Forecasting mortality with the Lee-Carter method

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Short course for the DAAD-CAPES(PROBAL) research collaboration

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Preliminaries

- Presentations
- All materials (slides & codes) available at https://github.com/ubasellini/LC-short-course
- Materials are (mostly) derived from a wider IDEM course jointly taught with Giancarlo Camarda (INED): https://github.com/ubasellini/IDEM117-AdvancesMortalityForecasting

Mortality forecasting

- Crucial for sustainability of pensions, insurances, elderly care; predicting population ageing and projecting populations; ...
- Until the 1980s, the methods used to forecast mortality were **deterministic**, based on mathematical formulae or expert judgment
- Revived interest in recent years following the introduction of the Lee-Carter method in 1992
- One of the firstly introduced **stochastic** mortality models $\Rightarrow$ the model "revolutionized probabilistic mortality and population forecasting" (Raftery 2023)

The Lee-Carter method (1992)

- Proposed in 1992 to model and forecast US mortality
- After 30+ years, Lee-Carter (LC) still widely employed by variety of users: governments, private companies, international organizations, ...
- The landmark model in mortality forecasting
- An **extrapolation** method:
  - model the mortality surface over age and time
  - extrapolate trends in the future, assuming that observed trends will continue
- Simplicity, robustness and objectivity have made the model so successful
- Nonetheless, some limitations of the model have stimulated several extensions over the years (see Basellini et al. (2023) for a recent comprehensive review)

The LC method

- A simple log-bilinear functional form for mortality rates $m_{x,t}$ at age x and time t

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t} \tag{1}$$

  where:
  - α_x is the general shape of log-mortality at age x
  - β_x is the rate of mortality improvement at age x
  - κ_t is the general level of mortality at time t
  - $\epsilon_{x,t}$ is the error term with mean 0 and variance σ^2_ϵ, reflecting residual age-specific influences not captured by the model
- Modelling log-rates $\Rightarrow$ fitted and forecast rates constrained to be positive
- Log transformation partially counters heteroscedasticity of observed rates

The LC method

- The model is undetermined: if $\boldsymbol{\theta}_1 = [\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}]$ is a solution, then for any scalar c:
  - $\boldsymbol{\theta}_2 = [\boldsymbol{\alpha} - \boldsymbol{\beta}c, \boldsymbol{\beta}, \boldsymbol{\kappa} + c]$ is also a solution
  - $\boldsymbol{\theta}_3 = [\boldsymbol{\alpha}, \boldsymbol{\beta}c, \boldsymbol{\kappa}/c]$ is also a solution
- Two constraints introduced to ensure model identification:

$$\sum_x \beta_x = 1 \qquad \text{and} \qquad \sum_t \kappa_t = 0 \tag{2}$$

The LC model: a schematic view

$$\ln\left(m_{x,t}\right) \simeq \alpha_x + \beta_x \kappa_t$$

$$\begin{pmatrix} \ln(m_{0,1960}) & \ln(m_{0,1961}) & \dots & \ln(m_{0,2018}) \\ \ln(m_{1,1960}) & \ln(m_{1,1961}) & \dots & \ln(m_{1,2018}) \\ \ln(m_{2,1960}) & \ln(m_{2,1961}) & \dots & \ln(m_{2,2018}) \\ \vdots & \vdots & \ddots & \vdots \\ \ln(m_{105,1960}) & \ln(m_{105,1961}) & \dots & \ln(m_{105,2018}) \end{pmatrix} \simeq$$

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{105} \end{pmatrix} + \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{105} \end{pmatrix} \begin{pmatrix} \kappa_{1960} & \kappa_{1961} & \dots & \kappa_{2018} \end{pmatrix}$$

$$\underbrace{59}_{\text{years}} \times \underbrace{106}_{\text{ages}} = \underbrace{6254}_{\text{cells}} \simeq \underbrace{106}_{\alpha_i} + \underbrace{106}_{\beta_i} + \underbrace{59}_{\kappa_j} - \underbrace{2}_{\text{constraints}} = \underbrace{269}_{\text{parameters}}$$

