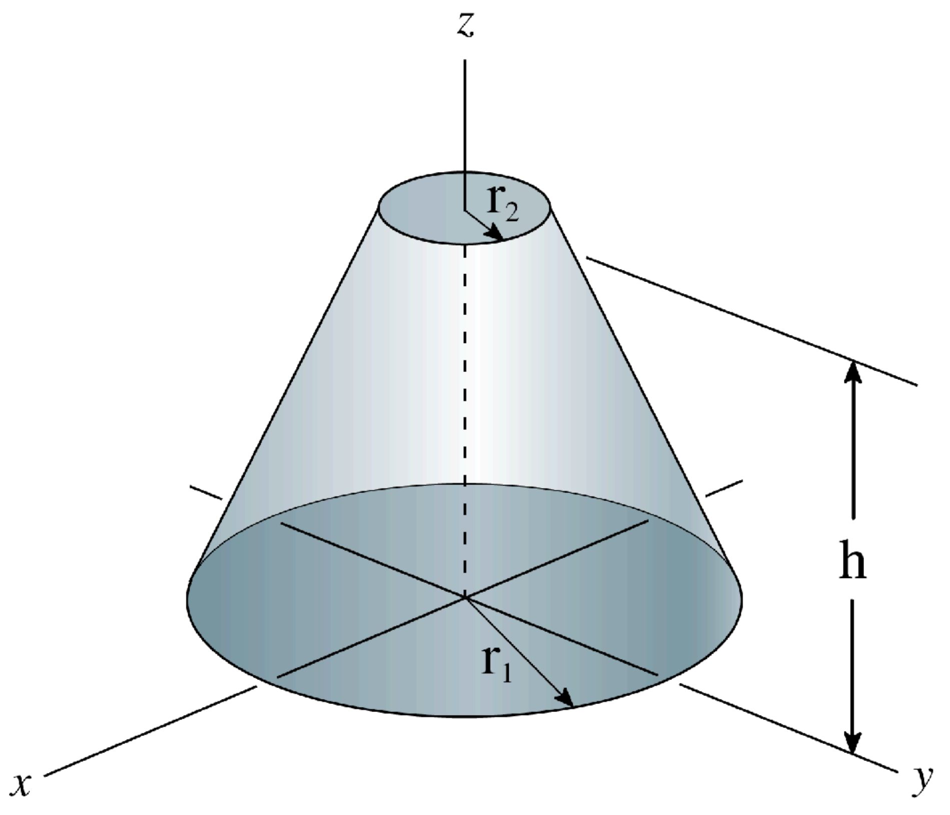
21-S-9.1-RP-03



Find the z coordinate, $\bar{z}$, of the centroid of a truncated cone with uniform density.

Find the volume of the cone.

Using similar triangles,

$$\frac{r_1 - r_2}{h} = \frac{r_1 - r(z)}{z}$$

$$\Rightarrow r(z) = \frac{r_1 h - zR}{h}$$

where $R = r_1 - r_2$

Since the area of a flat disc parallel to the xy plane is πr^2:

$$r(z)^2 = \frac{{r_1}^2 h^2 - 2 r_1 h z R + z^2 R^2}{h^2}$$

$$V = \int_V dV = \int_{z=0}^{z=h} \pi r(z)^2 dz = \frac{\pi}{h^2}\int_{z=0}^{z=h} (r_1{}^2h^2 - 2r_1hzR + z^2R^2)dz = \frac{\pi}{h^2}\left[r_1{}^2h^2z - r_1hRz^2 + \frac{z^3R^2}{3}\right]_0^h$$

$$\rightarrow V = \frac{\pi}{h^2}\cdot h^3\left(r_1{}^2 - r_1R + \frac{R^2}{3}\right) = \pi h\left(r_1{}^2 - r_1{}^2 + r_1r_2 + \frac{r_1{}^2 - 2r_1r_2 + r_2{}^2}{3}\right)$$

$$\Rightarrow V = \frac{\pi h}{3}(r_1{}^2 + r_1r_2 + r_2{}^2)$$

Find $\bar{z}$.

$$\bar{z} = \frac{\int_V z dV}{\int_V dV} = \frac{M_z}{V}$$

*Note: M_z is not actually the moment about the z axis, but rather is just a placeholder for the integral.

$$M_z = \int_{z=0}^{z=h} z\pi r(z)^2 dz = \frac{\pi}{h^2}\int_{z=0}^{z=h} (r_1{}^2h^2z - 2r_1hz^2R + z^3R^2) = \frac{\pi}{h^2}\left[r_1{}^2h^2\frac{z^2}{2} - 2r_1hR\frac{z^3}{3} + R^2\frac{z^4}{4}\right]_0^h$$

$$\rightarrow M_z = \frac{\pi}{h^2}\cdot h^4\left(\frac{r_1{}^2}{2} - \frac{2}{3}r_1R + \frac{R^2}{4}\right) = \frac{\pi h^2}{12}\left(6r_1{}^2 - 8r_1{}^2 + 8r_1r_2 + 3r_1{}^2 - 6r_1r_2 + 3r_2{}^2\right)$$

$$\Rightarrow M_z = \frac{\pi h^2}{12}(r_1{}^2 + 2r_1r_2 + 3r_2{}^2)$$

$$\bar{z} = \frac{h}{4}\cdot\frac{r_1{}^2 + 2r_1r_2 + 3r_2{}^2}{r_1{}^2 + r_1r_2 + r_2{}^2}$$