Forecasting mortality with the Lee-Carter method

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Guest lecture for the M.Sc. course "Mortality Analysis" taught by Prof. Roland Rau

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Preliminaries

- Presentations
- All materials (slides & codes) available at https://github.com/ubasellini/LC-short-course
- Materials are (mostly) derived from a wider IDEM course jointly taught with Giancarlo Camarda (INED): https://github.com/ubasellini/IDEM117-AdvancesMortalityForecasting

Mortality forecasting

- Crucial for sustainability of pensions, insurances, elderly care; predicting population ageing and projecting populations; ...
- Until the 1980s, the methods used to forecast mortality were **deterministic**, based on mathematical formulae or expert judgment
- Revived interest in recent years following the introduction of the Lee-Carter method in 1992
- One of the firstly introduced **stochastic** mortality models $\Rightarrow$ the model "revolutionized probabilistic mortality and population forecasting" (Raftery 2023)

The Lee-Carter method (1992)

- Proposed in 1992 to model and forecast US mortality
- After 30+ years, Lee-Carter (LC) still widely employed by variety of users: governments, private companies, international organizations, ...
- The landmark model in mortality forecasting
- An **extrapolation** method:
  - model the mortality surface over age and time
  - extrapolate trends in the future, assuming that observed trends will continue
- Simplicity, robustness and objectivity have made the model so successful
- Nonetheless, some limitations of the model have stimulated several extensions over the years (see Basellini et al. (2023) for a recent comprehensive review)

The LC method

- A simple log-bilinear functional form for mortality rates $m_{x,t}$ at age x and time t

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t} \qquad (1)$$

  where:

  - α_x is the general shape of log-mortality at age x
  - β_x is the rate of mortality improvement at age x
  - κ_t is the general level of mortality at time t
  - $\epsilon_{x,t}$ is the error term with mean 0 and variance σ^2_ϵ, reflecting residual age-specific influences not captured by the model

- Modelling log-rates $\Rightarrow$ fitted and forecast rates constrained to be positive
- Log transformation partially counters heteroscedasticity of observed rates

The LC method

- The model is undetermined: if $\boldsymbol{\theta}_1 = [\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}]$ is a solution, then for any scalar c:
  - $\boldsymbol{\theta}_2 = [\boldsymbol{\alpha} - \boldsymbol{\beta} c, \boldsymbol{\beta}, \boldsymbol{\kappa} + c]$ is also a solution
  - $\boldsymbol{\theta}_3 = [\boldsymbol{\alpha}, \boldsymbol{\beta} c, \boldsymbol{\kappa}/c]$ is also a solution

- Two constraints introduced to ensure model identification:

$$\sum_x \beta_x = 1 \qquad \text{and} \qquad \sum_t \kappa_t = 0 \tag{2}$$

The LC model: a schematic view

$$\ln\left(m_{x,t}\right) \simeq \alpha_x + \beta_x \kappa_t$$

$$\begin{pmatrix} \ln(m_{0,1960}) & \ln(m_{0,1961}) & \dots & \ln(m_{0,2018}) \\ \ln(m_{1,1960}) & \ln(m_{1,1961}) & \dots & \ln(m_{1,2018}) \\ \ln(m_{2,1960}) & \ln(m_{2,1961}) & \dots & \ln(m_{2,2018}) \\ \vdots & \vdots & \ddots & \vdots \\ \ln(m_{105,1960}) & \ln(m_{105,1961}) & \dots & \ln(m_{105,2018}) \end{pmatrix} \simeq$$

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{105} \end{pmatrix} + \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{105} \end{pmatrix} \begin{pmatrix} \kappa_{1960} & \kappa_{1961} & \dots & \kappa_{2018} \end{pmatrix}$$

$$\underbrace{59}_{\text{years}} \times \underbrace{106}_{\text{ages}} = \underbrace{6254}_{\text{cells}} \simeq \underbrace{106}_{\alpha_i} + \underbrace{106}_{\beta_i} + \underbrace{59}_{\kappa_j} - \underbrace{2}_{\text{constraints}} = \underbrace{269}_{\text{parameters}}$$

