HPI Hasso Plattner Institut
Digital Engineering · Universität Potsdam

MOIA

KIT
Karlsruhe Institute of Technology

Algorithm Bootcamp

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Day 3: Dynamic Programming

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Fleet Optimization

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Software Engineer
Fleet Optimization

Greedy vs. Dynamic Programming

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- optimal solution consists of optimal solutions for sub-problems

"Bellman's Optimality Principle"

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When is Dynamic Programming the right choice?

- optimal solution consists of optimal solutions for sub-problems

  "Bellman's Optimality Principle"

- many overlapping sub-problems

  if no overlap: use Divide & Conquer

Dynamic Programming

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3 steps for doing it:

1.) come up with a recursive approach to solve your problem

2.) store the solutions of sub-problems to avoid future recursive calls

3.) optional: generate solutions to sub-problems in a bottom-up fashion to get an iterative algorithm

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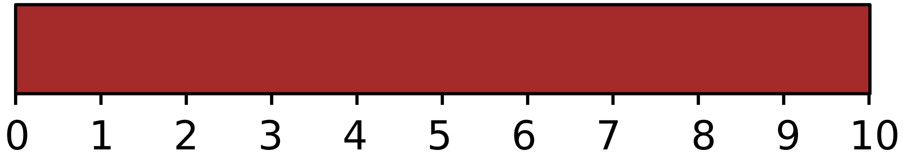

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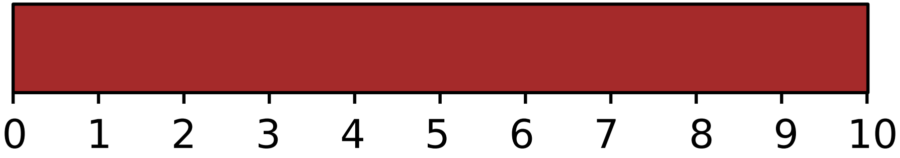

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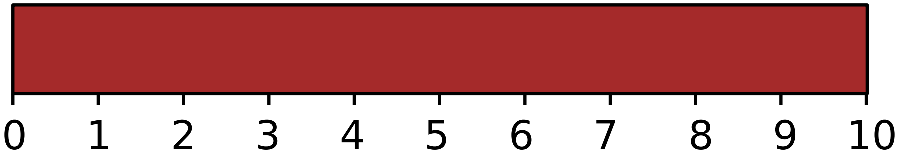

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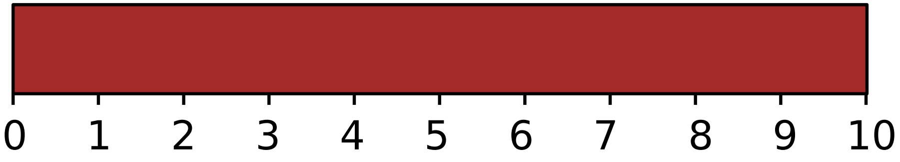

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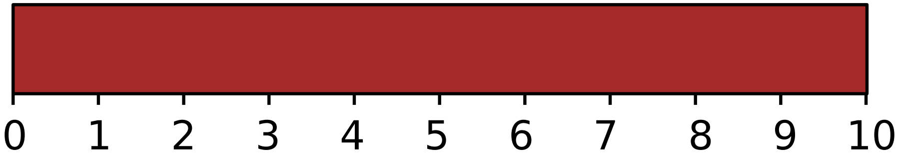

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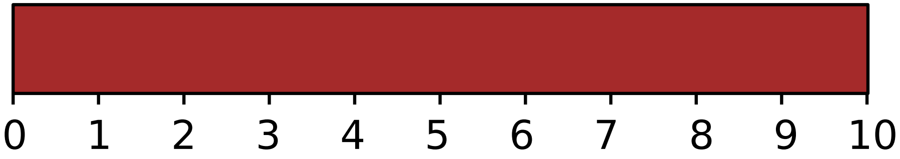

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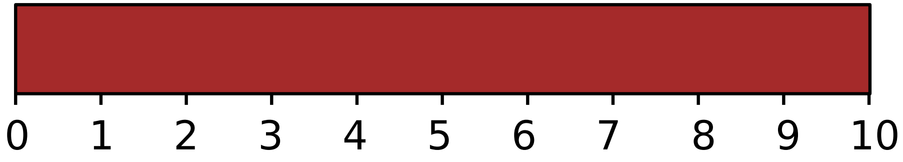

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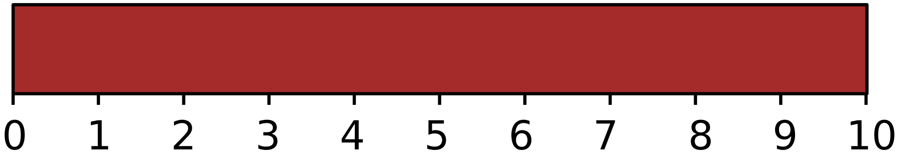

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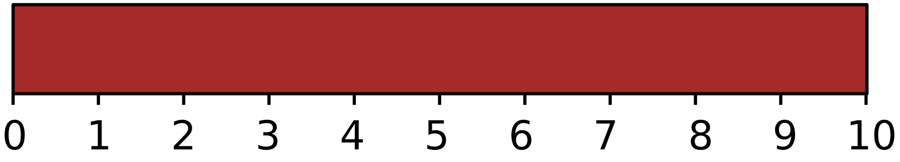
Input: 

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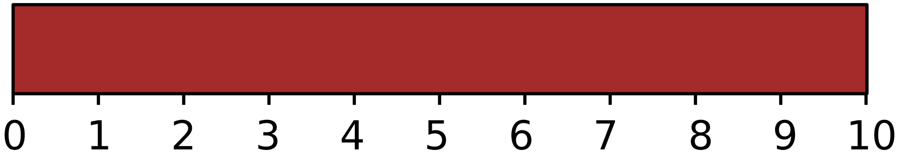

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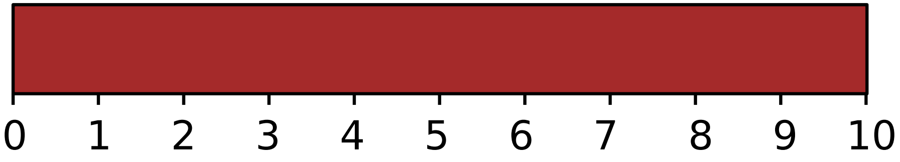

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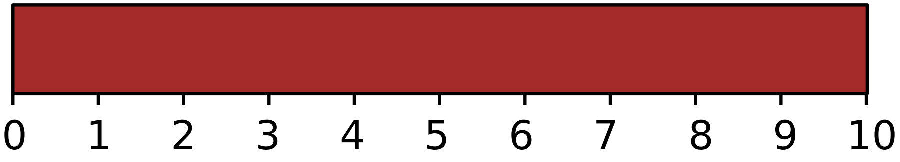

