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The headrest in a car is meant to cushion the human head and neck during a collision. The connection between the neck and the skull (shown here as point B) is a particularly vulnerable spot. During a collision, the force distribution when the head hits the headrest looks like $w(x) = C + Dx^2$ where *C* and *D* are unknown. If d_1 is d_1 meters, the resultant force is equal to F_R and is located at $x = X\ m$, what are *C* and *D*?

ANSWER:

We know that the equal for resultant force is,

$$F_R = \int_A^B w(x)\,dx = \int_0^{d_1} C + Dx^2\,dx = \left[Cx + \frac{D}{3}x^3\right]_0^{d_1} = C(d_1) + \frac{D}{3}(d_1)^3$$

We also know that *X* can be found by,

$$\int_A^B x \cdot w(x)\,dx = \int_0^{d_1} Cx + Dx^3\,dx = \left[\frac{C}{2}x^2 + \frac{D}{4}x^4\right]_0^{d_1} = \frac{C}{2}(d_1)^2 + \frac{D}{4}(d_1)^4$$

$$X = \frac{\int_A^B x \cdot w(x)\,dx}{\int_A^B w(x)\,dx} = \frac{\frac{C}{2}(d_1)^2 + \frac{D}{4}(d_1)^4}{C(d_1) + \frac{D}{3}(d_1)^3} = \frac{\frac{C}{2}(d_1) + \frac{D}{4}(d_1)^3}{C + \frac{D}{3}(d_1)^2}$$

$$X \cdot \left(C + \frac{D}{3}(d_1)^2\right) = \frac{C}{2}(d_1) + \frac{D}{4}(d_1)^3$$

$$C \cdot X + X \cdot \frac{D}{3}(d_1)^2 = \frac{C}{2}(d_1) + \frac{D}{4}(d_1)^3$$

$$C = \frac{X \cdot \frac{D}{3}(d_1)^2 + \frac{D}{4}(d_1)^3}{\frac{1}{2}(X - d_1)}$$

Then, we input the above derive equation for *A* into the equation for the resultant force

$$F_R = \frac{X \cdot \frac{D}{3}(d_1)^2 + \frac{D}{4}(d_1)^3}{\frac{1}{2}(X - d_1)}(d_1) + \frac{D}{3}(d_1)^3$$

$$D = \frac{F_R}{\left(\frac{\frac{X}{3}(d_1)^3 + \frac{1}{4}(d_1)^4}{\frac{1}{2}(X - d_1)} + \frac{(d_1)^3}{3}\right)}$$