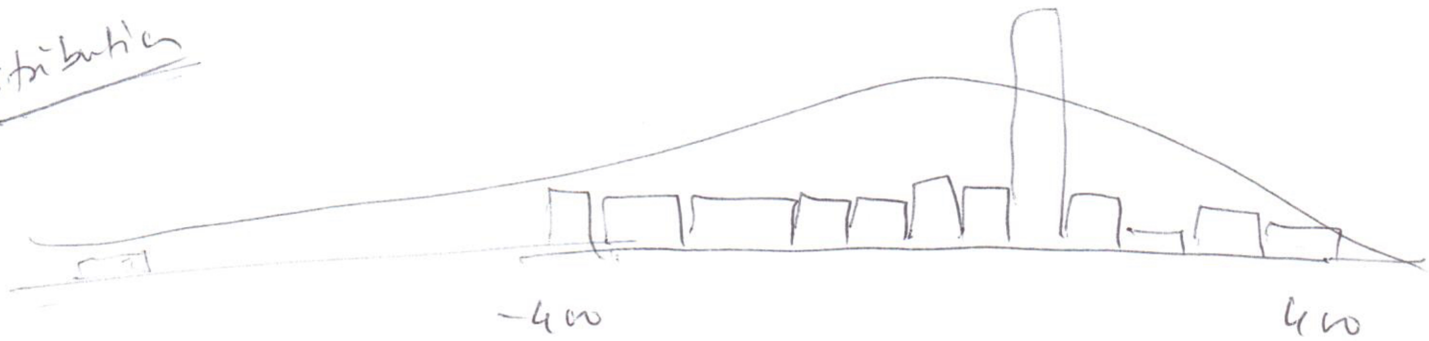
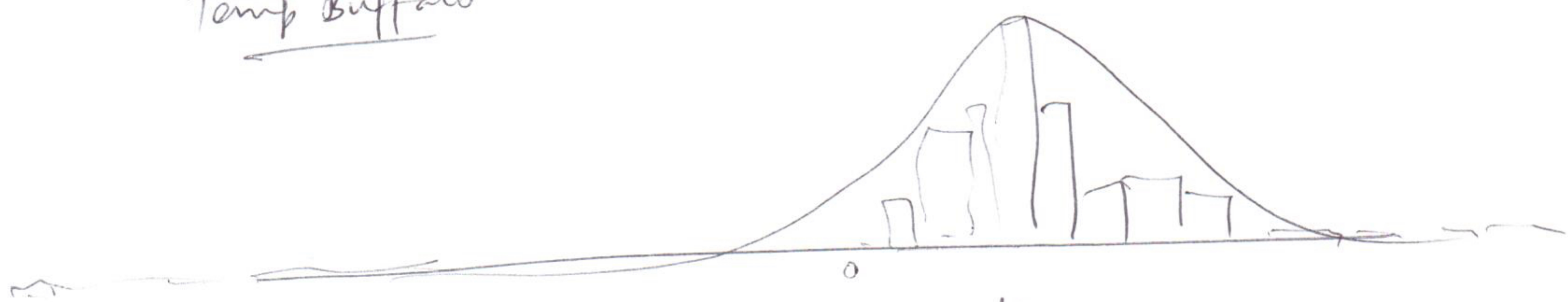
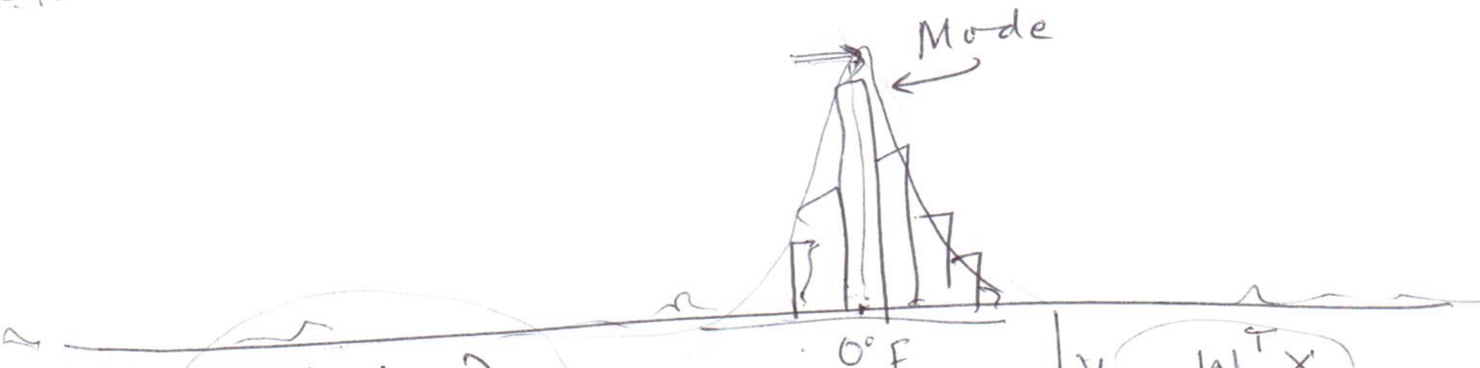
$\rightarrow$ -400 400