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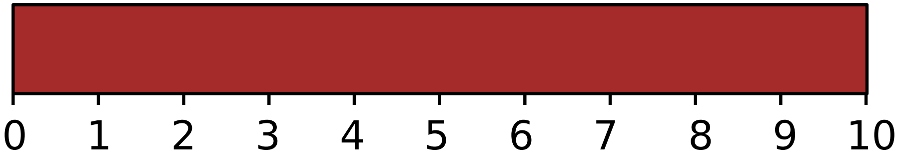

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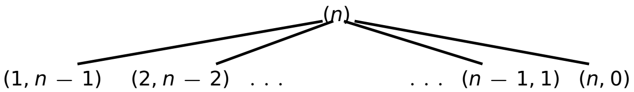

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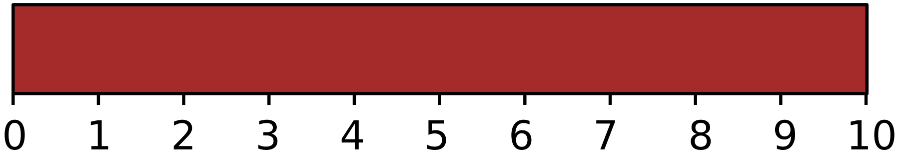

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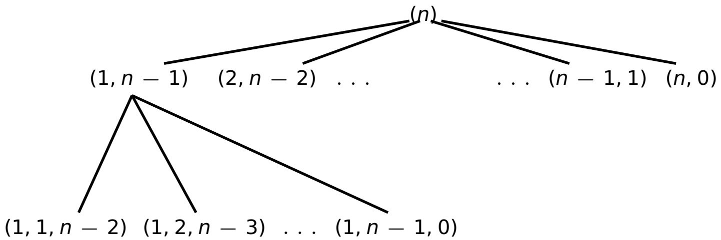

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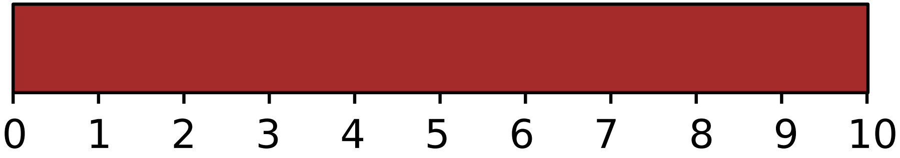

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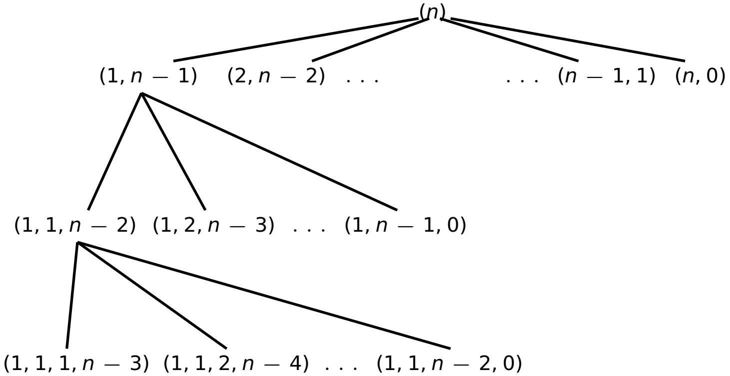

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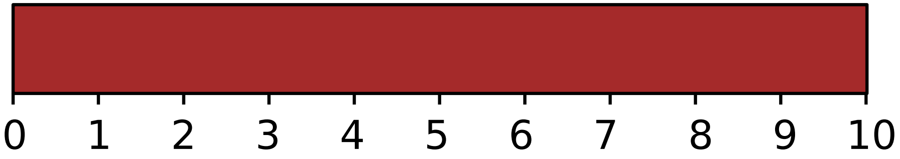

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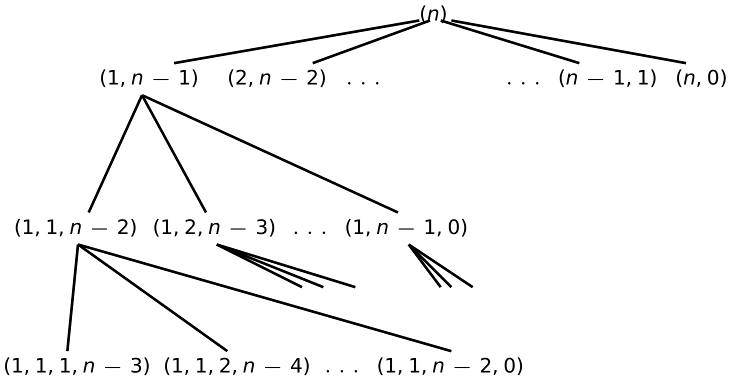

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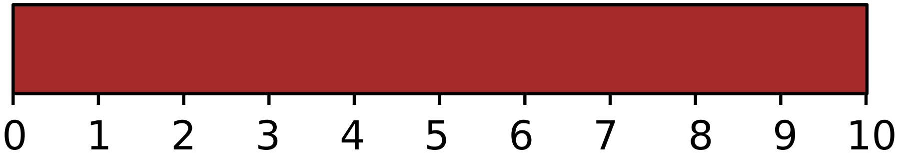

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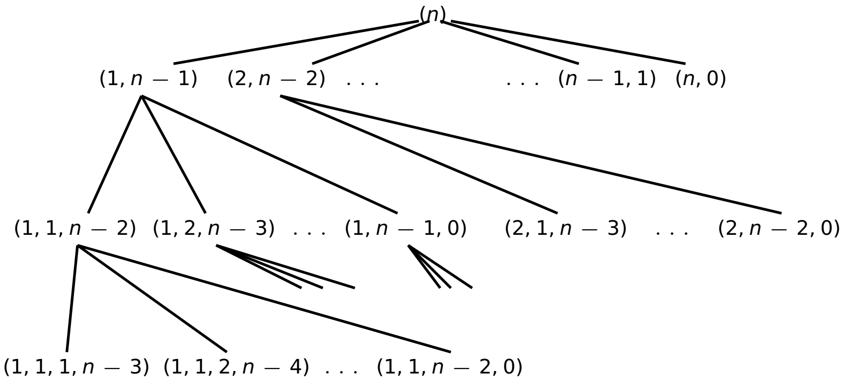

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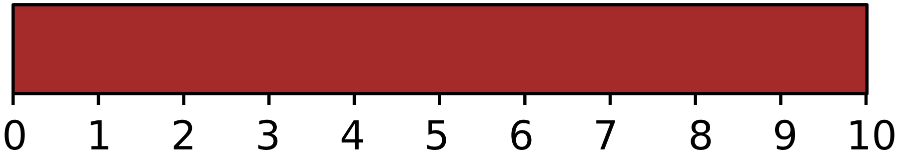

