

Modelling and inversion of electromagnetic data

Lindsey Heagy

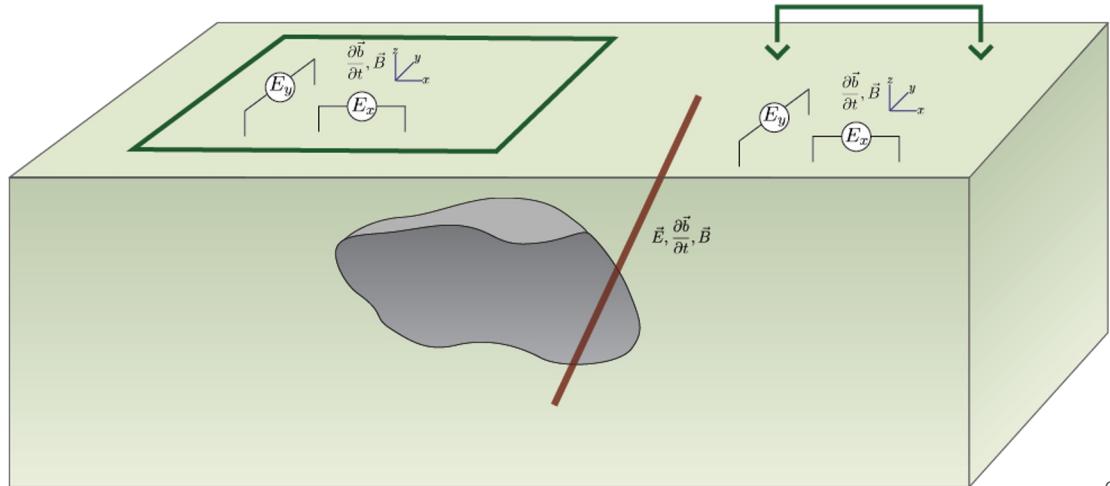
University of British Columbia - Geophysical Inversion Facility

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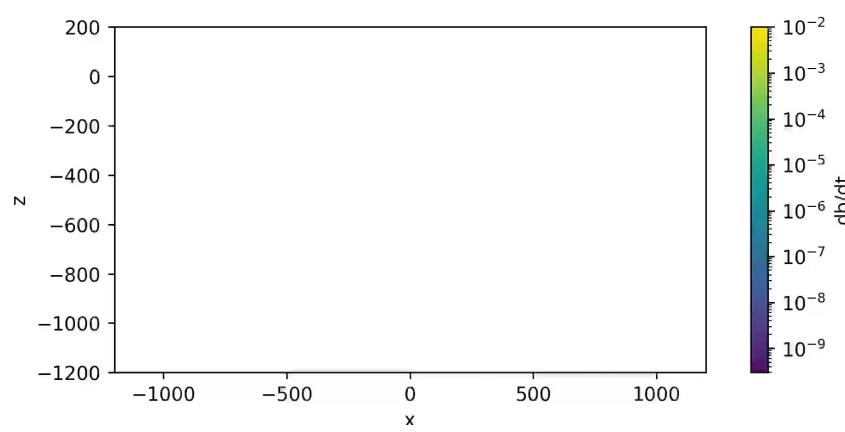
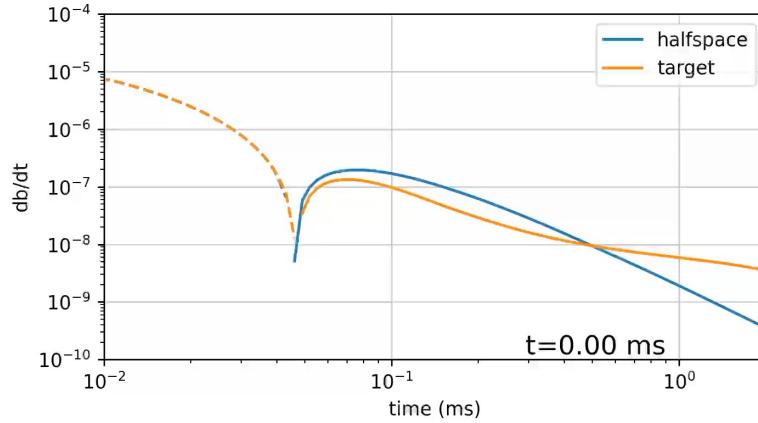
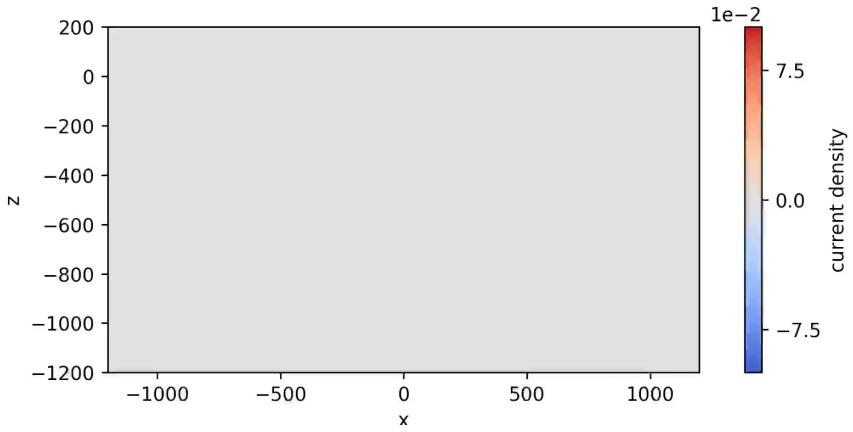
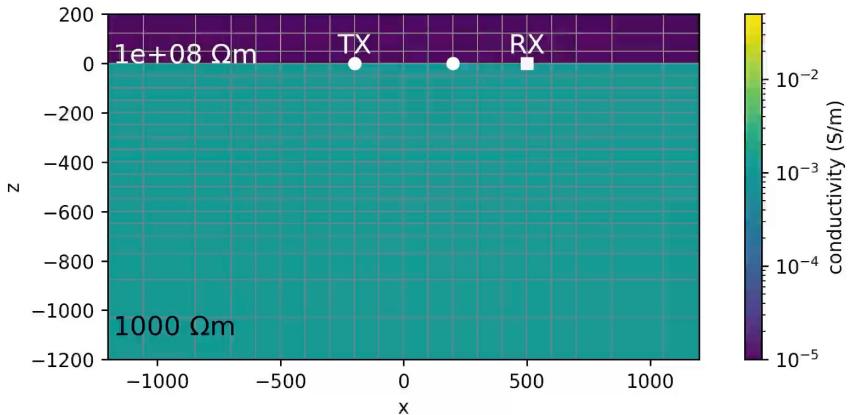
Ground and Borehole EM

setup:

