# Lab 5: SAT Solver

The questions below are due on Friday April 03, 2020; 04:00:00 PM.

#### You are not logged in.

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Note that this link will take you to an external site (https://shimmer.csail.mit.edu) to authenticate, and then you will be redirected back to this page.

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# 1) Preparation

This lab assumes you have Python 3.6 or later installed on your machine.

The following file contains code and other resources as a starting point for this lab: lab5.zip

Most of your changes should be made to lab.py, which you will submit at the end of this lab. Importantly, you should not

add any imports to the file.

You can also see and participate in online discussion about this lab in the "Lab 5" Category in the forum.

This lab is worth a total of 4 points. Your score for the lab is based on:

- answering the questions on this page (0.5 points)
- passing the test cases from test.py under the time limit (2 points), and
- a brief "checkoff" conversation with a staff member to discuss your code (1.5 points).

Note that passing all of the tests on the server will require that your code runs reasonably efficiently.

Please also review the collaboration policy before continuing.

The questions in sections 1-3 (inclusive) are due before tutorial, at 8:30am MIT time on Wednesday, 1 Apr. The other questions on this page (including your code submission) are due at 4pm MIT time on Friday, 3 Apr. Due to the quiz on 8 Apr, checkoffs are due at 4pm MIT time on Friday, 10 Apr.

# 2) Introduction

From recreational mathematics to standardized tests, one popular problem genre is *logic puzzles*, where some space of possible choices is described using a list of rules. A solution to the puzzle is a choice (often the one unique choice) that obeys the rules. This lab should give you the tools to make short work of any of those puzzles, assuming you have your trusty Python interpreter.

Here's an example of the kind of logic puzzle we have in mind.

The 6.009 staff were pleased to learn that grateful alumni had donated cupcakes for last week's staff meeting. Unfortunately, the cupcakes were gone when the staff showed up for the meeting! Who ate the cupcakes?

- 1. The suspects are Ana, Duane, Max, Valerie, and Tim the Beaver.
- 2. Whichever suspect ate any of the cupcakes must have eaten all of them.
- 3. The cupcakes included exactly two of the flavors chocolate, vanilla, and pickles.
- 4. Ana only eats pickles-flavored cupcakes.
- 5. Years ago, Max and Valerie made a pact that, whenever either of them eats cupcakes, they must share with the other one.
- 6. Duane feels strongly about flavor fairness and will only eat cupcakes if he can include at least 3 different flavors.

Let's translate the problem into Boolean logic, where we have a set of variables, each of which takes the value True or False. We write the rules as conditions over these variables. Here is each rule as a Python expression over Boolean variables. We include one variable for the guilt of each suspect, plus one variable for each potential flavor of cupcake.

In reading these rules, note that Python not binds more tightly than or, so that not p or q is the same as (not p) or q. It's also fine not to follow every last detail, as this rule set is just presented as one example of a general idea!

You may also find that some of our encoding choices don't match what you would come up with, such that our choices lead to longer or less comprehensible rules. We are actually intentionally forcing ourselves to adhere to a restricted format that we will explain shortly, and that will ultimately make the job of *solving* these kinds of problems more straightforward.

```
rule1 = (ana or duane or max_ or valerie or tim)
# At least one of them must have committed the crime! Here, one of these
# variables being True represents that person having committed the crime.
rule2 = ((not ana or not duane)
     and (not ana or not max_)
     and (not ana or not valerie)
     and (not ana or not tim)
     and (not duane or not max_)
     and (not duane or not valerie)
     and (not duane or not tim)
     and (not max_ or not valerie)
     and (not max_ or not tim)
     and (not valerie or not tim))
# At most one of the suspects is quilty. In other words, for any pair of
# suspects, at least one must be NOT guilty (so that we cannot possibly find
# two or more people guilty)
# Together, rule2 and rule1 guarantee that exactly one suspect is guilty.
rule3 = ((not chocolate or not vanilla or not pickles)
     and (chocolate or vanilla)
     and (chocolate or pickles)
     and (vanilla or pickles))
# Here is our rule that the cupcakes included exactly two of the flavors. Put
# another way: it can't be that all flavors were present, and among any pair of
# flavors, at least one was present.
rule4 = ((not ana or pickles)
     and (not ana or not chocolate)
     and (not ana or not vanilla))
# If Ana is guilty, this will evaluate to True only if only pickles-flavored
# cupcakes were present. If Ana is not guilty, this will always evaluate to
# True. This is our way of encoding the fact that, if Ana is quilty, only
# pickles-flavored cupcakes must have been present.
rule5 = (not max_ or valerie) and (not valerie or max_)
# If Max ate cupcakes without sharing with Valerie, the first case will fail to
# hold. Likewise for Valerie eating without sharing. Since Max and Valerie only
# eat cupcakes together, this rule excludes the possibility that only one of
# them ate cupcakes.
rule6 = ((not duane or chocolate)
     and (not duane or vanilla)
     and (not duane or pickles))
# If Duane is the culprit and we left out a flavor, the corresponding case here
# will fail to hold. So this rule encodes the restriction that Duane can only
# be guilty if all three types of cupcakes are present.
```

```
satisfied = rule1 and rule2 and rule3 and rule4 and rule5 and rule6
```