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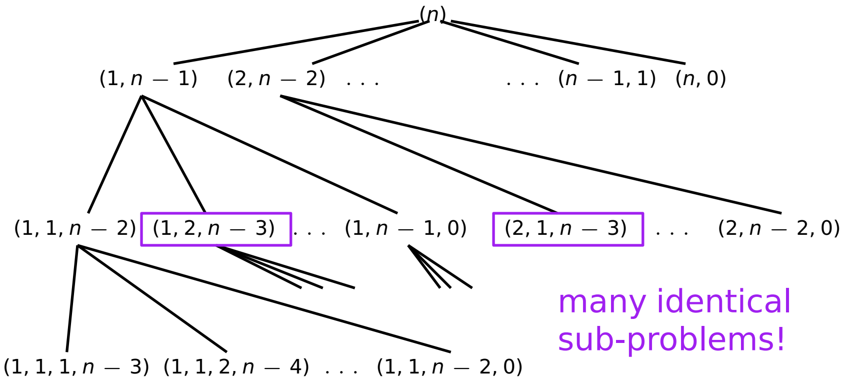

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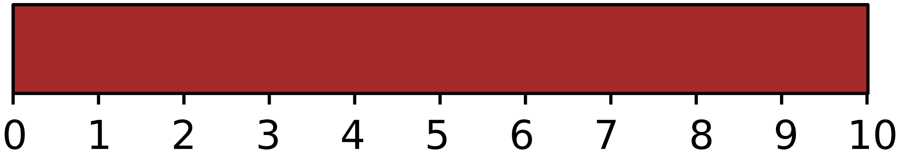
Input: 

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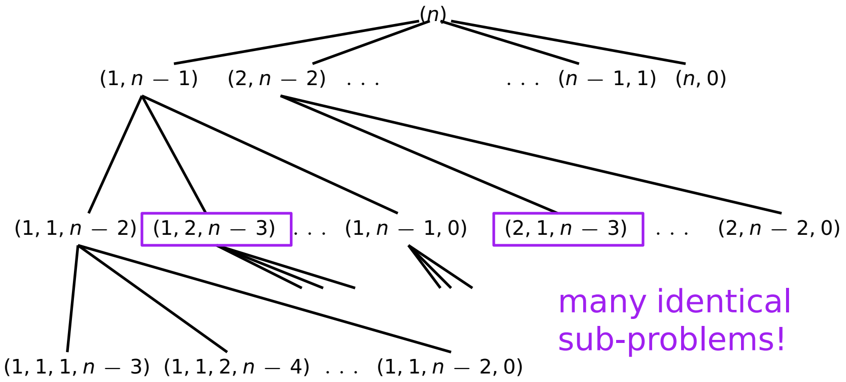

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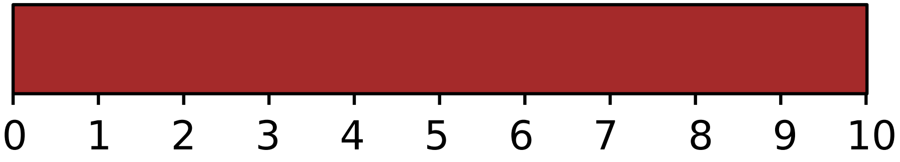

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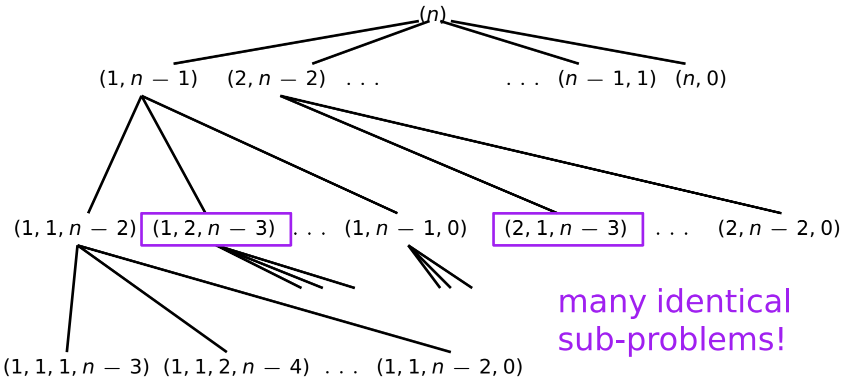

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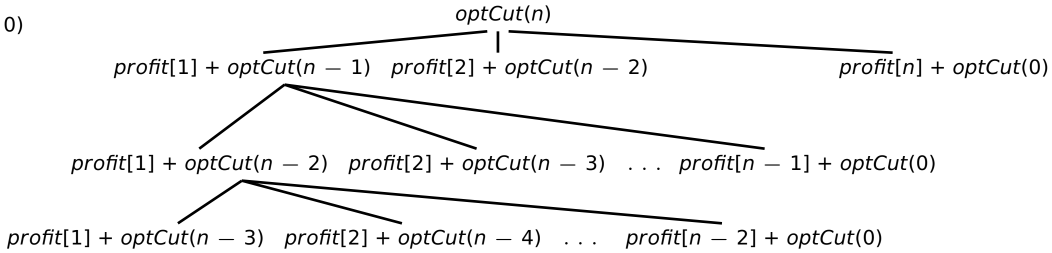

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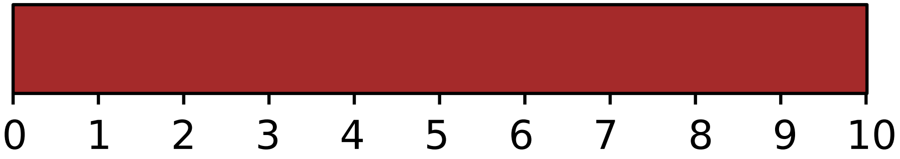

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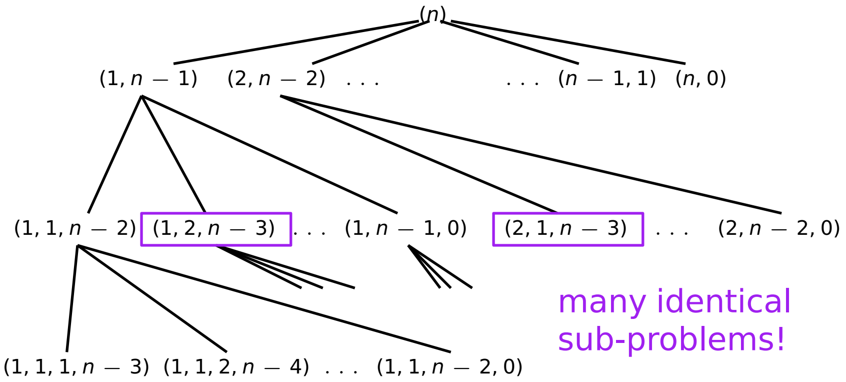