Distribution



Temp Buffalo





$P(y|x)$

$y = W^T x$

Coin

$P(\text{face is heads})$ $P(H)$

$P(\text{face is tails})$ $P(T)$ $P(H)+P(T)=1$

$P(H) = P(T) = 0.5$

Random variable

X

"Categorical"

$X \in$ one out of k categories.

"Continuous random variable"

$X \in$ min, max

$[-400, 400)$

$[-\infty, \infty]$

$[0, 1]$

$\boxed{\text{For any r.v. we have a probability distribution}}$

Joint random variable - Multivariate

Categorical

$X \rightarrow \{1, 2, 3, \ldots, k\}$

$$\boxed{\sum_{i=1}^{k} P(X=i) = 1}$$

$P(X=1)$
$P(X=2)$
$P(X=3)$ → Probability Distribution
$P(X=4)$
$\vdots$
$P(X=k)$

Cointoss :

$P(X=h) = 0.5$
$P(X=t) = 0.5$

$P(X=1) = 1/6$
$P(X=2) = 1/6$
$\vdots$
$P(X=6)$ $1/6$

$P(X=1)$ $1/3$
$P(X=2)$ $1/2$
$P(X=3)$ $1/6$

Binomial If I toss a coin ~~(heads)~~ n times, the number of heads I will get out of n

X : R.V = # heads out of n tosses

prob of heads for a single toss = p

X is ~~a~~ categorical and it can take values ~~from~~
$\{0, 1, \ldots, n\}$

Prob. distribution

$$\boxed{\begin{array}{c} P(X=0) \\ P(X=1) \\ \vdots \\ P(X=n) \end{array}}$$

$P(X = 0.3)$ $\qquad$ $P(\cancel{Y} = 0.4)$

$P_X(0.3)$ $\qquad$ $P_Y(0.3)$

$P(X = 40)$

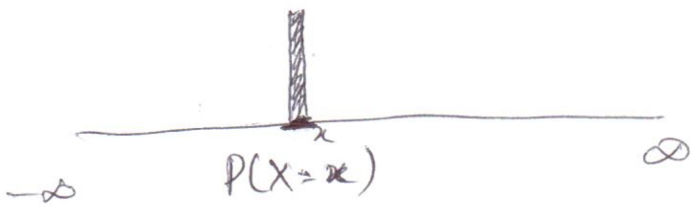
$$\boxed{\sum_{x \in -\infty}^{\infty} P(X=x) = 1}$$

$X \in \{-\infty, \infty\}$

$$\boxed{\int_{x=-\infty}^{\infty} P(X=x)\,dx = 1}$$

Probability Density — Continuous r.v.

