21-R-KIN-ZA-28 Solution

Question: Four slender rods with the same non-uniform, linear density of $\lambda = cx^2\ kg/m$ are attached to each other to form a square as shown. There is a pin connection at point A, and a cable attached to point B, that keeps the system at rest initially. The cable suddenly snaps at the ceiling, forcing the system to support its mass of $m_c\ kg$. If the length of each rod is $l\ m$, find the magnitude of angular acceleration of the system the instant the cable snaps, as well as the reaction forces at the pin connection.

Solution:

First, we find the mass of a single rod by integrating along its length.

$$m = \int dm = \int_0^l cx^2 dx = c/3\, l^3\ kg$$

Next, we find the center of gravity of the rod, using the general formula $r_G = \frac{1}{m}\int_m \vec{r} dm$, where

$dm = \rho dl$ and $r = x$.

$$COG = \frac{1}{m}\int_0^l x(cx^2)dx = c/4l^4/m = 3/4l$$

Now, to find the center of gravity of the entire object, we can use $r_G = \frac{1}{m}\Sigma_i r_i m_i$.

$$x_G = \frac{1}{m_{tot}}\Sigma_i x_i m_i = 1/(4m) * (0 + lm + 3/4lm + 3/4lm) = 5/8l$$

$$y_G = \frac{1}{m_{tot}}\Sigma_i y_i m_i = 1/(4m) * (0 + lm + lm/4 + lm/4) = 3/8l$$

To find the moment of inertia of one rod at $x = 0$ shown in the diagram we use the general equation

$I = \int_m r^2 dm$ where $r = x$ once again.

$$I_g = \int r^2 dm = \int_0^l x^2 3x^2 dx = 3/5l^5$$

To find the moment of inertia of the entire object about the center of gravity, we sum the individual moments of inertia about the center of gravity of the object using the parallel axis theorem.

$$I_G = \Sigma_i(I_g + md_i^2) = 2(I_g + m(2x_G^2)) + 2(I_g + m(x_G^2 + y_G^2))$$

We use the parallel axis theorem once again to find the moment of inertia about the pin at A.

$$I_A = I_G + 4m[\,(1/8l)^2 + (3/8l)^2\,]$$

We can now take the sum of forces and moments, which reveals that we have 5 unknowns (A_x, A_y, α, $a_{G,y}$, $and\ a_{G,x}$), and 3 equations.

$\Sigma F_x = A_x = 4ma_{G,x}$

$\Sigma F_y = A_y - 4mg - m_C g = 4ma_{G,y}$

$\Sigma M_A = I_A \alpha = - 4mg1/8l - m_C gl/2$

The final equation is derived from the expression $a_t = \alpha r$. We can ignore $a_n = v^2/r$ as the system is released from rest.

$a_G = \alpha\hat{k} \times r_{G/A}\hat{u}, \quad \vec{r}_{G/A} = 1/8l\hat{i} + 3/8l\hat{j}$

$a_{G,x} = 3/8\alpha l$

$a_{G,y} = - \alpha l/8$

This new equation solved for two unknowns, so we can solve the rest of the system.

$\alpha = [- 4mg1/8l - m_C gl/2\,]/I_A$

$A_x = 4ma_{G,x}$

$A_y = 4ma_{G,y} + 4mg + m_C g$