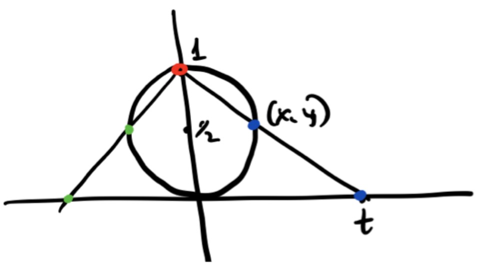
Projective geometry

A story that started 2400 years ago;
connects Euclid, Pappus, Pascal, and
the modern times.

Warm-up : stereographic projection.



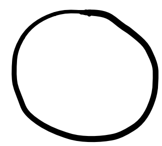
Problem 1 :
find (x, y) in terms
of t.

How to make a new mathematical object?

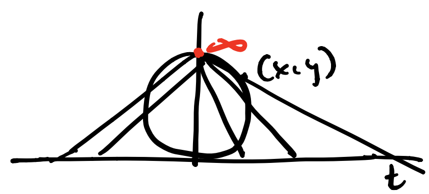
Axioms	Geometric realizations	Coordinates

projective line

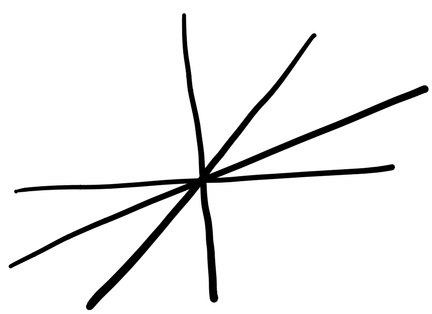
"a line + a point at ∞"

i) 

circle



2) The space of lines in the plane:



each line in the plane is a "point" on the projective line

$(x:y)$:

$(x:y) \sim (\lambda x, \lambda y)$

projective coordinates

Now, you can take the coordinates from any field [a set with addition and multiplication]

For example, $\mathbb{P}^1(\mathbb{R})$, $\mathbb{P}^1(\mathbb{F}_5)$

see problem set

So, $\mathbb{P}^1(\mathbb{R})$ is :

$$\{(x:y) \mid x, y \in \mathbb{R}\}/\sim$$

$$=$$

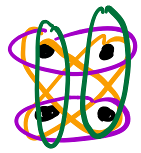
$\uparrow$ try to find a <u>representative</u>

The projective plane

- A collection of points

  With a collection of selected subsets, called lines

Example:



4 points, 6 lines

(here we have: every 2-point subset is selected)

an "affine plane over $\mathbb{F}_2$"

Must satisfy the axioms:

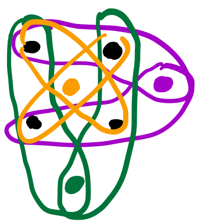
- There is exactly one line containing any given pair of points

- Any two lines have a common point

(4^{th} century BC)

Realizations

1) For our "plane" of 4 points:



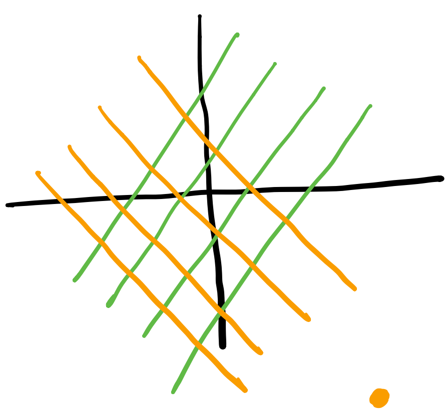
We added the intersection points of all pairs of lines

Problem: check that this satisfies the new axioms.

This is the smallest projective plane!

(Exer: assign projective coordinates to every point!)

2) What about the real projective plane?



• ← add "a point at ∞" for every family of parallel lines on the plane.

What would that look like?

Better construction:

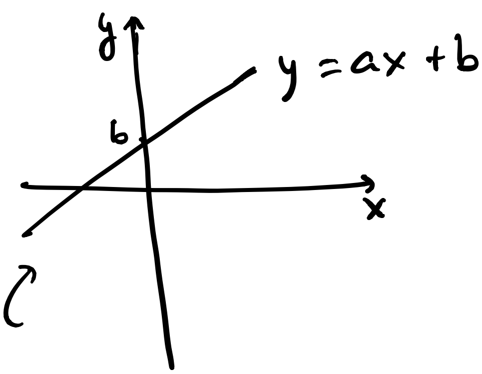
Make every line in $\mathbb{R}^3$ a point in this projective plane.

In coordinates: $(x:y:z)$

$$(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$$
for any $\lambda \in \mathbb{R}$.

What happens to some familiar equations?

- lines:



the line connecting $(0,b)$ with the point at ∞ given by a.