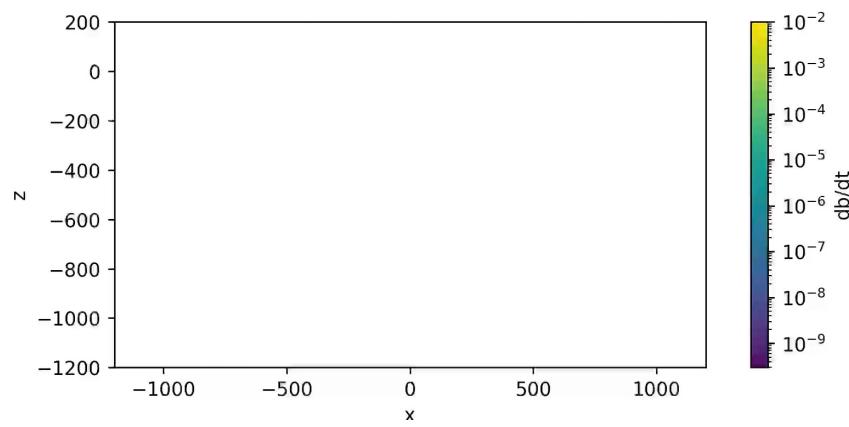
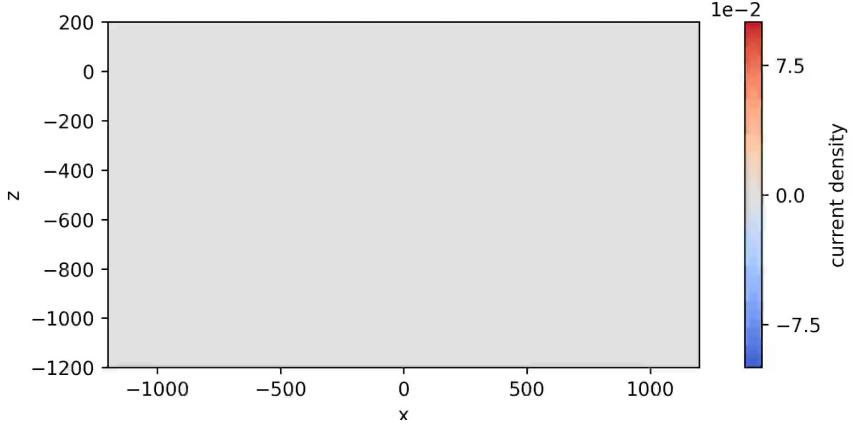
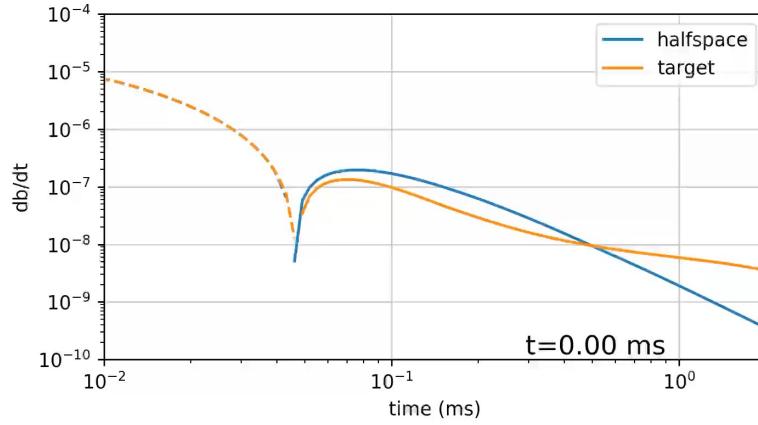
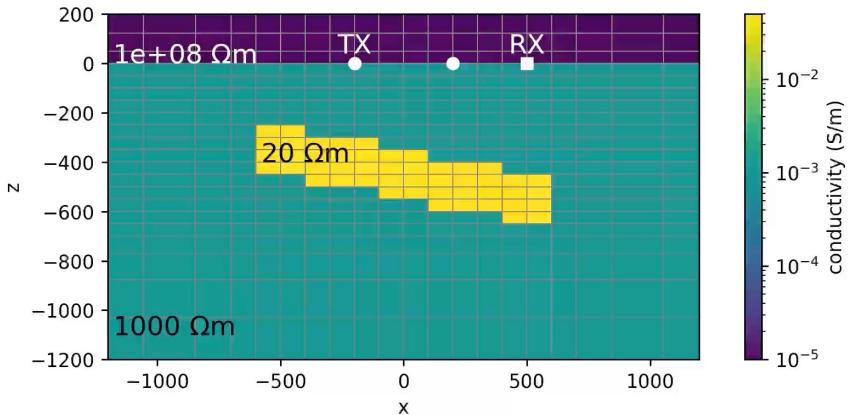
- grounded or inductive source
- measure E, B, $\frac{\partial \vec{B}}{\partial t}$
- x, y, z components
- on ground or in boreholes



Inductive source time domain response



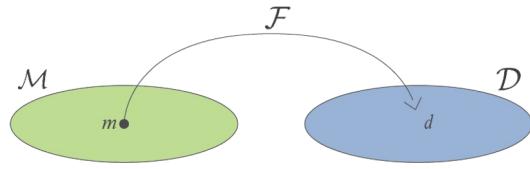
Inductive source time domain response



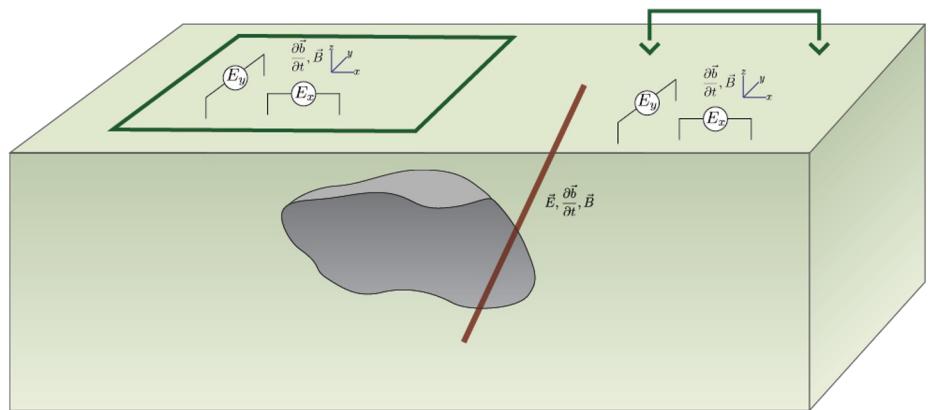
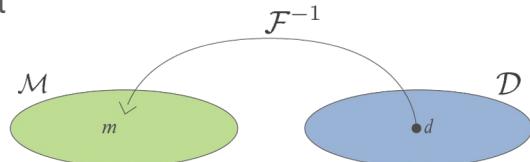
Statement of the inverse problem

Given

- observations: d_j^{obs} , $j = 1, \dots, N$
- uncertainties: ϵ_j
- ability to forward model: $\mathcal{F}[m] = d$



Find the Earth model that gave rise to the data



Electromagnetics: basic equations (quasi-static)

	Time	Frequency
Faraday's Law	$\nabla \times \vec{e} = -\frac{\partial \vec{b}}{\partial t}$	$\nabla \times \vec{E} = -i\omega \vec{B}$
Ampere's Law	$\nabla \times \vec{h} = \vec{j} + \frac{\partial \vec{d}}{\partial t}$	$\nabla \times \vec{H} = \vec{J} + i\omega \vec{D}$
No Magnetic Monopoles	$\nabla \cdot \vec{b} = 0$	$\nabla \cdot \vec{B} = 0$
Constitutive Relationships (non-dispersive)	$\vec{j} = \sigma \vec{e}$ $\vec{b} = \mu \vec{h}$ $\vec{d} = \epsilon \vec{e}$	$\vec{J} = \sigma \vec{E}$ $\vec{B} = \mu \vec{H}$ $\vec{D} = \epsilon \vec{E}$

* Solve with sources and boundary conditions

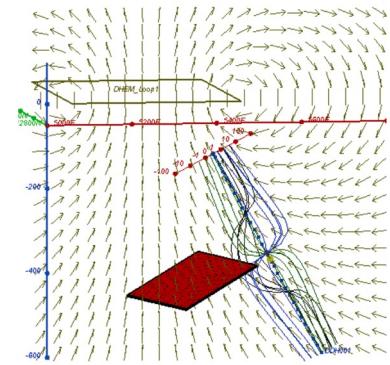
Forward modelling

Lots of choices...

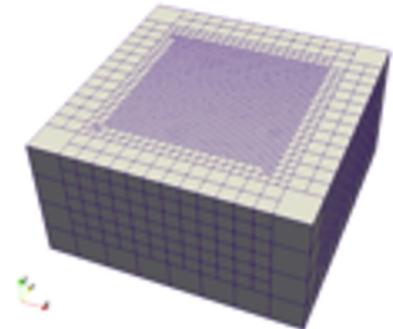
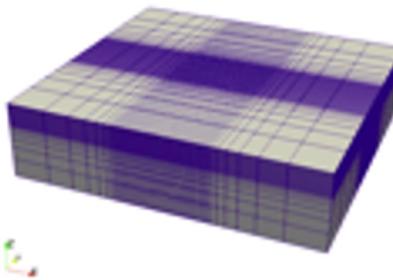
- plate modelling, 1D, cylindrically symmetric, 3D
- time or frequency
- different mesh choices
 - structured
 - semi-structured (e.g. OcTree)
 - unstructured (e.g. tetrahedral)
- discretization strategy
 - integral equation
 - finite difference
 - finite volume
 - finite element

plate
modelling

3D tensor



3D OcTree



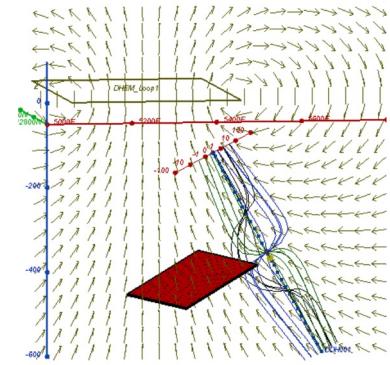
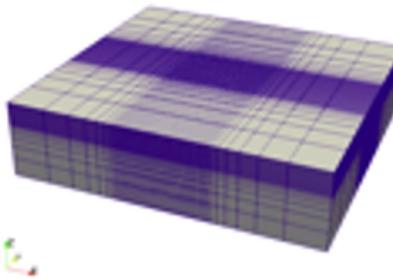
Forward modelling

