Chapter Outline

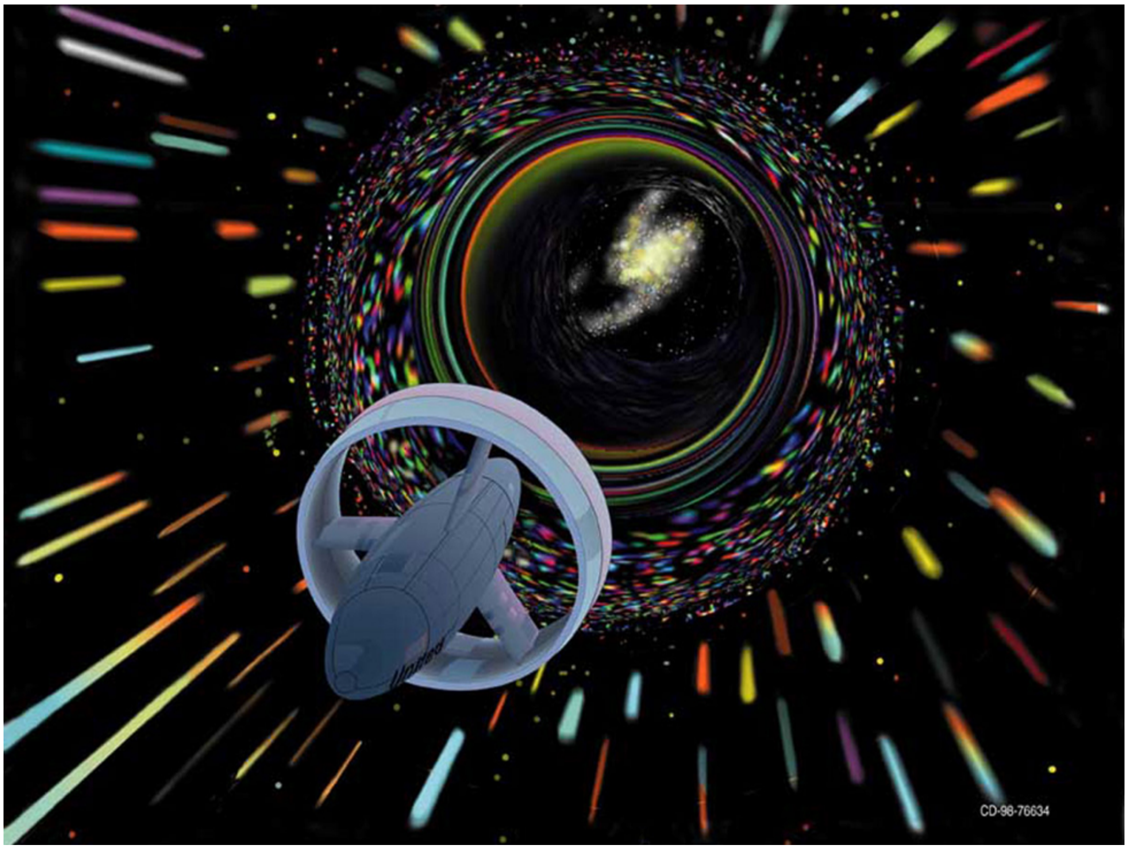


Figure 2.1 The vision of human exploration by the National Aeronautics and Space Administration (NASA) to distant parts of the universe illustrates the idea of space travel at high speeds. But, is there a limit to how fast a spacecraft can go? (credit: NASA)

Science fiction writers often imagine spaceships that can travel to far-off planets in distant galaxies. However, back in 1905, Albert Einstein showed that a limit exists to how fast any object can travel. The problem is that the faster an object moves, the more mass it attains (in the form of energy), according to the equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in Example 2.12.)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

Learning Objectives

2.1.1. Describe the tangent problem and how it led to the idea of a derivative.
2.1.2. Explain how the idea of a limit is involved in solving the tangent problem.
2.1.3. Recognize a tangent to a curve at a point as the limit of secant lines.
2.1.4. Identify instantaneous velocity as the limit of average velocity over a small time interval.
2.1.5. Describe the area problem and how it was solved by the integral.
2.1.6. Explain how the idea of a limit is involved in solving the area problem.
2.1.7. Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in Figure 2.2.

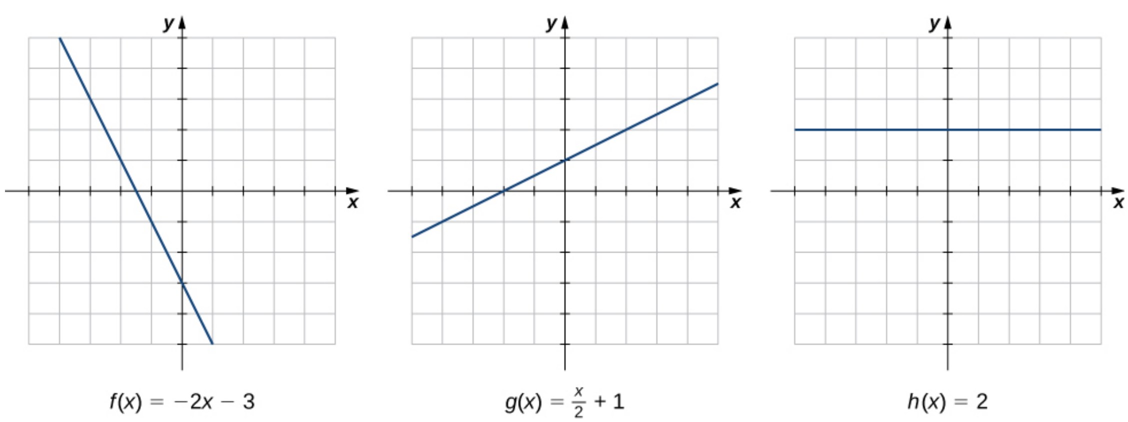


Figure 2.2 The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the *x*-axis, the *y*-coordinate decreases by 2 units. This rate of change is determined by the slope (–2) of the line. Similarly, the slope of 1/2 in the function $g(x)$ tells us that for every change in *x* of 1 unit there is a corresponding change in *y* of 1/2 unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (Figure 2.3). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

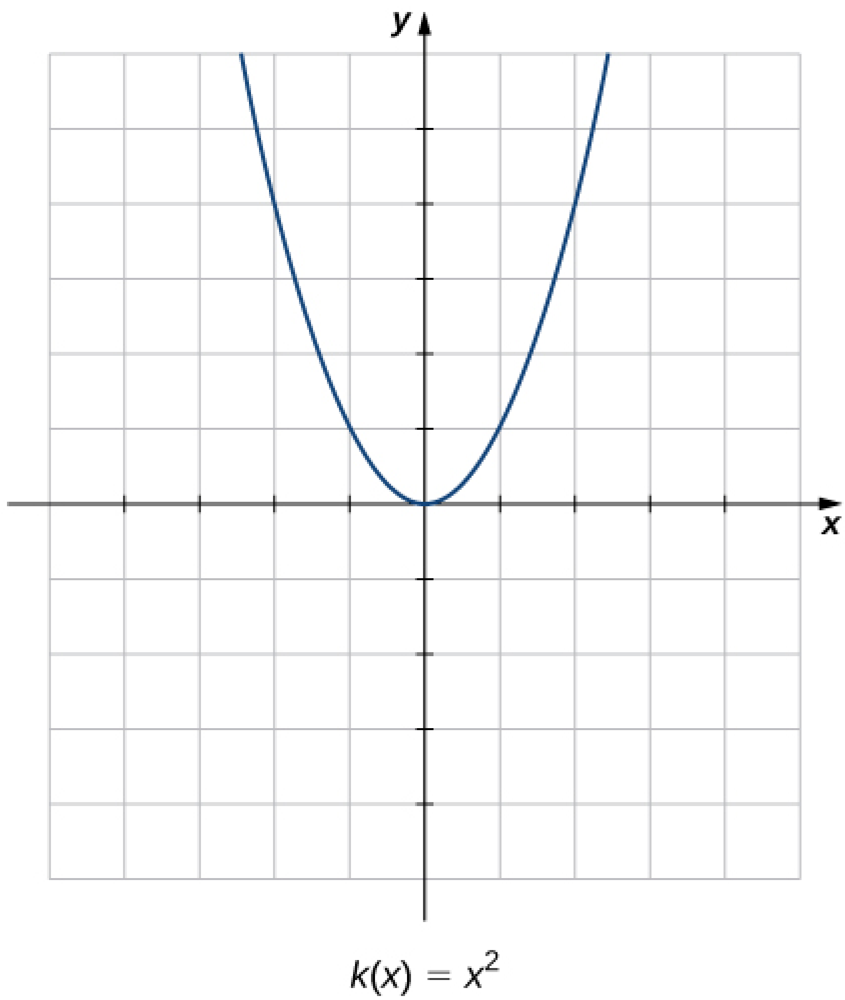


Figure 2.3 The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant** line. Figure 2.4 shows a secant line to a function $f(x)$ at a point $(a, f(a))$.

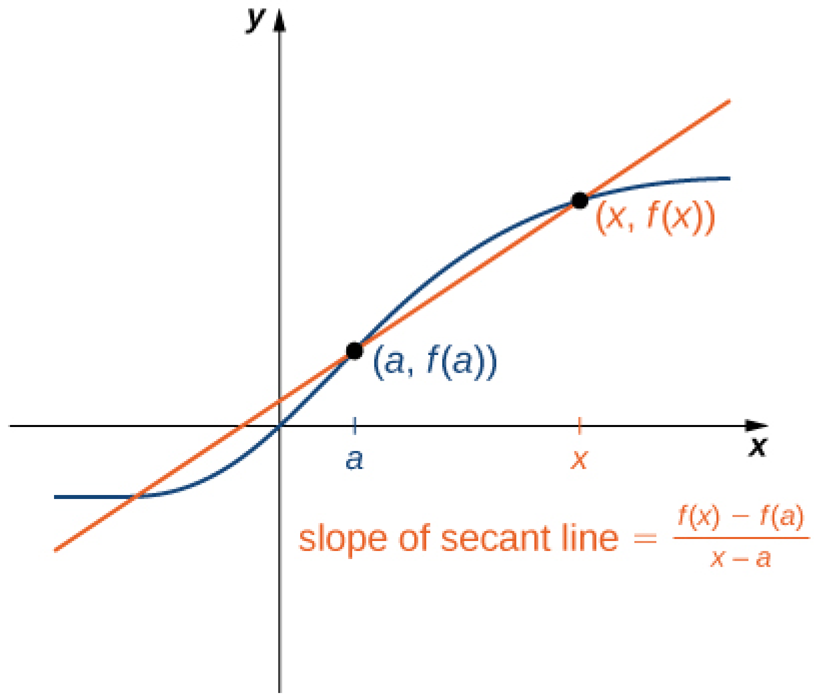


Figure 2.4 The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

DEFINITION

The **secant** to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}. \tag{2.1}$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close *x* is to *a*. As we see in Figure 2.5, if *x* is closer to *a*, the slope of the secant line is a better measure of the rate of change of $f(x)$ at *a*.