Model estimation

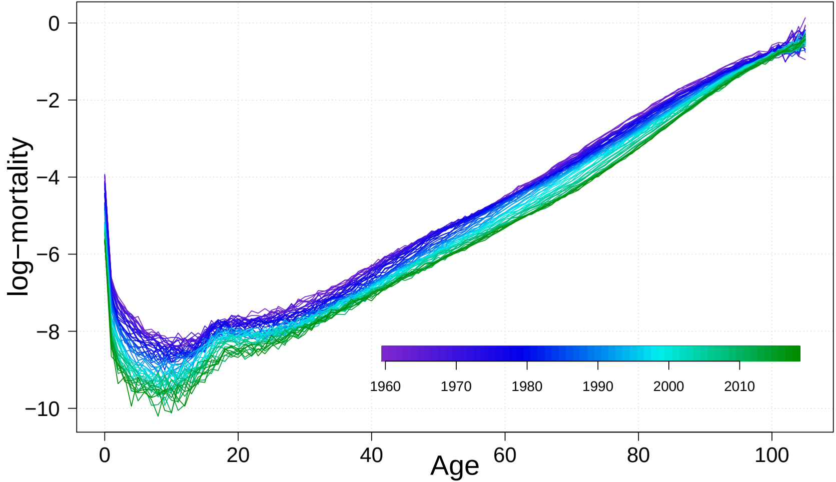
- The model is estimated by minimizing the residual sum of squares:

$$\sum_{x,t}\Big(\ln\left(m_{x,t}\right)-\alpha_x-\beta_x\kappa_t\Big)^2 \tag{3}$$

- A singular value decomposition (SVD) is employed to minimize Eq. (3) & derive an ordinary least squares (OLS) solution:
  - $\hat{\alpha}_x$ is the average of the observed $\ln\left(m_{x,t}\right)$

Estimating LC: an example

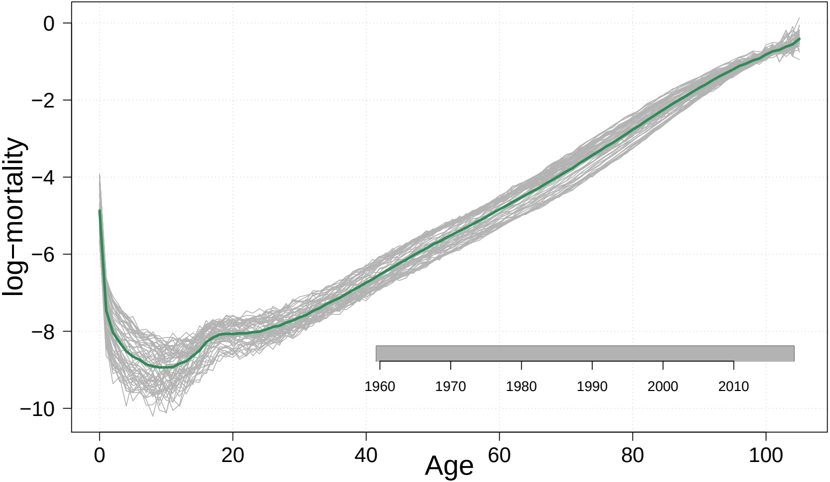
- observed mortality rates $\ln\left(m_{x,t}\right)$



Females aged 0–105+ in England & Wales, 1960–2018.
Source (all figures): Human Mortality Database (2021)

Estimating LC: an example

- $\hat{\alpha}_x$ = average of observed mortality rates $\ln(m_{x,t})$



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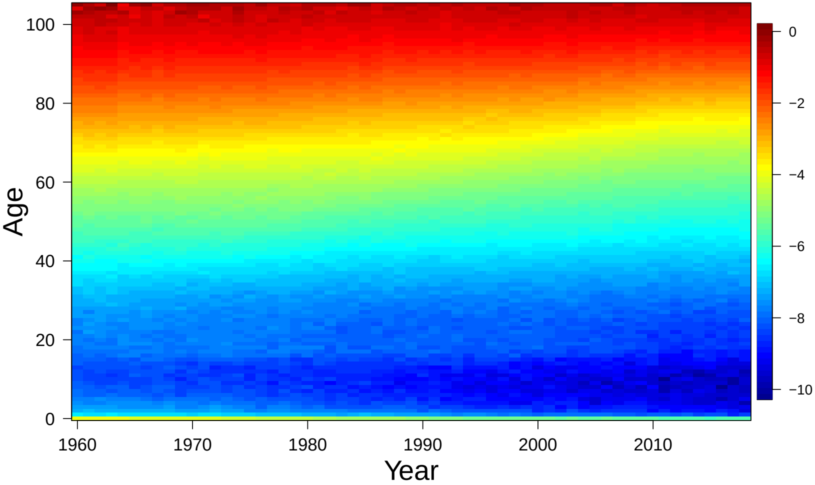
  - $\hat{\alpha}_x$ is the average of the observed $\ln\left(m_{x,t}\right)$
  - $\hat{\beta}_x$ and $\hat{\kappa}_t$ are derived from the first left- and right-singular vectors of the SVD of the matrix $\ln\left(m_{x,t}\right)-\hat{\alpha}_x$