Our focus today

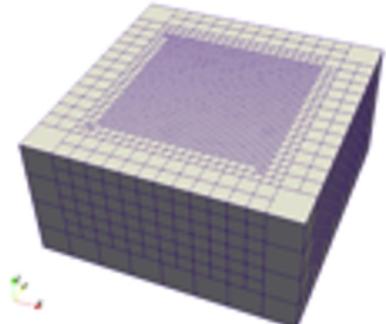
- plate modelling, 1D, cylindrically symmetric, **3D**
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- discretization strategy
 - integral equation
 - finite difference
 - **finite volume**
 - finite element

plate
modelling

3D tensor



3D OcTree



Solving the frequency domain equations

Use constitutive relations, reduce to two equations, one field, one flux

$$\nabla \times \vec{E} + i\omega \vec{B} = 0$$

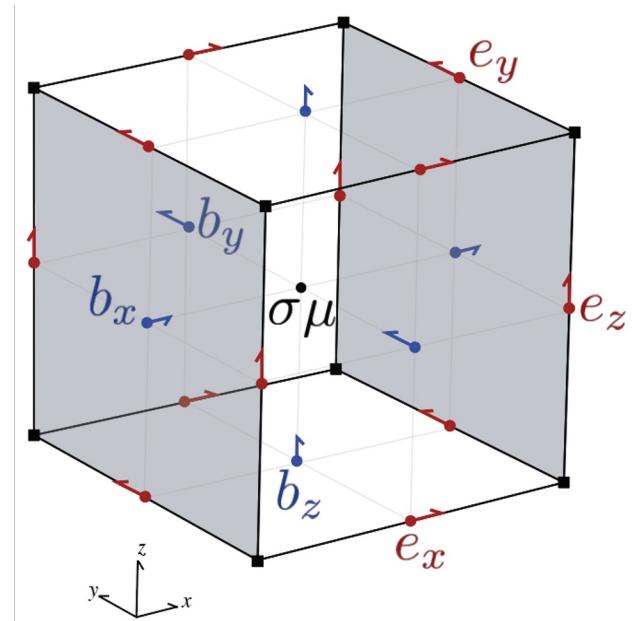
$$\nabla \times \mu^{-1} \vec{B} - \sigma \vec{E} = \vec{J}_s$$

Boundary conditions

$$\hat{n} \times \vec{B}|_{\partial\Omega} = 0$$

Staggered grid discretization

- Physical properties: cell centers
- Fields: edges
- Fluxes: faces



Solving the frequency domain equations

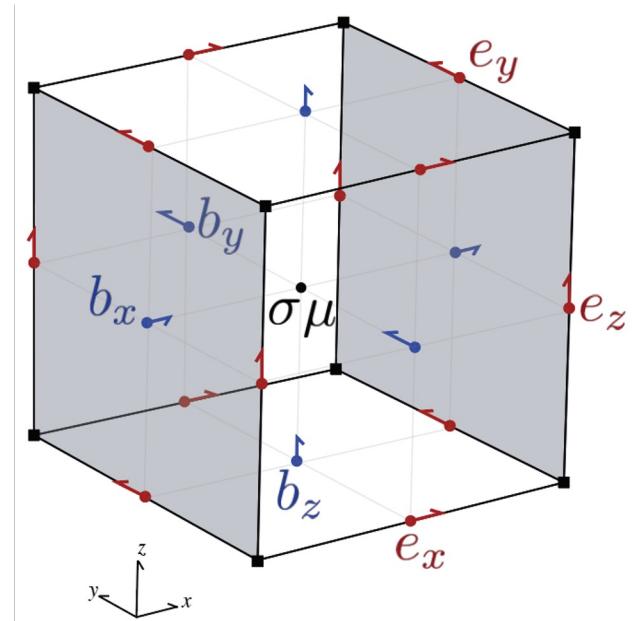
Discrete equations

$$\mathbf{C}\mathbf{e} + i\omega \mathbf{b} = 0$$

$$\mathbf{C}^\top \mathbf{M}_{\mu^{-1}}^f \mathbf{b} - \mathbf{M}_\sigma^e \mathbf{e} = \mathbf{M}^e \mathbf{j}_s$$

Eliminate \mathbf{b} to obtain a second-order system in \mathbf{e}

$$\underbrace{(\mathbf{C}^\top \mathbf{M}_{\mu^{-1}}^f \mathbf{C} + i\omega \mathbf{M}_\sigma^e)}_{\mathbf{A}(\sigma, \omega)} \underbrace{\mathbf{e}}_{\mathbf{u}} = \underbrace{-i\omega \mathbf{M}^e \mathbf{j}_s}_{\mathbf{q}(\omega)}$$



Solving the frequency domain equations

$$\underbrace{(\mathbf{C}^\top \mathbf{M}_{\mu^{-1}}^f \mathbf{C} + i\omega \mathbf{M}_\sigma^e)}_{\mathbf{A}(\sigma, \omega)} \underbrace{\mathbf{e}}_{\mathbf{u}} = \underbrace{-i\omega \mathbf{M}^e \mathbf{j}_s}_{\mathbf{q}(\omega)}$$

$\mathbf{A}(\sigma, \omega)$

- Complex
- Symmetric
- Factor once for each frequency (solve for multiple sources)
- Need to refactor on each model update
- Separable over frequencies

Solving the time domain equations

Use constitutive relations, reduce to two equations, one field, one flux

$$\nabla \times \vec{e} + \frac{\partial \vec{b}}{\partial t} = 0$$

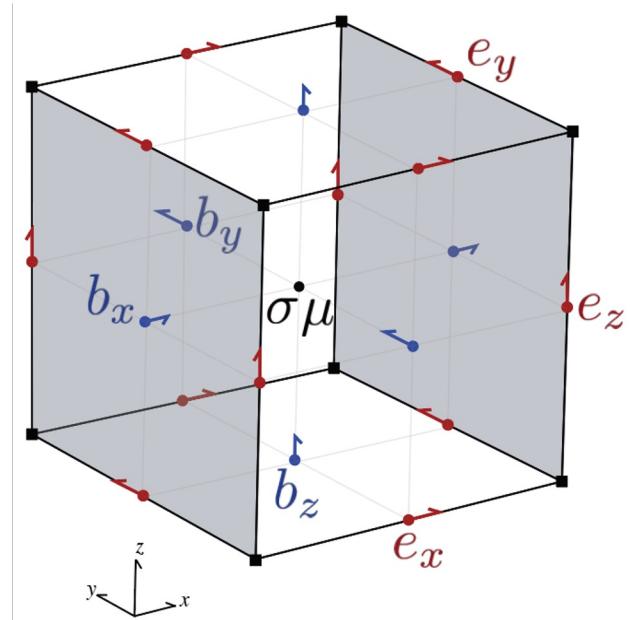
$$\nabla \times \mu^{-1} \vec{b} - \sigma \vec{e} = \vec{j}_s$$

Boundary conditions

$$\hat{n} \times \vec{b}|_{\partial\Omega} = 0$$

Staggered grid discretization

- Physical properties: cell centers
- Fields: edges
- Fluxes: faces



Solving the time domain equations

Semi-discretization in space

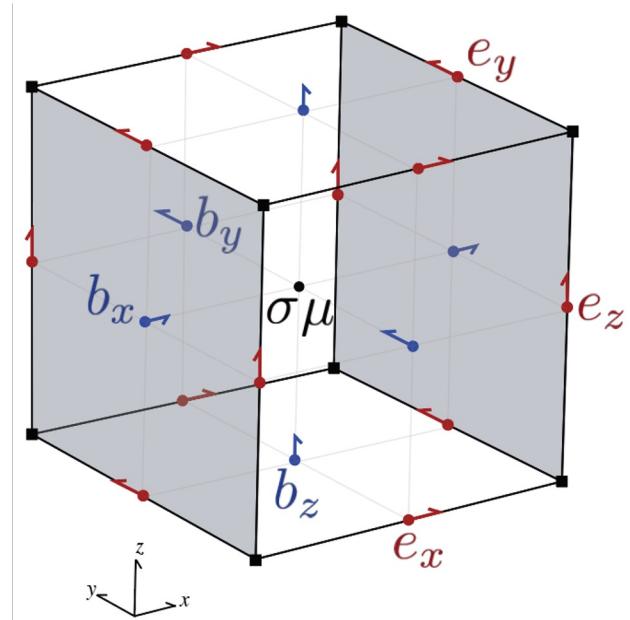
$$\mathbf{Ce} + \frac{\partial \mathbf{b}}{\partial t} = 0$$

$$\mathbf{C}^T \mathbf{M}_{\mu^{-1}}^f \mathbf{b} - \mathbf{M}_\sigma^e \mathbf{e} = \mathbf{M}^e \mathbf{j}_s$$

combine into a single, second order equation in \mathbf{e}

$$\mathbf{C}^T \mathbf{M}_{\mu^{-1}}^f \mathbf{Ce} + \mathbf{M}_\sigma^e \frac{\partial \mathbf{e}}{\partial t} = -\mathbf{M}^e \frac{\partial \mathbf{j}_s}{\partial t}$$

to solve, need to discretize in time...