$(x:y:z)$ $\rightsquigarrow \left(\frac{x}{z}, \frac{y}{z}\right)$ familiar coordinates

Its equation: $(x:y:z)$ such that

$$\frac{y-b}{a} = x$$

$$y - b = ax$$

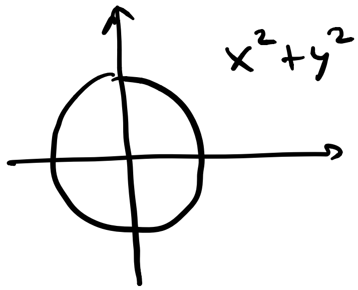
$$y - bz = ax$$

$$\frac{y}{z} = a\frac{x}{z} + b$$

$$y = ax + bz$$

given by homogeneous linear equations

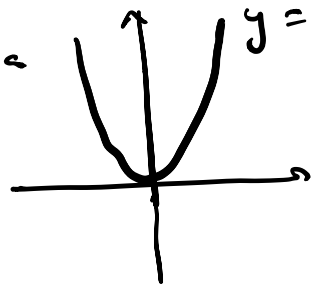
- (circles), ellipses, parabolas, hyperbolas:



$x^2+y^2=1$

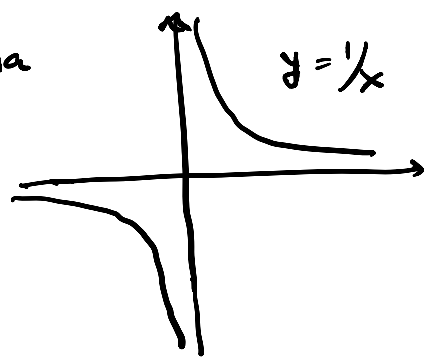
- special case of an <u>ellipse</u> $a^2x^2+b^2y^2=1$

- Parabola



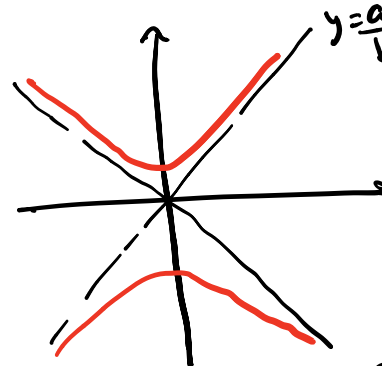
$y=ax^2$

- hyperbola



$y = 1/x$

$xy=1$



$y=\frac{a}{b}x$

$y=-\frac{a}{b}x$

$$\left(y-\frac{a}{b}x\right)\left(y+\frac{a}{b}x\right) = 1$$

$$y^2-\frac{a^2}{b^2}x^2=1$$

$$b^2y^2-a^2x^2=1$$

$$a^2x^2 + y^2b^2 = \underbrace{c^2z^2}$$

make it projective (homogeneous)

can go back to the ellipse by saying $x' = \frac{x}{z}$

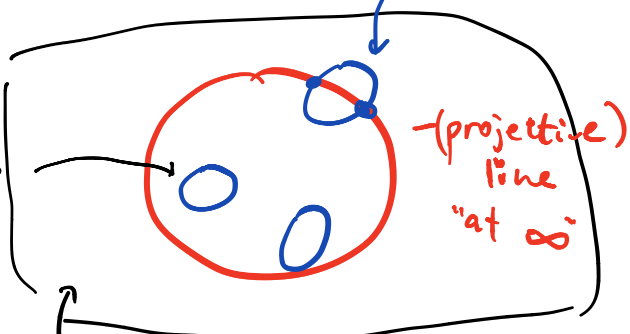
$y' = \frac{y}{z}$

becomes a hyperbola

(2 points at ∞ - corresponding to the asymptotes:

$y = \pm \frac{a}{b} x$

(every slope gives a point at ∞)



becomes an ellipse (no point at ∞)