Model estimation

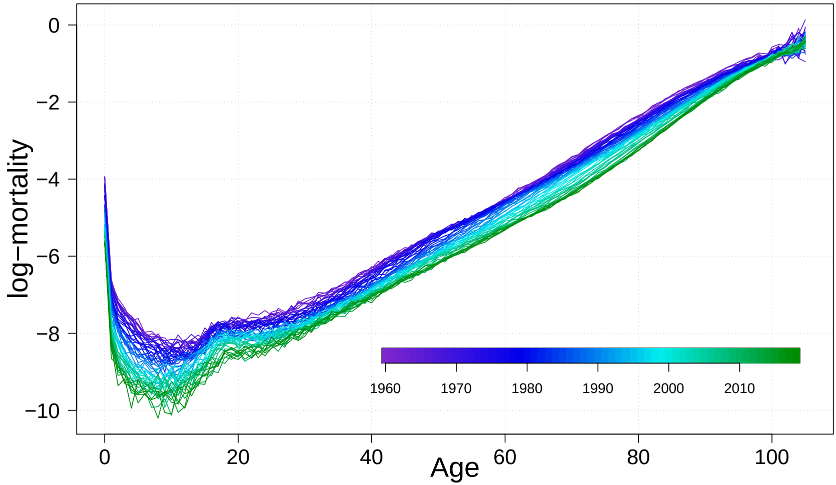
- The model is estimated by minimizing the residual sum of squares:

$$\sum_{x,t} \Big(\ln\left(m_{x,t}\right) - \alpha_x - \beta_x \kappa_t \Big)^2 \tag{3}$$

- A singular value decomposition (SVD) is employed to minimize Eq. (3) & derive an ordinary least squares (OLS) solution:
  - $\hat{\alpha}_x$ is the average of the observed $\ln\left(m_{x,t}\right)$

Estimating LC: an example

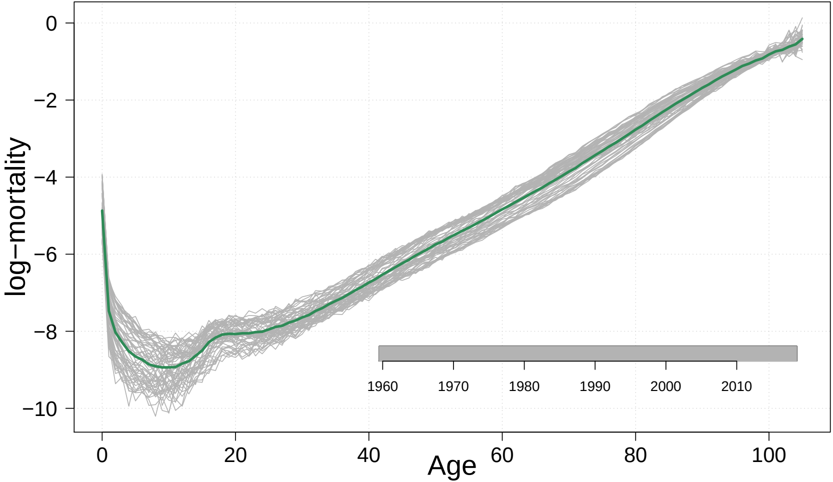
- observed mortality rates $\ln(m_{x,t})$



Females aged 0–105+ in England & Wales, 1960–2018.
Source (all figures): Human Mortality Database (2021)

