21-R-KIN-ZA-23 Solution

Question: The paraboloid shown has a density of $\rho = 3x\ kg/m^3$, and a length of $L = 1m$. The cross section is a circle, and the projection of the x-z axis follows the equation: $x = z^2$. A cone that has the same axis, and follows the equation $x^2 = y^2 + z^2$ is carved out from inside it. Find the radius of gyration k_x of the object.

Solution:
First we can find the mass and moment of inertia of the cone and paraboloid using the general equations $I = \frac{1}{2}\int r^2 dm$, and $dm = \rho dV$.

$$dm_{cone} = \rho dV = 3x\,\pi r^2 dx = 3\pi x^3 dx$$

$$I_{cone} = \frac{1}{2}\int r^2 dm = \frac{1}{2}3\pi\int_0^L x^5 dx = \frac{1}{2}3\pi\frac{1}{6}L^6 = 0.7854\ kg\cdot m^2$$

$$dm_{para} = \rho dV = 3x\pi r^2 dx = 3\pi x^2$$

$$I_{para} = \frac{1}{2}\int_0^L r^2 dm = \frac{1}{2}\pi 3\int_0^L x^3 dx = \frac{3}{8}\pi L^4 = 1.1781\ kg\cdot m^2$$

Now, we can subtract the MOI of the cone from the paraboloid to find the total MOI about the x axis.

$$I_x = I_{para} - I_{cone} = 1.1781 - 0.7854 = 0.3927\ kg\cdot m^2$$

We find total mass by subtracting the mass of the cone from the paraboloid as well.

$$m_{total} = \pi - \frac{3}{4}\pi = \pi/4$$

Using the equation $I = mk^2$ we can isolate k and solve.

$$k_x = \sqrt{\frac{I}{m}} = \sqrt{\frac{0.3927}{\pi/4}} = 0.7071\ m$$