projective plane

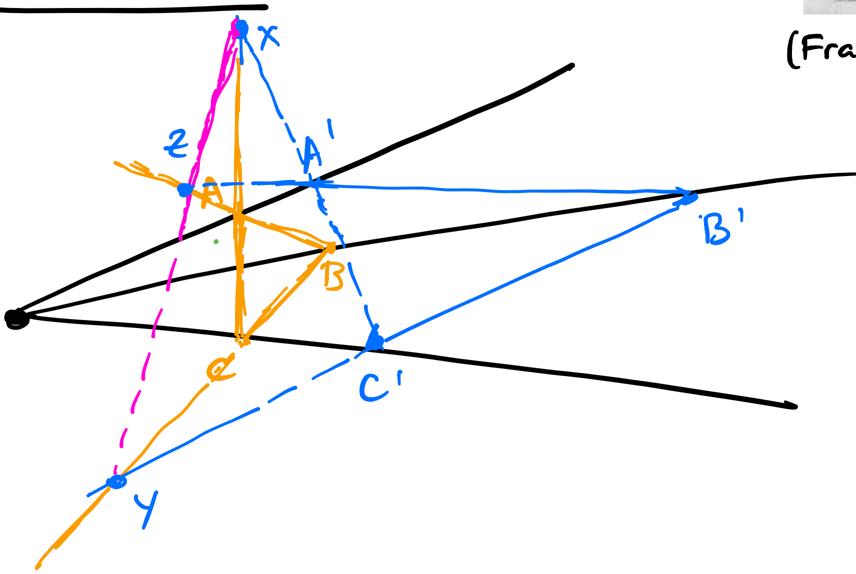
The cool stuff you can do with it

Girard Desargues

(France, 1591-1661)

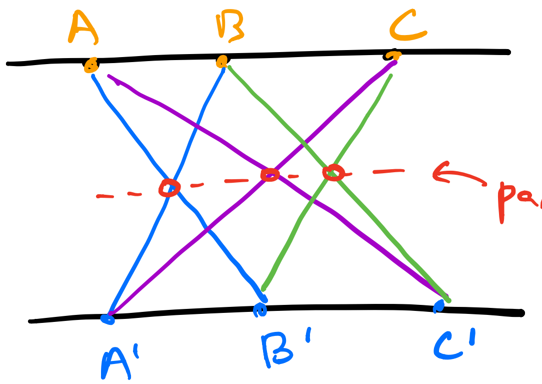
Desargues' Theorem

Start with 3 lines passing through a point



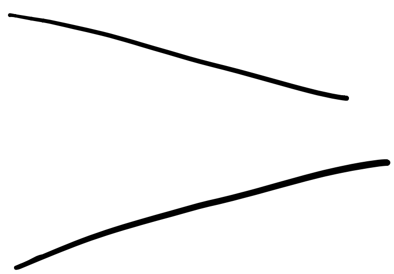
Pappus' Theorem

Pappus of Alexandria (~290 - 350AD)



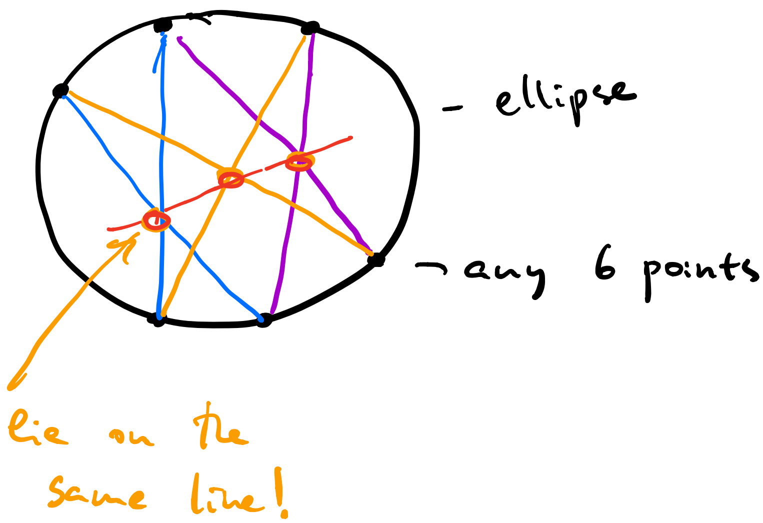
← parallel lines

← parallel to the black lines



← if these lines intersect, the red line passes through the intersection point.

Pascal's Hexagrammum Mysticum

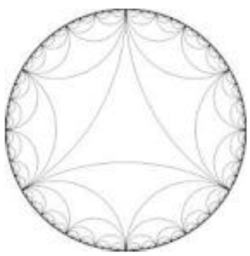


(Blaise Pascal, France, 1623–1662)

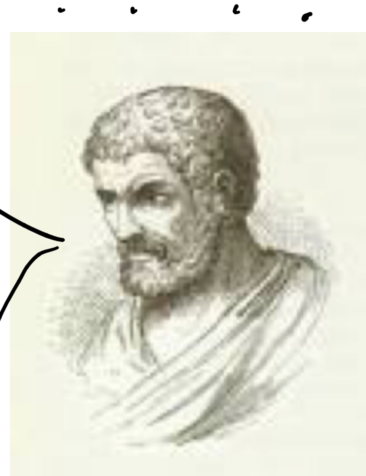
Pappus' Theorem is a special case of Pascal's Theorem!

Now what if instead of <u>NO</u> parallel lines, we require <u>two</u> 'parallel' lines through a given point?

Lobachewski, Minkowski, Einstein ...



?! ?!



How dare you?!