

4: LINEAR PROGRAMMING THE SIMPLEX METHOD



Rupinder Sekhon and Roberta Bloom
De Anza College

CHAPTER OVERVIEW

4: LINEAR PROGRAMMING THE SIMPLEX METHOD

This chapter covers principles of the simplex method to Linear Programming. After completing this chapter students should be able to: solve linear programming maximization problems using the simplex method and solve the minimization problems using the simplex method.

4.1: INTRODUCTION TO LINEAR PROGRAMMING APPLICATIONS IN BUSINESS, FINANCE, MEDICINE, AND SOCIAL SCIENCE

In this section, you will learn about real world applications of linear programming and related methods.

4.2: MAXIMIZATION BY THE SIMPLEX METHOD

The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

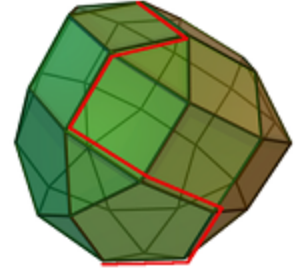
4.2.1: MAXIMIZATION BY THE SIMPLEX METHOD (EXERCISES)

4.3: MINIMIZATION BY THE SIMPLEX METHOD

In this section, we will solve the standard linear programming minimization problems using the simplex method. The procedure to solve these problems involves solving an associated problem called the dual problem. The solution of the dual problem is used to find the solution of the original problem. The dual problem is a maximization problem, which we learned to solve in the last section. We first solve the dual problem by the simplex method.

4.3.1: MINIMIZATION BY THE SIMPLEX METHOD (EXERCISES)

4.4: CHAPTER REVIEW



4.1: Introduction to Linear Programming Applications in Business, Finance, Medicine, and Social Science

Learning Objectives

In this section, you will learn about real world applications of linear programming and related methods.

The linear programs we solved in [Chapter 3](#) contain only two variables, x and y , so that we could solve them graphically. In practice, linear programs can contain thousands of variables and constraints. Later in this chapter we'll learn to solve linear programs with more than two variables using the simplex algorithm, which is a numerical solution method that uses matrices and row operations. However, in order to make the problems practical for learning purposes, our problems will still have only several variables. Now that we understand the main concepts behind linear programming, we can also consider how linear programming is currently used in large scale real-world applications.

Linear programming is used in business and industry in production planning, transportation and routing, and various types of scheduling. Airlines use linear programs to schedule their flights, taking into account both scheduling aircraft and scheduling staff. Delivery services use linear programs to schedule and route shipments to minimize shipment time or minimize cost. Retailers use linear programs to determine how to order products from manufacturers and organize deliveries with their stores. Manufacturing companies use linear programming to plan and schedule production. Financial institutions use linear programming to determine the mix of financial products they offer, or to schedule payments transferring funds between institutions. Health care institutions use linear programming to ensure the proper supplies are available when needed. And as we'll see below, linear programming has also been used to organize and coordinate life saving health care procedures.

In some of the applications, the techniques used are related to linear programming but are more sophisticated than the methods we study in this class. One such technique is called integer programming. In these situations, answers must be integers to make sense, and can not be fractions. Problems where solutions must be integers are more difficult to solve than the linear programs we've worked with. In fact, many of our problems have been very carefully constructed for learning purposes so that the answers just happen to turn out to be integers, but in the real world unless we specify that as a restriction, there is no guarantee that a linear program will produce integer solutions. There are also related techniques that are called non-linear programs, where the functions defining the objective function and/or some or all of the constraints may be non-linear rather than straight lines.

Many large businesses that use linear programming and related methods have analysts on their staff who can perform the analyses needed, including linear programming and other mathematical techniques. Consulting firms specializing in use of such techniques also aid businesses who need to apply these methods to their planning and scheduling processes.

When used in business, many different terms may be used to describe the use of techniques such as linear programming as part of mathematical business models. Optimization, operations research, business analytics, data science, industrial engineering and management science are among the terms used to describe mathematical modelling techniques that may include linear programming and related methods.

In the rest of this section we'll explore six real world applications, and investigate what they are trying to accomplish using optimization, as well as what their constraints might represent.

Airline Scheduling

Airlines use techniques that include and are related to linear programming to schedule their aircrafts to flights on various routes, and to schedule crews to the flights. In addition, airlines also use linear programming to determine ticket pricing for various types of seats and levels of service or amenities, as well as the timing at which ticket prices change.