Solving the time domain equations

First order backward difference (backward euler)

\mathbf{e}^{n+1} depends upon \mathbf{e}^n

$$\mathbf{C}^\top \mathbf{M}_{\mu^{-1}}^f \mathbf{C} \mathbf{e} + \mathbf{M}_\sigma^e \frac{\partial \mathbf{e}}{\partial t} = -\mathbf{M}^e \frac{\partial \mathbf{j}_s}{\partial t}$$

Time step: $\Delta t = t^{n+1} - t^n$

$$\mathbf{C}^\top \mathbf{M}_{\mu^{-1}}^f \mathbf{C} \mathbf{e}^{n+1} + \mathbf{M}_\sigma^e \frac{\mathbf{e}^{n+1} - \mathbf{e}^n}{\Delta t} = -\mathbf{M}^e \frac{\mathbf{j}_s^{n+1} - \mathbf{j}_s^n}{\Delta t}$$

Rearrange to solve for

$$\left(\mathbf{C}^\top \mathbf{M}_{\mu^{-1}}^f \mathbf{C} + \frac{1}{\Delta t} \mathbf{M}_\sigma^e \right) \mathbf{e}^{n+1} = \frac{1}{\Delta t} \mathbf{M}_\sigma^e \mathbf{e}^n - \frac{1}{\Delta t} \mathbf{M}^e (\mathbf{j}_s^{n+1} - \mathbf{j}_s^n)$$

Solving the time domain equations

Discrete equations

$$\underbrace{\left(\mathbf{C}^\top \mathbf{M}_{\mu^{-1}}^f \mathbf{C} + \frac{1}{\Delta t} \mathbf{M}_\sigma^e \right)}_{\mathbf{A}_{n+1}(\sigma, \Delta t)} \underbrace{\mathbf{e}^{n+1}}_{\mathbf{u}_{n+1}} = \underbrace{\frac{1}{\Delta t} \mathbf{M}_\sigma^e}_{\mathbf{B}_{n+1}(\sigma, \Delta t)} \underbrace{\mathbf{e}^n}_{\mathbf{u}_n} - \underbrace{\frac{1}{\Delta t} \mathbf{M}^e (\mathbf{j}_s^{n+1} - \mathbf{j}_s^n)}_{\mathbf{q}_{n+1}}$$

Arrange in a big matrix to solve

$$\begin{pmatrix} \mathbf{A}_0 & & & \\ \mathbf{B}_1 & \mathbf{A}_1 & & \\ & \mathbf{B}_2 & \mathbf{A}_2 & \\ & \ddots & \ddots & \\ & & \mathbf{B}_{n-1} & \mathbf{A}_{n-1} \\ & & & \mathbf{B}_n & \mathbf{A}_n \end{pmatrix} \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{n-1} \\ \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \vdots \\ \mathbf{q}_{n-1} \\ \mathbf{q}_n \end{pmatrix}$$

Solving the time domain equations

Solve with forward substitution

- Solve initial conditions for \mathbf{u}_0

$$\mathbf{A}_0 \mathbf{u}_0 = \mathbf{q}_0$$

- Propagate forward by solving

$$\mathbf{A}_{n+1} \mathbf{u}_{n+1} = -\mathbf{B}_n \mathbf{u}_n + \mathbf{q}_{n+1}$$

$$\begin{pmatrix} \mathbf{A}_0 & & & & \\ \mathbf{B}_1 & \mathbf{A}_1 & & & \\ & \mathbf{B}_2 & \mathbf{A}_2 & & \\ & & \ddots & \ddots & \\ & & & \mathbf{B}_{n-1} & \mathbf{A}_{n-1} \\ & & & & \mathbf{B}_n & \mathbf{A}_n \end{pmatrix} \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{n-1} \\ \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \vdots \\ \mathbf{q}_{n-1} \\ \mathbf{q}_n \end{pmatrix}$$

- Symmetric
- Need to solve many times
- Only changes if $\sigma, \Delta t$ changes → store factors & re-use

$$\underbrace{\left(\mathbf{C}^\top \mathbf{M}_{\mu^{-1}}^f \mathbf{C} + \frac{1}{\Delta t} \mathbf{M}_\sigma^e \right)}_{\mathbf{A}_{n+1}(\sigma, \Delta t)}$$

Frequency and Time domain simulations

Frequency

$$\mathbf{A}(\sigma, \omega)$$

- Complex, symmetric
- Factor for each frequency
- Inversion: sensitivity derivation straight-forward

Time

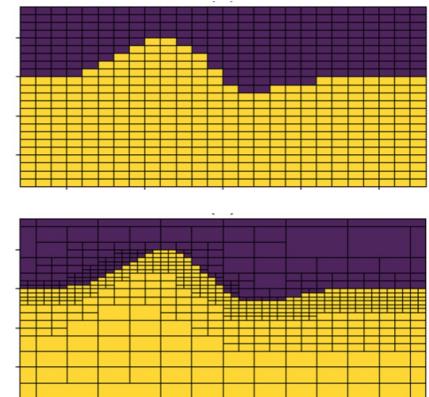
$$\mathbf{A}_{n+1}(\sigma, \Delta t)$$

- Real, symmetric
- Factor for each unique Δt
- Inversion: sensitivity derivation more involved

Setting up a forward simulation & user decisions

Mesh design

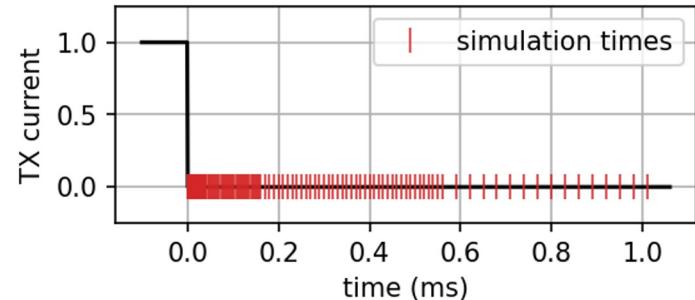
- minimum cell size
 - capture shortest time / highest frequency, highest conductivity
- domain extent
 - beyond skin depth / diffusion distance



Time discretization

- capture short-timescale variations near shut-off
 - needs to be finer than receiver times
- extend to latest time channel

trade-off between computation cost and accuracy



Forward modelling can address

Survey design:

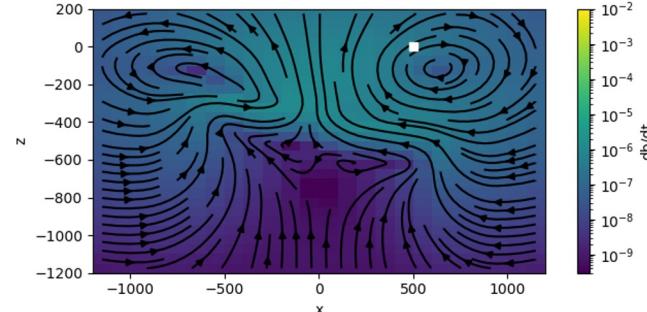
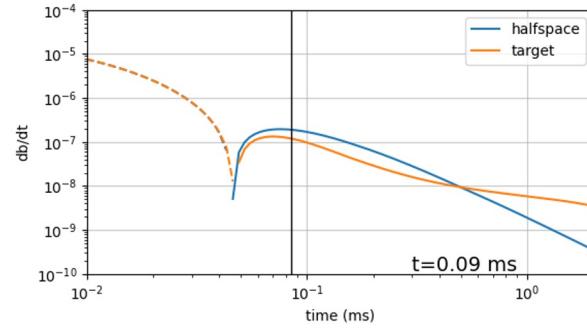
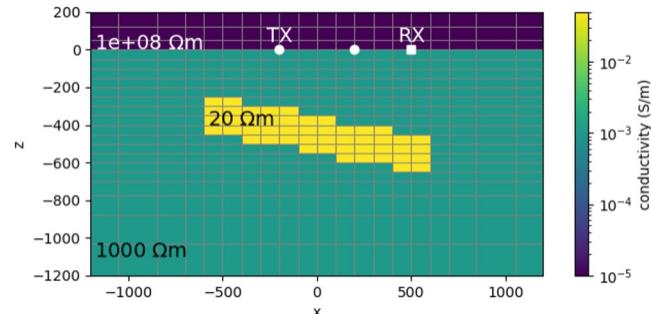
- source: coupling? base frequency?
- receivers: best to measure e , b , db/dt ?