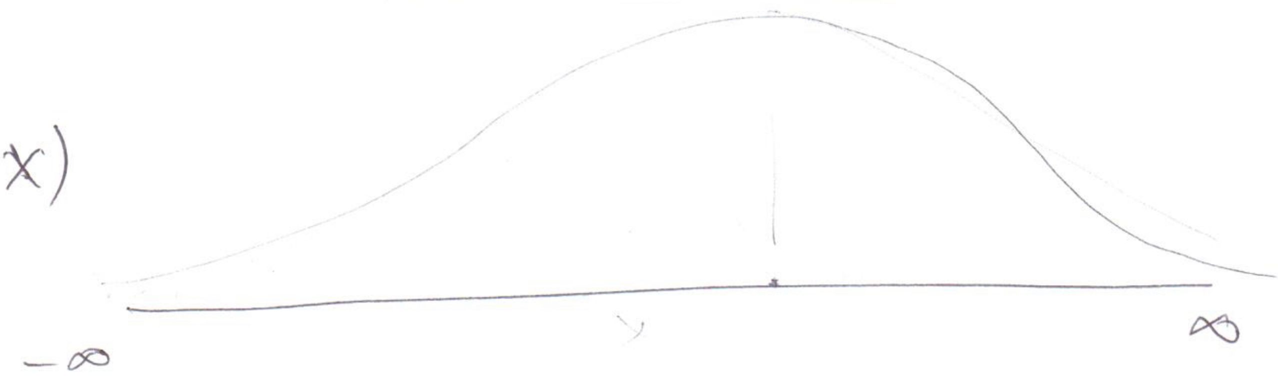
~~P(X x~~



$P(X > x - dx \text{ and } X < x + dx)$

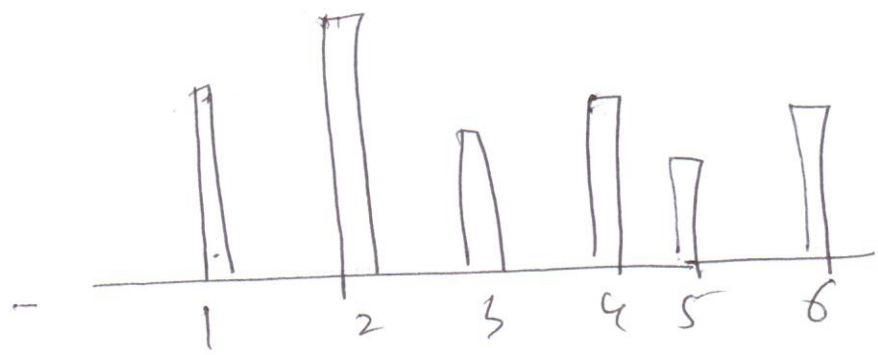
→ probability density at x $\quad \boxed{x \quad p(X=x) = p(x)}$

p(x)



Categorical

P(X)



X Y

$P(X=2, Y=3)$

Marginal → $P(X=3)$

$P(X=5 \mid Y=3)$

$P(X=3, \mid Y=1)$
$P(X=3, \mid Y=2)$
$P(X=3, \mid Y=3)$

Probability

Consider two random variables X $\{1, 2\}$ $\quad Y$ $\{1,2,3\}$

Prob. Distribution of X

0.7	0.3
$P(X=1)$	$P(X=2)$

Prob. Distr of Y

0.1	0.6	0.3
$P(Y=1)$	$P(Y=2)$	$P(Y=3)$

Joint prob. distribution $\underline{P(X, Y)}$

	Y=1	Y=2	Y=3
X=1	~~0.3~~	~~0.4~~	~~0.3~~
X=2	~~Y=1~~		

	Y=1	Y=2	Y=3
X=1	0.1	0.4	0.1
X=2	0.2	0.1	0.1

Marginal Distribution

$$P(X=1) = P(X=1|Y=1) + P(X=1|Y=2) + P(X=1|Y=3)$$
$$= 0.6$$
$$P(X=2) = 0.4$$

0.6	0.4

Conditional Distribution

$X|Y=1$ $\quad X|Y=2$ $\quad X|Y=3$

$$P(X=1|Y=1) = \boxed{0.1}$$
$$P(X=2|Y=1) = \boxed{0.2}$$

$Y=1 \mid X=2$

Bayes Rule

$$P(X=1 \mid Y=1) = \frac{P(Y=1 \mid X=1)P(X=1)}{P(Y=1)}$$

$$\boxed{P(\cancel{Y}=\cancel{y} \mid X=\cancel{x}) = \frac{P(X=x \mid \cancel{Y}=y)\,P(X=x)}{P(Y=y)}}$$