The process of scheduling aircraft and departure times on flight routes can be expressed as a model that minimizes cost, of which the largest component is generally fuel costs.

Constraints involve considerations such as:

- Each aircraft needs to complete a daily or weekly tour to return back to its point of origin.

- Scheduling sufficient flights to meet demand on each route.
- Scheduling the right type and size of aircraft on each route to be appropriate for the route and for the demand for number of passengers.
- Aircraft must be compatible with the airports it departs from and arrives at - not all airports can handle all types of planes.

A model to accomplish this could contain thousands of variables and constraints. Highly trained analysts determine ways to translate all the constraints into mathematical inequalities or equations to put into the model.

After aircraft are scheduled, crews need to be assigned to flights. Each flight needs a pilot, a co-pilot, and flight attendants. Each crew member needs to complete a daily or weekly tour to return back to his or her home base. Additional constraints on flight crew assignments take into account factors such as:

- Pilot and co-pilot qualifications to fly the particular type of aircraft they are assigned to
- Flight crew have restrictions on the maximum amount of flying time per day and the length of mandatory rest periods between flights or per day that must meet certain minimum rest time regulations.
- Numbers of crew members required for a particular type or size of aircraft.

When scheduling crews to flights, the objective function would seek to minimize total flight crew costs, determined by the number of people on the crew and pay rates of the crew members. However the cost for any particular route might not end up being the lowest possible for that route, depending on tradeoffs to the total cost of shifting different crews to different routes.

An airline can also use linear programming to revise schedules on short notice on an emergency basis when there is a schedule disruption, such as due to weather. In this case the considerations to be managed involve:

- Getting aircrafts and crews back on schedule as quickly as possible
- Moving aircraft from storm areas to areas with calm weather to keep the aircraft safe from damage and ready to come back into service as quickly and conveniently as possible
- Ensuring crews are available to operate the aircraft and that crews continue to meet mandatory rest period requirements and regulations.

Kidney Donation Chain

For patients who have kidney disease, a transplant of a healthy kidney from a living donor can often be a lifesaving procedure. Criteria for a kidney donation procedure include the availability of a donor who is healthy enough to donate a kidney, as well as a compatible match between the patient and donor for blood type and several other characteristics. Ideally, if a patient needs a kidney donation, a close relative may be a match and can be the kidney donor. However often there is not a relative who is a close enough match to be the donor. Considering donations from unrelated donor allows for a larger pool of potential donors. Kidney donations involving unrelated donors can sometimes be arranged through a chain of donations that pair patients with donors. For example a kidney donation chain with three donors might operate as follows:

- Donor A donates a kidney to Patient B.
- Donor B, who is related to Patient B, donates a kidney to Patient C.
- Donor C, who is related to Patient C, donates a kidney to Patient A, who is related to Donor A.

Linear programming is one of several mathematical tools that have been used to help efficiently identify a kidney donation chain. In this type of model, patient/donor pairs are assigned compatibility scores based on characteristics of patients and potential donors.

The objective is to maximize the total compatibility scores. Constraints ensure that donors and patients are paired only if compatibility scores are sufficiently high to indicate an acceptable match.

Advertisements in Online Marketing

Did you ever make a purchase online and then notice that as you browse websites, search, or use social media, you now see more ads related the item you purchased?

Marketing organizations use a variety of mathematical techniques, including linear programming, to determine individualized advertising placement purchases.

Instead of advertising randomly, online advertisers want to sell bundles of advertisements related to a particular product to batches of users who are more likely to purchase that product. Based on an individual's previous browsing and purchase

selections, he or she is assigned a “propensity score” for making a purchase if shown an ad for a certain product. The company placing the ad generally does not know individual personal information based on the history of items viewed and purchased, but instead has aggregated information for groups of individuals based on what they view or purchase. However, the company may know more about an individual’s history if he or she logged into a website making that information identifiable, within the privacy provisions and terms of use of the site.

The company’s goal is to buy ads to present to specified size batches of people who are browsing. The linear program would assign ads and batches of people to view the ads using an objective function that seeks to maximize advertising response modelled using the propensity scores. The constraints are to stay within the restrictions of the advertising budget.

Loans

A car manufacturer sells its cars through dealers. Dealers can offer loan financing to customers who need to take out loans to purchase a car. Here we will consider how car manufacturers can use linear programming to determine the specific characteristics of the loan they offer to a customer who purchases a car. In a future chapter we will learn how to do the financial calculations related to loans.