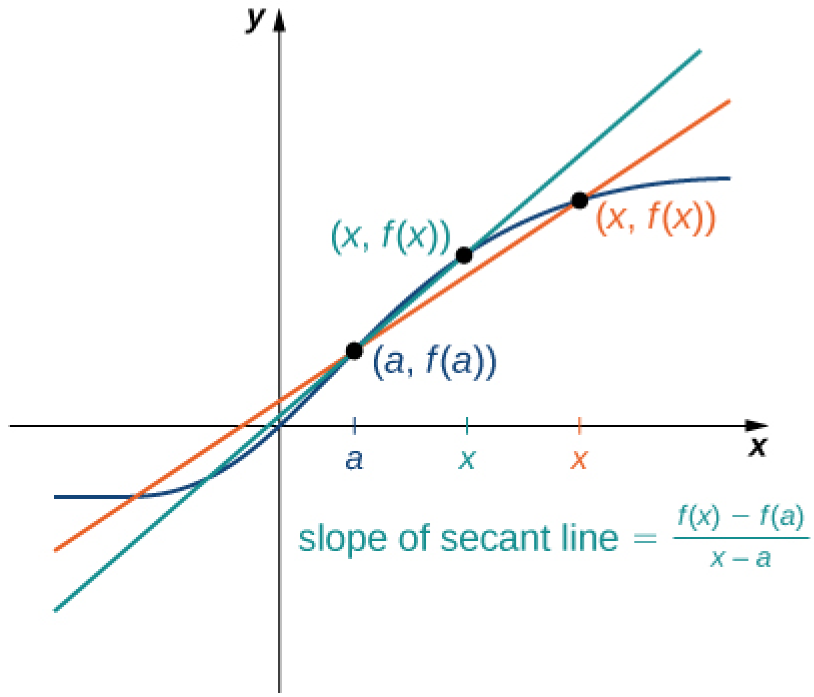


Figure 2.5 As *x* gets closer to *a*, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at *a*.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at *a* (Figure 2.6). The slope of the tangent line to the graph at *a* measures the rate of change of the function at *a*. This value also represents the derivative of the function $f(x)$ at *a*, or the rate of change of the function at *a*. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

MEDIA

For an interactive demonstration of the slope of a secant line that you can manipulate yourself, visit this applet (*Note:* this site requires a Java browser plugin): Math Insight.

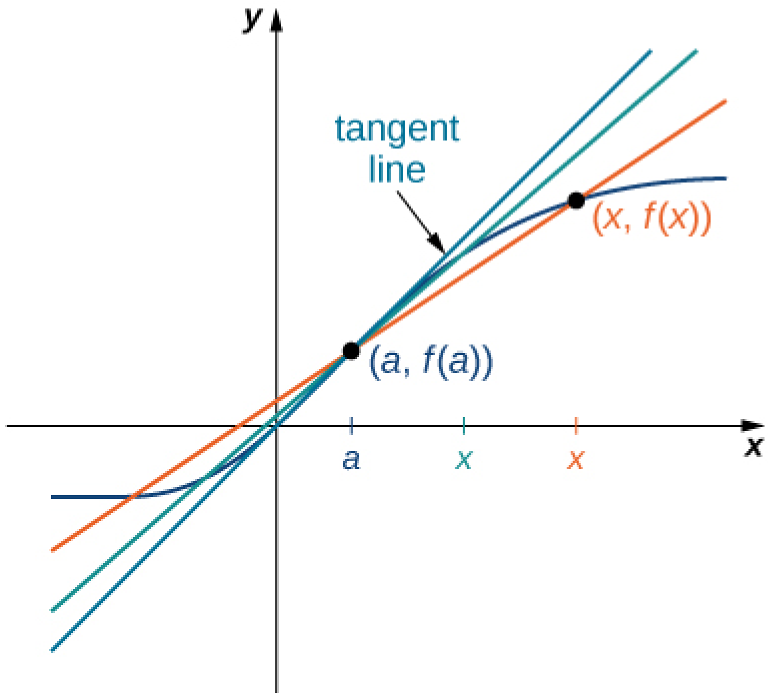


Figure 2.6 Solving the Tangent Problem: As *x* approaches *a*, the secant lines approach the tangent line.

Example 2.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

EXAMPLE 2.1

Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$
b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

[Show Solution]

CHECKPOINT 2.1

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $\left(\frac{5}{4}, \frac{25}{16}\right)$ on the graph of $f(x) = x^2$.

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time *t*. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

DEFINITION

Let $s(t)$ be the position of an object moving along a coordinate axis at time *t*. The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \tag{2.2}$$

As *t* is chosen closer to *a*, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point *a*, we let the *x*-values approach *a* in the slope of the secant line. Similarly, to find the instantaneous velocity at time *a*, we let the *t*-values approach *a* in the average velocity. This process of letting *x* or *t* approach *a* in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

DEFINITION

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of *t* become closer to *a*, provided such a value exists.

Example 2.2 illustrates this concept of limits and average velocity.

EXAMPLE 2.2

Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground *t* seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$
b. $[0.5, 0.51]$

[Show Solution]

CHECKPOINT 2.2

An object moves along a coordinate axis so that its position at time *t* is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the *x*-axis over an interval (Figure 2.8)?

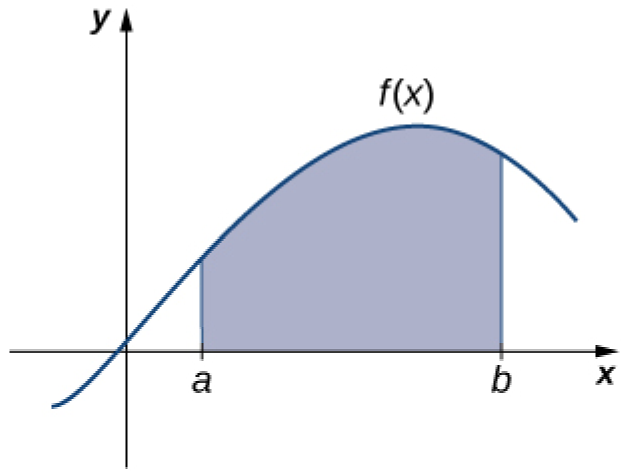


Figure 2.8 The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles (Figure 2.9).

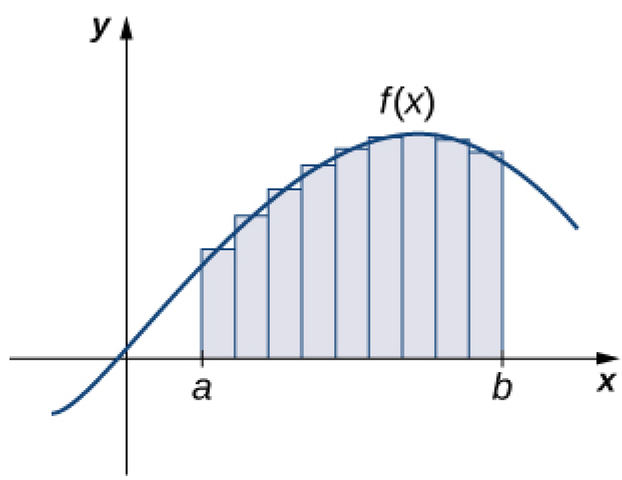


Figure 2.9 The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the x-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

EXAMPLE 2.3

Estimation Using Rectangles

Estimate the area between the *x*-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 2.10.

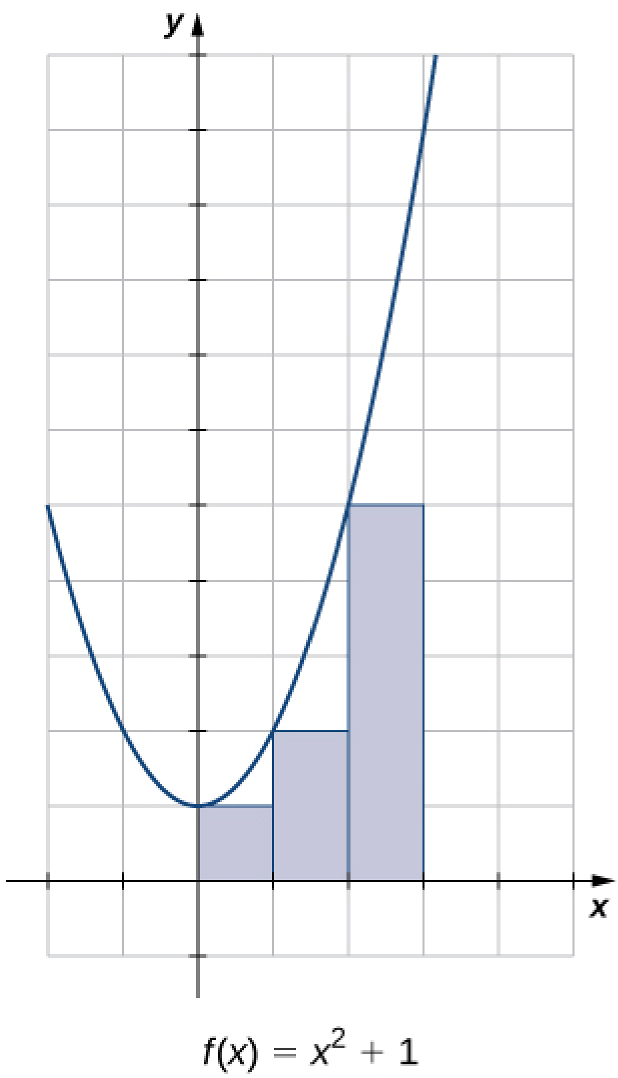
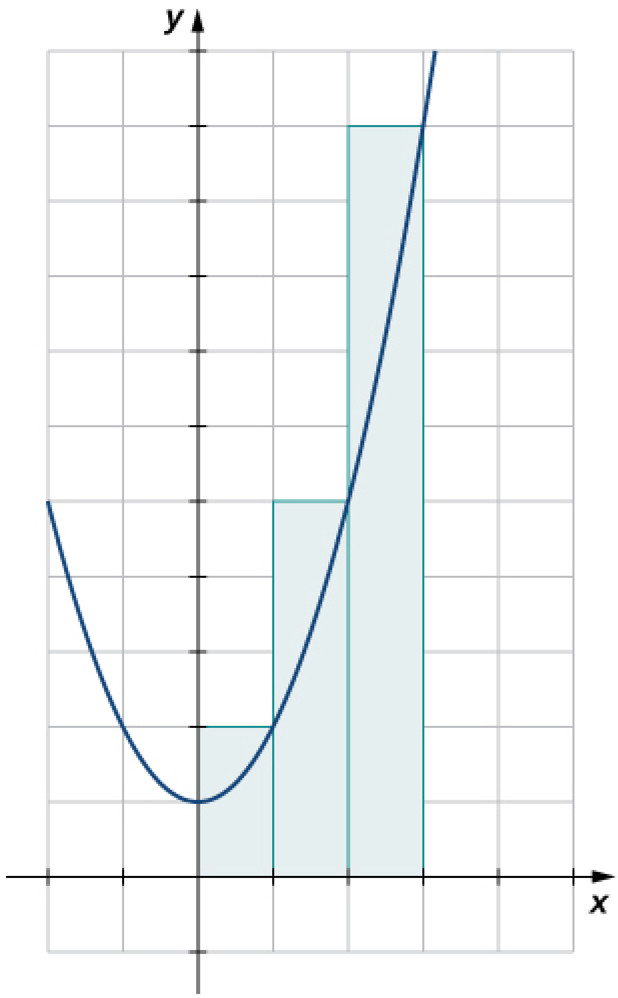


Figure 2.10 The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

[Show Solution]

CHECKPOINT 2.3

Estimate the area between the *x*-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown here:



$f(x) = x^2 + 1$

Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in Figure 2.11. These are only a few of the types of questions that can be asked and answered using **multivariable calculus**. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.