Estimating LC: an example

- $\hat{\alpha}_x$ = average of observed mortality rates $\ln(m_{x,t})$



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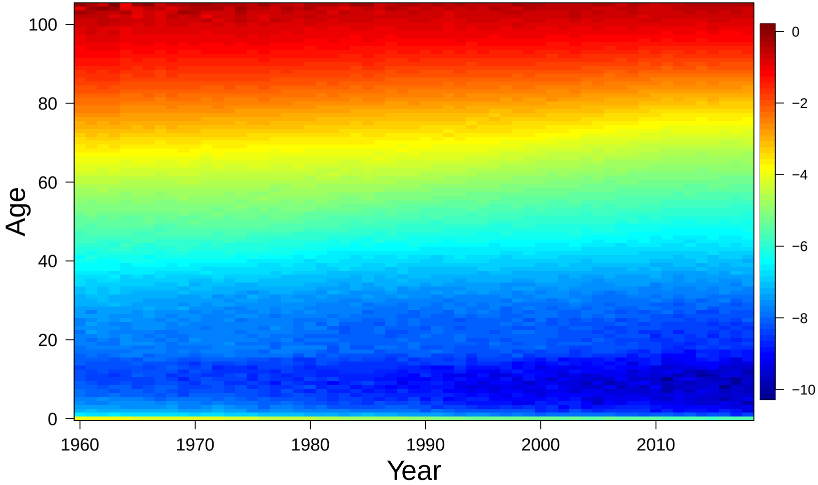
  - $\hat{\alpha}_x$ is the average of the observed $\ln\left(m_{x,t}\right)$
  - $\hat{\beta}_x$ and $\hat{\kappa}_t$ are derived from the first left- and right-singular vectors of the SVD of the matrix $\ln\left(m_{x,t}\right) - \hat{\alpha}_x$

Estimating LC: an example

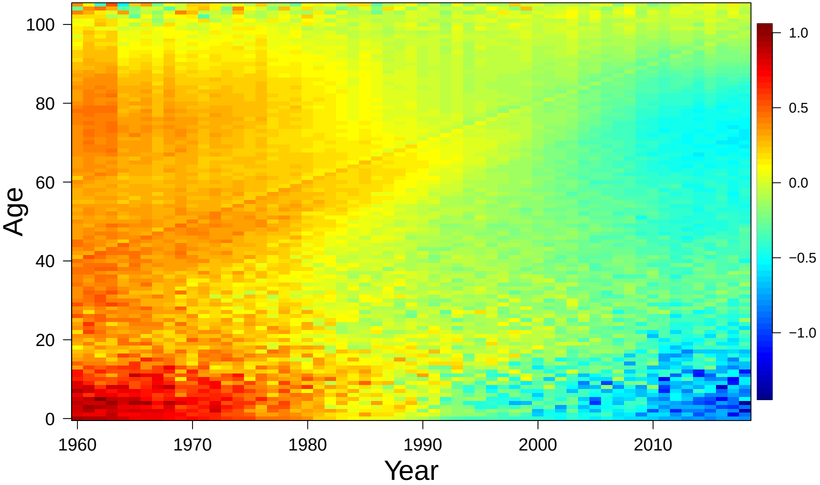
- $\boldsymbol{M} = (\ln(m_{x,t}))$: matrix of observed mortality rates



Females aged 0–105+ in England & Wales, 1960–2018.

Estimating LC: an example

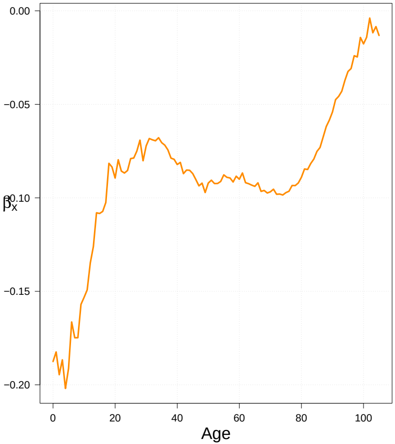
- $\tilde{M} = (\ln(m_{x,t}) - \hat{\alpha}_x)$: matrix of "centered" mortality rates

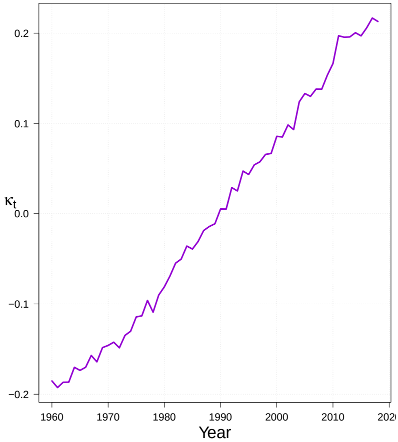


Females aged 0–105+ in England & Wales, 1960–2018.

