21-R-KIN-ZA-24 Solution

Question: The sphere shown has a non-uniform density of $\rho z\ kg/m^3$, and a radius of $R\ m$.
A thin, horizontal cylinder with a radius of $\alpha\ m$ is removed along the x axis of the sphere.
Assuming the cylinder just touches the ends of the sphere, find the radius of gyration k_z of the sphere.

Solution:
We start by finding the mass and moment of inertia of the sphere about the z axis. Using the equation of a sphere $x^2 + y^2 + (z - R)^2 = R^2$, and the fact that $r = x^2 + y^2$, we can solve for dm first. We approximate a small element of the sphere to be a cylinder.

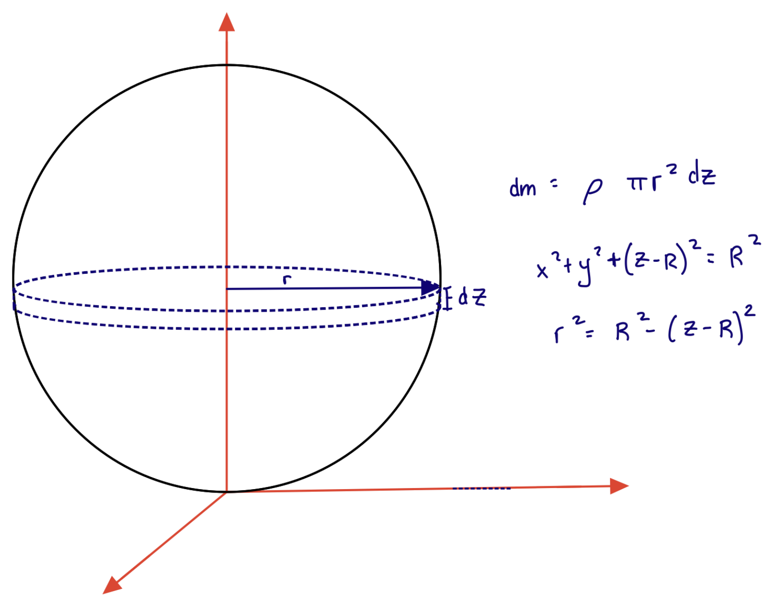
$$dm = \rho dV = \rho z * \pi r^2 dz = \rho\pi z(R^2 - (z - R)^2)dz$$

Plugging this into the moment of inertia equation $dI = \frac{1}{2}r^2 dm$ gives the final expression to be integrated.

$$dI = \frac{1}{2}\rho\pi z(R^2 - (z - R)^2)^2 dz = \frac{\rho\pi}{2}(4R^2z^3 - 4Rz^4 + z^5)dz$$

$$I_{sphere} = \frac{\rho\pi}{2}\int_0^{2R}(4R^2z^3 - 4Rz^4 + z^5)dz = \frac{\rho\pi}{2}R^6\frac{16}{15}\ kg \cdot m^2$$

$$m_{sphere} = \int_0^{2R}\rho\pi z(R^2 - (z - R)^2)dz = \rho\pi 16R^4(\frac{1}{3} - \frac{1}{4})\ kg$$

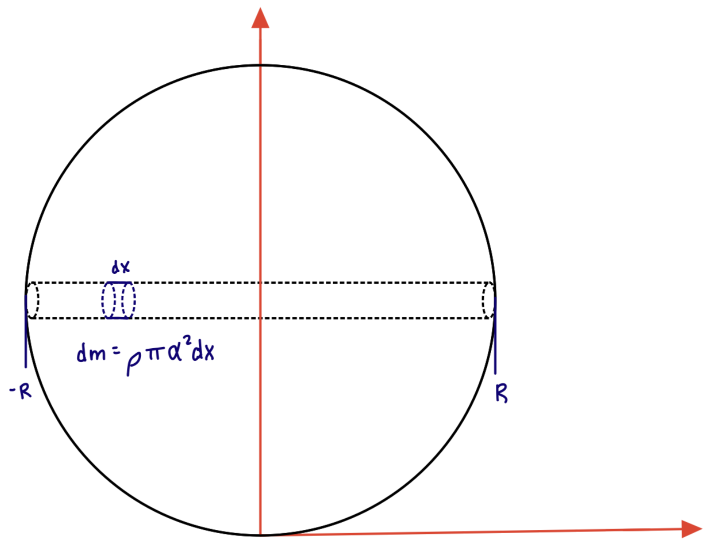


Now, we can solve for the moment of inertia and mass of the cylinder similarly. In this case we integrate along the length of the rod, approximating a small element to be a horizontal cylinder instead of a vertical one. The ‘r’ in the general MOI formula represents the x value, not the radius of the cylinder as it is taken about the z axis. Furthermore, as the cylinder is thin, we can approximate z to equal the radius of the sphere, which is constant.