Estimating LC: an example

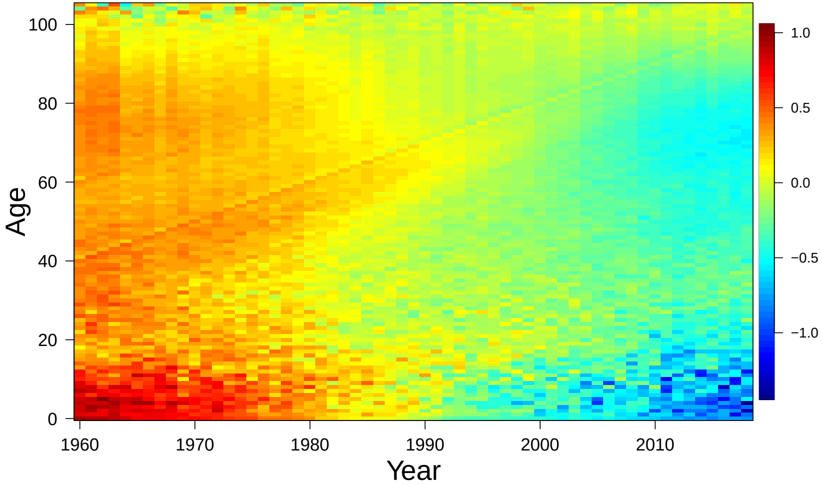
- $\boldsymbol{M} = (\ln(m_{x,t}))$: matrix of observed mortality rates



Females aged 0–105+ in England & Wales, 1960–2018.

Estimating LC: an example

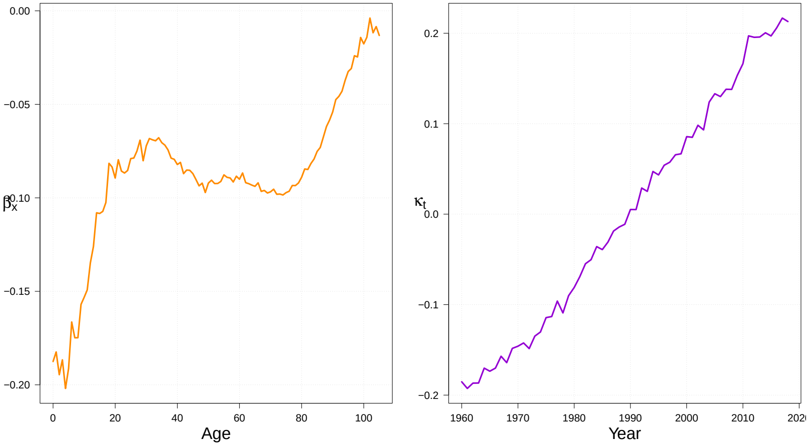
- $\tilde{M} = (\ln(m_{x,t}) - \hat{\alpha}_x)$: matrix of "centered" mortality rates



Females aged 0–105+ in England & Wales, 1960–2018.

Estimating LC: an example

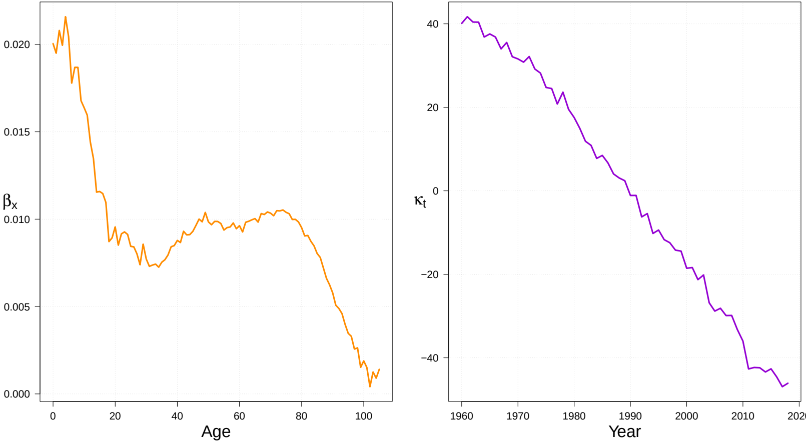
- From SVD of $\tilde{M}$



Females aged 0–105+ in England & Wales, 1960–2018.

Estimating LC: an example

- Adjusting the left-singular vector to sum to 1, and multiplying the right-singular vector for the leading singular value and the sum of the left-singular vector, we get $\hat{\beta}_x$ and $\hat{\kappa}_t$



Females aged 0–105+ in England & Wales, 1960–2018.