Estimating LC: an example

- From SVD of $\tilde{M}$

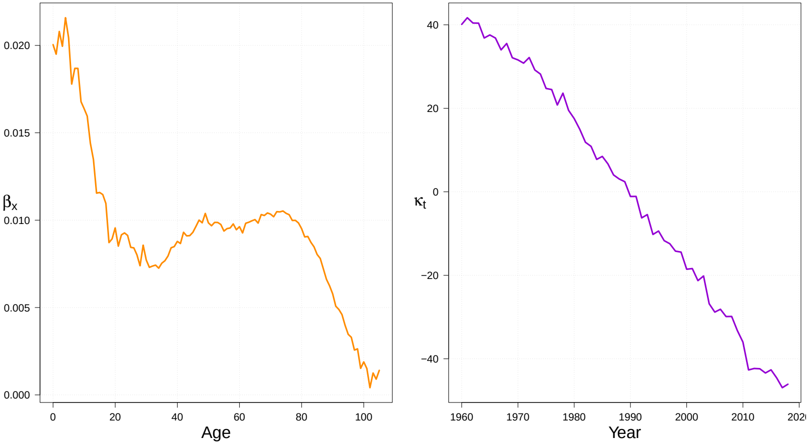




Females aged 0–105+ in England & Wales, 1960–2018.

Estimating LC: an example

- Adjusting the left-singular vector to sum to 1, and multiplying the right-singular vector for the leading singular value and the sum of the left-singular vector, we get $\hat{\beta}_x$ and $\hat{\kappa}_t$



Females aged 0–105+ in England & Wales, 1960–2018.

Model estimation

- The model is estimated by minimizing the residual sum of squares:

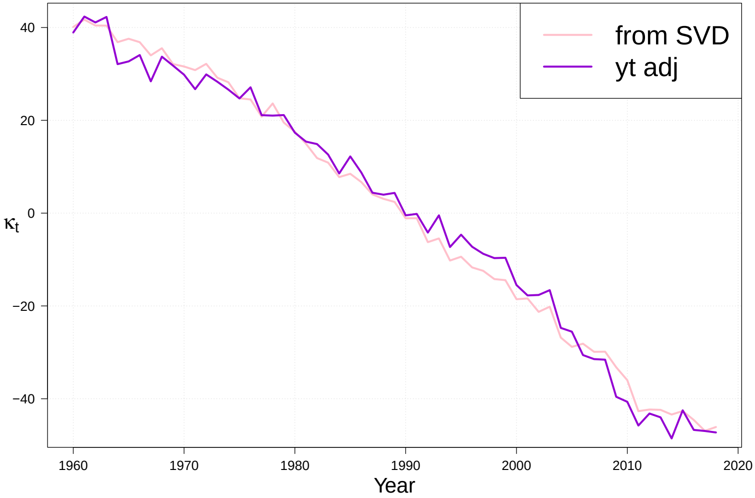
$$\sum_{x,t} \Big(\ln\left(m_{x,t}\right) - \alpha_x - \beta_x \kappa_t \Big)^2$$

- A singular value decomposition (SVD) is employed to minimize Eq. (3) & derive an ordinary least squares (OLS) solution:
  - $\hat{\alpha}_x$ is the average of the observed $\ln\left(m_{x,t}\right)$
  - $\hat{\beta}_x$ and $\hat{\kappa}_t$ are derived from the first left- and right-singular vectors of the SVD of the matrix $\ln\left(m_{x,t}\right) - \hat{\alpha}_x$
- In a second-step estimation, the parameter $\hat{\kappa}_t$ is adjusted so that the fitted deaths match the observed deaths in all years, i.e.