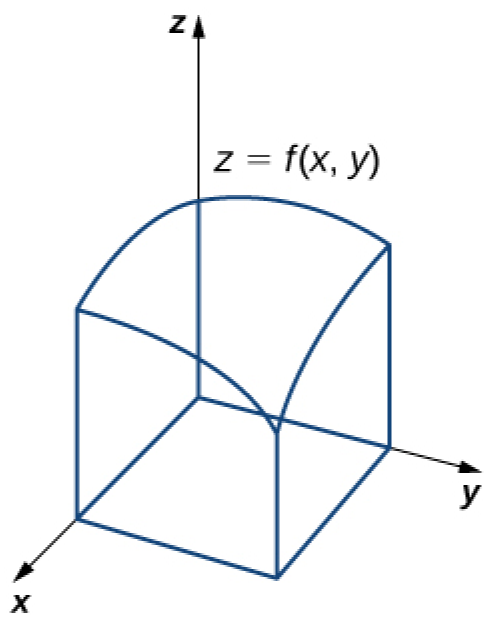


Figure 2.11 We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

Section 2.1 Exercises

For the following exercises, points $P(1, 2)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^2 + 1$.

1. **[T]** Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q(x, y)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	*y*	$Q(x, y)$	m_{sec}

x	*y*	$Q(x, y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

2. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to *f* at $x = 1$.

3. Use the value in the preceding exercise to find the equation of the tangent line at point *P*. Graph $f(x)$ and the tangent line.

For the following exercises, points $P(1, 1)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^3$.

4. **[T]** Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q(x, y)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	*y*	$Q(x, y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

5. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to *f* at $x = 1$.

6. Use the value in the preceding exercise to find the equation of the tangent line at point *P*. Graph $f(x)$ and the tangent line.

For the following exercises, points $P(4, 2)$ and $Q(x, y)$ are on the graph of the function $f(x) = \sqrt{x}$.

7. **[T]** Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q(x, y)$, and the slope of the secant line passing through points *P* and *Q*.

Round your answer to eight significant digits.

x	*y*	$Q(x, y)$	m_{sec}
4.1	a.	e.	i.
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

8. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to *f* at $x = 4$.

9. Use the value in the preceding exercise to find the equation of the tangent line at point *P*.

For the following exercises, points $P(1.5, 0)$ and $Q(\phi, y)$ are on the graph of the function $f(\phi) = \cos(\pi\phi)$.

10. **[T]** Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q(\varphi, y)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	*y*	$Q(\phi, y)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

11. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to *f* at $\varphi = 1.5$.

12. Use the value in the preceding exercise to find the equation of the tangent line at point *P*.

For the following exercises, points $P(-1, -1)$ and $Q(x, y)$ are on the graph of the function $f(x) = \frac{1}{x}$.

13. **[T]** Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q(x, y)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	*y*	$Q(x, y)$	m_{sec}
−1.05	a.	e.	i.
−1.01	b.	f.	j.
−1.005	c.	g.	k.
−1.001	d.	h.	l.

14. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to *f* at $x = -1$.

15. Use the value in the preceding exercise to find the equation of the tangent line at point *P*.

For the following exercises, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t) = 200 - 4.9t^2$, where position *s* is measured in meters and time *t* is measured in seconds. Round your answer to eight significant digits.

16. **[T]** Compute the average velocity of the ball over the given time intervals.
   a. $[4.99, 5]$
   b. $[5, 5.01]$
   c. $[4.999, 5]$
   d. $[5, 5.001]$

17. Use the preceding exercise to guess the instantaneous velocity of the ball at $t = 5$ sec.

For the following exercises, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time *t* seconds is $h(t) = 15t - 4.9t^2$.

18. **[T]** Compute the average velocity of the stone over the given time intervals.
   a. $[1, 1.05]$
   b. $[1, 1.01]$
   c. $[1, 1.005]$
   d. $[1, 1.001]$

19. Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

For the following exercises, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where *t* is measured in seconds.

20. **[T]** Compute the average velocity of the rocket over the given time intervals.
    a. $[9, 9.01]$
    b. $[8.99, 9]$
    c. $[9, 9.001]$
    d. $[8.999, 9]$

21. Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

For the following exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where *d* is the position in meters and *t* is the time elapsed, measured in seconds.

22. **[T]** Compute the average velocity of the runner over the given time intervals.
    a. $[1.95, 2.05]$
    b. $[1.995, 2.005]$
    c. $[1.9995, 2.0005]$
    d. $[2, 2.00001]$

23. Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

For the following exercises, consider the function $f(x) = |x|$.

24. Sketch the graph of *f* over the interval $[-1, 2]$ and shade the region above the *x*-axis.

25. Use the preceding exercise to find the aproximate value of the area between the *x*-axis and the graph of *f* over the interval $[-1, 2]$ using rectangles. For the rectangles, use the square units, and approximate both above and below the lines. Use geometry to find the exact answer.

For the following exercises, consider the function $f(x) = \sqrt{1 - x^2}$. (*Hint*: This is the upper half of a circle of radius 1 positioned at $(0, 0)$.)

26. Sketch the graph of *f* over the interval $[-1, 1]$.

27. Use the preceding exercise to find the aproximate area between the *x*-axis and the graph of *f* over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

For the following exercises, consider the function $f(x) = -x^2 + 1.$

28. Sketch the graph of *f* over the interval $[-1, 1]$.

29. Approximate the area of the region between the *x*-axis and the graph of *f* over the interval $[-1, 1]$.

Learning Objectives

2.2.1. Using correct notation, describe the limit of a function.
2.2.2. Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
2.2.3. Use a graph to estimate the limit of a function or to identify when the limit does not exist.
2.2.4. Define one-sided limits and provide examples.
2.2.5. Explain the relationship between one-sided and two-sided limits.
2.2.6. Using correct notation, describe an infinite limit.
2.2.7. Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

$$f(x) = \frac{x^2 - 4}{x - 2},\ g(x) = \frac{|x - 2|}{x - 2},\ \text{and } h(x) = \frac{1}{(x - 2)^2},$$

which are shown in Figure 2.12. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

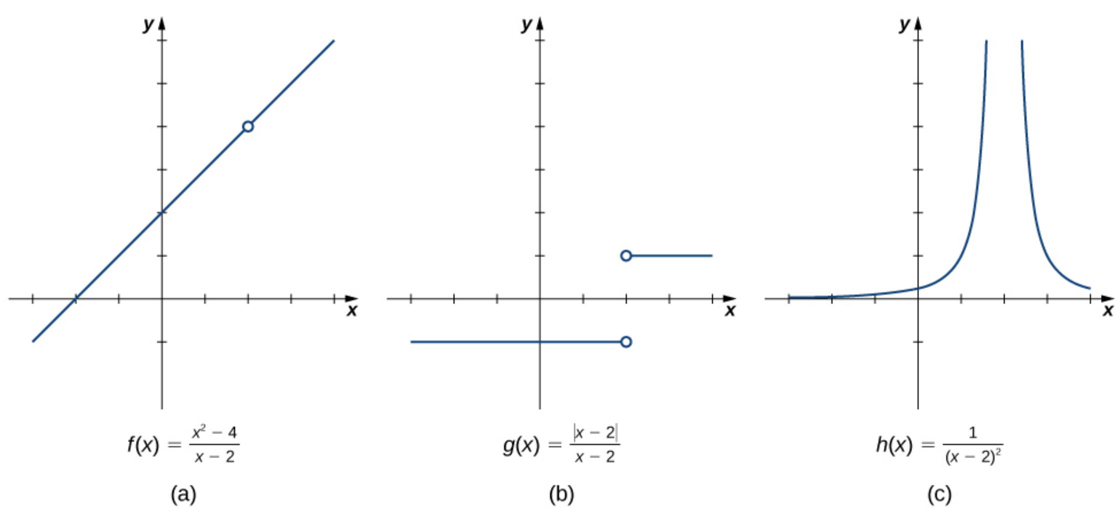


Figure 2.12 These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2,$ but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2.$ To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in Figure 2.12. As the values of *x* approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as *x* approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4.$$

From this very brief informal look at one limit, let's start to develop an **intuitive definition of the limit**. We can think of the limit of a function at a number *a* as being the one real number *L* that the functional values approach as the *x*-values approach *a,* provided such a real number *L* exists. Stated more carefully, we have the following definition:

DEFINITION

Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. If *all* values of the function $f(x)$ approach the real number *L* as the values of $x(\neq a)$ approach the number *a*, then we say that the limit of $f(x)$ as *x* approaches *a* is *L*. (More succinct, as *x* gets closer to *a*, $f(x)$ gets closer and stays close to *L*.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \tag{2.3}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

PROBLEM-SOLVING STRATEGY: EVALUATING A LIMIT USING A TABLE OF FUNCTIONAL VALUES

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of *x*-values—one set of values approaching *a* and less than *a*, and another set of values approaching *a* and greater than *a*. Table 2.1 demonstrates what your tables might look like.

x	$f(x)$		x	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.			Use additional values as necessary.	

Table 2.1 Table of Functional Values for $\lim_{x \to a} f(x)$

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a - 0.1), f(a - 0.01), f(a - 0.001).\,, f(a - 0.0001)$, and so on, and $f(a + 0.1), f(a + 0.01), f(a + 0.001), f(a + 0.0001)$, and so on. (*Note*: Although we have chosen the *x*-values $a \pm 0.1, a \pm 0.01, a \pm 0.001, a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)
3. If both columns approach a common *y*-value *L*, we state $\lim_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.
4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for *x*-values near *a* are in our window. We can use the trace feature to move along the graph of the function and watch the *y*-value readout as the *x*-values approach *a*. If the *y*-values approach *L* as our *x*-values approach *a* from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in Example 2.4.

EXAMPLE 2.4

Evaluating a Limit Using a Table of Functional Values 1

Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ using a table of functional values.

[Show Solution]

EXAMPLE 2.5

Evaluating a Limit Using a Table of Functional Values 2

Evaluate $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ using a table of functional values.

[Show Solution]

CHECKPOINT 2.4

Estimate $\lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1}$ using a table of functional values. Use a graph to confirm your estimate.