Model estimation

- The model is estimated by minimizing the residual sum of squares:

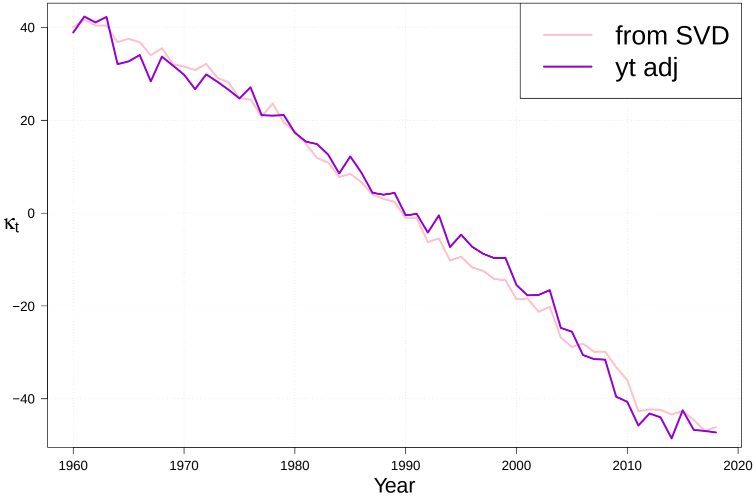
$$\sum_{x,t} \Big(\ln(m_{x,t}) - \alpha_x - \beta_x \kappa_t \Big)^2$$

- A singular value decomposition (SVD) is employed to minimize Eq. (3) & derive an ordinary least squares (OLS) solution:
  - $\hat{\alpha}_x$ is the average of the observed $\ln(m_{x,t})$
  - $\hat{\beta}_x$ and $\hat{\kappa}_t$ are derived from the first left- and right-singular vectors of the SVD of the matrix $\ln(m_{x,t}) - \hat{\alpha}_x$
- In a second-step estimation, the parameter $\hat{\kappa}_t$ is adjusted so that the fitted deaths match the observed deaths in all years, i.e.

$$\sum_x \hat{y}_{x,t} = \sum_x y_{x,t} \quad \text{for all } t$$

Estimating LC: an example

- Adjusting $\hat{\kappa}_t$ to match observed number of deaths at each year t



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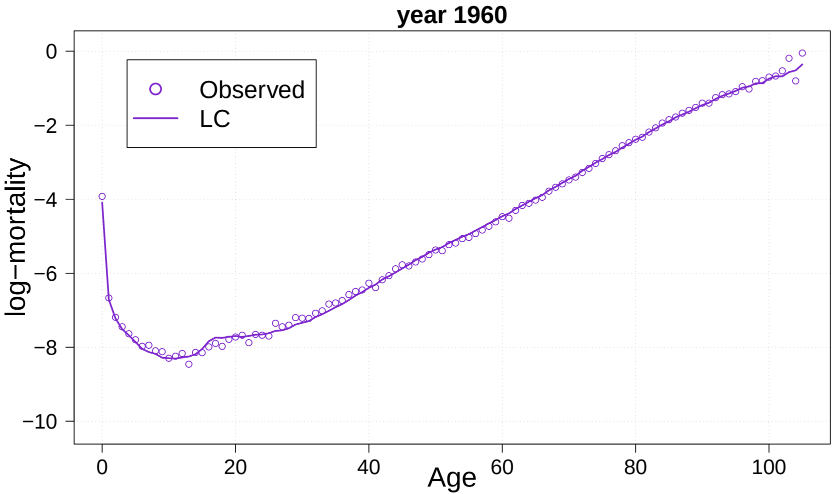
Estimating LC: an example



Females aged 0–105+ in England & Wales, 1960–2018.

Time-series analysis: a short introduction

- A *time-series* is a collection of observations made sequentially through time. Let us focus on discrete time series recorded at equal intervals of time
- Suppose we have an observed time series $y_1, y_2, \ldots, y_T$ and we wish to forecast future values such as y_{T+h}. The integer h is called *forecasting horizon*, and we denote by $\hat{y}_{T+h}$ the forecast made at time T for h steps ahead
- Forecasting methods may be broadly identified into three types:
  - *Judgemental forecasts*: based on subjective judgement and intuition
  - *Univariate methods*: forecasts depend on present and past values of a single series
  - *Multivariate methods*: forecasts depend, at least partly, on values of one or more additional variables (predictors)

Descriptive techniques

- In order to *forecast*, description and modelling of data is a prerequisite
- Always start from plotting your data!!
- This can help to identify two main sources of variation in many time series: i) trend, and ii) seasonal variation
- These variations are typically removed before time-series modelling via differencing $\Rightarrow$ helps to stabilize the **mean**
- The time plot may also help to decide whether a variable needs to be transformed $\Rightarrow$ helps to stabilize the **variance**