$$\sum_x \hat{y}_{x,t} = \sum_x y_{x,t} \quad \text{for all } t$$

Estimating LC: an example

- Adjusting $\hat{\kappa}_t$ to match observed number of deaths at each year t



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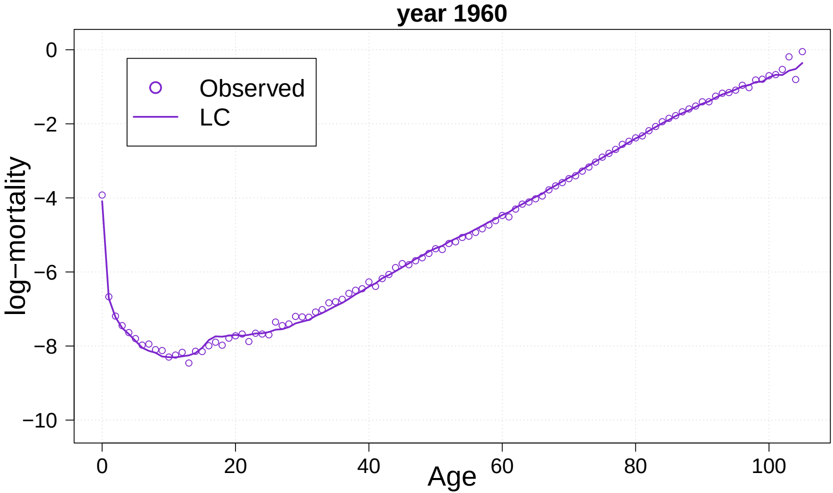
Estimating LC: an example



Females aged 0–105+ in England & Wales, 1960–2018.

Forecasting with LC

- Forecasting "made simple": choose an appropriate time-series model for $\hat{\kappa}_t$ and extrapolate it
- The forecast $\hat{\kappa}_{T+h}$ allows one to derive the entire age-pattern of mortality at time $T+h$, i.e.:

$$\ln\left(\hat{m}_{x,T+h}\right) = \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_{T+h}$$

- LC suggest a random walk model (i.e. ARIMA(0,1,0)) with drift:

$$\kappa_t = \kappa_{t-1} + c + e_t$$

  where c is a constant (drift) and e_t the error term (purely random process)

- For this time-series model: $\hat{\kappa}_{T+h|T} = \hat{\kappa}_T + c\,h$
- Simulated future trajectories of $\hat{\kappa}_{T+h}$ to construct prediction intervals for $\hat{m}_{x,T+h}$ and other summary measures (e.g., $\hat{e}_{0,T+h}$)
- (Coale and Guo (1989) adjustment for forecast rates at ages 85+)

Forecasting with LC: a schematic view

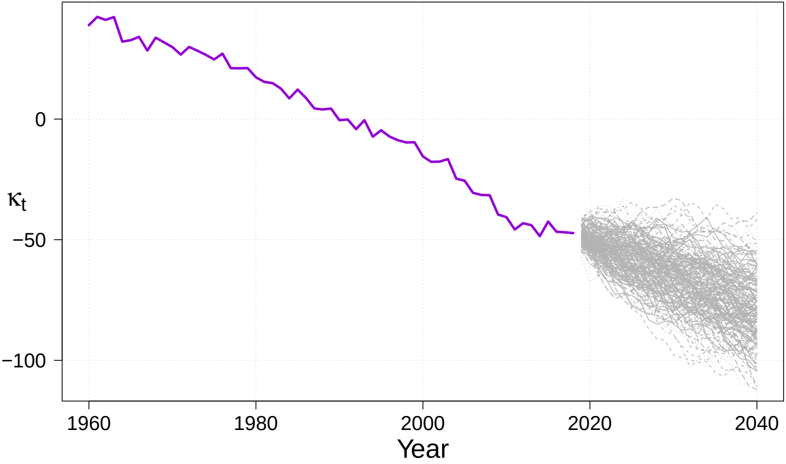
$$\ln\left(m_{x,t}\right) \simeq \alpha_x + \beta_x \kappa_t \qquad \ln\left(m_{x,T+h}\right) \simeq \alpha_x + \beta_x \kappa_{T+h}$$

$$\begin{pmatrix} \ln(m_{0,1960}) & \dots & \ln(m_{0,2018}) & \ln(m_{0,2019}) & \dots & \ln(m_{0,2040}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \ln(m_{105,1960}) & \dots & \ln(m_{105,2018}) & \ln(m_{105,2019}) & \dots & \ln(m_{105,2040}) \end{pmatrix} \simeq$$

$$\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{105} \end{pmatrix} + \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{105} \end{pmatrix} \begin{pmatrix} \kappa_{1960} & \dots & \kappa_{2018} & \kappa_{2019} & \dots & \kappa_{2040} \end{pmatrix}$$