A customer who applies for a car loan fills out an application. This provides the car dealer with information about that customer. In addition, the car dealer can access a credit bureau to obtain information about a customer’s credit score.

Based on this information obtained about the customer, the car dealer offers a loan with certain characteristics, such as interest rate, loan amount, and length of loan repayment period.

Linear programming can be used as part of the process to determine the characteristics of the loan offer. The linear program seeks to maximize the profitability of its portfolio of loans. The constraints limit the risk that the customer will default and will not repay the loan. The constraints also seek to minimize the risk of losing the loan customer if the conditions of the loan are not favorable enough; otherwise the customer may find another lender, such as a bank, which can offer a more favorable loan.

Production Planning and Scheduling in Manufacturing

Consider the example of a company that produces yogurt. There are different varieties of yogurt products in a variety of flavors. Yogurt products have a short shelf life; it must be produced on a timely basis to meet demand, rather than drawing upon a stockpile of inventory as can be done with a product that is not perishable. Most ingredients in yogurt also have a short shelf life, so can not be ordered and stored for long periods of time before use; ingredients must be obtained in a timely manner to be available when needed but still be fresh. Linear programming can be used in both production planning and scheduling.

To start the process, sales forecasts are developed to determine demand to know how much of each type of product to make.

There are often various manufacturing plants at which the products may be produced. The appropriate ingredients need to be at the production facility to produce the products assigned to that facility. Transportation costs must be considered, both for obtaining and delivering ingredients to the correct facilities, and for transport of finished product to the sellers.

The linear program that monitors production planning and scheduling must be updated frequently - daily or even twice each day - to take into account variations from a master plan.

Bike Share Programs

Over 600 cities worldwide have bikeshare programs. Although bikeshare programs have been around for a long time, they have proliferated in the past decade as technology has developed new methods for tracking the bicycles.

Bikeshare programs vary in the details of how they work, but most typically people pay a fee to join and then can borrow a bicycle from a bike share station and return the bike to the same or a different bike share station. Over time the bikes tend to migrate; there may be more people who want to pick up a bike at station A and return it at station B than there are people who want to do the opposite. In chapter 9, we’ll investigate a technique that can be used to predict the distribution of bikes among the stations.

Once other methods are used to predict the actual and desired distributions of bikes among the stations, bikes may need to be transported between stations to even out the distribution. Bikeshare programs in large cities have used methods related to linear programming to help determine the best routes and methods for redistributing bicycles to the desired stations once the

desired distributions have been determined. The optimization model would seek to minimize transport costs and/or time subject to constraints of having sufficient bicycles at the various stations to meet demand.

4.2: Maximization By The Simplex Method

Learning Objectives

In this section, you will learn to solve linear programming maximization problems using the Simplex Method:

1. Identify and set up a linear program in standard maximization form
2. Convert inequality constraints to equations using slack variables
3. Set up the initial simplex tableau using the objective function and slack equations
4. Find the optimal simplex tableau by performing pivoting operations.
5. Identify the optimal solution from the optimal simplex tableau.

In the last chapter, we used the geometrical method to solve linear programming problems, but the geometrical approach will not work for problems that have more than two variables. In real life situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very tedious. So we need a method that has a systematic algorithm and can be programmed for a computer. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the **simplex method**.

The simplex method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm which was supposed to be revolutionary, but as it turned out it is not any better than the simplex method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories developed Karmarkar's algorithm which has been proven to be four times faster than the simplex method for certain problems. But the simplex method still works the best for most problems.

The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

To learn the simplex method, we try a rather unconventional approach. We first list the algorithm, and then work a problem. We justify the reasoning behind each step during the process. A thorough justification is beyond the scope of this course.

We start out with an example we solved in the last chapter by the graphical method. This will provide us with some insight into the simplex method and at the same time give us the chance to compare a few of the feasible solutions we obtained previously by the graphical method. But first, we list the algorithm for the simplex method.

THE SIMPLEX METHOD

1. **Set up the problem.** That is, write the objective function and the inequality constraints.
2. **Convert the inequalities into equations.** This is done by adding one slack variable for each inequality.
3. **Construct the initial simplex tableau.** Write the objective function as the bottom row.
4. **The most negative entry in the bottom row identifies the pivot column.**
5. **Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.** The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.
6. **Perform pivoting to make all other entries in this column zero.** This is done the same way as we did with the Gauss-Jordan method.