Stationary stochastic processes

- A *stochastic* time series is one where future values can only partly be determined by past values
- A process is defined *stationary* if its properties do not change through time
- More formally, let y_t be the realization of the underlying random variable Y_t, and the observed time-series $\boldsymbol{y} = [y_1, y_2, \ldots, y_T]$ be a realization of the stochastic process
- A stochastic process is second-order stationary if its first and second moments are finite and do not change over time, i.e.:

$$\mathbb{E}[Y_t] = \mu$$
$$\mathrm{COV}[Y_t, Y_{t+k}] = \mathbb{E}[(Y_t - \mu)(Y_{t+k} - \mu)] = \gamma_k$$

  for all t and for $k = 0, 1, 2, \ldots$ (note, for $k = 0$, $\gamma_0 = \sigma^2$)
- In simpler words, a stationary series has constant mean, constant variance and no predictable patterns in the long-term

Differencing

- Powerful tool to stabilize the mean and obtain stationary time-series
- First-order differencing: $y'_t = y_t - y_{t-1}$, y'_t is the *change* between observations of y_t (composed by $T-1$ values)
- If y'_t still non-stationary, second-order differencing: $y''_t = y'_t - y'_{t-1} = y_t - 2y_{t-1} + y_t - 2$ (composed by $T-2$ values)
- Almost never necessary to go beyond y''_t
- For seasonal data, seasonal differencing: $y'_t = y_t - y_{t-m}$, where m is the number of seasons, y'_t is the change between one year to the next
- In addition to correlogram, two main tests for determining the required order of differencing:
  - Augmented Dickey Fuller test: H_0 data are non-stationary and non-seasonal
  - Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test : H_0 data are stationary and non-seasonal

Random walk

- Random walk models are widely used for non-stationary data (e.g. economic and financial data):

$$Y_t = Y_{t-1} + Z_t$$

  where Z_t is a purely random process (with zero mean and constant variance σ_z^2)

- Random walk is non-stationary (variance increases through time), but first-order differences (purely random process) is stationary
- Typical features:
  - sudden and unpredictable changes of direction
  - long periods of apparent trends up/down
- Forecasts for the random walk model are simply given by the value of the last observation (i.e. naïve forecast), i.e.:

$$\hat{y}_{T+h|T} = y_T$$

Random walk with drift

- A closely related model that allows first differences to have non-zero mean is the **random walk with drift**:

$$Y_t = c + Y_{t-1} + Z_t$$

where Z_t is a purely random process, and c is a constant
  - if $c > 0$, y_t will tend to drift upwards
  - if $c < 0$, y_t will tend to drift downwards
- The estimate of the drift c is given by the average of the changes between consecutive observations, i.e.

$$c = \frac{1}{T-1} \sum_{t=2}^{T} (y_t - y_{t-1}) = \frac{y_T - y_1}{T-1}$$

- Forecasts for the random walk model with drift are given by:

$$\hat{y}_{T+h|T} = y_T + c\,h$$

$\Rightarrow$ equivalent to drawing a line between the first and last observations, and extrapolating it into the future

Forecasting with LC

- Forecasting "made simple": choose an appropriate time-series model for $\hat{\kappa}_t$ and extrapolate it
- The forecast $\hat{\kappa}_{T+h}$ allows one to derive the entire age-pattern of mortality at time $T+h$, i.e.:

$$\ln\left(\hat{m}_{x,T+h}\right) = \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_{T+h}$$

- LC suggest a random walk model (i.e. ARIMA(0,1,0)) with drift:

$$\kappa_t = \kappa_{t-1} + c + e_t$$

  where c is a constant (drift) and e_t the error term (purely random process)
- For this time-series model: $\hat{\kappa}_{T+h|T} = \hat{\kappa}_T + c\,h$
- Simulated future trajectories of $\hat{\kappa}_{T+h}$ to construct prediction intervals for $\hat{m}_{x,T+h}$ and other summary measures (e.g., $\hat{e}_{0,T+h}$)
- (Coale and Guo (1989) adjustment for forecast rates at ages 85+)

Forecasting with LC: a schematic view

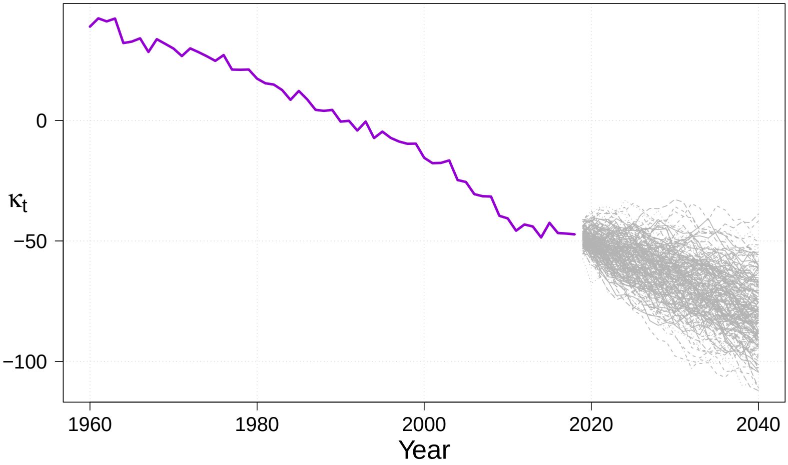
$$\ln\left(m_{x,t}\right) \simeq \alpha_x + \beta_x \kappa_t \qquad \ln\left(m_{x,T+h}\right) \simeq \alpha_x + \beta_x \kappa_{T+h}$$