Forecasting LC: an example

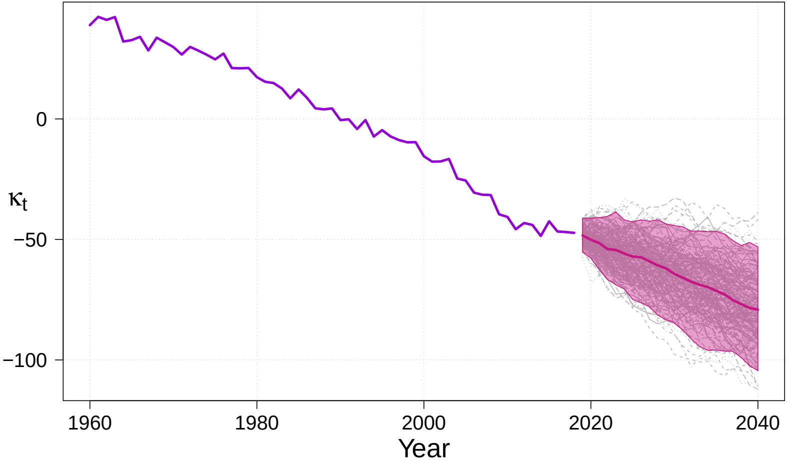
- Simulated future paths of $\hat{\kappa}_t$ from random walk with drift



Females aged 0–105+ in England & Wales, 1960–2018, forecast 2019–2040.

Forecasting LC: an example

- Simulated future paths of $\hat{\kappa}_t$ from random walk with drift $\Rightarrow$ prediction intervals from empirical percentiles



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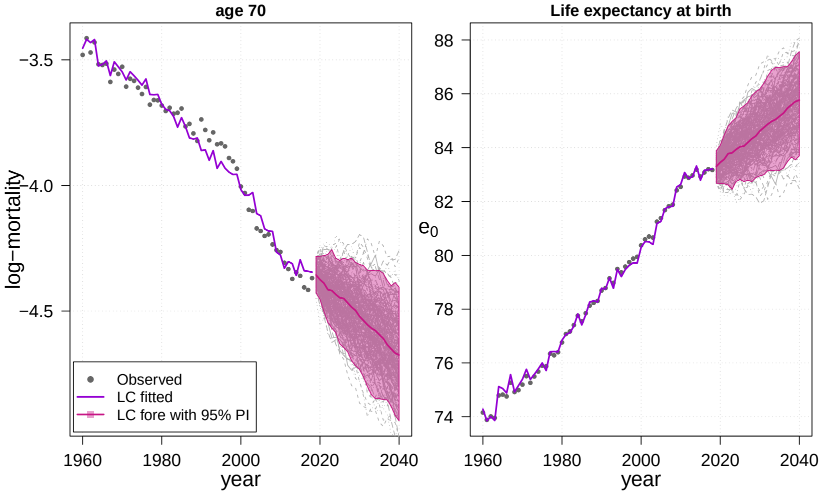
Forecasting LC: an example

- Prediction intervals from simulated future $\hat{\kappa}_{T+h}$



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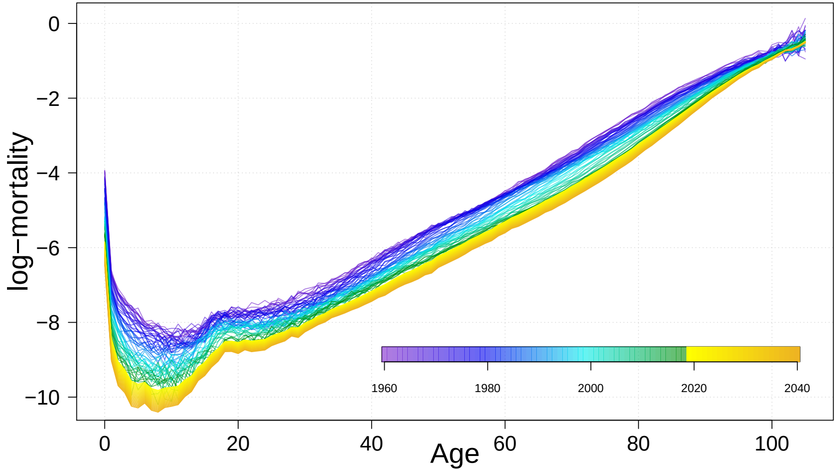
Forecasting LC: an example

- $\hat{\kappa}_{T+h}$ allows one to derive forecast rates $\hat{\boldsymbol{M}} = \left(\ln\left(\hat{m}_{x,T+h}\right)\right)$



