21-R-WE-ZA-35 Solution

Question: A disk of mass $m_{disk}\ kg$ and radius $r\ m$ is attached to two springs in parallel with the same spring constant of $k\ N/m$. A block of mass $m\ kg$ is hanging from the other side of the disk. If a moment of $M\theta\ N \cdot m$is applied to the system from rest, find angle the disk has turned in radians when is has reached an angular velocity of $\omega\ rad/s$. You can assume the disk moves in the direction shown, and the springs start at their unstretched length.

Solution:
There is no initial kinetic energy as it starts from rest. We can find the change in potential energy in terms of theta using the equation for the arclength of a circle: $s = \theta r$.

$T_1 = 0, \Delta h = t - (t - \theta r) = \theta r \Rightarrow V_{1\to2} = m_{block} g \Delta h$

We can find the kinetic energy in state two by finding the moment of inertia of the disk, and using Chassele's theorem to find the final velocity of the block.

$I_O = 1/2 m_{disk} r^2, v_{block} = \omega r$

$T_2 = 1/2 I_O \omega^2 + 1/2 v_{block}^2 m_{block}$

Integrating over the angle and expressing it in terms of theta gives the work done by the moment.

$U_M = \int_0^\theta M\theta \cdot d\theta = M/2\theta^2$

As the springs start at their unstretched length, $s_1 = 0$. s_2 is found using the arc length formula for a circle. This allows us to find the work done by the springs. As they both have the same unstretched length, we can rewrite both of them as one spring, with a spring constant of '2k'.

$s_1 = 0, s_2 = r\theta$

$U_k = - 1/2(2k)(s_2^2 - s_1^2) = - 1/2(2k)(r\theta)^2$

Putting this all together gives the quadratic equation shown below. Solving gives two roots, one of which is positive and the other is negative. As we know that the disk moves in the direction shown, the angle must be positive.

$0 + m_{block} g\theta r + M\theta^2/2 - kr^2\theta^2 = 1/2 I_O \omega^2 + 1/2 v_{block}^2 m_{block}$

$a = M/2 - kr^2, b = m_{block} gr, c = - T_2$

$\theta = (- b \pm \sqrt{b^2 - 4ac})/(2a) = (- b + \sqrt{b^2 - 4ac})/(2a)$