At this point, we see from Example 2.4 and Example 2.5 that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example 2.6, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

EXAMPLE 2.6

Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure 2.15, evaluate $\lim_{x \to -1} g(x)$.

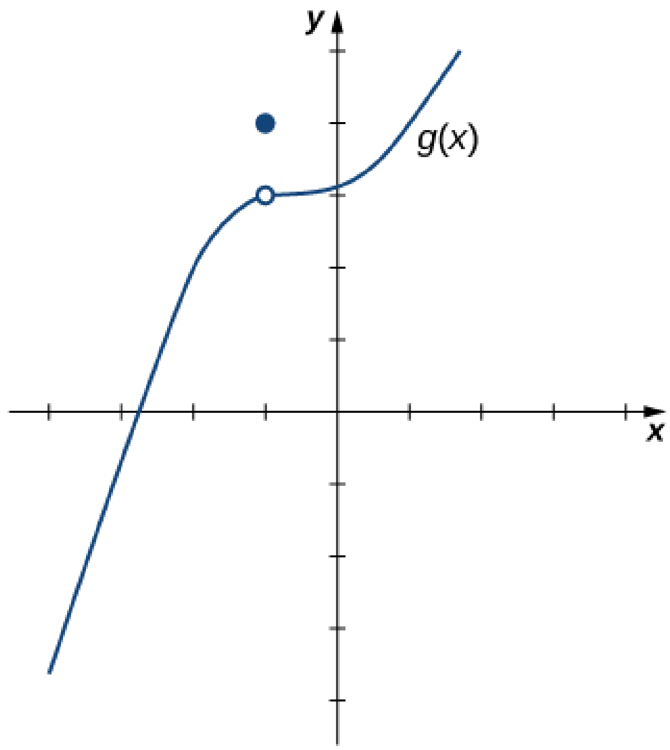


Figure 2.15 The graph of $g(x)$ includes one value not on a smooth curve.

[Show Solution]

Based on Example 2.6, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

CHECKPOINT 2.5

Use the graph of $h(x)$ in Figure 2.16 to evaluate $\lim_{x \to 2} h(x)$, if possible.

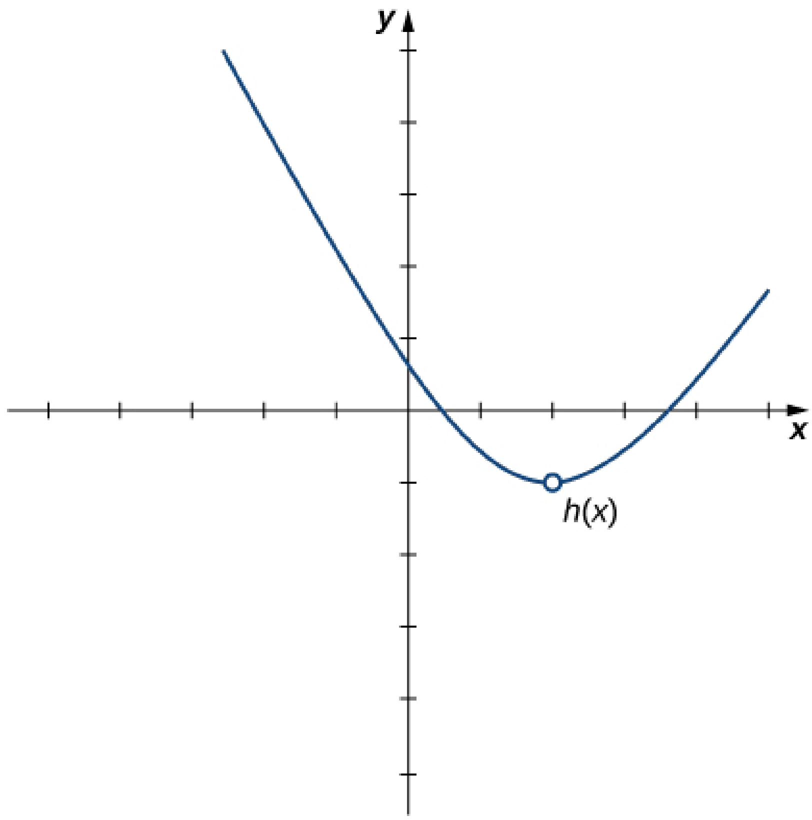


Figure 2.16

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

THEOREM 2.1

Two Important Limits

Let a be a real number and c be a constant.

i. $$\lim_{x \to a} x = a \quad \boxed{2.4}$$

ii. $$\lim_{x \to a} c = c \quad \boxed{2.5}$$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim\limits_{x \to a} x = a$.
ii. For the second limit, consider Table 2.4.

x	$f(x) = c$		x	$f(x) = c$
$a - 0.1$	c		$a + 0.1$	c
$a - 0.01$	c		$a + 0.01$	c
$a - 0.001$	c		$a + 0.001$	c
$a - 0.0001$	c		$a + 0.0001$	c

Table 2.4 Table of Functional Values for $\lim\limits_{x \to a} c = c$

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim\limits_{x \to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

EXAMPLE 2.7

Evaluating a Limit That Fails to Exist

Evaluate $\lim_{x \to 0} \sin(1/x)$ using a table of values.

[Show Solution]

CHECKPOINT 2.6

Use a table of functional values to evaluate $\lim_{x \to 2} \frac{|x^2-4|}{x-2}$, if possible.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x-2|/(x-2)$ introduced at the beginning of the section (see Figure 2.12(b)). As we pick values of *x* close to 2, $g(x)$ does not approach a single value, so the limit as *x* approaches 2 does not exist—that is, $\lim_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the *x*-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or *the negative side of* 2), $g(x) = -1.$ Thus, as *x* approaches 2 from the left, $g(x)$ approaches –1. Mathematically, we say that the limit as *x* approaches 2 from the left is –1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as *x* approaches 2 from the right (or *from the positive side*), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

DEFINITION

We define two types of **one-sided limits**.

Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (*c*, *a*), and let *L* be a real number. If the values of the function $f(x)$ approach the real number *L* as the values of *x* (where $x < a$) approach the number *a*, then we say that *L* is the limit of $f(x)$ as *x* approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L. \tag{2.6}$$

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let *L* be a real number. If the values of the function $f(x)$ approach the real number L as the values of *x* (where $x > a$) approach the number *a*, then we say that *L* is the limit of $f(x)$ as *x* approaches *a* from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L. \tag{2.7}$$

EXAMPLE 2.8

Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim\limits_{x \to 2^-} f(x)$

b. $\lim\limits_{x \to 2^+} f(x)$

[Show Solution]

CHECKPOINT 2.7

Use a table of functional values to estimate the following limits, if possible.

a. $\lim\limits_{x \to 2^-} \dfrac{|x^2 - 4|}{x - 2}$

b. $\lim\limits_{x \to 2^+} \dfrac{|x^2 - 4|}{x - 2}$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in Relating One-Sided and Two-Sided Limits.

THEOREM 2.2

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. Then,

$$\lim_{x \to a} f(x) = L. \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2,$ the third and final function introduced at the beginning of this section (see Figure 2.12(c)). From its graph we see that as the values of *x* approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x \to 2} h(x) = +\infty.$$

More generally, we define **infinite limits** as follows:

DEFINITION

We define three types of **infinite limits.**

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty. \tag{2.8}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty. \tag{2.9}$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty. \tag{2.10}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty. \tag{2.11}$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing *a*.

i. If the values of $f(x)$ increase without bound as the values of *x* (where $x \neq a$) approach the number *a*, then we say that the limit as *x* approaches *a* is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty. \tag{2.12}$$

ii. If the values of $f(x)$ decrease without bound as the values of *x* (where $x \neq a$) approach the number *a*, then we say that the limit as *x* approaches *a* is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty. \tag{2.13}$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at *a*, it must approach a real number *L* as *x* approaches *a*. That said, if, for example, $\lim_{x \to a} f(x) = +\infty,$ we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

EXAMPLE 2.9

Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x}$

b. $\lim_{x \to 0^+} \frac{1}{x}$

c. $\lim_{x \to 0} \frac{1}{x}$

[Show Solution]

CHECKPOINT 2.8

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim\limits_{x \to 0^-} \frac{1}{x^2}$
b. $\lim\limits_{x \to 0^+} \frac{1}{x^2}$
c. $\lim\limits_{x \to 0} \frac{1}{x^2}$

It is useful to point out that functions of the form $f(x) = 1/(x-a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (Figure 2.20). These limits are summarized in Infinite Limits from Positive Integers.

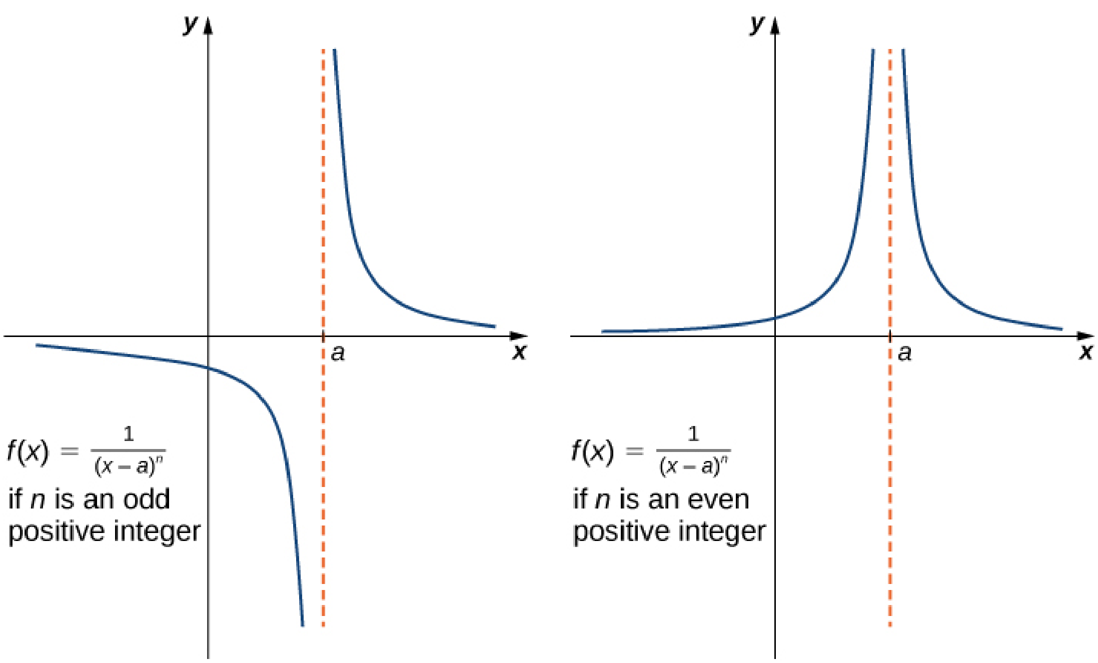


Figure 2.20 The function $f(x) = 1/(x-a)^n$ has infinite limits at a.

