Can you arrange the following sets from smallest to largest?

Integers ($\mathbb{Z}$)	
Even integers	
Multiples of 3	
Perfect squares	
Prime numbers	
Integer powers of 2	$\aleph_0$
Non-negative integers ($\mathbb{N}$)	
Pairs of integers ($\mathbb{Z} \times \mathbb{Z}$)	
Rational numbers ($\mathbb{Q}$)	
Functions from $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ to $\mathbb{Z}$	
Finite sets of integers	
Sets of integers ($2^{\mathbb{Z}}$)	
Functions from $\mathbb{Z}$ to $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$	
Functions from $\mathbb{Z}$ to $\mathbb{Z}$	
Real numbers ($\mathbb{R}$)	
Complex numbers ($\mathbb{C}$)	$2^{\aleph_0}$
Spacetime ($\mathbb{R}^4$)	
Functions from $\mathbb{Z}$ to $\mathbb{R}$	
Finite sets of real numbers	
Sets of real numbers ($2^{\mathbb{R}}$)	
Functions from $\mathbb{R}$ to $\mathbb{Z}$	$2^{2^{\aleph_0}}$
Functions from $\mathbb{R}$ to $\mathbb{R}$	
Sets of curves in $\mathbb{R}^3$	$2^{2^{2^{\aleph_0}}}$

Two sets A and B ***have the same cardinality***
(written $A \approx B$)
if and only if there is a bijection $f\colon A \to B$.

For any integer $n \geq 0$, define $[\boldsymbol{n}] = \{x \in \mathbb{Z} \mid 1 \leq x \leq n\} = \{1, 2, 3, \ldots, n\}$.

(For example: $[0] = \varnothing$, $[1] = \{1\}$, $[5] = \{1, 2, 3, 4, 5\}$.)

Every ***finite*** set X has the same cardinality as $[n]$
for some unique non-negative integer n.

If $X \approx [n]$, we ***define*** $|X| = n$.

Two sets A and B ***have the same cardinality***
(written $A \approx B$)
if and only if there is a bijection $f\colon A \to B$.

- So are there more even integers than odd integers?

- Are there more integers than even integers?

- Are there more even integers than integers divisible by 107?

- Are there more integers than positive integers?

Two sets A and B ***have the same cardinality***
(written $A \approx B$)
if and only if there is a bijection $f\colon A \to B$.

- How many positive rational numbers are there?

$$\begin{array}{ccccc}
1/1 & 1/2 & 1/3 & 1/4 & \ldots \\
2/1 & 2/2 & 2/3 & 2/4 & \ldots \\
3/1 & 3/2 & 3/3 & 3/4 & \ldots \\
4/1 & 4/2 & 4/3 & 4/4 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The following function is a bijection from $\mathbb{N}$ to $\mathbb{Q}$!

$$R(0) = 0,$$
$$R(n) = \frac{1}{2\lfloor R(n-1) \rfloor - R(n-1) + 1} \quad \text{for all integers } n \geq 1.$$

$$0, 1, \frac{1}{2}, 2, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, 3, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, 4, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \frac{3}{8}, \frac{8}{5}, \frac{5}{7}, \frac{7}{2}, \frac{2}{7}, \ldots$$

$$\boxed{\begin{array}{c}\text{Two sets } A \text{ and } B \textbf{\textit{ have the same cardinality}}\\ \text{(written } A \approx B\text{)}\\ \text{if and only if there is a bijection } f\colon A \to B.\end{array}}$$

- How many real numbers are in the interval $0 \leq x < 1$?

Theorem: $\mathbb{Z}^+ \not\approx [0,1)$

Proof: Suppose $f\colon \mathbb{Z} \to [0,1)$ is a bijection.

Write out a list containing each number $f(n)$ in decimal.

$$\begin{aligned}
f(1) &= 0.\underline{\mathbf{0}}00000000\ldots\\
f(2) &= 0.0\underline{\mathbf{1}}2345678\ldots\\
f(3) &= 0.31\underline{\mathbf{4}}159653\ldots\\
f(4) &= 0.271\underline{\mathbf{8}}28182\ldots\\
f(5) &= 0.1414\underline{\mathbf{2}}1400\ldots\\
f(6) &= 0.11235\underline{\mathbf{9}}549\ldots\\
&\vdots\\
&\Longrightarrow
\end{aligned}$$

Let x be any number with the following property:

For all n, the nth digit of $x \neq$ the nth digit of $f(n)$.

With the list above, we could take $x = 0.358214\ldots$

Then $x \neq f(n)$ for all positive integers n.

So f is not a bijection after all! $\square$

$$\boxed{\text{The real numbers are } \textbf{\textit{uncountable}}!}$$

A set X is ***infinite*** if and only if there is an injection $f\colon \mathbb{N} \to A$.

A set X is ***countable*** if and only if there is a function $f\colon A \to \mathbb{N}$.

Every function $f\colon \mathbb{N} \to \{0, 1\}$ defines an infinite string of bits.

So the set of functions is uncountable.

On the other hand, any *program* can be represented as a *finite* string of bits.

So the set of programs is countable.

$\boxed{\textbf{\textit{Almost all functions are uncomputable!}}}$