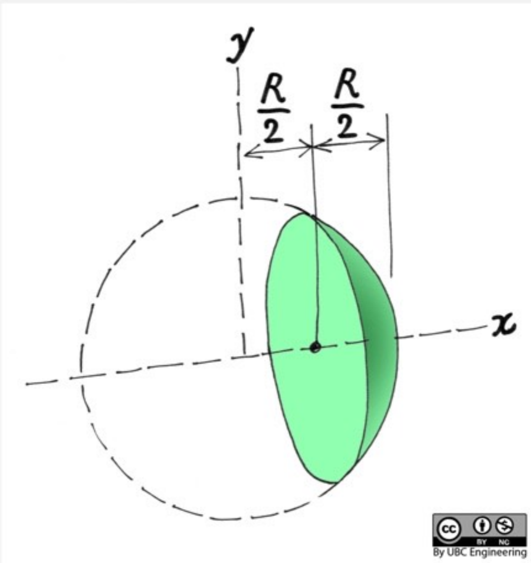
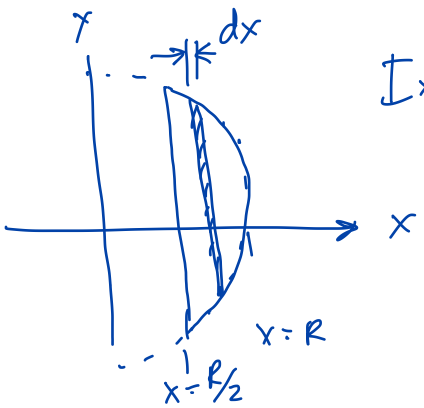
UBC-DYN-18-026

Find the mass moment of inertia about the x axis of the sphere segment shown below. Assume $m = 4\ kg$ and $R = 0.4\ m$.



$I_{xx} =$ ☐ $kg \cdot m^2$



$x^2 + y^2 = R^2$

$\Rightarrow y^2 = R^2 - x^2$

$$I_{xx} = \int_m dI = \int \tfrac{1}{2} y^2 dm \qquad dm = \rho \underbrace{y^2 \pi}_{\text{area}} \underbrace{dx}_{\text{thicknes}}$$

$$= \frac{1}{2}\int y^2 \rho y^2 \pi \, dx$$

$$= \frac{1}{2}\rho\pi \int y^4 dx$$

$$= \frac{1}{2}\rho\pi \int_{\frac{R}{2}}^{R} (R^2 - x^2)^2 dx$$

$$= \frac{1}{2}\rho\pi \int_{\frac{R}{2}}^{R} R^4 + x^4 - 2R^2x^2 \, dx$$

$$= \frac{1}{2}\rho\pi \left(R^4 x + \frac{x^5}{5} - \frac{2R^2x^3}{3}\right)\Bigg|_{\frac{R}{2}}^{R}$$

$$= \frac{1}{2}\rho\pi \left[\left(R^4(R) + \frac{R^5}{5} - \frac{2R^5}{3}\right) - \left(R^4\left(\frac{R}{2}\right) + \frac{\left(\frac{R}{2}\right)^5}{5} - \frac{2R^2\left(\frac{R}{2}\right)^3}{3}\right)\right]$$

$$dI = \frac{1}{2} r^2 dm$$

$$= \frac{1}{2} y^2 dm$$

$$= \frac{1}{2}\rho\pi\left[R^5 + \frac{R^5}{5} - \frac{2}{3}R^5 - \frac{R^5}{2} - \frac{R^5}{32\cdot 5} + \frac{2R^5}{3\cdot 8}\right]$$

$$= \frac{1}{2}\rho\pi R^5\left[\frac{480}{480} + \frac{96}{480} - \frac{320}{480} - \frac{240}{480} - \frac{3}{480} + \frac{40}{480}\right]$$

$$= \frac{1}{2}\rho\pi R^5 \frac{53}{480}$$

$$m = \iint_m dm = \int_{\frac{R}{2}}^{R} \rho y^2 \pi \, dx = \int_{\frac{R}{2}}^{R} \rho (R^2 - x^2)\pi \, dx = \rho\pi\left(R^2 x - \frac{x^3}{3}\right)_{\frac{R}{2}}^{R}$$

$$= \rho\pi\left(R^3 - \frac{R^3}{3} - \left(\frac{R^3}{2} - \frac{R^3}{24}\right)\right) = \rho\pi R^3\left(\frac{24}{24} - \frac{8}{24} - \frac{12}{24} + \frac{1}{24}\right)$$

$$= \rho\pi R^3 \frac{5}{24}$$

$$I_{xx} = \underbrace{\frac{1}{2}\rho\pi R^3\left(\frac{5}{24}\right)}_{m}\left(R^2 \frac{53}{\cancel{480}} \cdot \frac{\cancel{24}}{5}\right) = mR^2\frac{53}{200}$$