$P(Y=y \mid X=x)$

$$P(X=1, Y=1) = P(X=1)P(Y=1 \mid X=1)$$

$$P(Y=1 \mid X=1) = \frac{P(X=1, Y=1)}{P(X=1)}$$

$$P(X=1, Y=1) = P(Y=1)P(X=1 \mid Y=1)$$

$$P(Y=1 \mid X=1) = \frac{P(X=1 \mid Y=1)P(Y=1)}{P(X=1 \mid Y=1)P(Y=1) + P(X=1 \mid Y=0)P(Y=0)}$$

$$P(Y=0 \mid X=1) = \frac{P(X=1 \mid Y=0)P(Y=0)}{P(X=1 \mid Y=1)P(Y=1) + P(X=1 \mid Y=0)P(Y=0)}$$

$P(X=1 \mid Y=1) = 0.8$ ← Accuracy of the medical test.

$P(Y=1) = 0.004$ ← Prior prob of having cancer

$P(X=1 \mid Y=0) = 0.1$ ← False alarm rate of the test.

$P(Y=0) = 1 - P(Y=1) = 0.996$

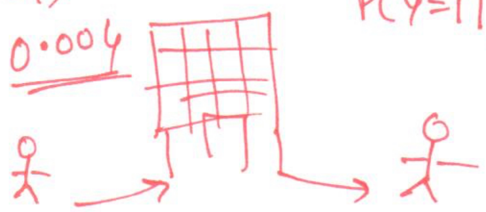
$$P(Y=1 \mid X=1) = \frac{0.8 * 0.004}{0.8 * 0.004 + 0.1 * 0.996}$$

Bayes' Rule

$$= \underline{0.031} \leftarrow$$

$P(Y=1) \longrightarrow P(Y=1 \mid \underline{X=1})$

$\underline{0.004}$ $\qquad$ $\underline{0.031}$



Evidence

Prior $\qquad$ Posterior

$$P(A \cap B) = \;\; \begin{cases} \text{I drew the good coin} & \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8} \\ \text{I drew the fake coin} & \frac{1}{2} * 1 * 1 = \frac{1}{2} \end{cases}$$

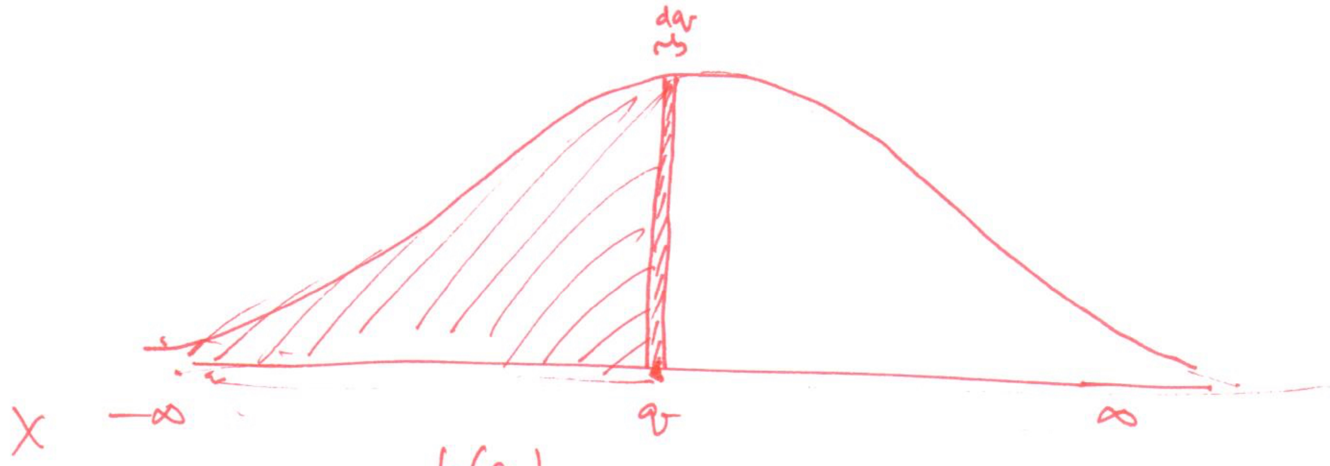
$$P(A \cap B) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