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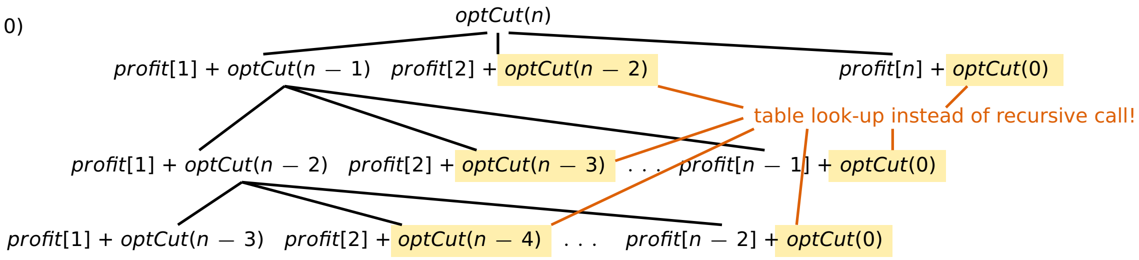

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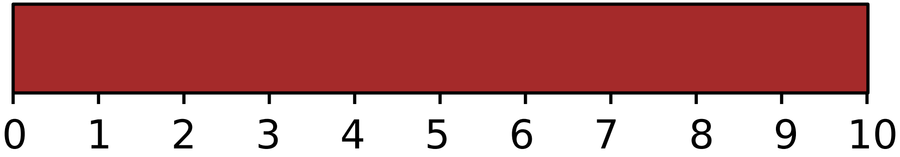

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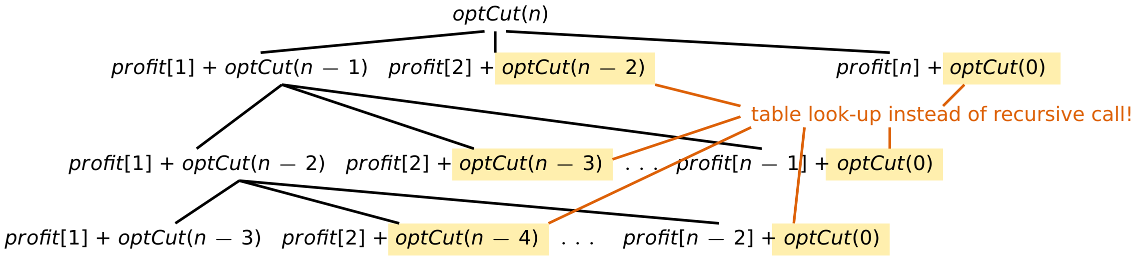

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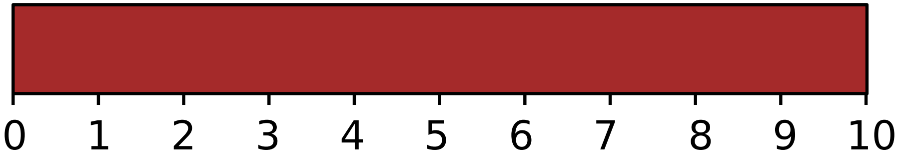

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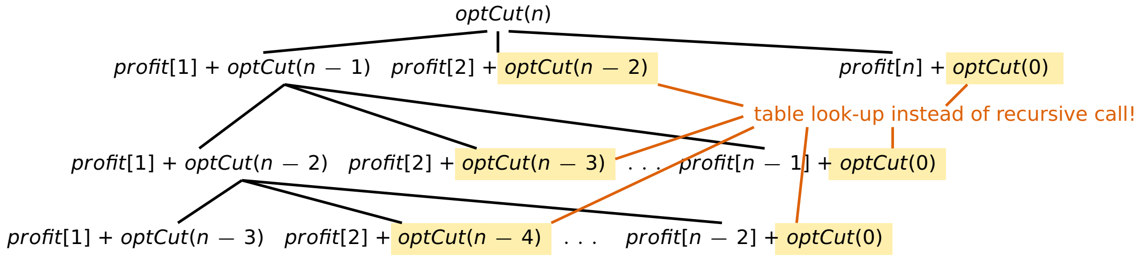

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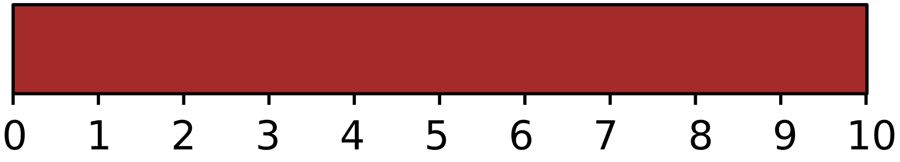

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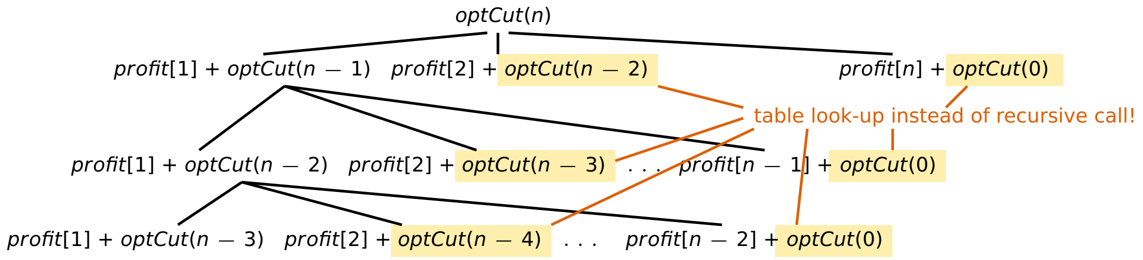

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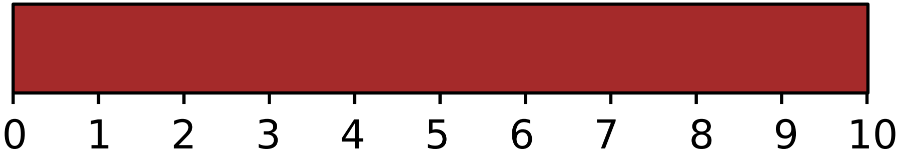
Input: 

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$optCut(n)$

$profit[1] + optCut(n-1)$ $profit[2] + optCut(n-2)$ $profit[n] + optCut(0)$

table look-up instead of recursive call!