Complexities:

- overburden
- multiple targets, interactions?

Value in looking at:

- currents, magnetic field
- and simulated data



Practical notes

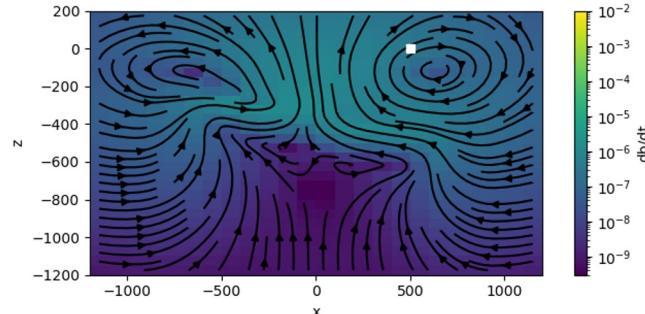
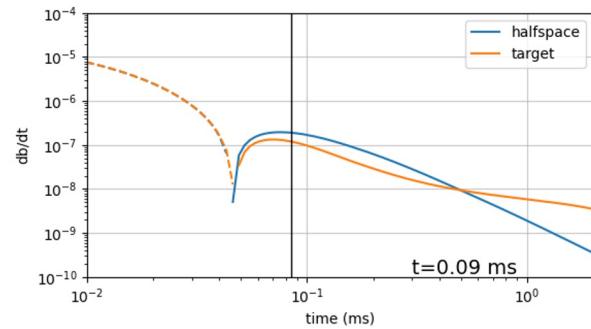
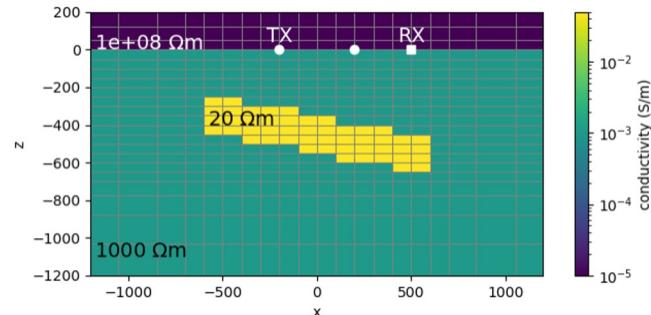
Computational time for this example

Tensor Mesh with...

- 62,000 cells
- 170 time steps, 4 unique Δt
- 1 source
→ **Forward simulation time: 57s**

Refine the mesh (~double number of cells)

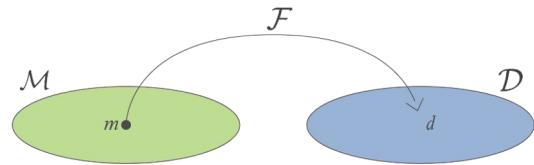
- 128,000 cells
- same time stepping
→ **Forward simulation time: 136s**



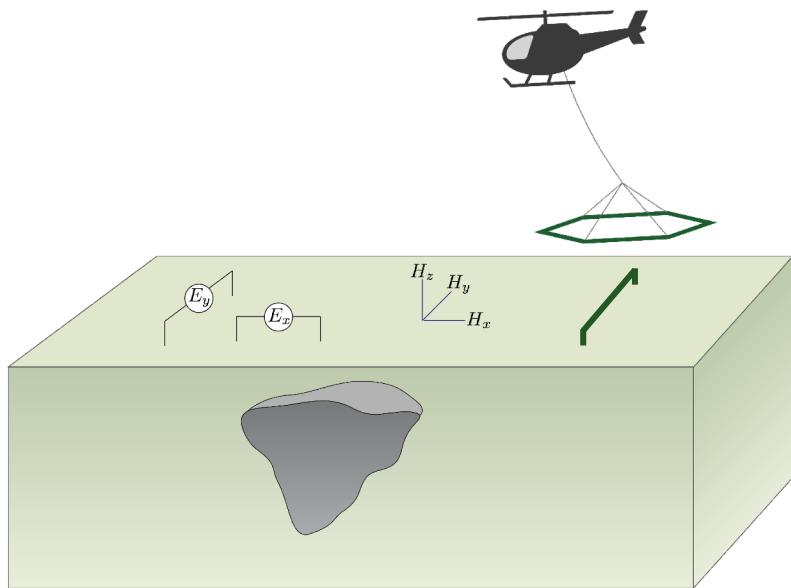
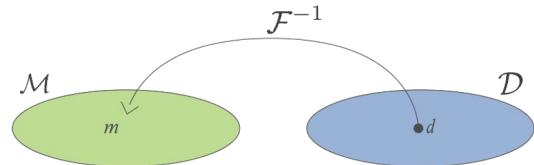
Statement of the inverse problem

Given

- observations: d_j^{obs} , $j = 1, \dots, N$
- uncertainties: ϵ_j
- ability to forward model: $\mathcal{F}[m] = d$



Find the Earth model that gave rise to the data



Need a framework for inverse problem

Tikhonov (deterministic)

Find a single “best” solution by solving optimization

$$\text{minimize} \quad \phi = \phi_d + \beta \phi_m$$

subject to $m_L < m < m_H$

$\left\{ \begin{array}{l} \phi_d: \text{data misfit} \\ \phi_m: \text{regularization} \\ \beta: \text{trade-off parameter} \\ m_L, m_H: \text{lower and upper bounds} \end{array} \right.$

Bayesian (probabilistic)

Use Bayes’ theorem

$$P(m|d^{obs}) \propto P(d^{obs}|m)P(m)$$

$\left\{ \begin{array}{l} P(m): \text{prior information about } m \\ P(d^{obs}|m): \text{probability about the data errors (likelihood)} \\ P(m|d^{obs}): \text{posterior probability for the model} \end{array} \right.$

Two approaches:

- Characterize $P(m|d^{obs})$
- Find a particular solution that maximizes $P(m|d^{obs})$
MAP: (maximum a posteriori) estimate

Flow chart for the inverse problem

What do we need for inversion?

