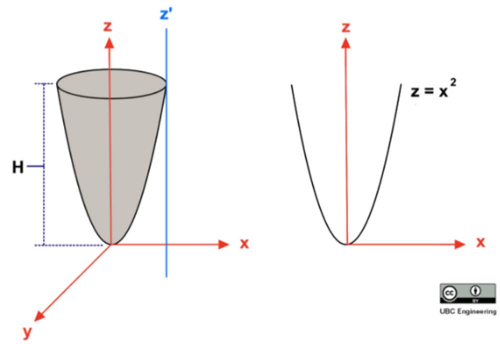
21-R-KIN-ZA-20 Solution



Question: The paraboloid shown has a density of $\rho = 900\ kg/m^3$ and a height of $H = 0.9m$. If the cross section is a circle, and the projection of the x-z axis follows the equation: $z = x^2$, find the moment of inertia about the z axis, and the z' axis.

Solution:
The infinitesimal volume of a small slice of the parabola, dm, can be written as the volume of a cylinder with a small height dz. The mass is written as the density times volume. Using the equation given, $x^2 = z$, we can see that when y = 0, x equals the radius of the paraboloid. We can express this completely in terms of z.

$$dV = \pi r^2 dz$$

$$dm = \rho dV = \rho \pi r^2 dz = \rho \pi z dz$$

Using the equation for moment of inertia of a disk with differential mass, $I = \int_m \frac{1}{2} r^2 dm$, we can find the infinitesimal moment of inertia by taking the derivative of both sides.

$$dI = 0.5\, r^2 dm = 0.5\, z * \rho\pi\, z\, dz = 0.5\, \rho\pi z^2\, dz$$

Plugging all values in, and integrating over the length z gives the moment of inertia about the z axis.

$$I_z = 0.5 \int_0^H \rho\pi z^2\, dz = 0.5\rho\pi \frac{1}{3} H^3 = \frac{1}{6} 900 * \pi * 0.9^3 = 343.5\ kg \cdot m^2$$

To find the mass, we integrate the expression for dm over the height H. We can then use the parallel axis theorem to find $I_{z'}$, where $d^2 = H$

$$m = \int_0^H \rho\pi z\, dz = \rho\pi \frac{1}{2} H^2$$

$$I_{z'} = I_z + md^2 = 343.5 + (\tfrac{1}{2}\rho\pi 0.9^2) * 0.9 = 1374.1\ kg \cdot m^2$$