7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.
8. Read off your answers. Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right hand corner.

Now, we use the simplex method to solve Example 3.1.1 solved geometrically in [section 3.1](#).

Example 4.2.1

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes \$40 an hour at Job I, and \$30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

In solving this problem, we will follow the algorithm listed above.

STEP 1. Set up the problem. Write the objective function and the constraints.

Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x , y , z etc. We use symbols x_1 , x_2 , x_3 , and so on.

Let

- x_1 = The number of hours per week Niki will work at Job I and
- x_2 = The number of hours per week Niki will work at Job II.

It is customary to choose the variable that is to be maximized as Z .

The problem is formulated the same way as we did in the last chapter.

$$\begin{array}{ll} \text{Maximize} & Z = 40x_1 + 30x_2 \\ \text{Subject to:} & x_1 + x_2 \leq 12 \\ & 2x_1 + x_2 \leq 16 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

STEP 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

For example to convert the inequality $x_1 + x_2 \leq 12$ into an equation, we add a non-negative variable y_1 , and we get

$$x_1 + x_2 + y_1 = 12$$

Here the variable y_1 picks up the slack, and it represents the amount by which $x_1 + x_2$ falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We rewrite the objective function $Z = 40x_1 + 30x_2$ as $-40x_1 - 30x_2 + Z = 0$.

After adding the slack variables, our problem reads

$$\begin{array}{ll} \text{Objective function} & -40x_1 - 30x_2 + Z = 0 \\ \text{Subject to constraints:} & x_1 + x_2 + y_1 = 12 \\ & 2x_1 + x_2 + y_2 = 16 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

STEP 3. Construct the initial simplex tableau. Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.) Write the objective function as the bottom row.

Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

The reader needs to observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1 = 0$ and $x_2 = 0$, we get

$$\begin{bmatrix} y_1 & y_2 & Z & | & C \\ 1 & 0 & 0 & | & 12 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

which reads

$$y_1 = 12 \quad y_2 = 16 \quad Z = 0$$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the **basic solution** associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

x_1	x_2	y_1	y_2	Z	
1	1	1	0	0	12 y_1
2	1	0	1	0	16 y_2
-40	-30	0	0	1	0 Z

STEP 4. The most negative entry in the bottom row identifies the pivot column.

The most negative entry in the bottom row is -40; therefore the column 1 is identified.

x_1	x_2	y_1	y_2	Z	
1	1	1	0	0	12 y_1
2	1	0	1	0	16 y_2
-40	-30	0	0	1	0 Z

↑

Question Why do we choose the most negative entry in the bottom row?

Answer The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1 , x_2 , x_3 etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function $Z = 40x_1 + 30x_2$, it will make more sense to increase the value of x_1 rather than x_2 . The variable x_1 represents the number of hours per week Niki works at Job I. Since Job I pays \$40 per hour as opposed to Job II which pays only \$30, the variable x_1 will increase the objective function by \$40 for a unit of increase in the variable x_1 .

STEP 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	$y_1 \quad 12 \div 1 = 12$
2	1	0	1	0	16	$y_2 \quad \leftarrow 16 \div 2 = 8$
-40	-30	0	0	1	0	Z
\uparrow						

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

Question Why do we find quotients, and why does the smallest quotient identify a row?

Answer When we choose the most negative entry in the bottom row, we are trying to increase the value of the objective function by bringing in the variable x_1 . But we cannot choose any value for x_1 . Can we let $x_1 = 100$? Definitely not! That is because Niki never wants to work for more than 12 hours at both jobs combined: $x_1 + x_2 \leq 12$. Can we let $x_1 = 12$? Again, the answer is no because the preparation time for Job I is two times the time spent on the job. Since Niki never wants to spend more than 16 hours for preparation, the maximum time she can work is $16 \div 2 = 8$.

Now you see the purpose of computing the quotients; using the quotients to identify the pivot element guarantees that we do not violate the constraints.

Question Why do we identify the pivot element?

Answer As we have mentioned earlier, the simplex method begins with a corner point and then moves to the next corner point always improving the value of the objective function. The value of the objective function is improved by changing the number of units of the variables. We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.

The variable whose units are being added is called the **entering variable**, and the variable whose units are being replaced is called the **departing variable**. The entering variable in the above table is x_1 , and it was identified by the most negative entry in the bottom row. The departing variable y_2 was identified by the lowest of all quotients.