$$P(A) = \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * 1 = \frac{3}{4}$$

$$P(B) = \frac{3}{4}$$

$$P(A)P(B) = \frac{9}{16}$$

$$\underline{P(A \cap B) \neq P(A)P(B)}$$



$p(q)$

~~pd~~ $p(q) = P(X \geq q - dq \text{ and } x \leq q + dq)$

CDF: $F(q)$

$F(\infty) = 1$

Let $X = \{1, 2, 3, 4, 5, 6\}$ $\boxed{1/6 \ 1/6 \ \cdots \ 1/6}$

$\mathbb{E}[X] = \sum x P(X = x)$

$= \left(1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6}\right)$

$\mathbb{E}[f(X)] = \sum f(x) P(X = x)$

If X - continuous

$\boxed{\mathbb{E}[f(X)] = \int f(x) \, p(\cancel{x=x}) \, dx}$

Binomial

X

$$P(X=k) = {}^{n}C_k = \frac{n!}{(n-k)!\,k!}$$

$$0 \leq k \leq n+1 \quad = \binom{n}{k} \theta^k (1-\theta)^{n-k}$$

Bernoulli

$n = 1$ $\quad$ θ - parameter

$\mathcal{X} = \{0, 1\}$

Gaussian

Univariate	Multivariate
$x \to$ scalar	$x \to \mathbb{R}^d$
μ, σ^2	$\mu \in \mathbb{R}^d$ $\quad \Sigma_{d \times d}$ $\quad \begin{bmatrix} var(x_1) & Cov(x_1,x_2) & \cdots \\ & \ddots & \\ & & \end{bmatrix}$
$pdf(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$	$pdf(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} \lvert\Sigma\rvert^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right]$

Learning Parameters given some samples.

$|\Sigma| \to$ determinant of Σ

X — Categorical $\qquad \mathcal{X}$

$X \rightarrow \{0, 1, 2, 3, 4, 5\}$

0.1	0.2	0.3	0.1	0.1	0.2

$$E[X] = \sum_{x \in \mathcal{X}} x \, P(X = x)$$

$$= 0 * 0.1 + 1 * 0.2 + 2 * 0.3 + 3 * 0.1 + 4 * 0.1 + 5 * 0.2$$

$$= .2 + .6 + .3 + .4 + 1$$

$$= \underline{2.5}$$

$$\boxed{E[f(X)] = \sum_{x \in \mathcal{X}} f(x) \, P(X = x)} \qquad f(X) = X^2$$

$$= 0^2 * 0.1 + 1^2 * 0.2 + \ldots$$

$$= \underline{\qquad}$$

X - Continuous $\qquad \mathcal{X}$ — infinite

$$E[X] = \int_{x \in \mathcal{X}} x \, p(x) \, dx$$

$$E[f(X)] = \int_{x \in \mathcal{X}} f(x) p(x) \, dx$$

$\Rightarrow$ sometimes can be tricky

Identities

X - categorical

$$\mathbb{E}[c] = \sum_{x \in X} c \, p(X=x) = c \underline{\sum_{x \in X} p(X=x)}$$

$$= \underline{c}$$

$$\mathbb{E}[f(X) + g(X)] = \mathbb{E}[f(X)] + \mathbb{E}[g(X)]$$

$X \rightarrow$ continuous

$$\mathbb{E}[c] = \int_{x \in X} c \, p(x) \, dx = c \int_{x \in X} p(x) \, dx$$

$$= \underline{\underline{c}}$$

$\mathbb{E}[X] \rightarrow$ mean μ $\longrightarrow$ First moment

$$\mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2X\mu + \mu^2]$$

$$= \mathbb{E}[X^2] - \mathbb{E}[2X\mu] + \mathbb{E}[\mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu \underline{\underline{\mathbb{E}[X]}} + \mu^2$$

$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2$$

$$= \underline{\mathbb{E}[X^2] - \mu^2}$$

$\rightarrow$ Variance $\longrightarrow$ Second Moment

σ^2 - variance

σ - standard deviation