The piece of code above is a Python program that will tell us whether a given assignment is consistent with the rules we have laid out. For example, if we had set the following variables (representing the hypothesis that Ana was guilty and that only Pickles-flavored cupcakes were present):

```
ana = True
duane = False
max_ = False
valerie = False
tim = False

pickles = True
vanilla = False
chocolate = False
```

and then run the code, the satisfied variable would be set to False (since rule3 would be False), indicating that this assignment did not satisfy the rules we had set out.

While code like the above could be useful in certain situations, it doesn't help us *solve* the problem (it only helps us check a possible solution). In this lab, we'll look at the problem of Boolean Satisfiability: our goal will be, given a description of Boolean variables and constraints on them (like that given above), to find a set of assignments that satisfies all of the given constraints.

### 2.1) Conjunctive Normal Form

In encoding the puzzle, we followed a very regular structure in our Boolean formulas, one important enough to have a common name: conjunctive normal form (CNF).

In this form, we say that a *literal* is a variable or the not of a variable. Then a *clause* is a multi-way or of literals, and a CNF formula is a multi-way and of clauses.

It's okay if this representation does not feel completely natural. Some people find this form to be "backwards" from the way they would otherwise think about these constraints. But forcing our constraints to be in this form can greatly simplify our problem. We'll try in this writeup to help you with the pieces of the lab involving converting expressions to CNF.

### 2.1.1) Python Representation

When we commit to representing problems in CNF, we can represent:

- a variable as a Python string
- a *literal* as a pair (a tuple), containing a variable and a Boolean value (False if not appears in this literal, True otherwise)
- a clause as a list of literals
- a formula as a list of clauses

For example, our puzzle from above can be encoded as follows, where again it is OK not to read through every last detail.

```
rule1 = [[('ana', True), ('duane', True), ('max', True),
          ('valerie', True), ('tim', True)]]
rule2 = [[('ana', False), ('duane', False)],
         [('ana', False), ('max', False)],
         [('ana', False), ('valerie', False)],
         [('ana', False), ('tim', False)],
         [('duane', False), ('max', False)],
         [('duane', False), ('valerie', False)],
         [('duane', False), ('tim', False)],
         [('max', False), ('valerie', False)],
         [('max', False), ('tim', False)],
         [('valerie', False), ('tim', False)]]
rule3 = [[('chocolate', False), ('vanilla', False), ('pickles', False)],
         [('chocolate', True), ('vanilla', True)],
         [('chocolate', True), ('pickles', True)],
         [('vanilla', True), ('pickles', True)]]
rule4 = [[('ana', False), ('pickles', True)],
        [('ana', False), ('chocolate', False)],
         [('ana', False), ('vanilla', False)]]
rule5 = [[('max', False), ('valerie', True)],
         [('valerie', False), ('max', True)]]
rule6 = [[('duane', False), ('chocolate', True)],
         [('duane', False), ('vanilla', True)],
         [('duane', False), ('pickles', True)]]
rules = rule1 + rule2 + rule3 + rule4 + rule5 + rule6
```

When we have formulated things in this way, the list rules contains a formula that encodes all of the constraints we need to satisfy.