$$\text{minimize } \phi = \phi_d + \beta\phi_m$$

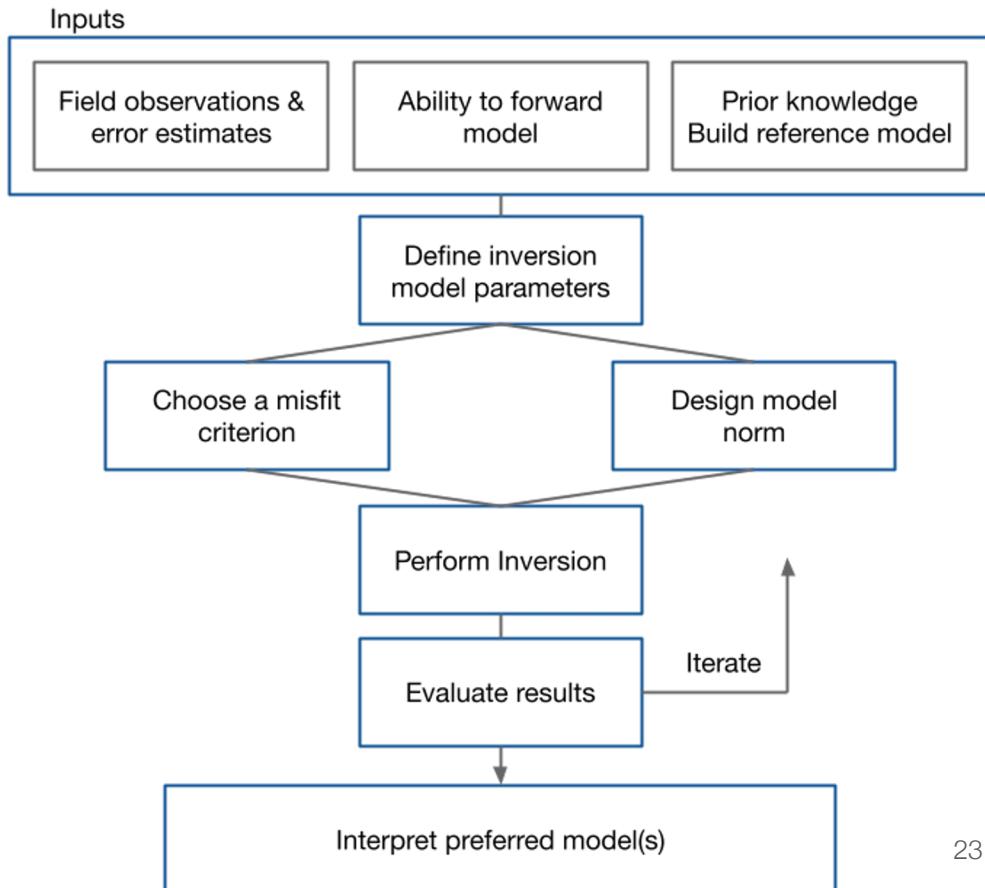
$$\text{subject to } m_L < m < m_H$$

From the simulation

- adjoint sensitivity times a vector
- sensitivity times a vector

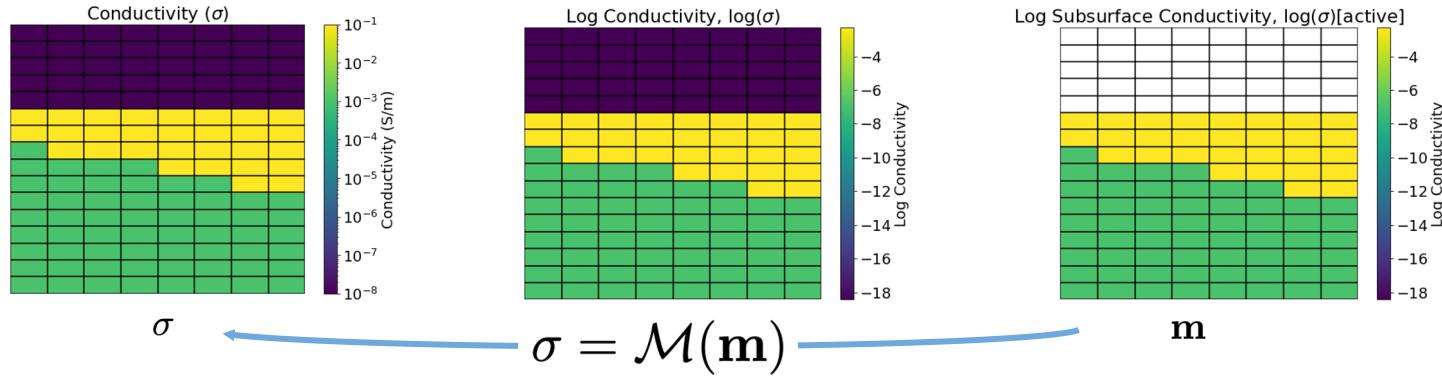
Inversion components:

- define a model norm
- perform optimization

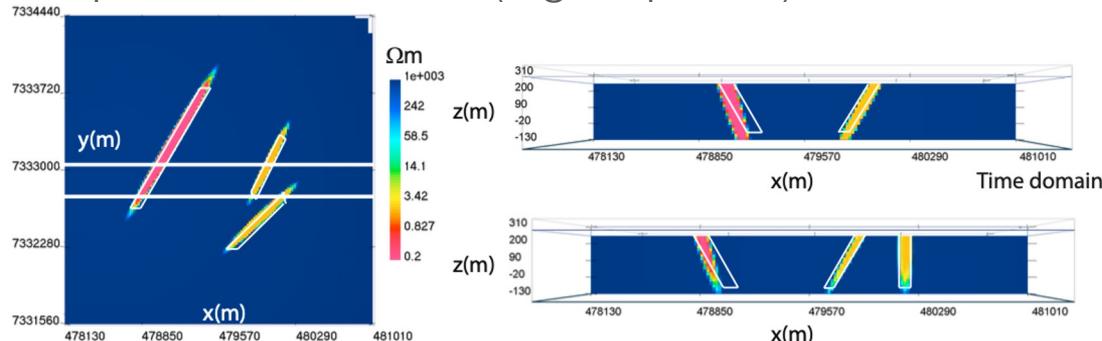


Defining the model model parameters

typically log conductivity below the surface of the earth



can also work with parametric models (e.g. ellipsoids)



Sensitivities

For inverse problem, also need sensitivities (and adjoint)

$$\begin{aligned}\mathbf{J} &= \frac{\partial \mathbf{d}^{\text{pred}}}{\partial \mathbf{m}} \\ &= \frac{\partial \mathbf{P}(\mathbf{u}, \omega)}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{m}}\end{aligned}$$

where the derivative of the fields (\mathbf{u}) is computed implicitly (requires a solve)

$$\frac{\partial \mathbf{A}(\sigma, \omega) \mathbf{u}^{\text{fixed}}}{\partial \mathbf{m}} + \mathbf{A}(\sigma, \omega) \frac{\partial \mathbf{u}}{\partial \mathbf{m}} = 0$$

\mathbf{J} is a large, dense matrix → compute products with a vector if memory-limited

Inversion as an optimization problem

$$\min_{\mathbf{m}} \phi(\mathbf{m}) = \phi_d(\mathbf{m}) + \beta \phi_m(\mathbf{m})$$

$$\text{s.t. } \phi_d \leq \phi_d^* \quad \mathbf{m}_L \leq \mathbf{m} \leq \mathbf{m}_U$$

data misfit

$$\phi_d = \|\mathbf{W}_d(\mathcal{F}(\mathbf{m}) - \mathbf{d}^{\text{obs}})\|^2$$

uncertainties captured in \mathbf{W}

$$\mathbf{W}_d = \text{diag}\left(\frac{1}{\epsilon}\right)$$

$$\epsilon_j = \%|d_j^{\text{obs}}| + \text{floor}$$

model norm (L2)

$$\phi_m = \alpha_s \int_V w_s (m - m_{\text{ref}})^2 dV + \alpha_x \int_V w_x \frac{d(m - m_{\text{ref}})}{dx}^2 dV$$



$$\phi_m = \alpha_s \|\mathbf{W}_s(\mathbf{m} - \mathbf{m}_{\text{ref}})\|^2 + \alpha_x \|\mathbf{W}_x(\mathbf{m} - \mathbf{m}_{\text{ref}})\|^2$$

Gauss Newton approach

$$\min_{\mathbf{m}} \phi(\mathbf{m}) = \phi_d(\mathbf{m}) + \beta \phi_m(\mathbf{m})$$

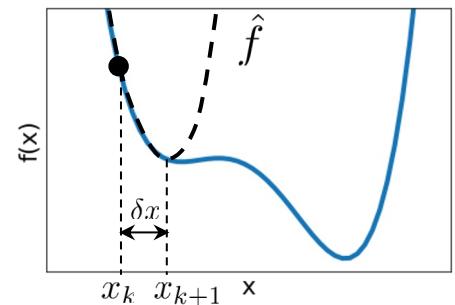
$$\text{s.t. } \phi_d \leq \phi_d^* \quad \mathbf{m}_L \leq \mathbf{m} \leq \mathbf{m}_U$$

Gradient

$$\mathbf{g}(\mathbf{m}) = \mathbf{J}^\top \mathbf{W}_d^\top \mathbf{W}_d (F[\mathbf{m}] - \mathbf{d}^{obs}) + \beta \mathbf{W}_m^\top \mathbf{W}_m (\mathbf{m} - \mathbf{m}_{ref})$$

Taylor expand → Gauss newton equations (solve with CG)

$$(\mathbf{J}^\top \mathbf{W}_d^\top \mathbf{W}_d \mathbf{J} + \beta \mathbf{W}_m^\top \mathbf{W}_m) \delta \mathbf{m} = -\mathbf{g}(\mathbf{m})$$