$$\begin{pmatrix} \ln(m_{0,1960}) & \dots & \ln(m_{0,2018}) & \ln(m_{0,2019}) & \dots & \ln(m_{0,2040}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \ln(m_{105,1960}) & \dots & \ln(m_{105,2018}) & \ln(m_{105,2019}) & \dots & \ln(m_{105,2040}) \end{pmatrix} \simeq$$

$$\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{105} \end{pmatrix} + \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{105} \end{pmatrix} \begin{pmatrix} \kappa_{1960} & \dots & \kappa_{2018} & \kappa_{2019} & \dots & \kappa_{2040} \end{pmatrix}$$

Forecasting LC: an example

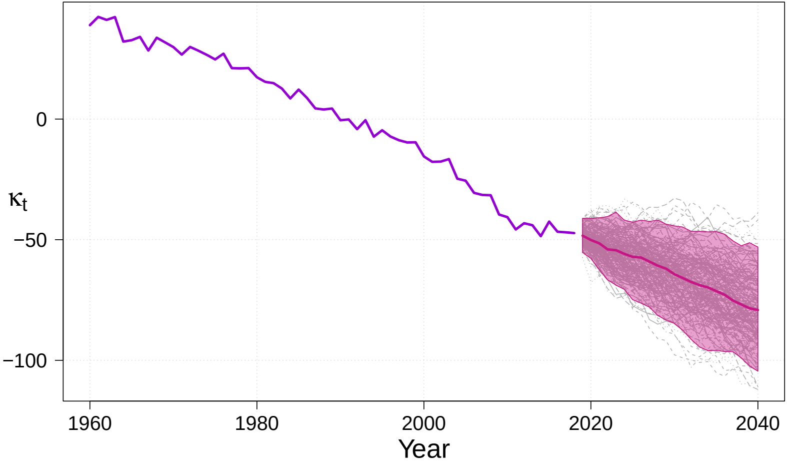
- Simulated future paths of $\hat{\kappa}_t$ from random walk with drift



Females aged 0–105+ in England & Wales, 1960–2018, forecast 2019–2040.

Forecasting LC: an example

- Simulated future paths of $\hat{\kappa}_t$ from random walk with drift $\Rightarrow$ prediction intervals from empirical percentiles



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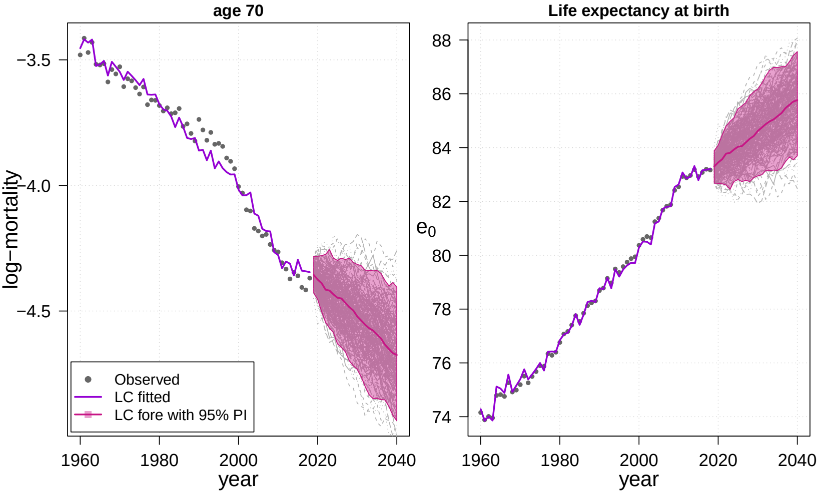
Forecasting LC: an example

