In-Class Activity #3

Vancouver Summer Program

Algorithms

(Tolerant colourings.) Consider a graph where every vertex has degree at most 25. We would like to colour the graph using two colours, *red* and *black*, such that no vertex has more than 12 adjacent vertices of the same colour as itself. To achieve this, your friend Mercator suggests the following scheme: For each vertex in the graph, toss a coin and if the coin toss results in a head then colour that vertex red otherwise colour it black. Label each vertex that has more than 12 adjacent vertices of the same colour with a *FixMe* label. Pick any vertex that has a FixMe label and call this vertex v. (If no such vertex exists then stop.) If v is coloured red then recolour it black; if v is coloured black then recolour it red. Update all the FixMe labels. Repeat the recolouring process until there is no vertex with a FixMe label.

Question: Show not only that Mercator's algorithm will stop but that it will do so in no more than $\Theta(m)$ colouring/recolouring and label update *iterations* where m is the number of edges in the graph.

(It is possible to show that the entire algorithm can run in $\Theta(m + n)$ worst-case time where n is the number of vertices but you do not have to include such a proof in your solution.)

Solution. Label edges that connect vertices of different colours as *diverse* edges. In a tolerant colouring, every vertex of degree greater than 12 will have at least one diverse edge, and a vertex with high degree should have more diverse edges. Therefore, to obtain a tolerant colouring we should increase the number of diverse edges.

With Mercator's algorithm, whenever a vertex is recoloured, the number of diverse edges strictly increases. (After a recolouring the number diverse edges associated with the vertex that has been recoloured will be greater than the number of non-diverse edges, leading to an in increase in the number of diverse edges.)

The number of edges in the graph is m therefore the algorithm will eventually terminate and the number of iterations is $O(m)$ because each iteration increases the number of diverse edges by at least 1.

(A generalization of this result is that we can use two colours and produce a tolerant graph such that each vertex has at most half its neighbours with the same colour as itself. We can do away with the degree 25 restriction, etc.)