THEOREM 2.3

Infinite Limits from Positive Integers

If *n* is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If *n* is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having *x*-coordinates very near to *a* are very close to the vertical line $x = a$. That is, as *x* approaches *a*, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a **vertical asymptote** of the graph. We formally define a vertical asymptote as follows:

DEFINITION

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.

$$\begin{aligned} \lim_{x \to a^-} f(x) &= +\infty \text{ or } -\infty \\ \lim_{x \to a^+} f(x) &= +\infty \text{ or } -\infty \\ &\text{or} \\ \lim_{x \to a} f(x) &= +\infty \text{ or } -\infty \end{aligned}$$

EXAMPLE 2.10

Finding a Vertical Asymptote

Evaluate each of the following limits using Infinite Limits from Positive Integers. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim\limits_{x \to -3^-} \frac{1}{(x+3)^4}$
b. $\lim\limits_{x \to -3^+} \frac{1}{(x+3)^4}$
c. $\lim\limits_{x \to -3} \frac{1}{(x+3)^4}$

[Show Solution]

CHECKPOINT 2.9

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$.

a. $\lim\limits_{x \to 2^-} \frac{1}{(x-2)^3}$
b. $\lim\limits_{x \to 2^+} \frac{1}{(x-2)^3}$
c. $\lim\limits_{x \to 2} \frac{1}{(x-2)^3}$

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

EXAMPLE 2.11

Behavior of a Function at Different Points

Use the graph of $f(x)$ in Figure 2.21 to determine each of the following values:

a. $\lim_{x\to-4^-} f(x)$; $\lim_{x\to-4^+} f(x)$; $\lim_{x\to-4} f(x)$; $f(-4)$
b. $\lim_{x\to-2^-} f(x)$; $\lim_{x\to-2^+} f(x)$; $\lim_{x\to-2} f(x)$; $f(-2)$
c. $\lim_{x\to1^-} f(x)$; $\lim_{x\to1^+} f(x)$; $\lim_{x\to1} f(x)$; $f(1)$
d. $\lim_{x\to3^-} f(x)$; $\lim_{x\to3^+} f(x)$; $\lim_{x\to3} f(x)$; $f(3)$

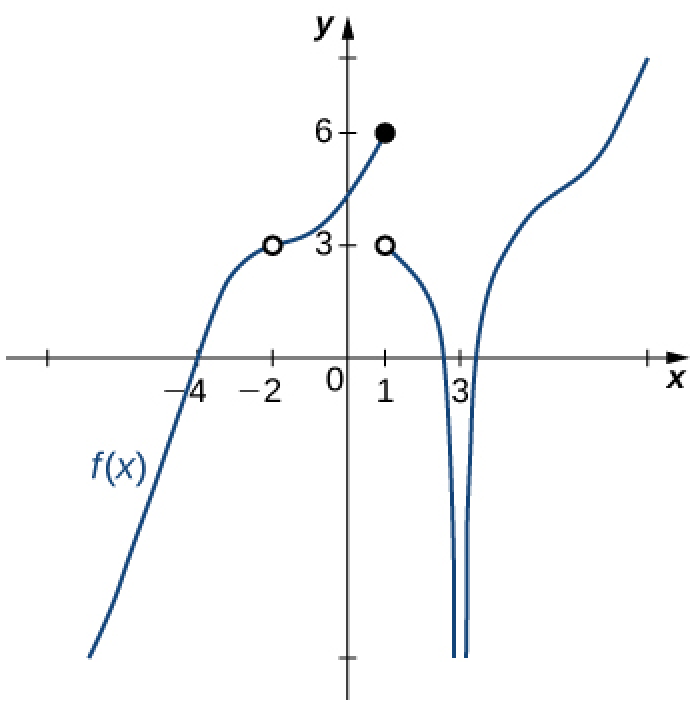
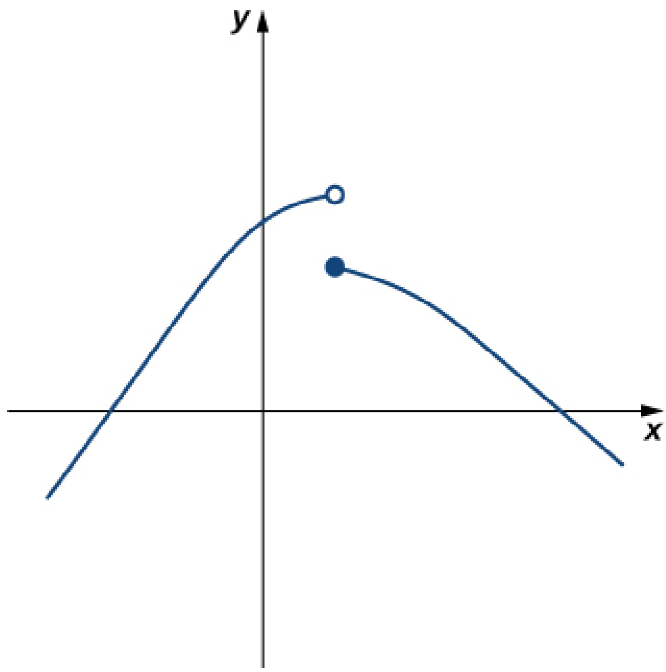


Figure 2.21 The graph shows $f(x)$.

[Show Solution]

CHECKPOINT 2.10

Evaluate $\lim_{x \to 1} f(x)$ for $f(x)$ shown here:



EXAMPLE 2.12

Chapter Opener: Einstein's Equation

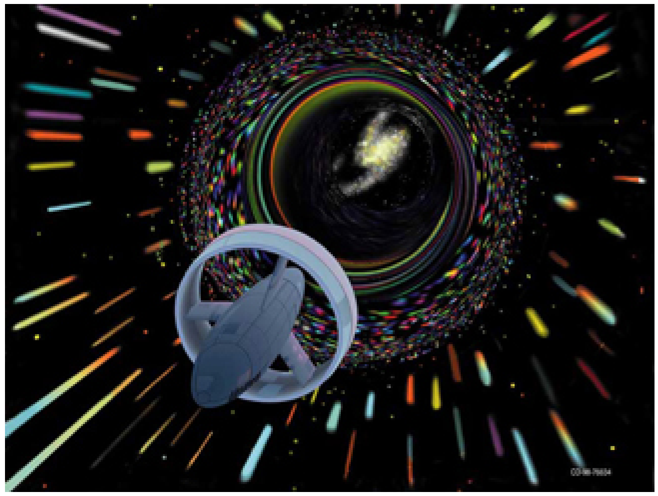


Figure 2.22 (credit: NASA)

In the chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object, what is the value of this bound?

[Show Solution]

Section 2.2 Exercises

For the following exercises, consider the function $f(x) = \frac{x^2 - 1}{|x - 1|}$.

30. **[T]** Complete the following table for the function. Round your solutions to four decimal places.

x	$f(x)$	x	$f(x)$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

31. What do your results in the preceding exercise indicate about the two-sided limit $\lim_{x \to 1} f(x)$? Explain your response.

For the following exercises, consider the function $f(x) = (1 + x)^{1/x}$.

32. **[T]** Make a table showing the values of *f* for $x = -0.01, -0.001, -0.0001, -0.00001$ and for $x = 0.01, 0.001, 0.0001, 0.00001$. Round your solutions to five decimal places.

x	$f(x)$	***x***	$f(x)$
–0.01	a.	0.01	e.
–0.001	b.	0.001	f.
–0.0001	c.	0.0001	g.
–0.00001	d.	0.00001	h.

33. What does the table of values in the preceding exercise indicate about the function $f(x) = (1 + x)^{1/x}$?

34. To which mathematical constant does the limit in the preceding exercise appear to be getting closer?

In the following exercises, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.

35. **[T]** $\lim_{x \to 0} \frac{\sin 2x}{x}$; $\pm 0.1, \pm 0.01, \pm 0.001, \pm .0001$

x	$\frac{\sin 2x}{x}$	***x***	$\frac{\sin 2x}{x}$
–0.1	a.	0.1	e.
–0.01	b.	0.01	f.
–0.001	c.	0.001	g.
–0.0001	d.	0.0001	h.

36. **[T]** $\lim_{x \to 0} \frac{\sin 3x}{x}$ ±0.1, ±0.01, ±0.001, ±0.0001

x	$\frac{\sin 3x}{x}$	*x*	$\frac{\sin 3x}{x}$
–0.1	a.	0.1	e.
–0.01	b.	0.01	f.
–0.001	c.	0.001	g.
–0.0001	d.	0.0001	h.

37. Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x \to 0} \frac{\sin ax}{x}$ for *a*, a positive real value.

[T] In the following exercises, set up a table of values to find the indicated limit. Round to eight digits.

38. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6}$

x	$\frac{x^2-4}{x^2+x-6}$	*x*	$\frac{x^2-4}{x^2+x-6}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

39. $\lim_{x \to 1} (1 - 2x)$

x	$1 - 2x$	*x*	$1 - 2x$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

40. $\lim\limits_{x\to 0} \frac{5}{1-e^{1/x}}$

x	$\frac{5}{1-e^{1/x}}$	x	$\frac{5}{1-e^{1/x}}$
–0.1	a.	0.1	e.
–0.01	b.	0.01	f.
–0.001	c.	0.001	g.
–0.0001	d.	0.0001	h.

41. $\lim\limits_{z\to 0} \frac{z-1}{z^2(z+3)}$

z	$\frac{z-1}{z^2(z+3)}$	z	$\frac{z-1}{z^2(z+3)}$
–0.1	a.	0.1	e.
–0.01	b.	0.01	f.
–0.001	c.	0.001	g.
–0.0001	d.	0.0001	h.