#### 2.1.2) Examples

```
Consider this Boolean formula.

c and (a or d) and (not b or a) and (not a or e or not d)

Write an equivalent CNF formula (as a Python literal), in the format used in the lab (e.g., [[('a', True), ('b', False)], [('c', True)]]).

This question is due on Wednesday April 01, 2020 at 08:30:00 AM.
```

Now, consider this Boolean formula (which is **not** in CNF).

```
(a and b) or (c and not d)
```

This expression looks innocuous, but translating it into CNF is actually a non-trivial exercise! It turns out, though, that this expression does have a representation in CNF as:

```
(a or c) and (a or not d) and (b or c) and (b or not d)
```

or, in our representation, as:

```
[[('a', True), ('c', True)], [('a', True), ('d', False)], [('b', True), ('c', True)], [('b', True), ('d', False)]]
```

Notice that the above expression will be true if a and b are both true, or if c is true and d is false.

```
Now, try your hand at it. Consider the following Boolean Formula (not in CNF):

a and (not b or (c and d))

Write an equivalent CNF formula (as a Python literal), in the format used in the lab (e.g., [[('a', True), ('b', False)], [('c', True)]]).

This question is due on Wednesday April 01, 2020 at 08:30:00 AM.
```

## 3) SAT Solver

A classic tool that works on Boolean formulas is a **satisfiability solver**, or SAT solver. Given a formula, either the solver finds Boolean variable values that make the formula true, or the solver indicates that no solution exists. In this lab, you will write a SAT solver that can solve puzzles like ours, as in:

```
>>> print(satisfying_assignment(rules))
{'valerie': False, 'max': False, 'chocolate': False, 'duane': False,
   'ana': False, 'pickles': True, 'tim': True, 'vanilla': True}
```

The return value of satisfying\_assignment is a dictionary mapping variables to the Boolean values that have been inferred for them (or None if no valid mapping exists).

So, we can see that, in our example above, Tim the Beaver is guilty and has a taste for vanilla and pickles!

It turns out that there are other possible answers that have Tim enjoying other flavors, but it also turns out that Tim is the

uniquely determined culprit. How do we know? The SAT solver fails to find an assignment when we add an additional rule proclaiming Tim's innocence.

```
>>> print(satisfying_assignment(rules + [[('tim', False)]]))
None
```

### 3.1) The Naive Approach

There's one straightforward, brute-force way to solve Boolean puzzles: enumerate all possible Boolean assignments to the variables. Evaluate the rules on each assignment, returning the first assignment that works. Unfortunately, this process can take prohibitively long to run! For a concrete illustration of why, consider this Python code to generate all sequences of Booleans of a certain length.

When we have N Boolean variables in our puzzle, the possible assignments can be represented as length-N sequences of Booleans.

```
def all_bools_generator(length):
    if length == 0:
        yield []
    else:
        for v in all_bools_generator(length-1):
            yield [True] + v
            yield [False] + v

def all_bools(length):
    return list(all_bools_generator(length))
```

Here's an example output.

```
>>> all_bools(3)
[[True, True, True], [False, True, True], [True, False, True],
[False, False, True], [True, True, False], [False, True, False],
[True, False, False], [False, False, False]]
```

We could get more ambitious and try to generate longer sequences.

```
>>> len(all_bools(3))
8
>>> len(all_bools(4))
16
>>> len(all_bools(5))
32
>>> len(all_bools(6))
64
>>> len(all_bools(20))
1048576
>>> len(all_bools(25))
# Python runs for long enough that we give up!
```

It's actually quite expensive even to run through all Boolean sequences of nontrivial lengths, let alone to test each sequence against the rules. The reason is that there are  $2^N$  length-N Boolean sequences, and that kind of exponential function grows quite quickly as the length N of our mappings grows.

