21-R-WE-ZA-33 Solution

Question: A rod of length $l\ m$ and mass $m\ kg$ is connected to a pin and attached to a spring that is also pinned with a spring constant of $k\ N/m$. If the rod starts at $\theta = 30^{\circ}$ from rest, and a moment of $M\theta\ N \cdot m$ is applied in the direction shown, find the angular velocity when $\theta = 90^{\circ}$. The unstretched length of the spring is $l_{unstretched} = l/4\ m$, and you can neglect the mass of the spring.

Solution:
Use the general equation to write kinetic energy in each state, using the moment of inertia about the pin. We can also calculate the moment of inertia for a slender rod about its end.

$$T_1 = 0, T_2 = \frac{1}{2} I_{rod}\omega^2, I_{rod} = \frac{1}{3} ml^2$$

Use the angle and the length of the bar to find gravitational potential energy in each state.

$$V_1 = \frac{l}{2} mg\ sin\ 30, V_2 = \frac{l}{2} mg$$

Integrate the moment applied over the angle to find the work done.

$$U_M = \int_{\pi/6}^{\pi/2} M\theta\ d\theta = \frac{M}{2}\left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{6}\right)^2\right)$$

Find s1 and s2 for the work done by the spring using geometry and subtracting by the springs' unstretched length.

$$s_1 = l\ sin\ 30 - l_{unstretched}$$

$$s_2 = \sqrt{l^2 + l^2 cos^2 30} - l_{unstretched} = l\sqrt{1 + cos^2 30} - l_{unstretched}$$

For the work done by the spring we use the general equation. As the spring starts out stretched in state 1 and stretches further in state 2, the force of the spring opposes the direction of motion. This means that the spring does negative work.

$$U_k = -\frac{1}{2} k(s_2^{\ 2} - s_1^{\ 2})$$

Putting this all together, we can isolate and solve for the angular velocity in state 2.

$$T_1 + V_1 + U_M + U_k = T_2 + V_2$$

$$\omega = \left((V_1 + U_M + U_k - V_2)2/I_{rod} \right)^{0.5}$$