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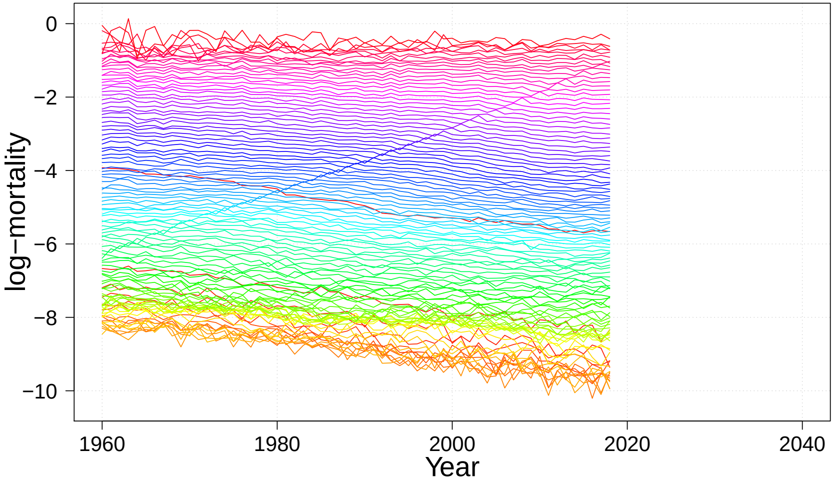
Forecasting LC: an example



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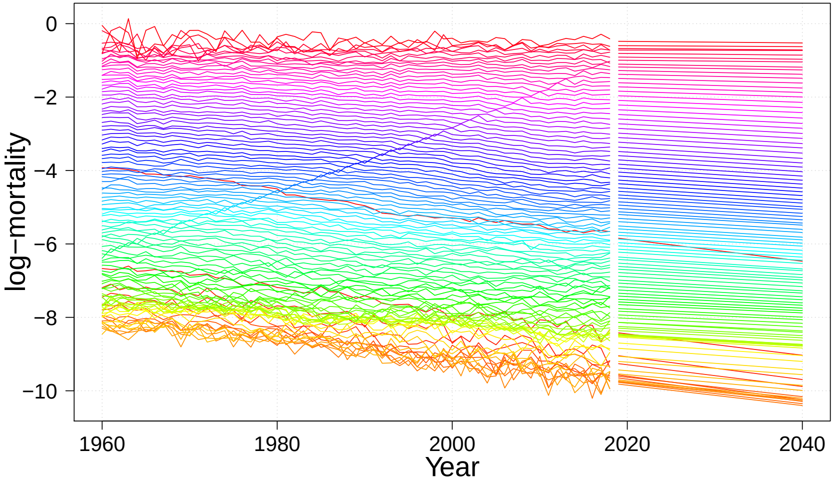

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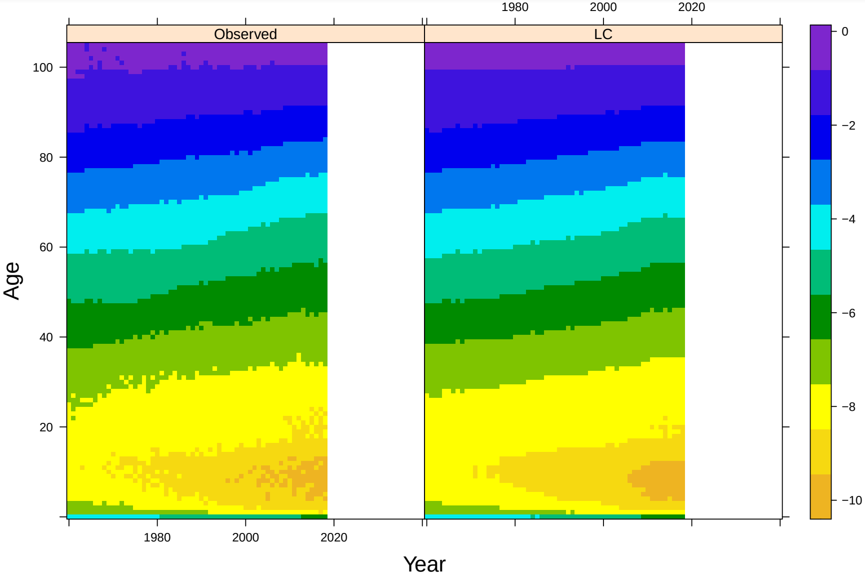

