

21-R-KIN-ZA-20 Solution

Question: The paraboloid shown has a density of $\rho = 900 \text{ kg/m}^3$ and a height of $H = 0.9\text{m}$. If the cross section is a circle, and the projection of the x-z axis follows the equation: $z = x^2$, find the moment of inertia about the z axis, and the z' axis.

Solution:

The infinitesimal volume of a small slice of the parabola, dm , can be written as the volume of a cylinder with a small height dz . The mass is written as the density times volume. Using the equation given, $x^2 = z$, we can see that when $y = 0$, x equals the radius of the paraboloid. We can express this completely in terms of z .

$$\begin{aligned}dV &= \pi r^2 dz \\dm &= \rho dV = \rho \pi r^2 dz = \rho \pi z dz\end{aligned}$$

Using the equation for moment of inertia of a disk with differential mass, $I = \int_m \frac{1}{2} r^2 dm$, we can find the infinitesimal moment of inertia by taking the derivative of both sides.

$$dI = 0.5 r^2 dm = 0.5 z * \rho \pi z dz = 0.5 \rho \pi z^2 dz$$

Plugging all values in, and integrating over the length z gives the moment of inertia about the z axis.

$$I_z = 0.5 \int_0^H \rho \pi z^2 dz = 0.5 \rho \pi \frac{1}{3} H^3 = \frac{1}{6} 900 * \pi * 0.9^3 = 343.5 \text{ kg} \cdot \text{m}^2$$

To find the mass, we integrate the expression for dm over the height H . We can then use the parallel axis theorem to find $I_{z'}$, where $d^2 = H$

$$\begin{aligned}m &= \int_0^H \rho \pi z dz = \rho \pi \frac{1}{2} H^2 \\I_{z'} &= I_z + md^2 = 343.5 + \left(\frac{1}{2} \rho \pi 0.9^2\right) * 0.9 = 1374.1 \text{ kg} \cdot \text{m}^2\end{aligned}$$