21-R-WE-ZA-38 Solution

Question: A disk of mass m_{disk} kg, outer radius r_o m, and inner radius r_i m is attached to a block of mass m kg and two springs in series each with a constant of k N/m starting from their unstretched length. A force of $(1 + F\phi)N$ is applied to a cable wound around its inner radius at an angle of θ° with the horizontal, where ϕ is the angle turned by the disk. Find the mass of a block wound around the inner radius that would provide the disk with the same kinetic energy after rotating it the same angle, if the final velocity is v_f m/s and it is released from rest.

Solution:

We know the kinetic energy in state 2 of the system as we know the final velocity. The initial kinetic energy is 0.

$I_{disk} = \frac{1}{2}m_{disk}r_o^{\ 2}$

$T_2 = \frac{1}{2}I_{disk}(v_f/r_2)^2 + \frac{1}{2}mv_f^{\ 2}$

$T_1 = 0$

Adding the springs in series gives a net spring constant of k/2. The distance compressed is equal to $r\theta$.

$U_k =- \frac{1}{2}(\frac{k}{2})(r_2\phi)^2$

The weight of the block does negative work.

$V_{1\rightarrow 2} =- mgr_1\phi$

We integrate to find the work done by the applied force in terms of the angle the disk has turned.

$U_F = \int_0^{\phi}(1 + F\phi)cos\theta r_1 d\phi = r_1 cos\theta[\phi + F/2\phi^2]$

Putting this all together, we can solve for the angle the disk has turned at state 2.

$T_1 + V_{1\rightarrow 2} + U_k + U_F = T_2$

$0 - mgr_1\phi - \frac{1}{2}(\frac{k}{2})(r_2\phi)^2 + r_1cos\theta[\phi + F/2\phi^2] = \frac{1}{2}I_{disk}(v_f/r_2)^2 + \frac{1}{2}mv_f^{\ 2}$

$\phi^2(r_1cos\theta F/2 - \frac{k}{4}r_2^{\ 2}) + \phi(r_1cos\theta - r_1mg) - T_2 = 0$

$\phi = (-(r_1cos\theta - r_1mg) + \sqrt{(r_1cos\theta - r_1mg)^2 - 4(r_1cos\theta F/2 - \frac{k}{4}r_2^{\ 2})(-T_2)})/(2(r_1cos\theta F/2 - \frac{k}{4}r_2^{\ 2}))$

Equating work done by the weight of a block to the final kinetic energy allows us to isolate and solve for mass.

$mgr_1\phi = T_2 \Rightarrow m = T_2/(gr_1\phi)$