$profit[1] + optCut(n-2)$ $profit[2] + optCut(n-3)$ ... $profit[n-1] + optCut(0)$

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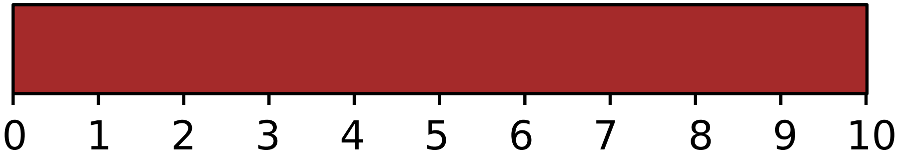

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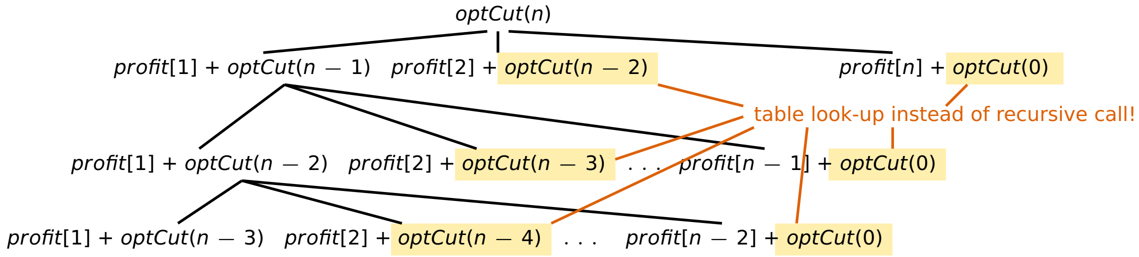
table look-up

length	0	1	2	3	4	5	6	7	8	9	10
optCut	0	0	1								



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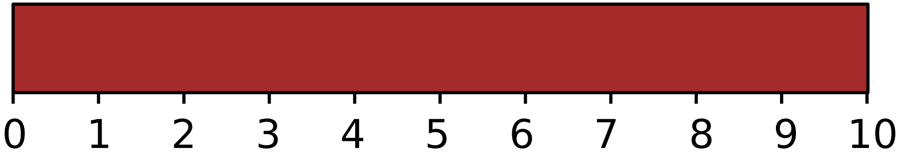

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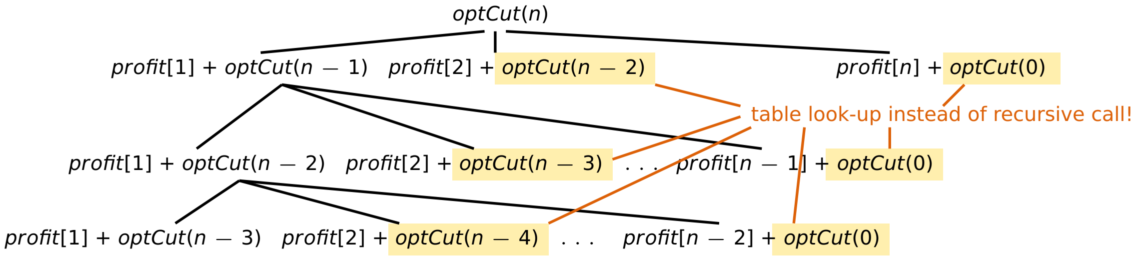

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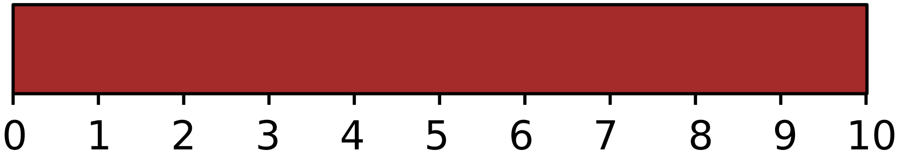

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length	0	1	2	3	4	5	6	7	8	9	10
optCut	0	0	1	3	5						

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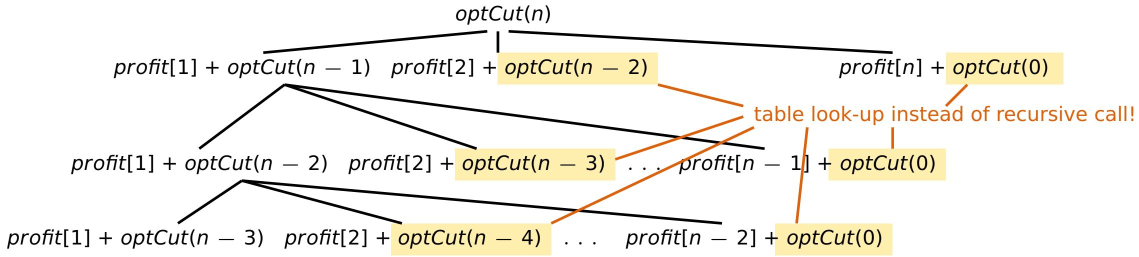

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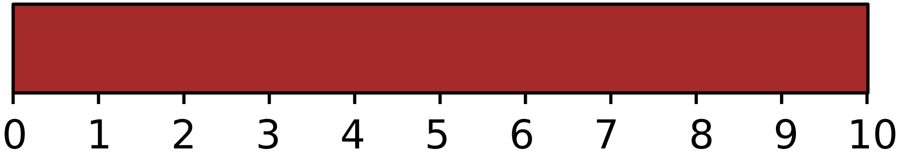

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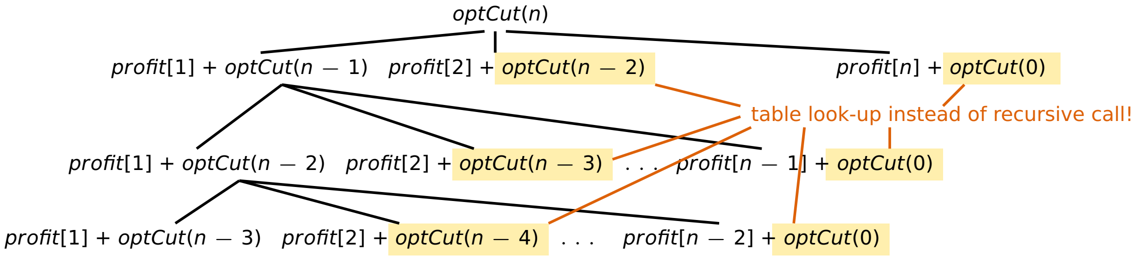

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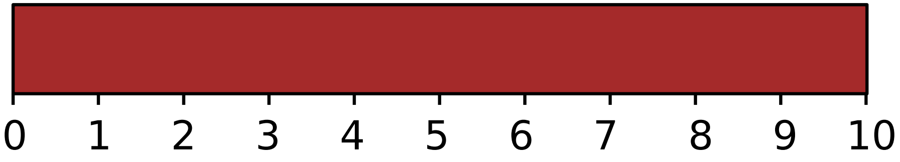

