21-R-KIN-SS-58

Find the center of mass of a half cone with radius r and height h.

Solution

Begin by finding the volume of the shape

$$\begin{aligned} V &= \frac{1}{2} \cdot \frac{1}{3} \pi r^2 h \\ &= \frac{1}{6} \pi r^2 h \end{aligned}$$

Using a thin half disk as a mass element, we can integrate along the x axis. Since the mass is thin, the centroid of each element is x. The center of mass of a half disk is $\frac{4r}{3\pi}$.

For convenience in integrating, shift the coordinate system to begin at the vertex of the cone so that the equation for the projection in the xy plane is given by the equation $y = \frac{r}{h}x$.

The moment in the x and y directions are:

$$\begin{aligned} \Sigma M_x &= \int_0^h \bar{x}_{\text{element}} \mathrm{d}V \\ &= \int_0^h x \cdot \frac{1}{2} \pi y^2 \mathrm{d}x \\ &= \int_0^h x \cdot \frac{1}{2} \pi \left(\frac{r}{h} x \right)^2 \mathrm{d}x \\ &= \frac{1}{8} \pi r^2 h^2 \end{aligned}$$

$$\begin{aligned} \Sigma M_y &= \int_0^h \bar{y}_{\text{element}} \mathrm{d}V \\ &= \int_0^h \frac{4y}{3\pi} \cdot \frac{1}{2} \pi y^2 \mathrm{d}x \\ &= \int_0^h \frac{2}{3} \left(\frac{r}{h} x \right)^3 \mathrm{d}x \\ &= \frac{1}{6} r^3 h \end{aligned}$$

So the centroids are:

$$\begin{aligned} \bar{x} &= \frac{\Sigma M_x}{V} \\ &= \frac{3h}{4} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{\Sigma M_y}{V} \\ &= \frac{r}{\pi} \end{aligned}$$

Shifting the x coordinate to align with the question,

$$\bar{x} = h - \frac{3h}{4} = \frac{h}{4}$$