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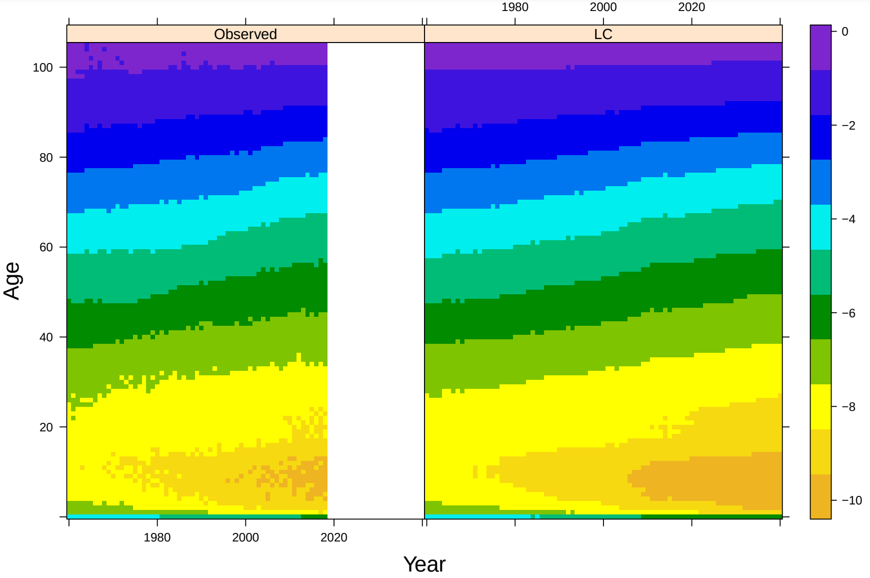

Females aged 0–105+ in England & Wales, 1960–2018, forecast 2019–2040.

The LC model: a summary

Advantages:

- simple & powerful method: forecast rates derived by modeling single time index by standard time-series model
- linear time index very often captures well historical decline in mortality
- stochastic model $\Rightarrow$ probabilistic intervals
- extrapolative approach, no expert opinions
- more accurate than previous methodologies

Disadvantages:

- "jump-off" bias
- Normality assumption (from SVD)
- jagged fitted and forecast age profile, lacking smoothness
- fixed age-pattern of mortality decline
- rigid structure

$\Rightarrow$ several extensions proposed to overcome some of these issues

Some LC extensions

Disadvantages:

- ~~"jump-off" bias~~ ⇒ observed jump-off rates (Lee and Miller 2001)
- ~~Normality assumption (from SVD)~~ ⇒ Poisson LC (Brouhns et al. 2002)
- ~~jagged fitted and forecast age profile, lacking smoothness~~ ⇒ Smooth LC (Delwarde et al. 2007)
- ~~fixed age-pattern of mortality decline~~ ⇒ Li et al. (2013)
- rigid structure

Other LC extensions (single population)

- Booth et al. (2002): adjusting κ_t to match the age-at-death distribution & determining optimal fitting period
- Renshaw and Haberman (2003): adding more than one principal components, i.e. $\ln(m_{x,t}) = \alpha_x + \sum_k \beta_x^k \kappa_t^k$
- Koissi et al. (2006): residual bootstrap to include parameter uncertainty in forecasts
- Renshaw and Haberman (2006): including cohort effects, i.e. $\ln(m_{x,t}) = \alpha_x + \beta_x^{(1)} \kappa_t + \beta_x^{(0)} \gamma_{t-x}$
- Hyndman and Ullah (2007): smooth underlying data (functional data) & additional principal components
- Camarda and Basellini (2021): smoothing, decomposing and forecasting the three components of mortality (childhood, early-adulthood and senescence), i.e. $m_{x,t} = \sum_k \exp\left(\alpha_x^k + \beta_x^k \kappa_t^k\right)$
- for a comprehensive review, see Basellini et al. (2023)

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Thank you for your attention !!

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