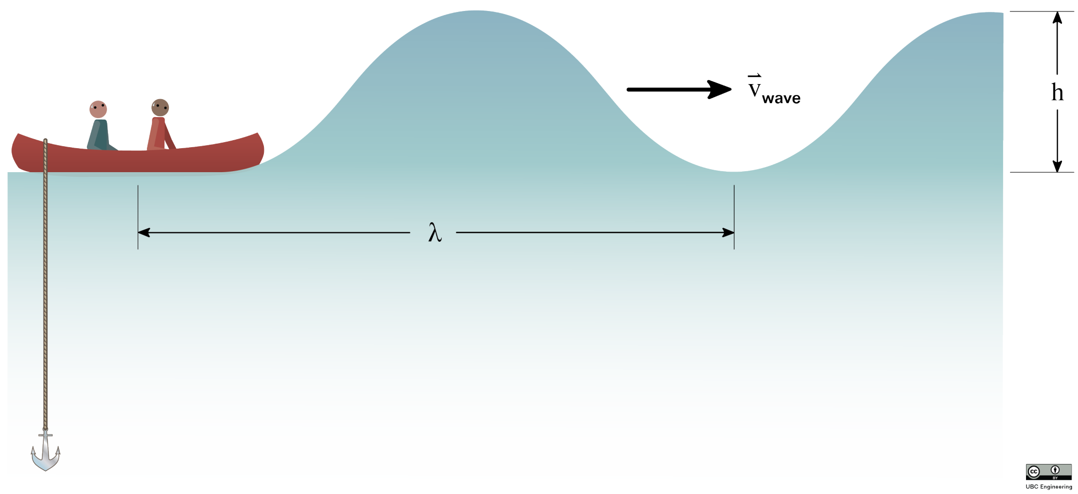
22-R-VIB-TW-47



A boat is sitting anchored in the water, when a bird lands on the boat and then takes off, causing the boat to start bobbing up and down due to its buoyancy force. In addition to this, the boat is also going over waves of height $h = 0.5$ m, wavelength $\lambda = 5$ m, and speed $v_{\text{wave}} = 3$ m/s. Given that the cross-sectional area of the bottom of the boat is $A = 2.5$ m^2 and the mass of the boat is 100 kg, what is the amplitude of the boat's steady-state vertical oscillations?
Note that the buoyancy force is given by $F_b = V_D g\rho$ where V_D is the volume of the liquid displaced and ρ is the density of the liquid. (Use $g = 9.81$ m/s^2 and $\rho = 1000$ kg/m^3 and assume the cross-sectional area stays constant)

Solution:
Let's begin by solving for the natural frequency and writing the sum of forces to get a differential equation in the form of Hooke's law

$$F_b = V_D g\rho = Ahg\rho$$
$$\sum F_y: \ ma = mg + hAg\rho$$
$$m\ddot{h} - hAg\rho = mg$$

This equation is non-homogeneous, making it slightly harder to solve. We can get around this by redefining our coordinate system so that we are centered at the equilibrium point that the boat will oscillate about

$$mg = -h_{eq}Ag\rho$$
$$h_{eq} = -\frac{mg}{Ag\rho}$$
$$\text{let } y = h - h_{eq}$$
$$\ddot{y} = \ddot{h}$$
$$m\ddot{y} - (y + h_{eq})Ag\rho = mg$$
$$m\ddot{y} - yAg\rho + \frac{mg}{Ag\rho}Ag\rho = mg$$

$$
\begin{aligned}
&m\ddot{y} - yAg\rho = 0 \\
&\ddot{y} - \omega_n^2 y = 0 \\
&\Rightarrow \omega_n^2 = \frac{Ag\rho}{m} \\
&\omega_n = \sqrt{\frac{Ag\rho}{m}} = \sqrt{\frac{(2.5)(9.81)(1000)}{100}} = 15.66 \text{ [Hz]}
\end{aligned}
$$

Now we need to find the frequency and amplitude of the forced vibration
Looking at the wave, we can see that the amplitude is $F_0 = \frac{h}{2} = 0.25$ [m]

$$
\begin{aligned}
&v_w = \lambda f \\
&2\pi v_w = \lambda \omega_0 \\
&\omega_0 = \frac{2\pi v_w}{\lambda} = \frac{2\pi(3)}{5} = 3.77 \text{ [Hz]}
\end{aligned}
$$

Now we can use the values we calculated and plug them into the following formula (given in the textbook) to get the amplitude of the steady-state oscillations

$$
\begin{aligned}
X &= \frac{\delta_0}{1 - (\frac{\omega_0}{\omega_n})^2} \\
&= \frac{h/2}{1 - (\frac{\omega_0}{\omega_n})^2} = \frac{0.25}{1 - (\frac{3.77}{15.66})^2} = 0.265 \text{ [m]}
\end{aligned}
$$