STEP 6. Perform pivoting to make all other entries in this column zero.

In chapter 2, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a 1 in the location of the pivot element, and then making all other entries zeros in that column. So now our job is to make our pivot element a 1 by dividing the entire second row by 2. The result follows.

x_1	x_2	y_1	y_2	Z	
1	1	1	0	0	12
1	1/2	0	1/2	0	8
-40	-30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by -1 and add it to row 1. We get

x_1	x_2	y_1	y_2	Z	
0	1/2	1	-1/2	0	4
1	1/2	0	1/2	0	8
-40	-30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row.

x_1	x_2	y_1	y_2	Z	
0	1/2	1	-1/2	0	4
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2 = 0$ and $y_2 = 0$, we obtain $x_1 = 8$, $y_1 = 4$, and $z = 320$. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.

$$\left[\begin{array}{ccc|c} x_1 & y_1 & Z & C \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 320 \end{array} \right]$$

We can restate the solution associated with this matrix as $x_1 = 8$, $x_2 = 0$, $y_1 = 4$, $y_2 = 0$ and $z = 320$. At this stage of the game, it reads that if Niki works 8 hours at Job I, and no hours at Job II, her profit Z will be \$320. Recall from Example 3.1.1 in [section 3.1](#) that $(8, 0)$ was one of our corner points. Here $y_1 = 4$ and $y_2 = 0$ mean that she will be left with 4 hours of working time and no preparation time.

STEP 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.

Since there is still a negative entry, -10, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step, instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.

x_1	x_2	y_1	y_2	Z		
0	1/2	1	-1/2	0	4	$y_1 \leftarrow 4 \div 1/2 = 8$
1	1/2	0	1/2	0	8	$x_1 \quad 8 \div 1/2 = 16$
0	-10	0	20	1	320	Z

↑

We make the pivot element 1 by multiplying row 1 by 2, and we get

x_1	x_2	y_1	y_2	Z	
0	1	2	-1	0	8
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

Now to make all other entries as zeros in this column, we first multiply row 1 by -1/2 and add it to row 2, and then multiply row 1 by 10 and add it to the bottom row.

x_1	x_2	y_1	y_2	Z	
0	1	2	-1	0	8
1	0	-1	1	0	4
0	0	20	10	1	400

We no longer have negative entries in the bottom row, therefore we are finished.

Question Why are we finished when there are no negative entries in the bottom row?

Answer The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400 \quad \text{or}$$

$$z = 400 - 20y_1 - 10y_2$$

Since all variables are non-negative, the highest value Z can ever achieve is 400, and that will happen only when y_1 and y_2 are zero.

STEP 8. Read off your answers.

We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a 1 and all other entries zeros. Since the columns labeled y_1 and y_2 are not such columns, we arbitrarily choose $y_1 = 0$, and $y_2 = 0$, and we get

$$\left[\begin{array}{ccc|c} x_1 & x_2 & Z & C \\ 0 & 1 & 0 & 8 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 400 \end{array} \right]$$

The matrix reads $x_1 = 4$, $x_2 = 8$ and $z = 400$.

The final solution says that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to \$400. Since both slack variables are zero, it means that she would have used up all the working time, as well as the preparation time, and none will be left.

4.2.1: Maximization By The Simplex Method (Exercises)

SECTION 4.2 PROBLEM SET: MAXIMIZATION BY THE SIMPLEX METHOD

Solve the following linear programming problems using the simplex method.

1)

$$\begin{array}{ll}\text{Maximize} & z = x_1 + 2x_2 + 3x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 + 3x_3 \leq 18 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

2)

$$\begin{array}{ll}\text{Maximize} & z = x_1 + 2x_2 + x_3 \\ \text{subject to} & x_1 + x_2 \leq 3 \\ & x_2 + x_3 \leq 4 \\ & x_1 + x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

3) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and \$20 of capital, and each acre of corn requires 16 hours of labor and \$40 of capital. The farmer has at most 800 hours of labor and \$2400 of capital available. If the profit from an acre of wheat is \$80 and from an acre of corn is \$100, how many acres of each crop should she plant to maximize her profit?

SECTION 4.2 PROBLEM SET: MAXIMIZATION BY THE SIMPLEX METHOD

Solve the following linear programming problems using the simplex method.