Sensitivity (**J**) requires forward modelling

- large, dense matrix (expensive to store)
- compute matrix-vector products when memory-limited $\mathbf{J}\mathbf{v}, \mathbf{J}^\top \mathbf{v}$

Practical notes

One inversion iteration requires we solve:

$$\mathbf{g}(\mathbf{m}) = \underline{\mathbf{J}^\top \mathbf{W}_d^\top \mathbf{W}_d} (\underline{F[\mathbf{m}] - \mathbf{d}^{obs}}) + \beta \mathbf{W}_m^\top \mathbf{W}_m (\mathbf{m} - \mathbf{m}_{ref})$$

$$(\underline{\mathbf{J}^\top \mathbf{W}_d^\top \mathbf{W}_d \mathbf{J}} + \beta \mathbf{W}_m^\top \mathbf{W}_m) \delta \mathbf{m} = -\mathbf{g}(\mathbf{m})$$

Number of forward calculations per iteration:

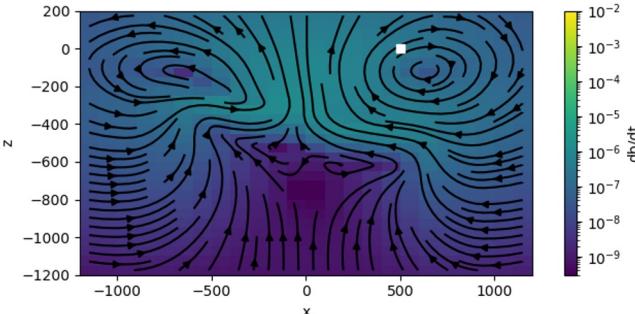
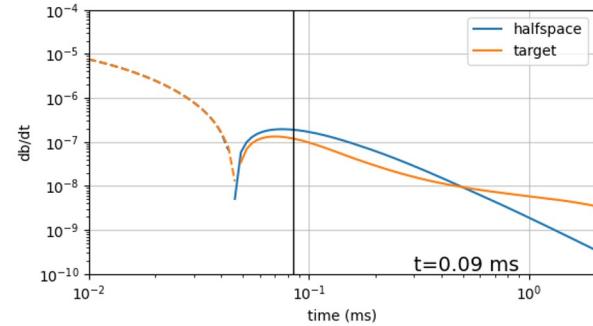
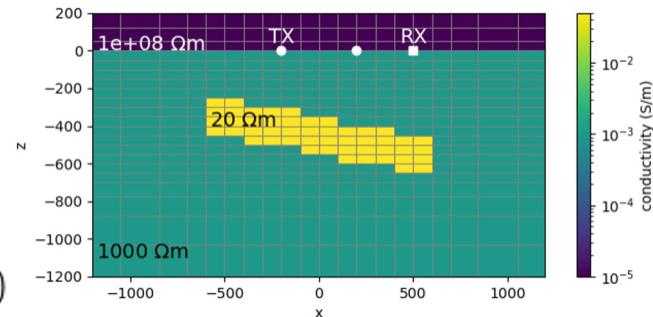
$$2(N_{CG} + 1) \sim 20$$

62,000 cells $\sim 57\text{s}$

Total computation time:

$$N_{iter} \cdot 2(N_{CG} + 1) \cdot T$$

~ 10 ~ 20 $\sim 1 \text{ min (small prob)}$
 $\sim 3.5 \text{ hours}$



3D inversions with EM

Challenges: computational cost

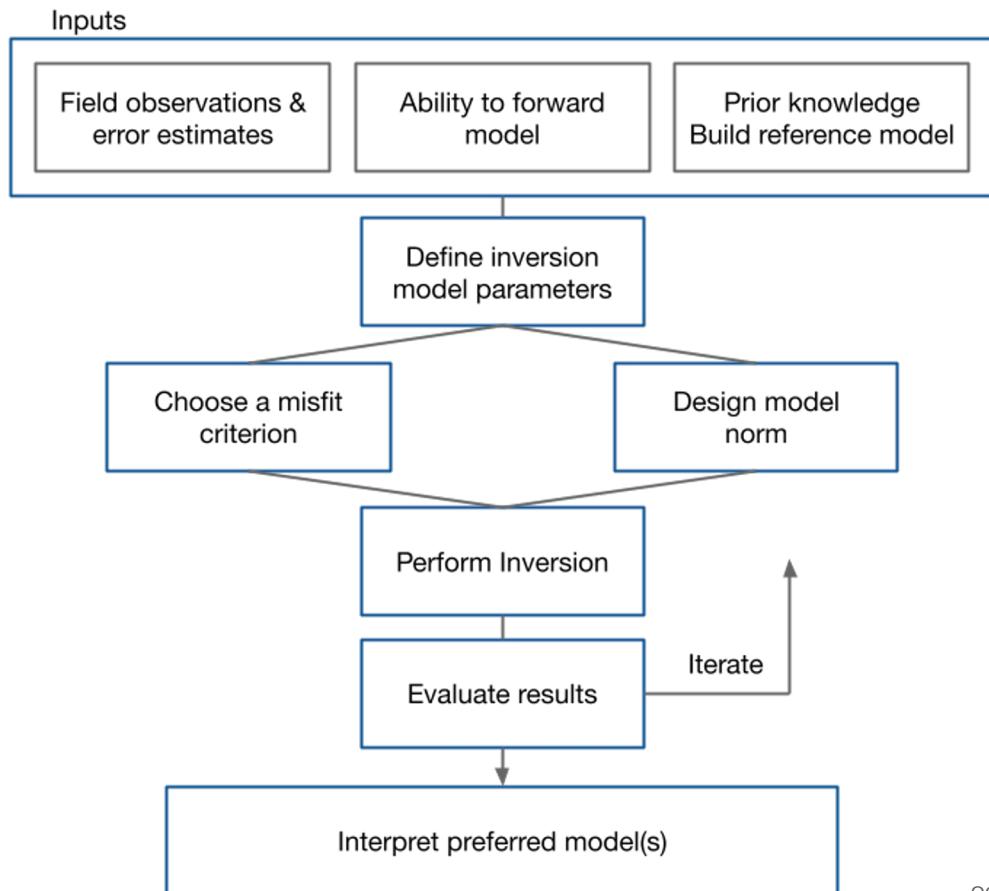
- need many forward simulations to run an inversion

Non-uniqueness

- all inversions non-unique
- impacted by survey geometry

Role of a-priori information

- reference models
- model parameterization
- ...



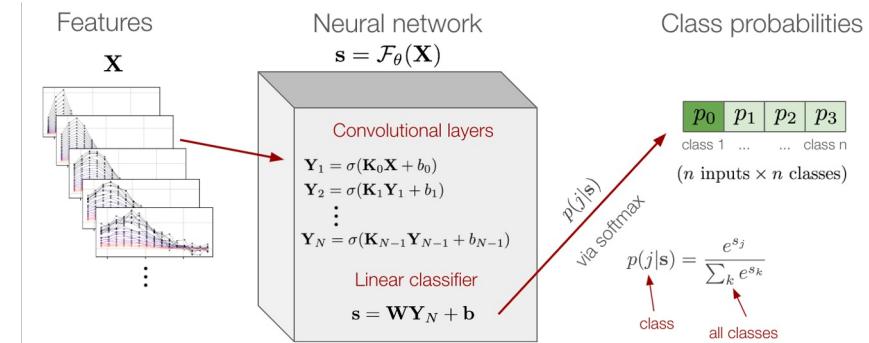
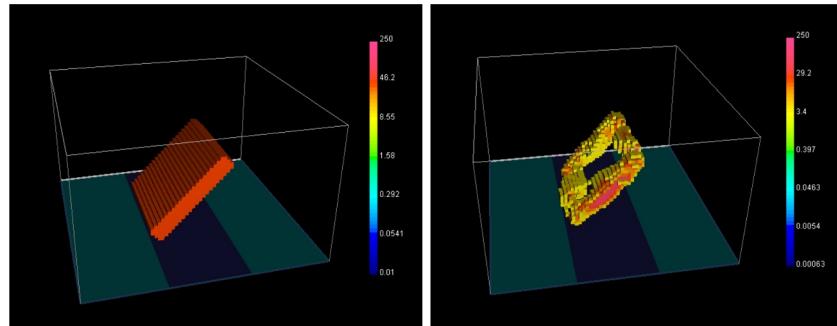
Research avenues