- Prediction intervals from simulated future $\hat{\kappa}_{T+h}$



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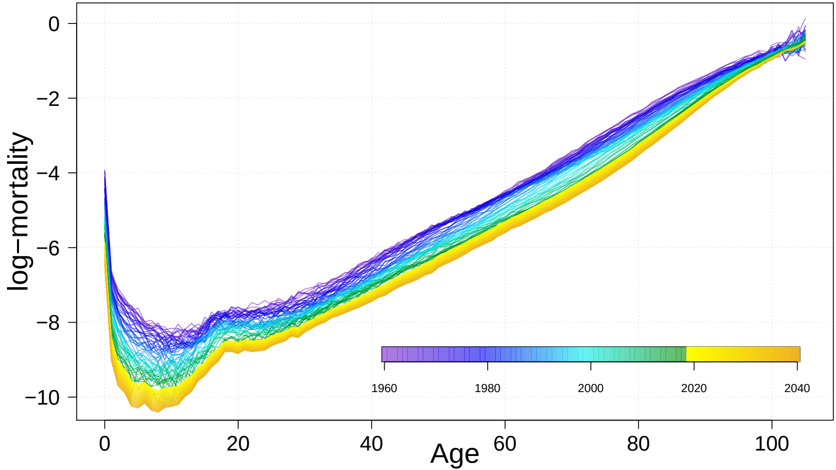
Forecasting LC: an example

- $\hat{\kappa}_{T+h}$ allows one to derive forecast rates $\hat{\boldsymbol{M}} = (\ln(\hat{m}_{x,T+h}))$



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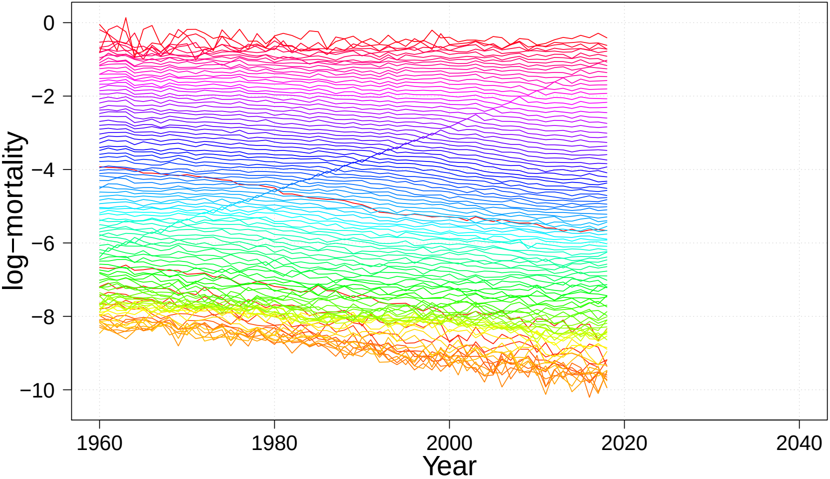
Forecasting LC: an example



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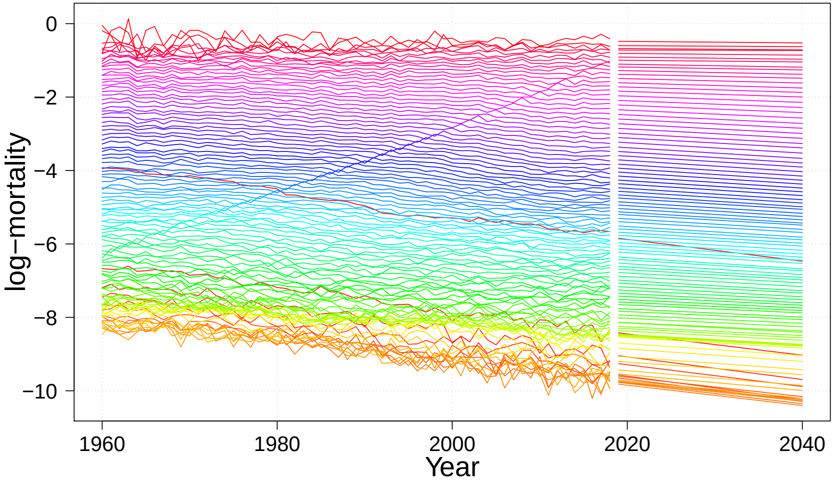

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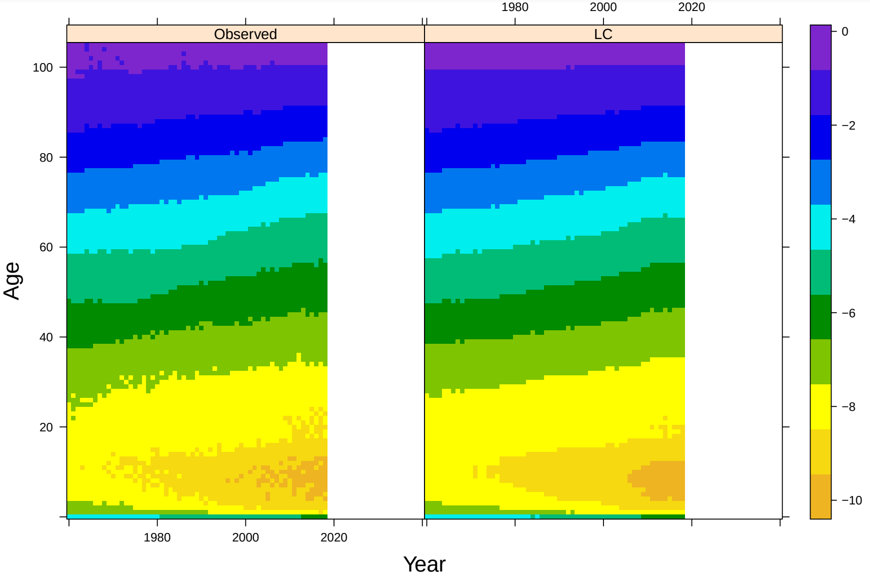

