21-R-KM-ZA-05 Solution

Question: The ball and socket joints at A and C support a rotating assembly. The assembly is rotating at an angular velocity of $\omega_{AC} = \omega \, rad/s$, and an angular acceleration of $\alpha_{AC} = \alpha \, rad/s^2$ about the axis of the shaft AC. Find the velocity and acceleration of the point D in Cartesian vector form using the coordinate system given.

The following dimensions are known:
$d1 = d_1 m, d2 = d_2 m, d3 = d_3 m, d4 = d_4 m$

Solution: First, we can find the unit vector of the direction from point C to point A, $\widehat{u}_{AC}$. Multiplying this by the angular velocity magnitude will give the angular velocity vector $\widehat{\omega}_{AC}$. As we know that point A has a velocity of 0, we will use this as our reference point. Using the distance between points A and D, and the angular velocity, we can find the velocity of the point D.

$$\vec{u}_{AC} = \left(d_1 \hat{i} - d_2 \hat{j} + 0\hat{k} \right) \frac{1}{\sqrt{d_1^2 + d_2^2}}$$

$$\vec{\omega} = \omega \vec{u}_{AC}$$

$$\vec{r}_{D/A} = \left(-d_1 \hat{i} + d_3 \hat{j} + d_4 \hat{k} \right)$$

$$\vec{V}_D = \cancelto{0}{\vec{V}_A} + \vec{\omega} \times \vec{r}_{D/A} = \frac{\omega}{\sqrt{d_1^2 d_2^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d_1 & -d_2 & 0 \\ -d_1 & d_3 & d_4 \end{vmatrix}$$

$$V_D = \frac{\omega}{\sqrt{d_1^2 + d_2^2}} \left[\hat{i}\left(-d_2 d_4\right) - \hat{j}\left(d_1 d_4\right) + \hat{k}\left(d_1 d_3 - d_1 d_2\right) \right]$$

Next, we write the acceleration equation for point D with respect to point A, and cancel the acceleration of A. We already know the unit vector for point C with respect to point A. Multiplying this by the angular acceleration magnitude will give the angular velocity vector $\widehat{\alpha}_{AC}$. Taking the cross product between $\widehat{\alpha}_{AC}$ *and* $\widehat{r}_{D/A}$ gives the values for the first term in the acceleration equation.

$$\vec{a}_D = \cancel{\vec{a}_A}^{\,0} + \vec{\alpha}_{AC} \times \vec{r}_{D/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{D/A})$$

$$\vec{\alpha}_{AC} = \alpha_{AC} \vec{u}_{AC}$$

$$\vec{\alpha}_{AC} \times \vec{r}_{D/A} = \frac{\alpha}{\sqrt{d_1^2 + d_2^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d_1 & -d_2 & 0 \\ -d_1 & d_3 & d_4 \end{vmatrix}$$

$$= \frac{\alpha}{\sqrt{d_1^2 + d_2^2}} \left[\hat{i}\left(-d_2 d_4\right) - \hat{j}\left(d_1 d_4\right) + \hat{k}\left(d_1 d_3 - d_1 d_2\right) \right]$$

To evaluate the second term, we cannot simplify the acceleration equation for planar kinematics as the problem contains 3 dimensions. The first cross product $\widehat{\omega}_{AC} \times \widehat{r}_{D/A}$ is equal to the velocity of point D. We can then simply calculate the cross product between the angular velocity of AC, and the velocity of D as shown. This gives values for the second term in the acceleration equation. Finally, adding the values for the first and second terms gives acceleration of D in terms of its x, y, and z components.

$$\vec{\omega} \times \vec{r}_{D/A} = \vec{V}_D = \left[V_{Dx}\hat{i} + V_{Dy}\hat{j} + V_{Dk}\hat{k} \right]$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{D/A}) = \frac{\omega}{\sqrt{d_1^2 + d_2^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d_1 & -d_2 & 0 \\ V_{Dx} & V_{Dy} & V_{Dk} \end{vmatrix}$$

$$= \frac{\omega}{\sqrt{d_1^2 + d_2^2}} \left[\hat{i}\left(-d_2 V_{Dk}\right) - \hat{j}\left(d_1 V_{Dk}\right) + \hat{k}\left(d_1 V_{Dy} + d_2 V_{Dx}\right) \right]$$

$$a_{Dx} = \frac{\alpha + \omega}{\sqrt{d_1^2 + d_2^2}} \left[-d_2 d_4 - d_2 V_{Dk} \right]$$

$$a_{Dy} = \frac{\alpha + \omega}{\sqrt{d_1^2 + d_2^2}} \left[-d_1 d_4 - d_1 V_{Dk} \right]$$

$$a_{Dk} = \frac{\alpha + \omega}{\sqrt{d_1^2 + d_2^2}} \left[\left(d_1 d_3 - d_1 d_2\right) + \left(d_1 V_{Dy} + d_2 V_{Dx}\right) \right]$$