3D inversion with highly conductive targets

- Challenging sensitivities, modelling
- Role for parametric inversions

Incorporating in a-priori information

- Different norms in the regularization
- Petrophysically and Geologically guided inversions (PGI)
- Opportunities for machine learning



SimPEG

```
[1]: from simpeg import electromagnetics
```

forward simulation and inversion tools

- frequency and time
- 1D, 3D, cylindrically symmetric

inversions

- standard L2
- different norms, PGI
- parametric approaches

The screenshot shows a web browser displaying the official website for SimPEG. The header features a green navigation bar with links for WHY, ABOUT, CONTACT, and CONTRIBUTE. Below the header is a stylized 3D cube composed of colored cubes (purple, blue, green, red). The main content area has a light gray background. It includes the title "Simulation and Parameter Estimation in Geophysics" and a brief description: "An open source python package for simulation and gradient based parameter estimation in geophysical applications." A section titled "Geophysical Methods" lists various methods: Gravity, Magnetics, Direct current resistivity, Induced polarization, Electromagnetics (with sub-options for Time domain, Frequency domain, and Natural source (e.g., magnetization)), and Inversion. To the right of this text is a 3D visualization of a geological model within a wireframe grid. The model shows various subsurface structures with color-coded properties (blue, red, green) and includes a grid of small dots representing data points or sensors.

<https://simpeg.xyz>

Resources

GIF tools cookbook

SimPEG user tutorials

The screenshot shows a documentation page for the "GIF tools cookbook". The top navigation bar includes links for "Docs", "10. Comprehensive Workflows", "10.7.1. Understanding Surface UTEM Anomalies", "GIF Documentation", and "Edit on GitHub". The main content area is titled "10.7.1. Understanding Surface UTEM Anomalies". It discusses the importance of understanding what fields are measured, how data are plotted, and anomalies for basic structures. It mentions investigating a UTEM anomaly produced by a block in a half-space, determining coordinate systems, and sign conventions for field collected data, and operations required to transform raw data into UBC-GIF convention.

10.7.1.1. Introduction to Surface UTEM

10.7.1.1.1. Survey Geometry and Fundamental Concept

During a UTEM survey, large inductive loop sources are placed on the Earth's surface. Depending on how each loop is coupled with the target, receivers may be located inside or outside the loop. Data may be collected for several sets of loops and receivers in order to excite the target from multiple directions.

Survey geometry for target near center of the loop (left) and near edge of the loop (right).

During field data collection, standard UTEM systems measure directional components of a highly regulated triangular waveform; see below. However, this is equivalent to measuring the directional components of $B(t)$ for a corresponding step-like waveform. The principal idea of the

Left sidebar navigation:

- 1. Quick-start guide
- 2. Installation
- 3. Basic Functionality
- 4. Object Functionality
- 5. UBC File formats
- 6. Fundamentals of Inversions
- 7. Coordinates, Sign Conventions and Units: A Quick Guide
- 8. Recipes
- 9. A to Z Examples

10. Comprehensive Workflows

- 10.1. Magnetics: From surface TMI data to sparse-norm inversion
- 10.2. DCIP: From UBC-GIF/XYZ to DC and IP inversion
- 10.3. MT: From EDI data to inversion
- 10.4. ZTEM: From XYZ data to inversion
- 10.5. MobileMT: From XYZ data to inversion
- 10.6. Joint MT-ZTEM inversion

10.7. Surface UTEM: From AMIRA TEM data file to inversion (ongoing research)

- 10.7.1. - Understanding UTEM anomalies
 - 10.7.1.1. Introduction to Surface UTEM
 - 10.7.1.2. Plotting UTEM Data

<https://giftoolscookbook.readthedocs.io>

The screenshot shows a "User Tutorials" page for SimPEG. The top navigation bar includes links for "Search", "K", "Learn More", and icons for CC BY, CC SA, and a download arrow. The main content area is titled "3D Forward Simulation for On-Time Large-Loop Data". It is authored by Devin Cowan from the University of British Columbia. The text explains that the chapter publishes code comparisons and validations for time-domain electromagnetic modeling packages, using the `simpeg.electromagnetics.time_domain` module for modeling time-domain data.

In this chapter, we publish code comparisons and validations for time-domain electromagnetic modeling packages. In SimPEG, the `simpeg.electromagnetics.time_domain` module is used for modeling time-domain data.

On-Time Currents and Fields

Off-Time Currents and Fields

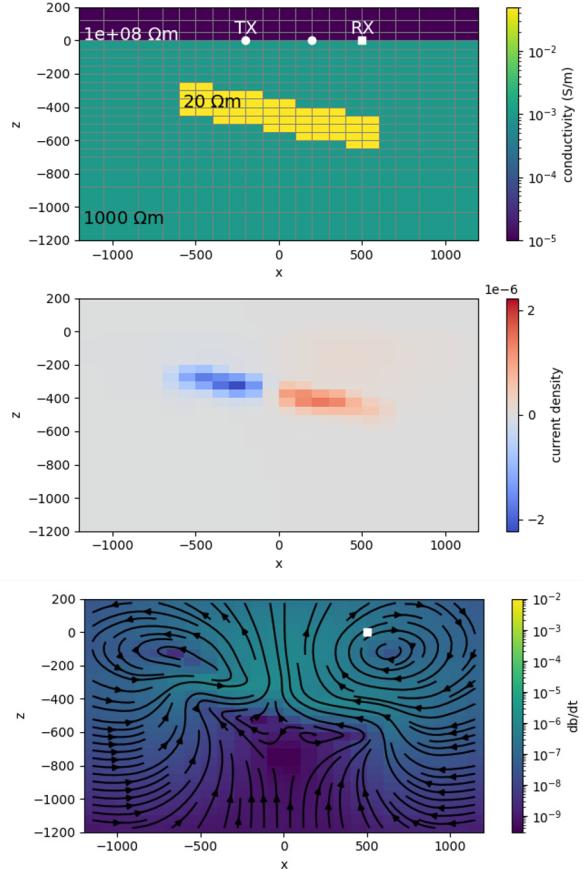
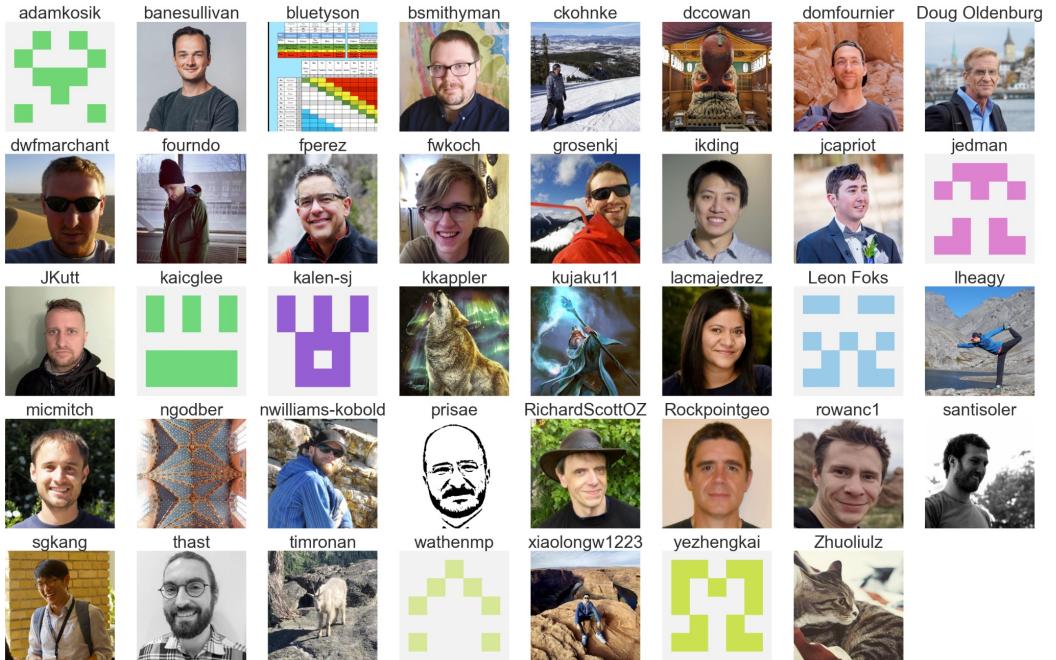
Schematic illustrating the physics of the TEM method for an inductive source.

<https://simpeg.xyz/user-tutorials>

Thank you!



simpeg



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