21-R-WE-ZA-40 Solution

Question: A rod of mass $m\ kg$, and length $x\ m$ pinned at a height $h\ m$ above the ground is attached to a spring with a constant $k\ N/m$. If the rod is released from rest from the position shown, find the length of the spring after the end of the rod has reached the smallest height, and bounced back up. The spring starts at its unstretched length $l_{unstretched} = h\ m$.

Solution:

We can first find the potential energy in state 2 by finding the length the spring is initially compressed. In states 1 and 2 the rod is not moving so there is no kinetic energy

$T_1 = T_2 = 0$

Gravitational potential energy in states 1 and 2 are found using a datum on the ground. $l_1$represents the length of the spring in state 2.

$V_1 = mgh$

$V_2 = mg(l_1 + \frac{h-l_1}{2})$

We can find the work done by the spring using its unstretched length and length in state 2. The spring does negative work initially.

$U_{k\ 1\to2} =- \frac{1}{2}k((l_1 - h)^2 - 0)$

Putting this all together we can solve a quadratic equation to find the initial compressed length of the spring.

$V_1 + U_{k\ 1\to2} = V_2$

$0 = (k/2)l_1^{\ 2} + (mg - \frac{mg}{2} - kh)l_1 + (\frac{mgh}{2} - mgh\ + \frac{1}{2}kh^2)$

$l_1 = (-\ b + \sqrt{b^2 - 4ac})\frac{1}{2a}$

Next we write the energy equation for states 2 and 3, when the rod has reached its maximum height after state 2. The length of the spring in state 3 is written as l_2.

$V_2 = mg(l_1 + \frac{h-l_1}{2})$

$V_3 = mg(l_2 + \frac{h-l_2}{2})$

$U_{k\ 2\to3} = \frac{1}{2}k((l_2 - h)^2 - (l_1 - h)^2)$

The kinetic energy in state 3 is also 0. Putting this all together we get another quadratic equation and we can solve for l_2 similarly.

$T_3 = 0$

$V_2 + U_{k\ 2\to3} = V_2$

$0 = (k/2)l_2^{\ 2} + (mg/2 - mg - kh)l_2 + (V_2 + h^2k/2 - k(l_1 - h)^2/2 - mgh/2)$

$l_2 = (-\ b + \sqrt{b^2 - 4ac})/(2a)$