$\boxed{\textbf{\textit{Cardinality}}\ |S| \ = \text{number of elements in } S}$

- $|\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}| = 4$
- $|\{1, 2, 1, 3, 1, 2, 1\}| = 3$
- $|\varnothing| = 0$
- $|\{\varnothing\}| = 1$
- $|\{\varnothing,\ \{\varnothing\}\}| = 2$
- $|\mathbb{Z}| = \infty$

Caveat! Hic sunt dracones!

Inclusion-Exclusion

$$\boxed{|A \cup B| \; = \; |A| + |B| - |A \cap B| \;\text{ for all finite sets } A \text{ and } B.}$$

Each element in $A \cup B$ is counted once on both sides.

$$\boxed{|A \cup B| + |A \cap B| \; = \; |A| + |B| \;\text{ for all finite sets } A \text{ and } B.}$$

Each element in $A \oplus B$ is counted once on both sides.
Each element in $A \cup B$ is counted twice on both sides.

For all finite sets A, B, C:

$$\begin{aligned}
|A \cup B \cup C| &= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| \\
&= \big(|A| + |B| - |A \cap B|\big) + |C| \\
&\qquad - \big(|A \cap C| + |B \cap C| - |(A \cap B) \cap (B \cap C)|\big) \\
&= |A| + |B| + |C| \\
&\qquad - |A \cap B| - |A \cap C| - |B \cap C| \\
&\qquad + |A \cap B \cap C|
\end{aligned}$$

For all finite sets A, B, C, D:

$$\begin{aligned}
&|A \cup B \cup C \cup D| = \\
&\qquad |A| + |B| + |C| + |D| \\
&\qquad - |A \cap B| - |A \cap C| - |B \cap C| - |A \cap D| - |B \cap D| - |C \cap D| \\
&\qquad + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
&\qquad - |A \cap B \cap C \cap D|
\end{aligned}$$

$\boxed{\textbf{\textit{Power set}}\ \mathbf{2^S} = \{A \mid A \subseteq S\} = \text{the set of all subsets of } S}$

- $2^{\{\bigstar\}} =$
- $2^{\{T,F\}} =$
- $2^{\{r,g,b\}} =$
- $2^{\varnothing} =$
- $2^{2^{\varnothing}} =$
- $2^{2^{2^{\varnothing}}} =$

Theorem: $\boxed{\textit{For any finite set } A\textit{, we have } |2^A| = 2^{|A|}.}$

Proof: Let A be an arbitrary finite set.

Assume that for any proper subset $Z \subset A$, we have $|2^Z| = 2^{|Z|}$.

There are two cases to consider: Either A is empty or not.

- If $A = \varnothing$, then $|2^A| = |\{\varnothing\}| = 1$ and $2^{|A|} = 2^0 = 1$.
- Suppose $A \neq \varnothing$.

  Let x be an arbitrary element of A, and let $Z = A \setminus \{x\}$.

  The inductive hypothesis implies that $|2^Z| = 2^{|Z|}$.

  For each subset $Y \subseteq Z$, both Y and $Y \cup \{a\}$ are subsets of A.

  Thus, A has $2^{|Z|}$ distinct subsets that do *not* contain x.

  For all subsets $U, V \subseteq Z$, if $U \neq V$, then $U \cup \{x\} \neq V \cup \{x\}$.

  Thus, A also has $2^{|Z|}$ distinct subsets that *do* contain x.

  Finally, $|Z| = |A| - 1$.

  So overall, A has $2^{|Z|} + 2^{|Z|} = 2^{|Z|+1} = 2^{|A|}$ subsets.

$\square$

$\boxed{\textbf{\textit{Cartesian Product}}\ \boldsymbol{A \times B} = \{(a,b) \mid a \in A \ \wedge \ b \in B\}}$

Each element $(a,b) \in A \times B$ is called an ***ordered pair***.

- $\{1,2\} \times \{a,b,c\} = \{(1,a),\ (1,b),\ (1,c),\ (2,a),\ (2,b),\ (2,c)\}$

- Standard deck of 52 cards:

$$\{A,2,3,4,5,6,7,8,9,10,J,Q,K\} \times \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$$

- $\{1,2\} \times \{1,2\} = \{(1,1),\ (1,2),\ (2,1),\ (2,2)\}$

$$(1,2) \neq \{1,2\}$$

$\{1,2\} = \{2,1\}$, but $(1,2) \neq (2,1)$

$\{1,1\} = \{1\}$, but $(1,1) \neq \{1\} \neq 1$.

- $\varnothing \times X =$

$$\boxed{\boldsymbol{A^2 = A \times A}}$$

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} =$ the standard Euclidean plane.

Identities

These you can prove by grinding definitions (hint, hint, hint):

- $(A \cap B) \times C = (A \times C) \cap (B \times C)$
- $(A \cup B) \times C = (A \times C) \cup (B \times C)$
- $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$
- $A \times B = B \times A \iff A = B$

Note that set difference is sometimes denoted as \ in place of −.

This you can prove by induction (hint, hint, hint):

- $|A \times B| = |A| \cdot |B|$ if A and B are finite.