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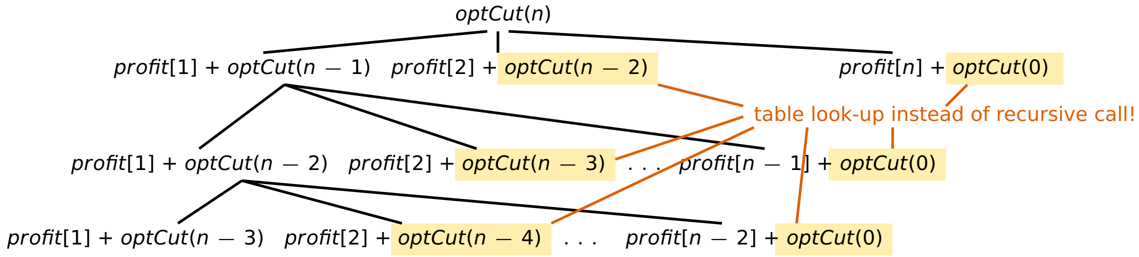

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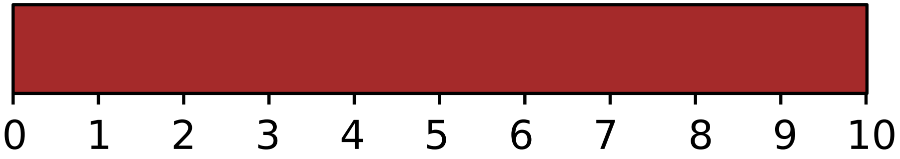

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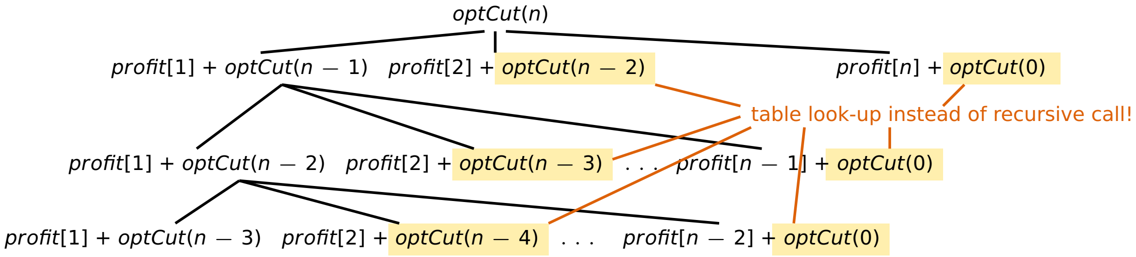

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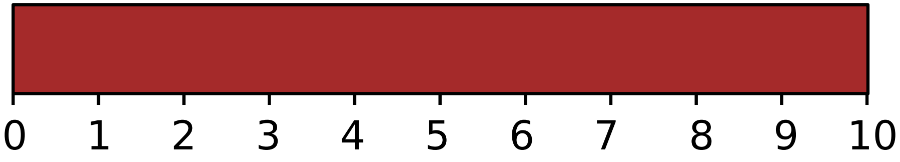

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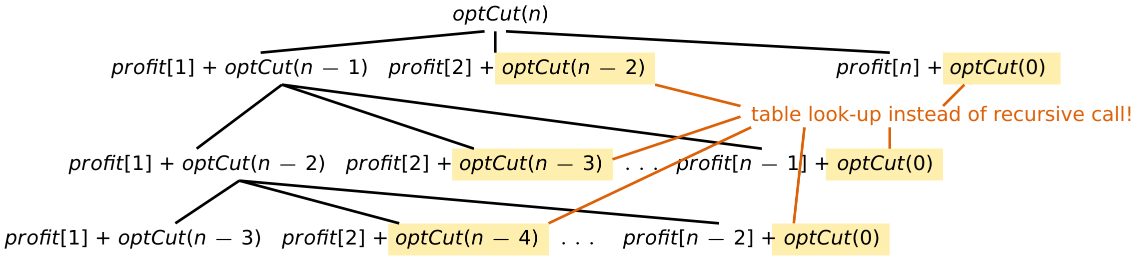

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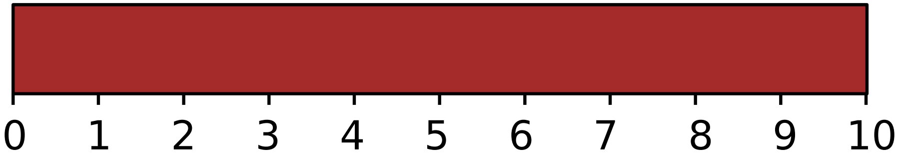

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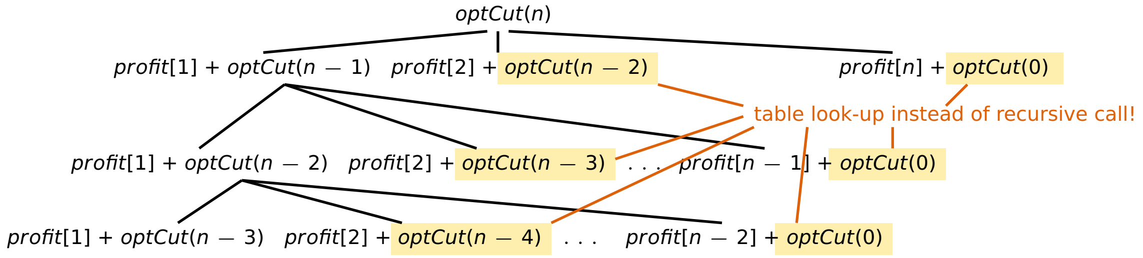
optCut	0	0	1	3	5	6	6	8	10	11	



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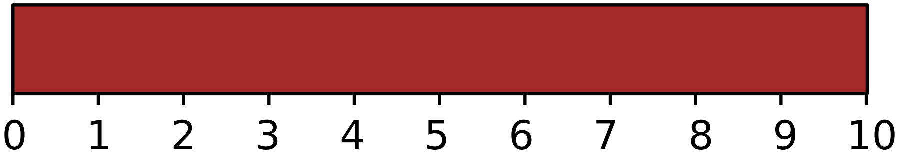

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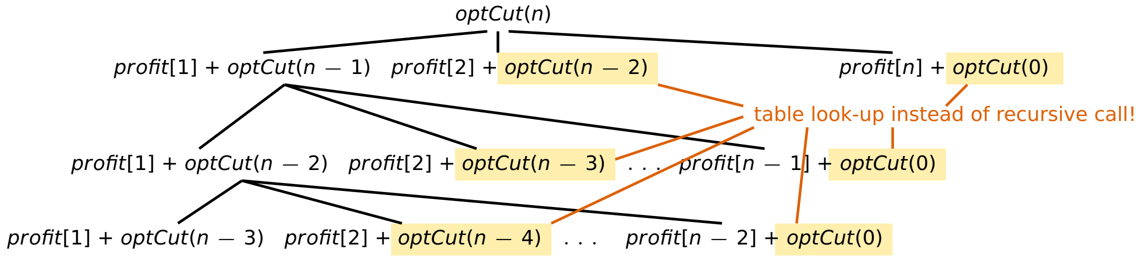