42. $\lim\limits_{t\to 0^+} \frac{\cos t}{t}$

t	$\frac{\cos t}{t}$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

43. $\lim_{x \to 2} \frac{1-\frac{2}{x}}{x^2-4}$

x	$\frac{1-\frac{2}{x}}{x^2-4}$	x	$\frac{1-\frac{2}{x}}{x^2-4}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

[T] In the following exercises, set up a table of values and round to eight significant digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

44. $\lim_{\theta \to 0} \sin\left(\frac{\pi}{\theta}\right)$

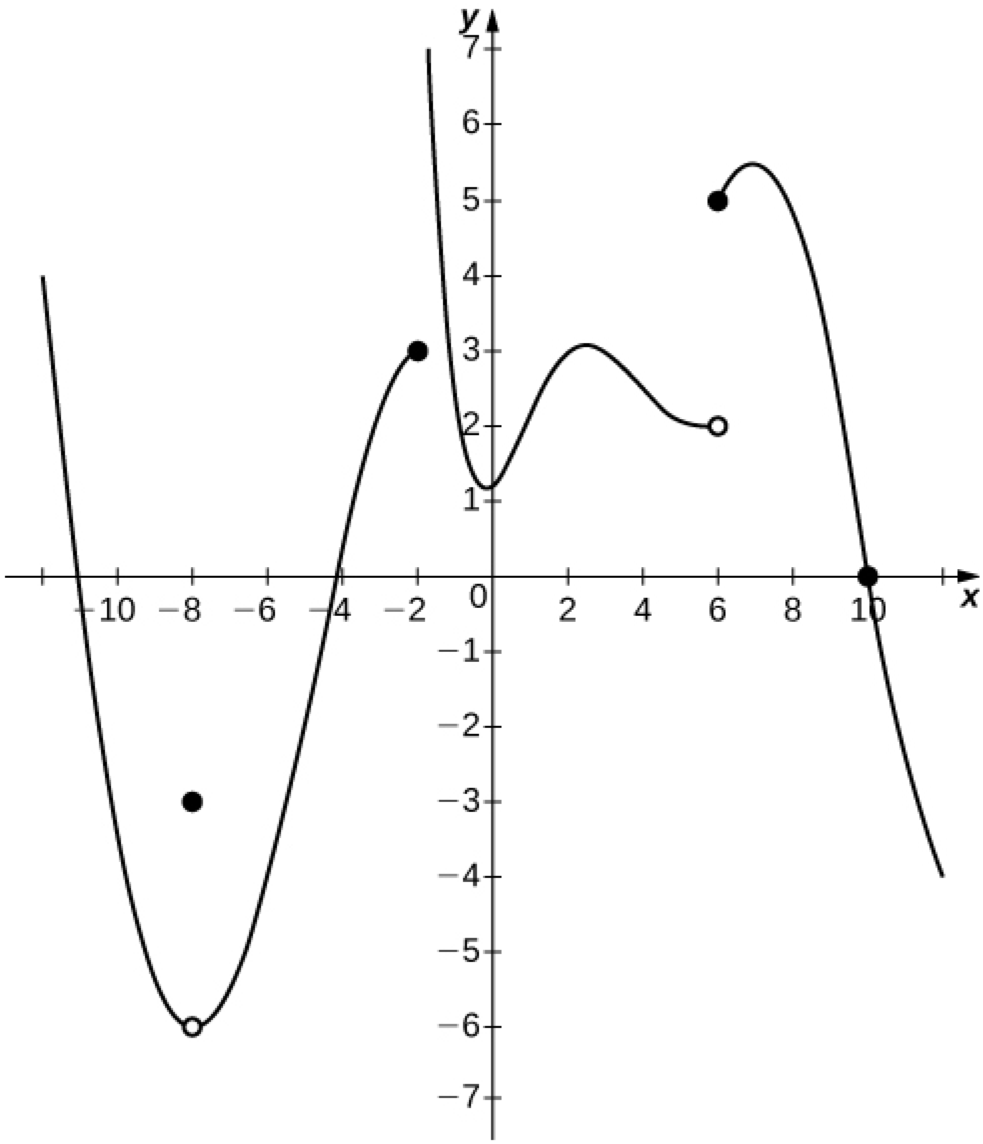
θ	$\sin\left(\frac{\pi}{\theta}\right)$	θ	$\sin\left(\frac{\pi}{\theta}\right)$
–0.1	a.	0.1	e.
–0.01	b.	0.01	f.
–0.001	c.	0.001	g.
–0.0001	d.	0.0001	h.

45. $\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$

a	$\frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$

a	$\frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

In the following exercises, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.