4) A factory manufactures chairs, tables and bookcases each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 600 hours; the second at most 500 hours; and the third at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing; a table needs 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; and a bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is \$20 per unit for a chair, \$30 for a table, and \$25 for a bookcase, how many units of each should be manufactured to maximize profit?

5). The Acme Apple company sells its Pippin, Macintosh, and Fuji apples in mixes. Box I contains 4 apples of each kind; Box II contains 6 Pippin, 3 Macintosh, and 3 Fuji; and Box III contains no Pippin, 8 Macintosh and 4 Fuji apples. At the end of the season, the company has altogether 2800 Pippin, 2200 Macintosh, and 2300 Fuji apples left. Determine the maximum number of boxes that the company can make.

4.3: Minimization By The Simplex Method

Learning Objectives

In this section, you will learn to solve linear programming minimization problems using the simplex method.

1. Identify and set up a linear program in standard minimization form
2. Formulate a dual problem in standard maximization form
3. Use the simplex method to solve the dual maximization problem
4. Identify the optimal solution to the original minimization problem from the optimal simplex tableau.

In this section, we will solve the standard linear programming minimization problems using the simplex method. Once again, we remind the reader that in the standard minimization problems all constraints are of the form $ax + by \geq c$.

The procedure to solve these problems was developed by Dr. John Von Neuman. It involves solving an associated problem called the **dual problem**. To every minimization problem there corresponds a dual problem. The solution of the dual problem is used to find the solution of the original problem. The dual problem is a maximization problem, which we learned to solve in the last section. We first solve the dual problem by the simplex method.

From the final simplex tableau, we then extract the solution to the original minimization problem.

Before we go any further, however, we first learn to convert a minimization problem into its corresponding maximization problem called its **dual**.

Example 4.3.1

Convert the following minimization problem into its dual.

$$\begin{array}{ll} \text{Minimize} & Z = 12x_1 + 16x_2 \\ \text{Subject to:} & x_1 + 2x_2 \geq 40 \\ & x_1 + x_2 \geq 30 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

Solution

To achieve our goal, we first express our problem as the following matrix.

$$\begin{array}{cc|c} 1 & 2 & 40 \\ 1 & 1 & 30 \\ \hline 12 & 16 & 0 \end{array}$$

Observe that this table looks like an initial simplex tableau without the slack variables. Next, we write a matrix whose columns are the rows of this matrix, and the rows are the columns. Such a matrix is called a **transpose** of the original matrix. We get:

$$\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 1 & 16 \\ \hline 40 & 30 & 0 \end{array}$$

The following maximization problem associated with the above matrix is called its dual.

$$\begin{array}{ll} \text{Maximize} & Z = 40y_1 + 30y_2 \\ \text{Subject to:} & y_1 + y_2 \leq 12 \\ & 2y_1 + y_2 \leq 16 \\ & y_1 \geq 0; y_2 \geq 0 \end{array}$$

Note that we have chosen the variables as y's, instead of x's, to distinguish the two problems.

Example 4.3.2

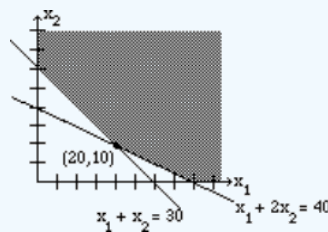
Solve graphically both the minimization problem and its dual maximization problem.

Solution

Our minimization problem is as follows.

$$\begin{array}{ll} \text{Minimize} & Z = 12x_1 + 16x_2 \\ \text{Subject to:} & x_1 + 2x_2 \geq 40 \\ & x_1 + x_2 \geq 30 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

We now graph the inequalities:

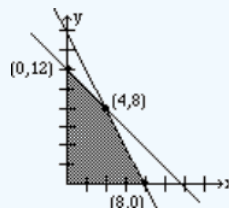


We have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (20, 10) gives the lowest value for the objective function and that value is 400.

Now its dual is:

$$\begin{array}{ll} \text{Maximize} & Z = 40y_1 + 30y_2 \\ \text{Subject to:} & y_1 + y_2 \leq 12 \\ & 2y_1 + y_2 \leq 16 \\ & y_1 \geq 0; y_2 \geq 0 \end{array}$$

We graph the inequalities:



Again, we have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (4, 8) gives the highest value for the objective function, with a value of 400.