Lots of logic puzzles can lead to hundreds of Boolean variables. Are we out of luck if we want Python to do all the work? Worry not! In this lab, you will implement a SAT solver that uses a much smarter algorithm than this brute-force enumeration of all assignments, thanks to the magic of backtracking search.

## 3.2) A Nicer Approach

Instead of enumerating all assignments, we will ask you to implement a more clever approach for Boolean satisfiability. One such approach is outlined below (with some of the details intentionally omitted):

We start by picking an arbitrary variable x from our formula F. We then construct a related formula  $F_1$ , which does not involve x but incorporates all the consequences of setting x to be True. We then try to solve  $F_1$ . If it produces a successful result, we can combine that result with information about x being True to produce our answer to the original problem.

If we could not solve  $F_1$ , we should try setting x to be False instead. If no solution exists in either of the above cases, then the formula F cannot be satisfied.

#### 3.2.1) Updating Expressions

A key operation here is updating a formula to model the effect of a variable assignment. As an example, consider this starting formula.

```
(a or b or not c) and (c or d)
```

If we learn c = True, then the formula should be updated as follows.

```
(a or b)
```

We removed not c from the first clause, because we now know conclusively that that literal is False. Conversely, we can remove the second clause, because with c true, it is assured that the clause will be satisfied.

Note a key effect of this operation: variable d has disappeared from the formula, so we no longer need to consider values for d.

In general, this approach often saves us from an exponential enumeration of variable values, because we learn that, in some branches of the search space, some variables are actually irrelevant to the problem.

This pruning will show up in the assignments that your SAT solver returns: you are allowed (but not required) to omit variables that turn out to be irrelevant.

If we had instead learned that c was False, we would update the formula instead as follows:

d

How did we get there? Note that, with c being False, the first clause is already satisfied. The second clause, though, will only be True if d is True.

#### 3.2.2) Examples

Consider this CNF formula, in the form we use in this lab:

```
[
    [('a', True), ('b', True), ('c', True)],
    [('a', False), ('f', True)],
    [('d', False), ('e', True), ('a', True), ('g', True)],
    [('h', False), ('c', True), ('a', False), ('f', True)],
]
```

Write a CNF formula for the case where a = True, without using variable a. Write the string 'falsifies' if this assignment falsifies the formula, and write the string 'truthifies' if this assignment renders the formula true.

This question is due on Wednesday April 01, 2020 at 08:30:00 AM.

Write a CNF formula for the case where a = False, without using variable a. Write the string 'falsifies' if this assignment falsifies the formula, and write the string 'truthifies' if this assignment renders the formula true.

This question is due on Wednesday April 01, 2020 at 08:30:00 AM.

Write a CNF formula for the case where a = True and f = True, without using variables a or f. Write the string 'falsifies' if this assignment falsifies the formula, and write the string 'truthifies' if this assignment renders the formula true.

This question is due on Wednesday April 01, 2020 at 08:30:00 AM.

Write a CNF formula for the case where a = True and f = False, without using variables a or f. Write the string 'falsifies' if this assignment falsifies the formula, and write the string 'truthifies' if this assignment renders the formula true.

This question is due on Wednesday April 01, 2020 at 08:30:00 AM.

# 4) Implementation

Implement the function satisfying\_assignment as described in lab.py. Your function should take as input a CNF formula (in the form described throughout this writeup). It should return a dictionary mapping variable names to Boolean values if there exists such an assignment that satisfies the given formula. If no such assignment exists, it should return None.

### 4.1) Optimizations

A couple of further optimizations are likely to be necessary in order to pass all of the test cases quickly enough on the server:

- In the procedure described above, if setting the value of x immediately leads to a contradiction, we can immediately discard that possibility (rather than waiting for a later step in the recursive process to notice the contradiction).
- At the start of any call to your procedure, check if the formula contains any length-one clauses ("unit" clauses). If such a clause [(x, b)] exists, then we may set x to Boolean value b, just as we do in the True and False cases of the outline above. However, we know that, if this setting leads to failure, there is no need to backtrack and also try x = not b (because the unit clause alone tells us exactly what the value of x must be)!

