Written Assignment #3

Vancouver Summer Program 2019 – Algorithms – UBC

- You should work with a partner.
- You must typeset your solutions.
- Submit your work using Gradescope by **10:00 p.m. on Tuesday, July 30**.
- **Notation.** $\mathbb{N} = \{1, 2, \dots\} \subset \{0, 1, 2, \dots\} = \mathbb{Z}_+$, and $\mathbb{R}_+ = [0, \infty)$.

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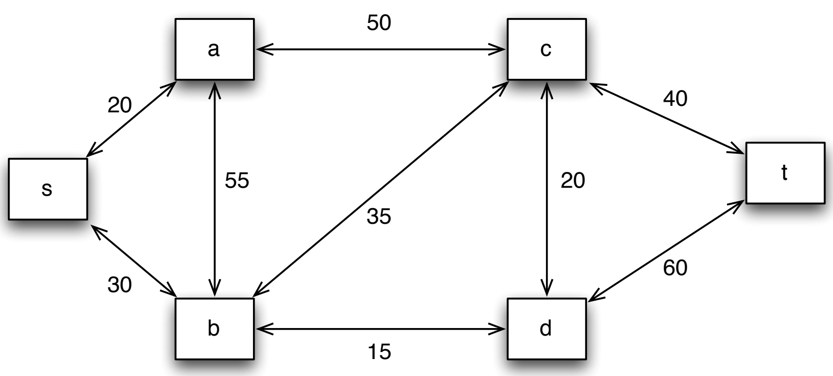
1. (Applying graph algorithms: 5 points.)

   (a) (1 point) Find the shortest path between s and t in the following graph.



   (b) We have three containers whose sizes are 10 Liters (L), 7 L, and 4 L, respectively. The 7 L and 4 L containers start out full of water, but the 10 L container is initially empty. We are allowed one type of operation: pouring the contents of one container into another, stopping only when the source container is empty or the destination container is full. We want to know if there is a sequence of pourings that leaves exactly 2 L in the 7 or 4 L container.

      (i) (2 point) Model this as a graph problem: give a precise definition of the graph involved by clearly explaining the vertices and e<dges, and state the specific question about this graph that needs to be answered.

      (ii) (1 point) What algorithm should be applied to solve the problem?

      (iii) (1 point) Find the answer by applying the algorithm.

2. (Some simple graph properties: 5 points.) Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Which of the following propositions must be true? Provide a short proof or counterexample in each case.

   (a) (1 point) $2e/v \leq M$

   (b) (1 point) $2e/v \geq m$

   (c) (1 point) There exists a simple path (includes no cycles) of length at least m.

   (d) (1 point) $m > 2$ implies that G is connected

(e) (1 point) In every (simple) graph there is a path from any vertex of odd degree to some other vertex of odd degree.

3. (BONUS. Some more graph properties: 2 points.) Let m be a positive integer and consider a graph G^* with $2m$ vertices: $v_1, \ldots, v_{2m}$. An edge exists between vertices v_i and v_j if and only if $(i - j \equiv 1 \mod 2m) \vee (i - j \equiv 2m - 1 \mod 2m) \vee (i - j \equiv m \mod 2m)$.

   Note that $x \equiv y \mod n$ if and only if $x = kn + y$ for some integer k. As examples, $25 \equiv 5 \mod 20$, $29 \equiv -1 \mod 30$ and $29 \equiv 29 \mod 30$.

   (a) (1 point) For each $j \in \{2, \ldots, 2m\}$, what is the distance between v_1 and v_j? The *distance* between two vertices of a graph is the number of edges on the shortest path that connects the two vertices. (Derive an expression in terms of i, j and m. You will have to consider a few cases.)

   (b) (1 point) A graph G is k-edge-connected if and only if one has to remove k edges to disconnect the graph. Prove that G^* is not 4-edge-connected: you can remove three or fewer edges to disconnect the graph.