The reader may recognize that Example 4.3.2 above is the same as Example 3.1.1, in [section 3.1](#). It is also the same problem as Example 4.1.1 in [section 4.1](#), where we solved it by the simplex method.

We observe that the minimum value of the minimization problem is the same as the maximum value of the maximization problem; in Example 4.3.2 the minimum and maximum are both 400. This is not a coincidence. We state the duality principle.

The Duality Principle

The Duality Principle

The objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.

Our next goal is to extract the solution for our minimization problem from the corresponding dual. To do this, we solve the dual by the simplex method.

Example 4.3.3

Find the solution to the minimization problem in Example 4.3.1 by solving its dual using the simplex method. We rewrite our problem.

$$\begin{array}{ll} \text{Minimize} & Z = 12x_1 + 16x_2 \\ \text{Subject to:} & x_1 + 2x_2 \geq 40 \\ & x_1 + x_2 \geq 30 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

Solution

$$\begin{array}{ll} \text{Maximize} & Z = 40y_1 + 30y_2 \\ \text{Subject to:} & y_1 + y_2 \leq 12 \\ & 2y_1 + y_2 \leq 16 \\ & y_1 \geq 0; y_2 \geq 0 \end{array}$$

Recall that we solved the above problem by the simplex method in Example 4.1.1, [section 4.1](#). Therefore, we only show the initial and final simplex tableau.

The initial simplex tableau is

y_1	y_2	x_1	x_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

Observe an important change. Here our main variables are y_1 and y_2 and the slack variables are x_1 and x_2 .

The final simplex tableau reads as follows:

y_1	y_2	x_1	x_2	Z	
0	1	2	-1	0	8
1	0	-1	1	0	4
0	0	20	10	1	400

A closer look at this table reveals that the x_1 and x_2 values along with the minimum value for the minimization problem can be obtained from the last row of the final tableau. We have highlighted these values by the arrows.

y_1	y_2	x_1	x_2	Z	
0	1	2	-1	0	8
1	0	-1	1	0	4
0	0	20	10	1	400
		↑	↑	↑	

We restate the solution as follows:

The minimization problem has a minimum value of 400 at the corner point (20, 10)

We now summarize our discussion.

Minimization by the Simplex Method

1. Set up the problem.
2. Write a matrix whose rows represent each constraint with the objective function as its bottom row.
3. Write the transpose of this matrix by interchanging the rows and columns.
4. Now write the dual problem associated with the transpose.
5. Solve the dual problem by the simplex method learned in [section 4.1](#).
6. The optimal solution is found in the bottom row of the final matrix in the columns corresponding to the slack variables, and the minimum value of the objective function is the same as the maximum value of the dual.

4.3.1: Minimization By The Simplex Method (Exercises)

SECTION 4.3 PROBLEM SET: MINIMIZATION BY THE SIMPLEX METHOD

In problems 1-2, convert each minimization problem into a maximization problem, the dual, and then solve by the simplex method.

1)

$$\begin{aligned} \text{Minimize } z &= 6x_1 + 8x_2 \\ \text{subject to } 2x_1 + 3x_2 &\geq 7 \\ 4x_1 + 5x_2 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2)

$$\begin{aligned} \text{Minimize } z &= 5x_1 + 6x_2 + 7x_3 \\ \text{subject to } 3x_1 + 2x_2 + 3x_3 &\geq 10 \\ 4x_1 + 3x_2 + 5x_3 &\geq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

SECTION 4.3 PROBLEM SET: MINIMIZATION BY THE SIMPLEX METHOD

In problems 3-4, convert each minimization problem into a maximization problem, the dual, and then solve by the simplex method.

3)

$$\begin{aligned} \text{Minimize } z &= 4x_1 + 3x_2 \\ \text{subject to } x_1 + x_2 &\geq 10 \\ 3x_1 + 2x_2 &\geq 24 \\ x_1, x_2 &\geq 0 \end{aligned}$$

4) A diet is to contain at least 8 units of vitamins, 9 units of minerals, and 10 calories. Three foods, Food A, Food B, and Food C are to be purchased. Each unit of Food A provides 1 unit of vitamins, 1 unit of minerals, and 2 calories. Each unit of Food B provides 2 units of vitamins, 1 unit of minerals, and 1 calorie. Each unit of Food C provides 2 units of vitamins, 1 unit of minerals, and 2 calories. If Food A costs \$3 per unit, Food B costs \$2 per unit and Food C costs \$3 per unit, how many units of each food should be purchased to keep costs at a minimum?