Thus, you can begin your function with a loop that repeatedly finds unit clauses, if any, and propagates their consequences through the formula. Propagating the effects of one unit clause may reveal further unit clauses, whose later propagations may themselves reveal more unit clauses, and so on.

Implementing function satisfying\_assignment in this way should allow your code to pass the Test\_1\_Sat and Test\_2\_SudokuPuzzle test cases in time. The Test\_2\_SudokuPuzzle tests will use your SAT solver to solve sudoku puzzles, which can also be expressed in the CNF form! These test cases are large, so consider making smaller test cases to debug them.

You are free to add additional optimizations beyond what we laid out above, or even make broader changes to the algorithm, so long as you avoid "hard coding" for rather specific SAT problems (except for base cases like empty formulas).

## 5) Scheduling by Reduction

Now that we have a fancy new SAT solver, let's look at applying it to a new problem!

In general, it's possible to write a new implementation of backtracking search for each new problem we encounter, but another strategy is to *reduce* a new problem to one that we already know how to solve well. Boolean satisfiability is a popular target for reductions, because a lot of effort has gone into building fast SAT solvers. In this last part of the lab, you will implement a reduction to SAT from a scheduling problem.

In particular, we are interested in the real-life problem of assigning 6.009 students to different sessions for taking a quiz. Each student is available for only some of the sessions, but each session has limited capacity. We want to find a schedule (assignment of students to sessions) that respects all the constraints.

Please implement the function boolify\_scheduling\_problem(student\_preferences, session\_capacities), as described both below and in lab.py:

- The argument student\_preferences is a dictionary mapping a student name (string) to a set of session names (strings) when that student is available.
- Argument session\_capacities is a dictionary mapping each session name to a positive integer for how many students can fit in that session.
- The function returns a CNF formula encoding the schedule problem, as we explain next.

Here's an example call:

In English, Alice is available for the sessions in the basement and penthouse, Bob is available only for the session in the kitchen, etc. The basement can fit 1 student, the kitchen 2 students, and the penthouse 4 students. In this case, one legal schedule would be Alice in the basement, Bob in the kitchen, Charles in the kitchen, and Dana in the penthouse.

Your job is to translate such inputs into CNF formulas, such that your SAT solver can then find a legal schedule (or confirm that none exists).

The CNF formula you output should mention only Boolean variables named like student\_session, where student is the name of a student, and where session is the name of a session. The variable student\_session should be True if and only if that student is assigned to that session (for example, the variable Bob\_kitchen should be True if Bob is in the kitchen, and False otherwise.

The CNF clauses you include should enforce exactly the following rules (which are discussed in more detail below):

- 1. Students are only assigned to sessions included in their preferences.
- 2. Each student is assigned to exactly one session.
- 3. No session has more assigned students than it can fit.

Our requirement for this part of the lab is that your code *should not* solve the optimization problem. Rather, you should implement a translation to CNF formulas (which satisfying\_assignment can then solve).

During the checkoff, our staff will try to make sure you've followed this rule and not merely, say, implemented a backtracking search directly for the scheduling problem, to output trivial CNF problems that directly encode answers.

## 5.1) Encoding the Rules

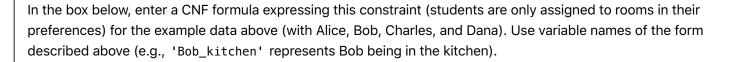
Turning these rules into CNF formulas is a tricky task. We'll try to provide some guidance for thinking about each of these rules in the following sections (though if you get stuck, please don't hesitate to ask via the Forum).

Note that each of these rules can be expressed as its own CNF formula, and the AND of these three rules represents the

overall formula we need to solve. As such, you may wish to write a helper function to generate the formula for each of these rules, and to use those helper functions in boolify\_scheduling\_problem. The examples below also make nice test cases (simpler than many we will test your code on)!

#### 5.1.1) Students Only In Desired Rooms

For each student, we need to guarantee that they are given a room that they selected as one of their preferences. In the example above, for example, we know that Charles must be in the basement or the kitchen, and that Alice must be in the basement or in the penthouse.