Example 2: Subset Sum

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Input: natural numbers $a_1, \dots, a_n, s$
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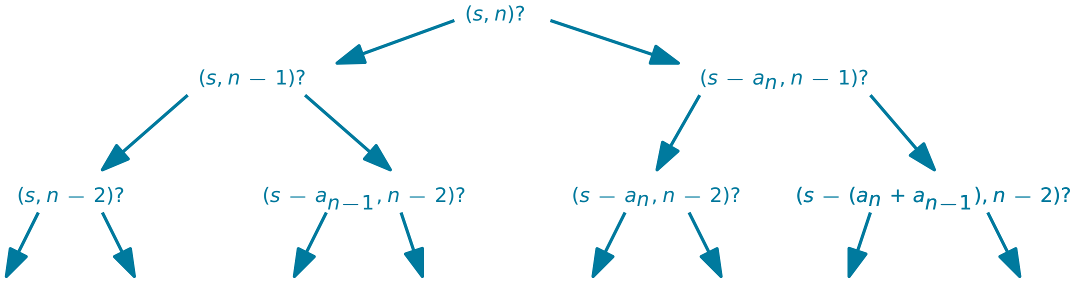
2.) recursion + save solutions for sub-problems (k, i)



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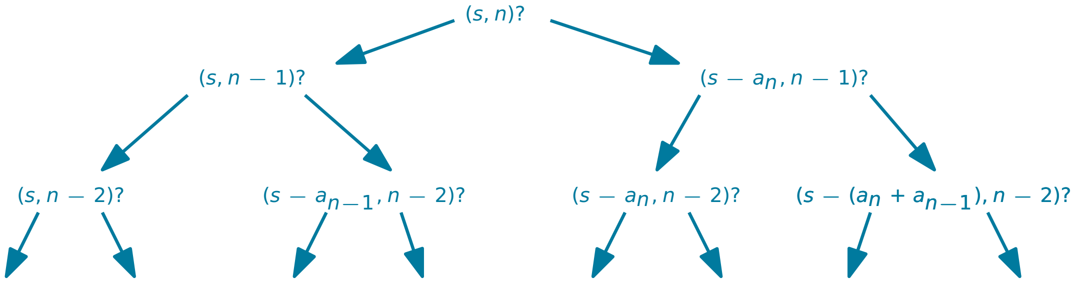
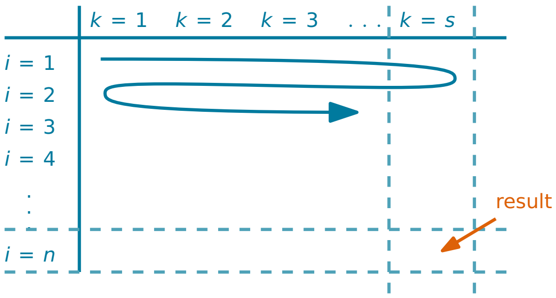


3.) iterative bottom-up:



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Why not something like: sumPossible(k) if for some $1 \leq i \leq n$: sumPossible($k - a_i$)
for $1 \leq i \leq n$: sumPossible(a_i) = TRUE

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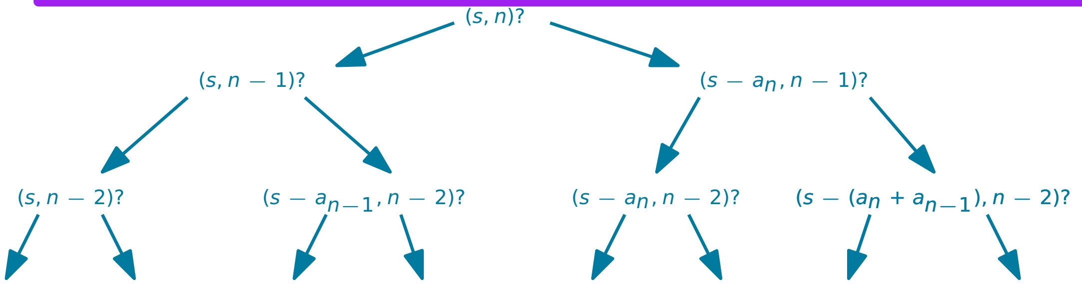

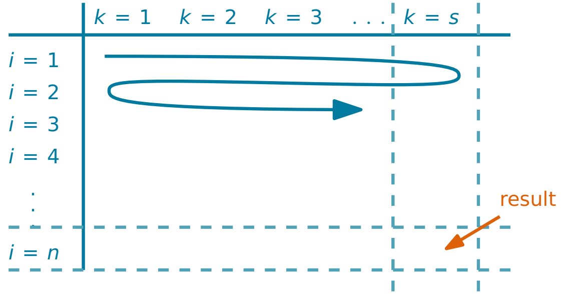

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Answer: numbers could be chosen more than once!

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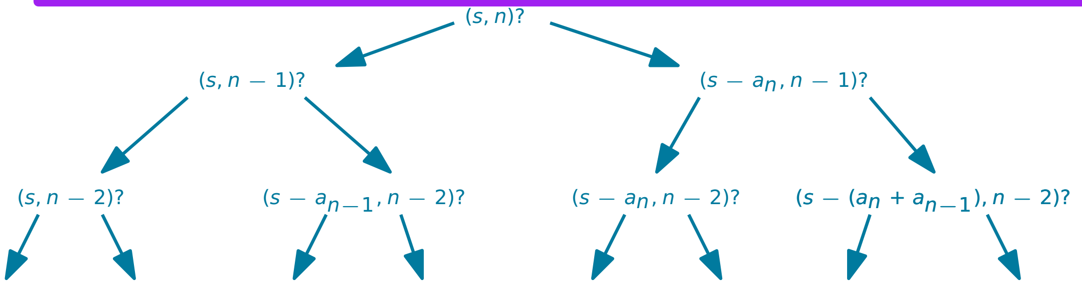

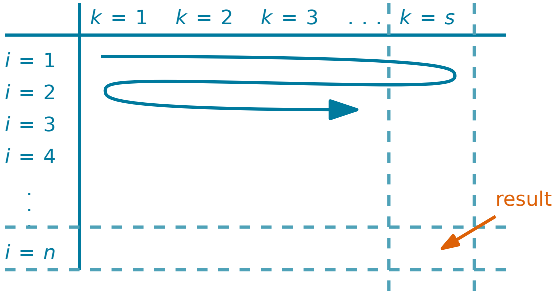

Example 2: Subset Sum

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Input: natural numbers $a_1, \ldots, a_n, s$
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Why not something like: sumPossible(k) if for some $1 \leq i \leq n$: sumPossible($k - a_i$)
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(s, n)?