$$dI \;=\; \frac{1}{2}r^2 dm \;=\; \frac{1}{2}x^2 \rho\pi\alpha^2 z\, dx \;=\; \frac{\rho\pi}{2}R\alpha^2\, x^2 dx$$

$$I_{rod} = \frac{\rho\pi}{2}R\alpha^2 \int_{-R}^{R} x^2 dx \;=\; \rho\pi R\alpha^2 \frac{1}{6}\; \frac{2}{8}(2R)^3\; kg \cdot m^2$$

$$m_{rod} = \int_{-R}^{R} \rho R\alpha^2 \pi dx \;=\; \rho\pi R\alpha^2 (2R)$$



Finally, using the general equation $I = mk^2$ we can plug in values and solve for k_z.

$$k_z = \;[\,(I_{sphere} - I_{rod})/(m_{sphere} - m_{rod})\,]^{0.5}\; m$$

The work done to solve each integral is shown below.

SPHERE

$R, \rho z, \alpha$

$x^2+y^2+(z-R)^2 = R^2$

$r^2 = R^2-(z-R)^2$

$= R^2 - (z^2-2zR+R^2)$

$= 2zR - z^2$

$dm = \rho dV$

$= \rho z \left[\pi r^2 dz\right]$

$(2zR-z^2)^2$

$(2zR-z^2)(2zR-z^2)$

$4z^2R^2 - 2z^3R - 2z^3R + z^4$

$= 4z^2R^2 - 4z^3R + z^4$

$dI = \frac{1}{2} r^2 dm = \frac{1}{2}\rho z \pi r^2 dz \; r^2$

$= \frac{1}{2}\rho\pi \; z r^4 dz$

$= \frac{\rho\pi}{2} z\left[R^2-(z-R)^2\right]^2 dz$

$I = \frac{\rho\pi}{2}\int_0^{2R}\left(4z^3R^2 - 4z^4R + z^5\right)dz$

$= \frac{\rho\pi}{2}\left[\cancel{4}R^2 \cdot \frac{1}{\cancel{4}}(2R)^4 - 4R\cdot\frac{1}{5}(2R)^5 + \frac{1}{6}(2R)^6\right]$

$= \frac{\rho\pi R^6}{2}\left[16 - \frac{4}{5}(32) + \frac{64}{6}\right]$

$\boxed{I_{sphere} = \frac{\rho\pi R^6}{2}\left(\frac{16}{15}\right)}$

$m = \int \rho z \pi r^2 dz$

$= \rho\pi \int_0^{2R} z\,(2zR - z^2)\,dz$

$= \rho\pi\left[2R\cdot\frac{1}{3}(2R)^3 - \frac{1}{4}(2R)^4\right]$

$= \rho\pi R^4\left[\frac{2}{3}\cdot 8 - \frac{16}{4}\right]$

$\boxed{m_{sphere} = \rho\pi R^4\left(\frac{4}{3}\right)}$

ROD

$z = R$

$dm = \rho z \pi \alpha^2 dx = \rho R \pi \alpha^2 dx$

$I = \int \frac{1}{2} r^2 dm$

$= \int \frac{1}{2} x^2 \rho R \pi \alpha^2 dx$

$= \frac{1}{2}\rho\pi R\alpha^2 \int_{-R}^{R} x^2 dx$

$= \frac{\rho\pi R\alpha^2}{2}\cdot\frac{1}{3}\left[R^3-(-R^3)\right]$

$\boxed{I_{rod} = \frac{1}{3}\rho\pi R\alpha^2 R^3}$

$m = \int_{-R}^{R} \rho\pi R\alpha^2 dx$

$\boxed{m_{rod} = \rho\pi R\alpha^2 (2R)}$