4.4: Chapter Review

SECTION 4.4 PROBLEM SET: CHAPTER REVIEW

Solve the following linear programming problems using the simplex method.

1)

$$\begin{array}{ll} \text{Maximize} & z = 5x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 12 \\ & 2x_1 + x_2 \leq 16 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

2)

$$\begin{array}{ll} \text{Maximize} & z = 5x_1 + 8x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 30 \\ & 3x_1 + x_2 \leq 30 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

3)

$$\begin{array}{ll} \text{Maximize} & z = 2x_1 + 3x_2 + x_3 \\ \text{subject to} & 4x_1 + 2x_2 + 5x_3 \leq 32 \\ & 2x_1 + 4x_2 + 3x_3 \leq 28 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

4)

$$\begin{array}{ll} \text{Maximize} & z = x_1 + 6x_2 + 8x_3 \\ \text{subject to} & x_1 + 2x_2 \leq 1200 \\ & 2x_2 + x_3 \leq 1800 \\ & 4x_1 + x_3 \leq 3600 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

5)

$$\begin{array}{ll} \text{Maximize} & z = 6x_1 + 8x_2 + 5x_3 \\ \text{subject to} & 4x_1 + x_2 + x_3 \leq 1800 \\ & 2x_1 + 2x_2 + x_3 \leq 2000 \\ & 4x_1 + 2x_2 + x_3 \leq 3200 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

6)

$$\begin{array}{ll} \text{Minimize} & z = 12x_1 + 10x_2 \\ \text{subject to} & x_1 + x_2 \geq 6 \\ & 2x_1 + x_2 \geq 8 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

7)

$$\begin{array}{ll} \text{Minimize} & z = 4x_1 + 6x_2 + 7x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 \\ & x_1 + 2x_2 + x_3 \geq 30 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

8)

$$\begin{array}{ll} \text{Minimize} & z = 40x_1 + 48x_2 + 30x_3 \\ \text{subject to} & 2x_1 + 2x_2 + x_3 \geq 25 \\ & x_1 + 3x_2 + 2x_3 \geq 30 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

9) An appliance store sells three different types of ovens: small, medium, and large. The small, medium, and large ovens require, respectively, 3, 5, and 6 cubic feet of storage space; a maximum of 1,000 cubic feet of storage space is available. Each oven takes 1 hour of sales time; there is a maximum of 200 hours of sales labor time available for ovens. The small, medium, and large ovens require, respectively, 1, 1, and 2 hours of installation time; a maximum of 280 hours of installer labor for ovens is available monthly.

If the profit made from sales of small, medium and large ovens is \$50, \$100, and \$150, respectively, how many of each type of oven should be sold to maximize profit, and what is the maximum profit?

SECTION 4.4 PROBLEM SET: CHAPTER REVIEW

10) A factory manufactures three products, A, B, and C. Each product requires the use of two machines, Machine I and Machine II. The total hours available, respectively, on Machine I and Machine II per month are 180 and 300. The time requirements and profit per unit for each product are listed below.

	A	B	C
Machine I	1	2	2
Machine II	2	2	4
Profit	20	30	40

How many units of each product should be manufactured to maximize profit, and what is the maximum profit?

11) A company produces three products, A, B, and C, at its two factories, Factory I and Factory II. Daily production of each factory for each product is listed below.

	Factory I	Factory II
Product A	10	20
Product B	20	20
Product C	20	10

The company must produce at least 1000 units of product A, 1600 units of B, and 700 units of C. If the cost of operating Factory I is \$4,000 per day and the cost of operating Factory II is \$5000, how many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

12) For his classes, Professor Wright gives three types of quizzes, objective, recall, and recall-plus. To keep his students on their toes, he has decided to give at least 20 quizzes next quarter.

The three types, objective, recall, and recall-plus quizzes, require the students to spend, respectively, 10 minutes, 30 minutes, and 60 minutes for preparation, and Professor Wright would like them to spend at least 12 hours(720 minutes) preparing for these quizzes above and beyond the normal study time.

An average score on an objective quiz is 5, on a recall type 6, and on a recall-plus 7, and Dr. Wright would like the students to score at least 130 points on all quizzes.

It takes the professor one minute to grade an objective quiz, 2 minutes to grade a recall type quiz, and 3 minutes to grade a recall-plus quiz.

How many of each type should he give in order to minimize his grading time?