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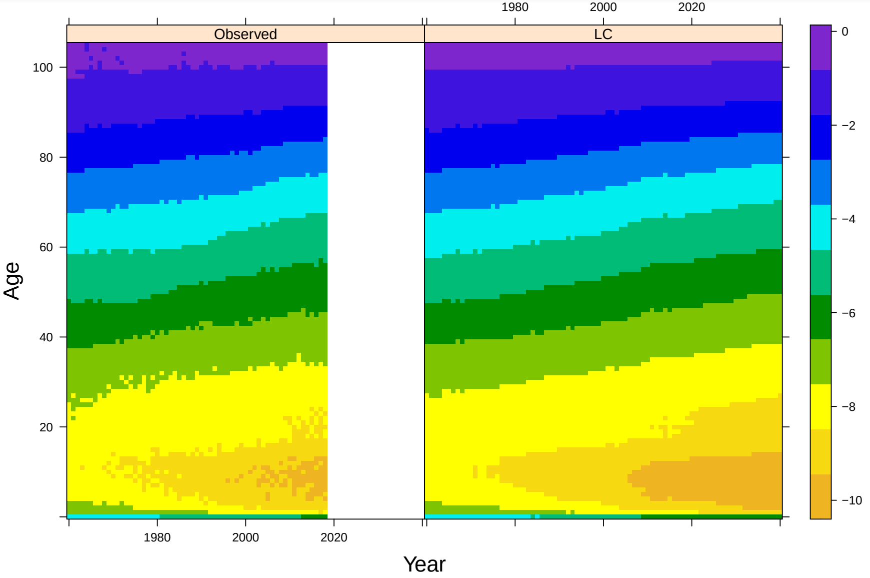

Females aged 0–105+ in England & Wales, 1960–2018, forecast 2019–2040.

The LC model: a summary

Advantages:

- simple & powerful method: forecast rates derived by modeling single time index by standard time-series model
- linear time index very often captures well historical decline in mortality
- stochastic model ⇒ probabilistic intervals
- extrapolative approach, no expert opinions
- more accurate than previous methodologies

Disadvantages:

- "jump-off" bias
- Normality assumption (from SVD)
- jagged fitted and forecast age profile, lacking smoothness
- fixed age-pattern of mortality decline
- rigid structure

⇒ several extensions proposed to overcome some of these issues

Some LC extensions

Disadvantages:

- ~~"jump-off" bias~~ $\Rightarrow$ observed jump-off rates (Lee and Miller 2001)
- ~~Normality assumption (from SVD)~~ $\Rightarrow$ Poisson LC (Brouhns et al. 2002)
- ~~jagged fitted and forecast age profile, lacking smoothness~~ $\Rightarrow$ Smooth LC (Delwarde et al. 2007)
- ~~fixed age-pattern of mortality decline~~ $\Rightarrow$ Li et al. (2013)
- rigid structure

Other LC extensions (single population)

- Booth et al. (2002): adjusting κ_t to match the age-at-death distribution & determining optimal fitting period
- Renshaw and Haberman (2003): adding more than one principal components, i.e. $\ln(m_{x,t}) = \alpha_x + \sum_k \beta_x^k \kappa_t^k$
- Koissi et al. (2006): residual bootstrap to include parameter uncertainty in forecasts
- Renshaw and Haberman (2006): including cohort effects, i.e. $\ln(m_{x,t}) = \alpha_x + \beta_x^{(1)} \kappa_t + \beta_x^{(0)} \gamma_{t-x}$
- Hyndman and Ullah (2007): smooth underlying data (functional data) & additional principal components
- Camarda and Basellini (2021): smoothing, decomposing and forecasting the three components of mortality (childhood, early-adulthood and senescence), i.e. $m_{x,t} = \sum_k \exp\left(\alpha_x^k + \beta_x^k \kappa_t^k\right)$
- for a comprehensive review, see Basellini et al. (2023)

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Thank you for your attention !!

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