule

$$g'(h(x)) = 2(5x - 1)$$
$$h'(x) = 5$$
$$\frac{d}{dx} g(h(x)) = 2(5x - 1) \cdot 5 = 50x - 10$$

Derivative of $h(g(x))$

- Directly

$h(g(x)) = 5x^2 + 9$

$\frac{d}{dx}(h(g(x))) = 10x$

- Using Chain rule

$h'(g(x)) = 5$

$g'(x) = 2x$

$\frac{d}{dx} h(g(x)) = 5.2x = 10x$

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b) $g(x) = x^3$, $h(z) = (z-1)(z+1)$

Sol:

Derivative of $g(h(z))$

- Directly

$g(h(z)) = z^6 - 3z^4 + 3z^2 - 1$

$\frac{d}{dx}(g(h(z))) = 6z^5 - 12z^3 + 6z$

- Using Chain rule

$g'(h(z)) = 3((z-1)(z+1))^2$

$h'(z) = 2z$

$\frac{d}{dz} g(h(z)) = 3(z^2-1)^2 \, 2z$

$= 6z(z^4 - 2z^2 + 1)$

$= 6z^5 - 12z^3 + 6z$

"

Derivative of $h(g(x))$.

- Directly

$h(g(x)) = x^6 - 1$

$\frac{d}{dx} h(g(x)) = 6x^5$

- Using chain rule

$h'(g(x)) = 2(x^3)$. $\cancel{3x^2}$

$g'(x) = 3x^2$

$\frac{d}{dx} h(g(x)) = 2x^3 . 3x^2 = 6x^5$

c) $g(x) = 4x+2$, $h(z) = \frac{1}{4}(z-2)$

Solution.

For $g(h(z))$;

$g'(h(z)) = 4$

$h'(z) = \frac{1}{4}$

$\frac{d}{dx} g(h(z)) = \cancel{4} \times \frac{1}{\cancel{4}} = 1$

For $h(g(x))$;

$h'(g(x)) = \frac{1}{4}$

$g'(x) = 4$

$\frac{d}{dx} h(g(x)) = \frac{1}{4} \times 4 = 1$

b) $f(x) = \begin{cases} x^3, & x \leq 1 \\ x, & x > 1 \end{cases}$

For $x \leq 1$ & $x > 1$;

$f(1) = x^3 = (1)^3 = 1$

$f(1) = x = 1$

Continuous

Differentiating both x^3 & x

$3x^2$ & 1

$3(1)^2$ & 1

$3 \neq 1$

Not differentiable

Possible Derivatives

A) Not possible - it is of a straight li-

B) Not possible - it is of a straight line

C) C seems to be right as the line seems like that of a parabola & aligns with the zero

D) Seems wrong as the derivative seems both beneath & to the right of where the original graph a f(x)'s vertex is.

The Other graph

Option B as it seems the best option for a derivative of a straight line.