21-R-KIN-ZA-18 Solution

Question: A cone of height $h_{cone} = 5\ cm$ and a uniform density of $\rho_{cone} = 1000\ kg/m^3$ has a small cylinder of diameter $d_i = 2\ cm$ carved out from inside it. You can assume the top of the inner cylinder just touches the edge of the cone. The cone is attached to a cylinder with the same outer diameter of $d_o = 4\ cm$. The cylinder of length $l = 10\ cm$ has a non-uniform density of $\lambda_{cylinder} = 4z^2 + 6\ kg/m$. Find the center of gravity of the object.

Solution:
By symmetry, we know that the centre of gravity is on the z axis.
$x_G = 0,\ y_G = 0$

We are able to find the equation of the cone using the height and length given. We know that the general equation follows: $x^2 + y^2 = \frac{r^2}{h^2}z^2$, so plugging in the values and neglecting the height above the x-y plane gives the following.

$x^2 + y^2 = \frac{r^2}{h^2}(z)^2 = \frac{4}{25}z^2$

We also know that $x^2 + y^2 = r^2$, so we can plug in the inner radius, and solving for z gives the distance between the point of the cone and the top of the inner cylinder. Neglecting the height above the x-y plane allows us to solve for this value directly, and we can subtract this from the height of the cone to find the height of the inner cylinder.

$x^2 + y^2 = r_i^2 = 1^2$

$h_{inner\ cyl} = h - r_i * h/r_o = 5 - 1 * 5/2 = 2.5\ cm$

We can use the mass of a solid cone with the same dimensions, and the inner cylinder to find the mass of the cone with the cylinder carved out.

$m_{full\ cone} = \frac{1}{3}\pi r_o^2 h_{cone} * \rho_{cone} = \frac{1}{3}\pi 2^2 * 5 * 1000/1000000 = 0.0209\ kg$

$m_{inner\ cyl} = \rho_{cone}/1000000 * \pi r_i^2 h_{inner\ cyl} = 1000/1000000 * \pi 1^2 2.5 = 0.00785\ kg$

$m_{cone} = m_{full\ cone} - m_{inner\ cyl} = 0.01305\ kg$

Now using the equation for the center of gravity $z_G = \frac{\Sigma_i r_i m_i}{m}$, we can find the center of gravity of the hollow cone. As expected, it's above the cylinder but below the halfway point of the cone.

$z_{G,\ cone} = [((h_{cone}/4 + l)m_{full\ cone}) - ((h_{inner\ cyl}/2 + l)m_{inner\ cyl})]/m_{cone} = 11.25\ cm$

Integrating the linear density over the length along the z axis gives the mass of the cylinder. Plugging this into the equation for the centre of gravity gives the center of gravity in the z direction of the cylinder. As expected, it's slightly greater than the halfway point of the cylinder.

$$z_{G,\,cyl} = \frac{\int_0^l z dm}{\int_z^l dm},\ m = \int_0^l \rho(z)\, dz \Rightarrow dm = \rho(z) dz$$

$$z_{G,\,cyl} = \frac{\int_0^l z(4z^2+6)dz}{\int_0^l (4z^2+6)dz} = \frac{l^4+3l^2}{4/3l^3+6l} = \frac{0.1^4+3(0.1^2)}{4*0.1^3/3+(6*0.1)} = 5.006\ cm$$

The denominator of that expression is the total mass of the cylinder.

$$m_{cyl} = \frac{4}{3}l^3 + 6l = 0.601\ kg$$

Using the equation for the centre of gravity again gives the center of gravity for the whole object.

$$z_G = [(z_{cyl} * m_{cyl}) + (z_{cone} * m_{cone})]/[m_{cone} + m_{cyl}] = 0.0514\ m = 5.14\ cm$$