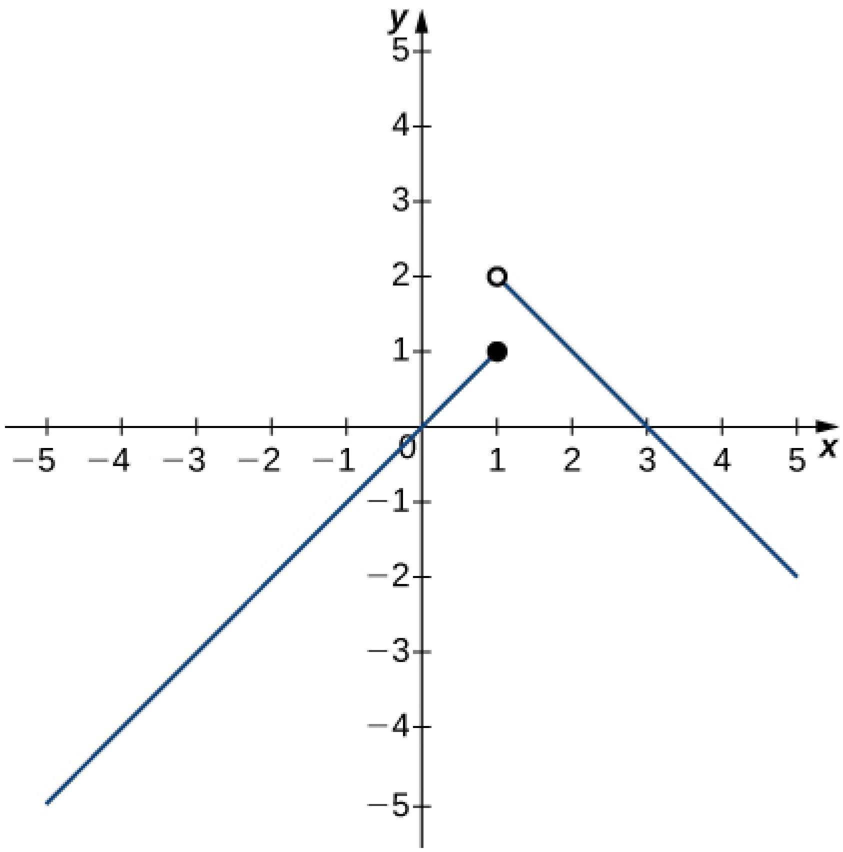
46. $\lim_{x \to 10} f(x) = 0$

47. $\lim_{x \to -2^+} f(x) = 3$

48. $\lim_{x \to -8} f(x) = f(-8)$

49. $\lim_{x \to 6} f(x) = 5$

In the following exercises, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



50. $\lim_{x \to 1^-} f(x)$

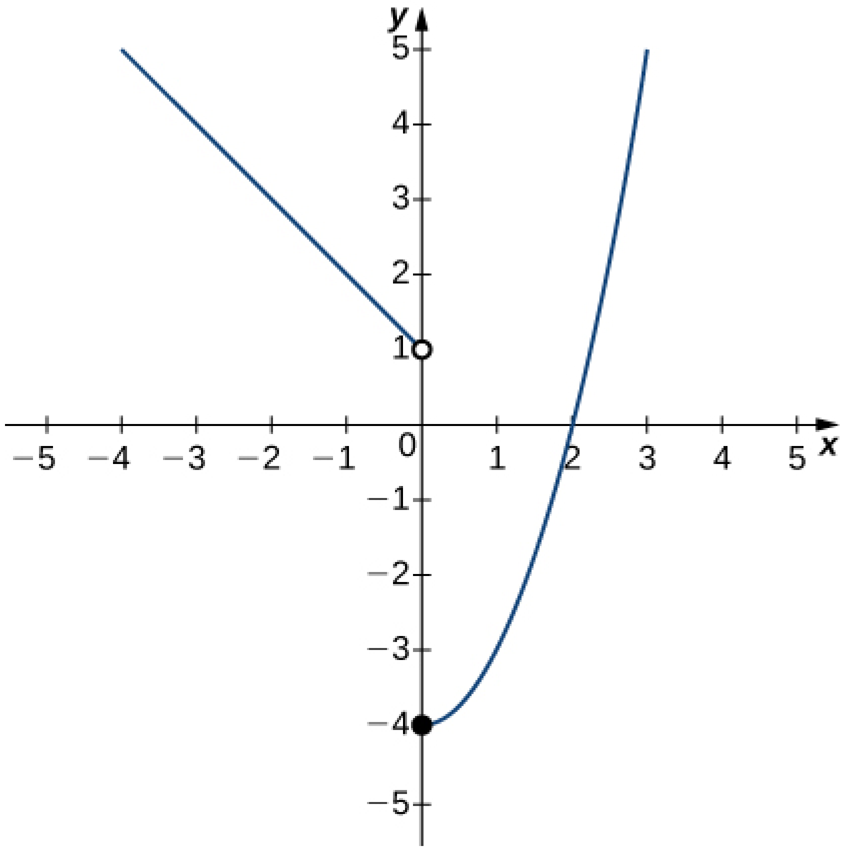
51. $\lim_{x \to 1^+} f(x)$

52. $\lim_{x \to 1} f(x)$

53. $\lim_{x \to 2} f(x)$

54. $f(1)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



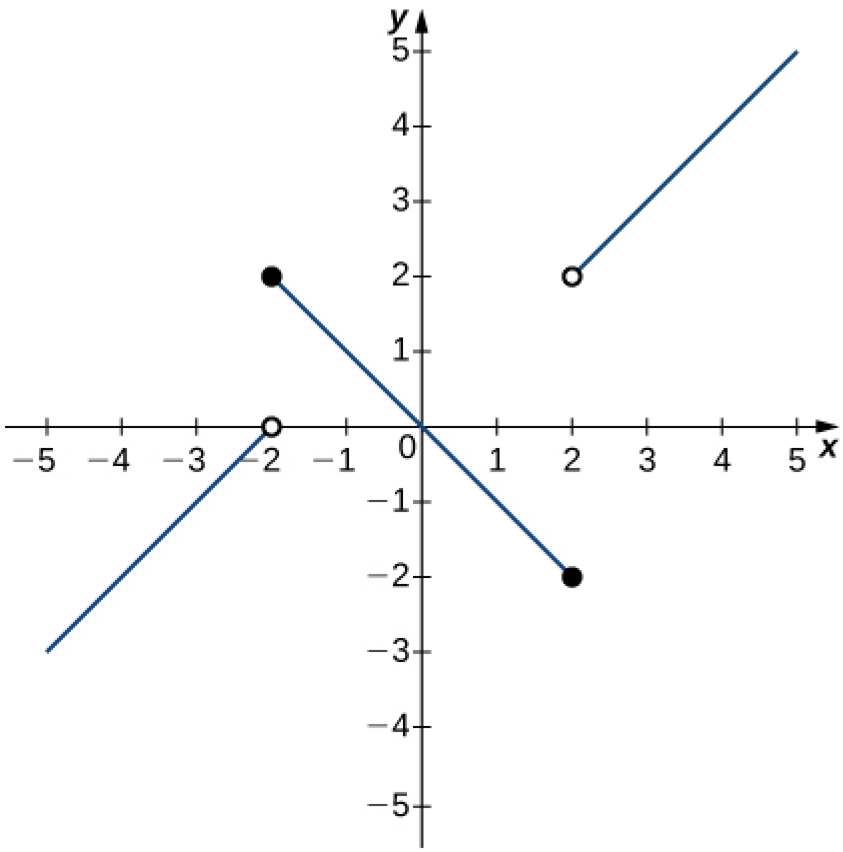
55. $\lim_{x \to 0^-} f(x)$

56. $\lim_{x \to 0^+} f(x)$

57. $\lim_{x \to 0} f(x)$

58. $\lim_{x \to 2} f(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



59. $\lim_{x \to -2^-} f(x)$

60. $\lim_{x \to -2^+} f(x)$

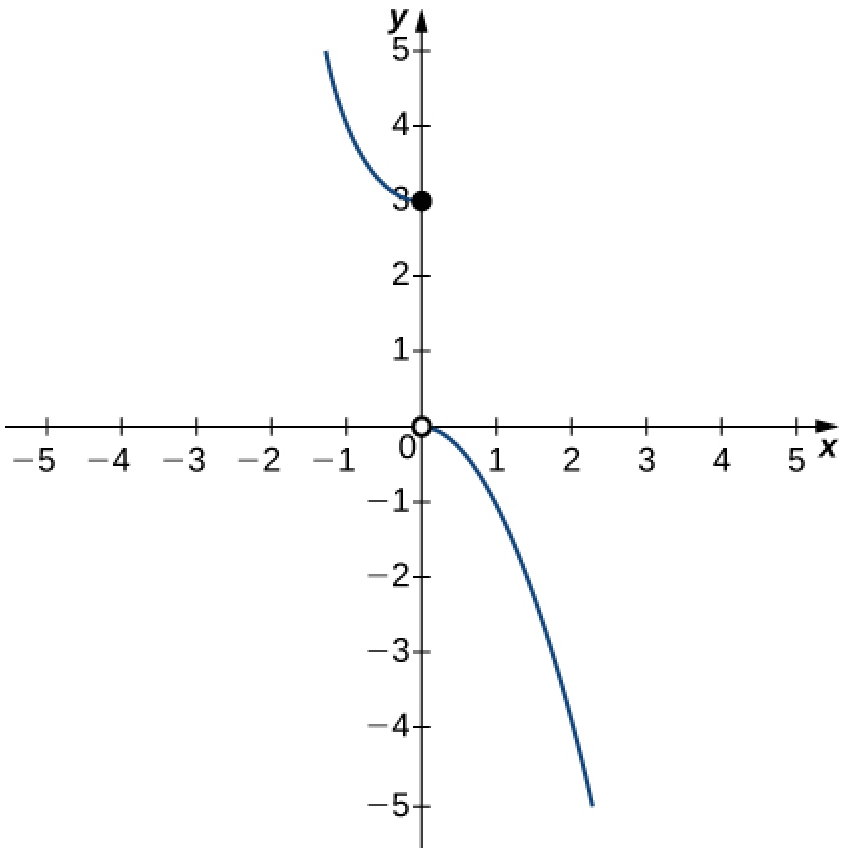
61. $\lim_{x \to -2} f(x)$

62. $\lim_{x \to 2^-} f(x)$

63. $\lim_{x \to 2^+} f(x)$

64. $\lim_{x \to 2} f(x)$

In the following exercises, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.

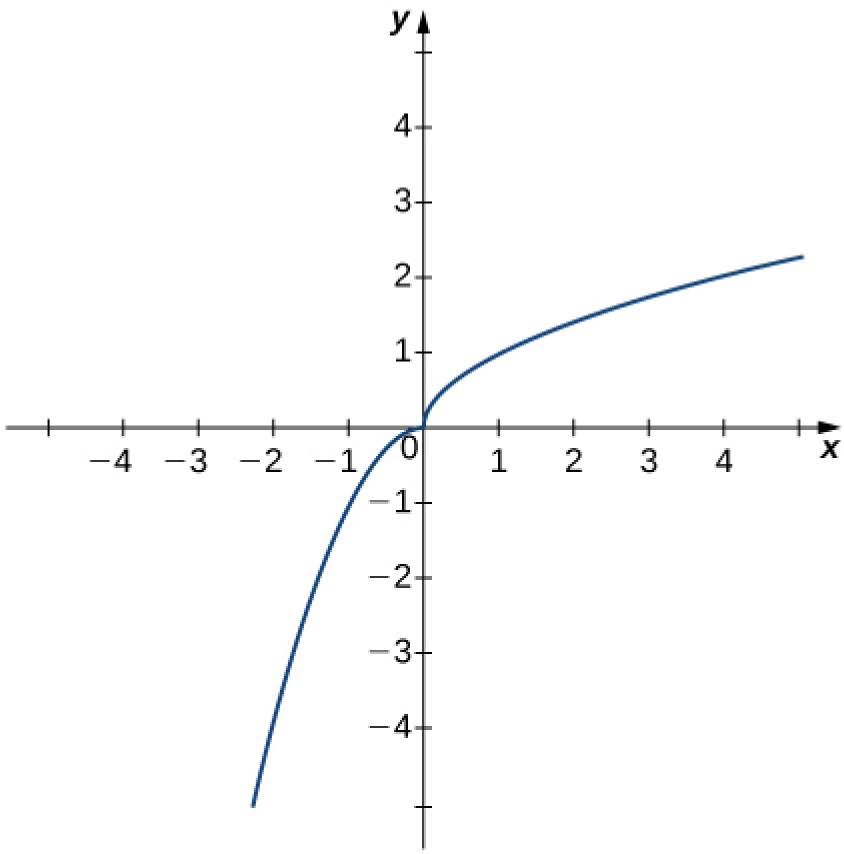


65. $\lim_{x \to 0^-} g(x)$

66. $\lim_{x \to 0^+} g(x)$

67. $\lim_{x \to 0} g(x)$

In the following exercises, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.

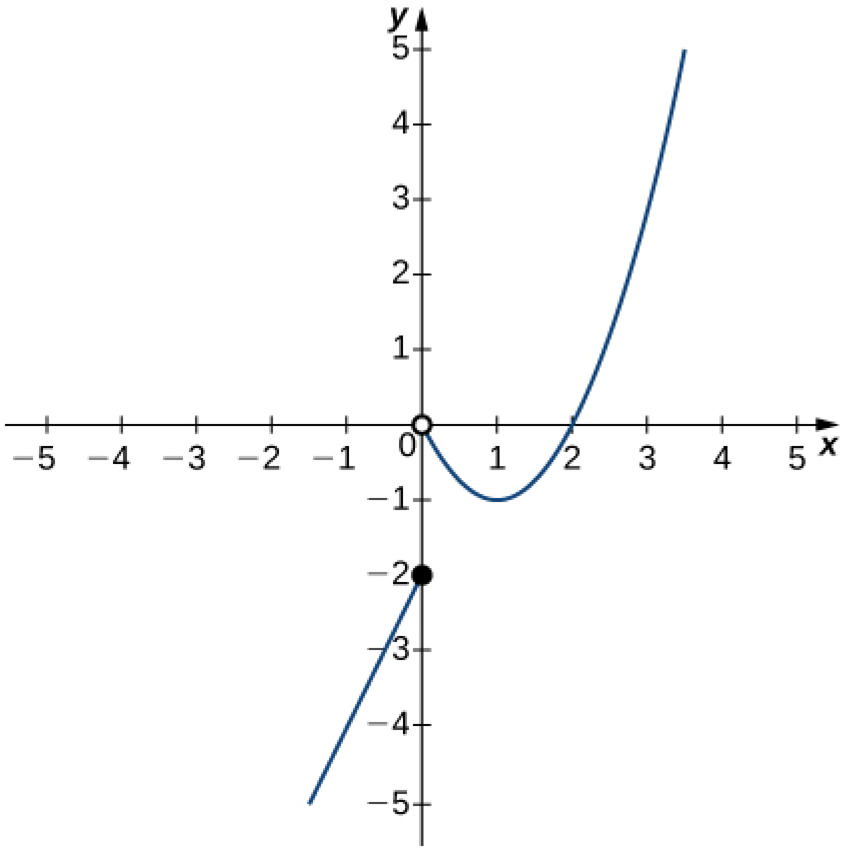


68. $\lim_{x \to 0^-} h(x)$

69. $\lim_{x \to 0^+} h(x)$

70. $\lim_{x \to 0} h(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



71. $\lim_{x \to 0^-} f(x)$

72. $\lim_{x \to 0^+} f(x)$

73. $\lim_{x \to 0} f(x)$

74. $\lim_{x \to 1} f(x)$

75. $\lim_{x \to 2} f(x)$

In the following exercises, sketch the graph of a function with the given properties.

76. $\lim_{x \to 2} f(x) = 1, \lim_{x \to 4^-} f(x) = 3, \lim_{x \to 4^+} f(x) = 6, f(4)$ is not defined.

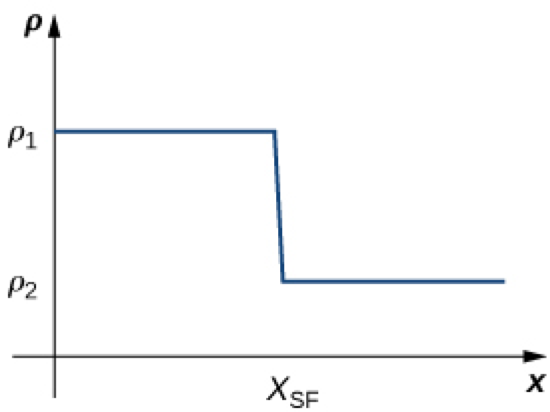
77. $\lim_{x \to -\infty} f(x) = 0, \lim_{x \to -1^-} f(x) = -\infty,$
$\lim_{x \to -1^+} f(x) = \infty, \lim_{x \to 0} f(x) = f(0), f(0) = 1, \lim_{x \to \infty} f(x) = -\infty$

78. $\lim_{x \to -\infty} f(x) = 2, \lim_{x \to 3^-} f(x) = -\infty, \lim_{x \to 3^+} f(x) = \infty, \lim_{x \to \infty} f(x) = 2, f(0) = \frac{-1}{3}$

79. $\lim_{x \to -\infty} f(x) = 2, \lim_{x \to -2} f(x) = -\infty, \lim_{x \to \infty} f(x) = 2, f(0) = 0$

80. $\lim_{x \to -\infty} f(x) = 0, \lim_{x \to -1^-} f(x) = \infty, \lim_{x \to -1^+} f(x) = -\infty,$
$f(0) = -1, \lim_{x \to 1^-} f(x) = -\infty, \lim_{x \to 1^+} f(x) = \infty, \lim_{x \to \infty} f(x) = 0$

81. Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, *x*, is shown here. We are mainly interested in the location of the front of the shock, labeled x_{SF} in the diagram.



a. Evaluate $\lim_{x \to x_{SF}^+} \rho(x)$.
b. Evaluate $\lim_{x \to x_{SF}^-} \rho(x)$.
c. Evaluate $\lim_{x \to x_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

82. A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where *x* is the position in meters of the runner and *t* is time in seconds. What is $\lim_{t \to 2} x(t)$? What does it mean physically?