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Clever idea of (k, i) approach:
Fixed order of adding elements to solution restricts possibilities!

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Answer: to avoid double-counting solutions

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Why optional?

Rod Cutting Again

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Input: rod of length n, prices for different lengths of the rod
Task: find the optimal way to cut the rod to maximize the profit

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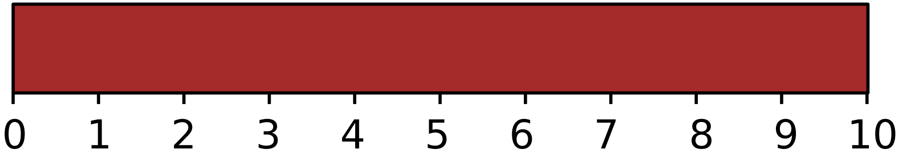

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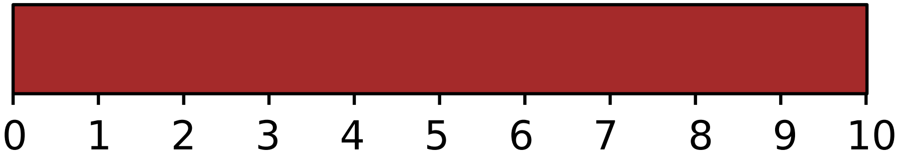

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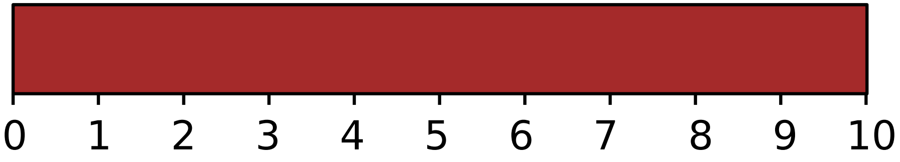

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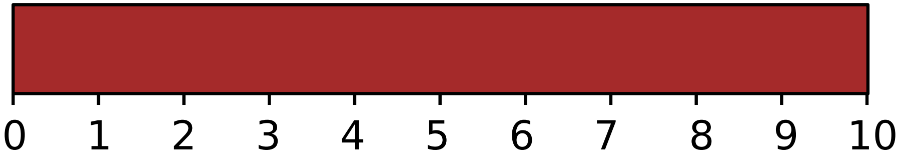
Input: 

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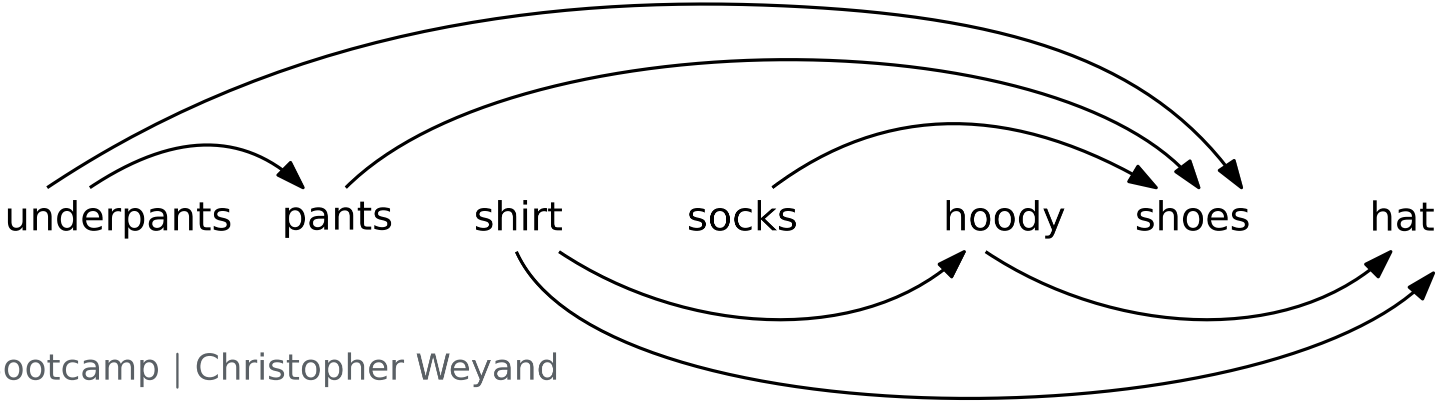
Reverse Topological Order!



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Per Subtree T_x with root x:

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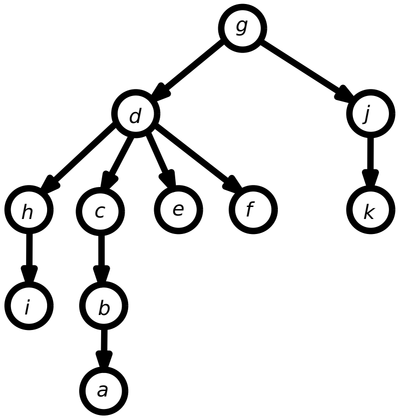

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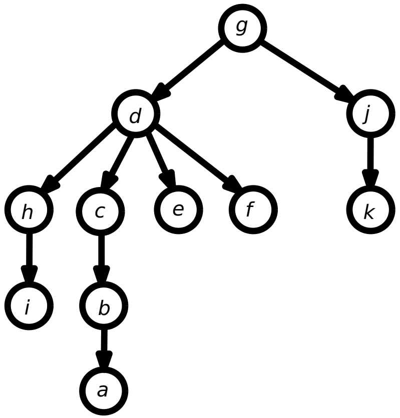

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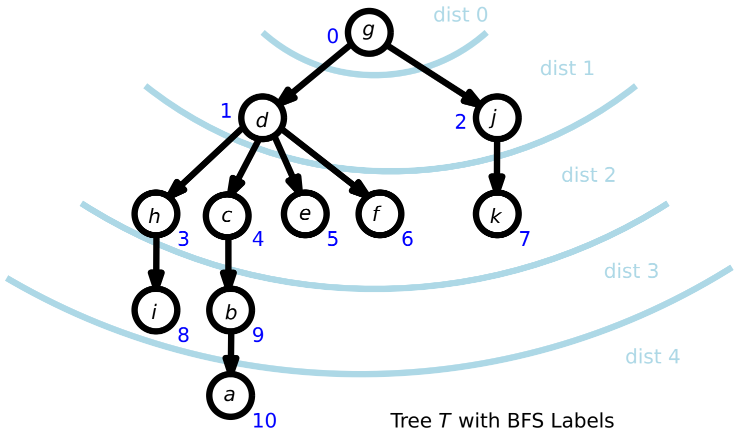

Tree T with BFS Labels

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