This question is due on Friday April 03, 2020 at 04:00:00 PM.

#### **Check Yourself:**

How could you generate this formula from the arguments given to boolify\_scheduling\_problem?

### 5.1.2) Each Student In Exactly One Session

This rule is a little bit trickier, but it may help to separate it into two pieces:

- each student must be in at least one session, and
- each student must be in at most one session.

We can generate formulae for each of these conditions, and combine them to construct the overall formula corresponding to this rule.

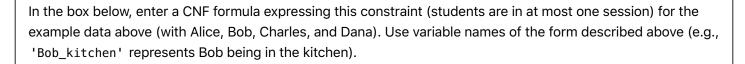
In fact, the first bullet (that each student be assigned to at least one session) is redundant with our first condition (that each student be assigned to a session in their preferences).

So let's turn our attention to making sure that each student is assigned to at most one session. This one is a bit trickier, particularly since we need to put things in CNF. One flip of perspective that can be helpful is that, if we need each student to be in at most one session, that means that for any pair of sessions, any given student can be in only one session.

For example, one clause in this expression will say that Bob cannot be in both the kitchen and the basement. That is, we cannot have both Bob\_kitchen and Bob\_basement be True (or, to phrase it a different way, at least one of them must be False).

Note that there is a corresponding clause for every other pair of rooms; and each of these clauses has a corresponding clause for Alice (and the other students). For purposes of this question, include *all* rooms for each student, regardless of

their preferences.



This question is due on Friday April 03, 2020 at 04:00:00 PM.

#### **Check Yourself:**

How could you generate this formula from the arguments given to boolify\_scheduling\_problem?

#### 5.1.3) No Oversubscribed Sections

This last rule is also fairly tricky, and it maybe requires a bit of a shift of perspective to express this constraint in CNF. However, it is similar to the previous rule in some ways.

We can think about this as: if a given room can contain N students, then in every possible group of N+1 students, there must be at least one student who is *not* in the given room.

For example, since the kitchen holds 2 people, we would need to consider all possible groups of 3 students and make sure that at least one of them is *not* in that room. For purposes of this question, include *all* groups of students, regardless of their preferences.

What about the penthouse? It has enough room for everyone, so there is no need even to include it in the constraints (it doesn't constrain our decision in any way)!

In the box below, enter a CNF formula expressing this constraint (no oversubscribed sections) for the example data above (with Alice, Bobe, Charles, and Dana). Use variable names of the form described above (e.g., 'Bob\_kitchen' represents Bob being in the kitchen).

This question is due on Friday April 03, 2020 at 04:00:00 PM.

#### **Check Yourself:**

How could you generate this formula from the arguments given to boolify\_scheduling\_problem?

## 6) UI

We have provided a browser UI for this lab, so you can see your code solving scheduling problems in action! When you have implemented both main functions of the lab, you may run python3 server.py and navigate to http://localhost:6009/. Running a test case on the UI involves generating a CNF formula and searching for a satisfying assignment for it, done by calling your functions. Enjoy!

## 7) Code Submission

Once you have debugged your code locally and are satisfied, upload your lab.py below:

Select File No file selected

This question is due on Friday April 03, 2020 at 04:00:00 PM.

# 8) Checkoff

Once you are finished with the code, please come to a tutorial, lab session, or office hour and add yourself to the queue asking for a checkoff. You must be ready to discuss your code and test cases in detail before asking for a checkoff.

You should be prepared to demonstrate your code (which should be well-commented, should avoid repetition, and should make good use of helper functions). In particular, be prepared to discuss:

- In English, the general recipe for finding assignments that satisfy a given CNF formula.
- In English, your strategy for updating a formula based on setting an variable to be True or False.
- Your implementation of satisfying\_assignment, including any helper functions.
- In English, the general recipe you came up with for translating scheduling problems into CNF formulas (including all three rules).
- Your code for boolify\_scheduling\_problem to implement that recipe.

## 8.1) **Grade**

You have not yet received this checkoff. When you have completed this checkoff, you will see a grade here.