***t* (sec)**	***x* (m)**
1.75	4.5
1.95	6.1

t (sec)	*x* (m)
1.99	6.42
2.01	6.58
2.05	6.9
2.25	8.5

Learning Objectives

2.3.1. Recognize the basic limit laws.
2.3.2. Use the limit laws to evaluate the limit of a function.
2.3.3. Evaluate the limit of a function by factoring.
2.3.4. Use the limit laws to evaluate the limit of a polynomial or rational function.
2.3.5. Evaluate the limit of a function by factoring or by using conjugates.
2.3.6. Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated in Two Important Limits and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

THEOREM 2.4

Basic Limit Results

For any real number *a* and any constant *c*,

i. $$\lim_{x \to a} x = a \tag{2.14}$$

ii. $$\lim_{x \to a} c = c \tag{2.15}$$

EXAMPLE 2.13

Evaluating a Basic Limit

Evaluate each of the following limits using Basic Limit Results.

a. $\lim\limits_{x \to 2} x$
b. $\lim\limits_{x \to 2} 5$

[Show Solution]

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

THEOREM 2.5

Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing *a*. Assume that *L* and *M* are real numbers such that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let *c* be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + M$

Difference law for limits: $\lim\limits_{x \to a} (f(x) - g(x)) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim\limits_{x \to a} cf(x) = c \cdot \lim\limits_{x \to a} f(x) = cL$

Product law for limits: $\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim\limits_{x \to a} (f(x))^n = \left(\lim\limits_{x \to a} f(x)\right)^n = L^n$ for every positive integer *n*.

Root law for limits: $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)} = \sqrt[n]{L}$ for all *L* if *n* is odd and for $L \geq 0$ if *n* is even and $f(x) \geq 0$.

We now practice applying these limit laws to evaluate a limit.

EXAMPLE 2.14

Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate $\lim_{x \to -3} (4x + 2)$.

[Show Solution]

EXAMPLE 2.15

Using Limit Laws Repeatedly

Use the limit laws to evaluate $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$.

[Show Solution]

CHECKPOINT 2.11

Use the limit laws to evaluate $\lim_{x \to 6} (2x - 1)\sqrt{x + 4}$. In each step, indicate the limit law applied.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x \to a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of *a* and for all rational functions at all values of *a* for which the rational function is defined.

THEOREM 2.6

Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let *a* be a real number. Then,

$$\lim_{x \to a} p(x) = p(a)$$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

To see that this theorem holds, consider the polynomial $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$. By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned} \lim_{x \to a} p(x) &= \lim_{x \to a} \left(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0\right) \\ &= c_n\left(\lim_{x \to a} x\right)^n + c_{n-1}\left(\lim_{x \to a} x\right)^{n-1} + \cdots + c_1\left(\lim_{x \to a} x\right) + \lim_{x \to a} c_0 \\ &= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 \\ &= p(a). \end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Example 2.16 applies this result.

EXAMPLE 2.16

Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

[Show Solution]

CHECKPOINT 2.12

Evaluate $\lim_{x \to -2} \left(3x^3 - 2x + 7\right)$.

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x \to a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a, f(x) = g(x)$ over some open interval containing *a*, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

To understand this idea better, consider the limit $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

The function

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x - 1)(x + 1)}{x - 1} \end{aligned}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in Figure 2.24.

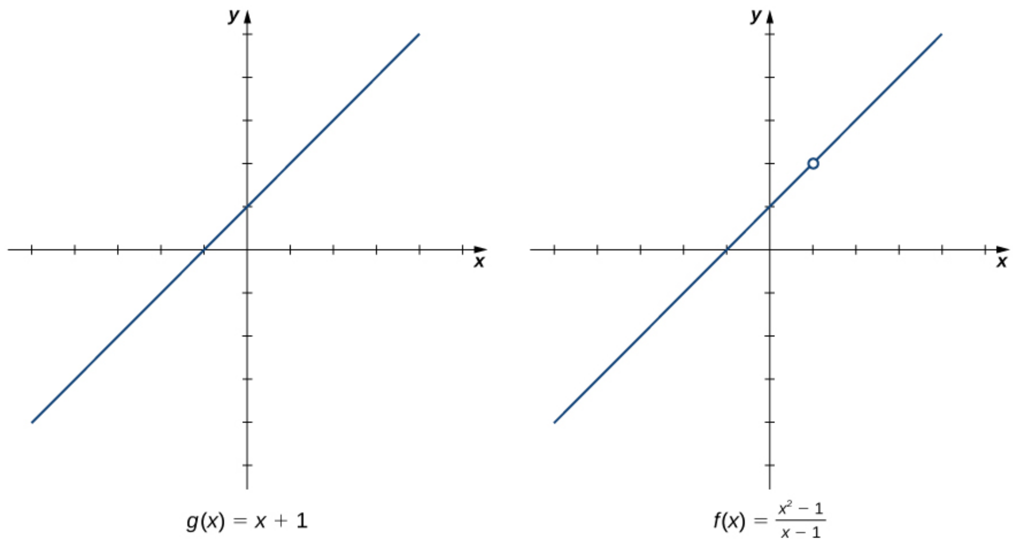


Figure 2.24 The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} \\ &= \lim_{x \to 1} (x+1) \\ &= 2. \end{aligned}$$

The limit has the form $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$, where $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

PROBLEM-SOLVING STRATEGY: CALCULATING A LIMIT WHEN $F(X)/G(X)$ HAS THE INDETERMINATE FORM 0/0

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing *a*. To do this, we may need to try one or more of the following steps:

    a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
    b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and

denominator by the conjugate of the expression involving the square root.

c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.

3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. Example 2.17 illustrates the factor-and-cancel technique; Example 2.18 shows multiplying by a conjugate. In Example 2.19, we look at simplifying a complex fraction.

EXAMPLE 2.17

Evaluating a Limit by Factoring and Canceling

Evaluate $\lim\limits_{x \to 3} \frac{x^2-3x}{2x^2-5x-3}$.

[Show Solution]

CHECKPOINT 2.13

Evaluate $\lim\limits_{x \to -3} \frac{x^2+4x+3}{x^2-9}$.

EXAMPLE 2.18

Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim\limits_{x \to -1} \frac{\sqrt{x+2}-1}{x+1}$.

[Show Solution]

CHECKPOINT 2.14

Evaluate $\lim\limits_{x \to 5} \dfrac{\sqrt{x-1}-2}{x-5}$.

EXAMPLE 2.19

Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim\limits_{x \to 1} \dfrac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

[Show Solution]

CHECKPOINT 2.15

Evaluate $\lim\limits_{x \to -3} \dfrac{\frac{1}{x+2}+1}{x+3}$.

Example 2.20 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

EXAMPLE 2.20

Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim\limits_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right)$.

[Show Solution]

CHECKPOINT 2.16

Evaluate $\lim\limits_{x \to 3} \left(\frac{1}{x-3} - \frac{4}{x^2-2x-3} \right)$.

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim\limits_{x \to a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b, a); for a limit of the form $\lim\limits_{x \to a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a, c). Example 2.21 illustrates this point.

EXAMPLE 2.21

Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim\limits_{x \to 3^-} \sqrt{x-3}$

b. $\lim\limits_{x \to 3^+} \sqrt{x-3}$

[Show Solution]

In Example 2.22 we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

EXAMPLE 2.22

Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x - 3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$
c. $\lim_{x \to 2} f(x)$

[Show Solution]

CHECKPOINT 2.17

Graph $f(x) = \begin{cases} -x - 2 \text{ if } x < -1 \\ 2 \text{ if } x = -1 \\ x^3 \text{ if } x > -1 \end{cases}$ and evaluate $\lim_{x \to -1^-} f(x)$.

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at *a*.

EXAMPLE 2.23

Evaluating a Limit of the Form $K/0, K \neq 0$ Using the Limit Laws

Evaluate $\lim\limits_{x \to 2^-} \dfrac{x-3}{x^2-2x}$.

[Show Solution]

CHECKPOINT 2.18

Evaluate $\lim\limits_{x \to 1} \dfrac{x+2}{(x-1)^2}$.

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem**, proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by “squeezing” a function, with a limit at a point *a* that is unknown, between two functions having a common known limit at *a*. Figure 2.27 illustrates this idea.

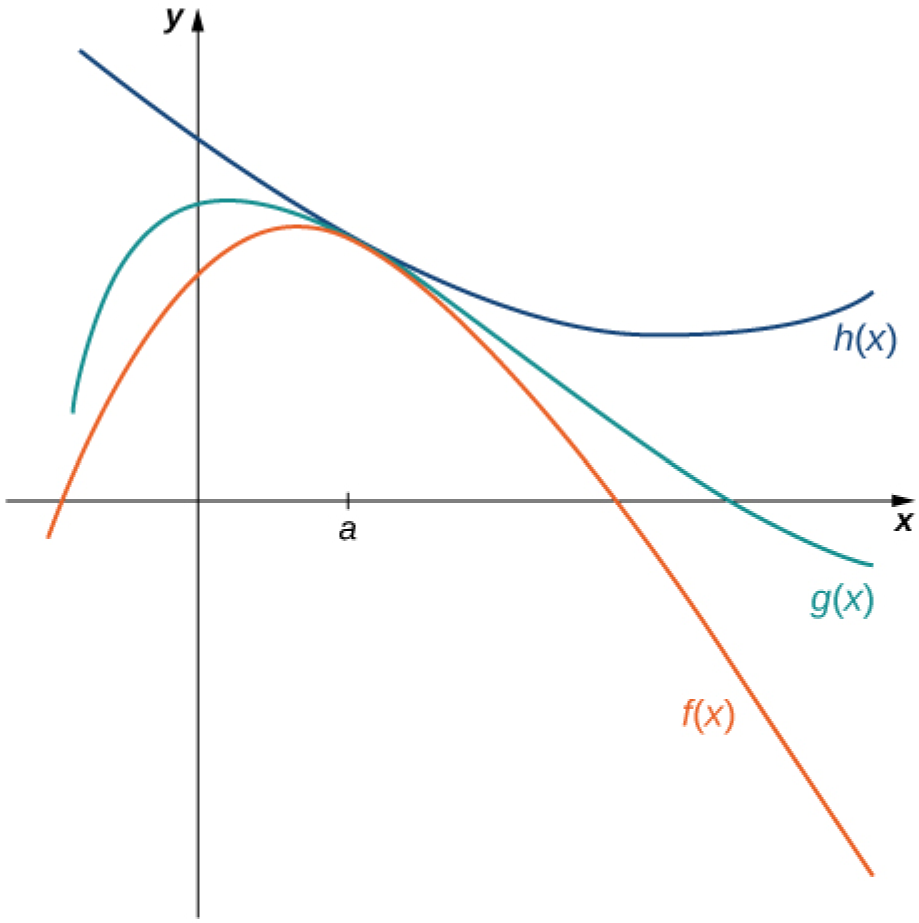


Figure 2.27 The Squeeze Theorem applies when $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

THEOREM 2.7

The Squeeze Theorem

Let $f(x), g(x),$ and $h(x)$ be defined for all $x \neq a$ over an open interval containing *a*. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing *a* and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where *L* is a real number, then $\lim_{x \to a} g(x) = L.$

EXAMPLE 2.24

Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim\limits_{x \to 0} x \cos x$.

[Show Solution]

CHECKPOINT 2.19

Use the squeeze theorem to evaluate $\lim\limits_{x \to 0} x^2 \sin \frac{1}{x}$.

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim\limits_{\theta \to 0} \sin \theta$. Consider the unit circle shown in Figure 2.29. In the figure, we see that $\sin \theta$ is the *y*-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}$, $0 < \sin \theta < \theta$.

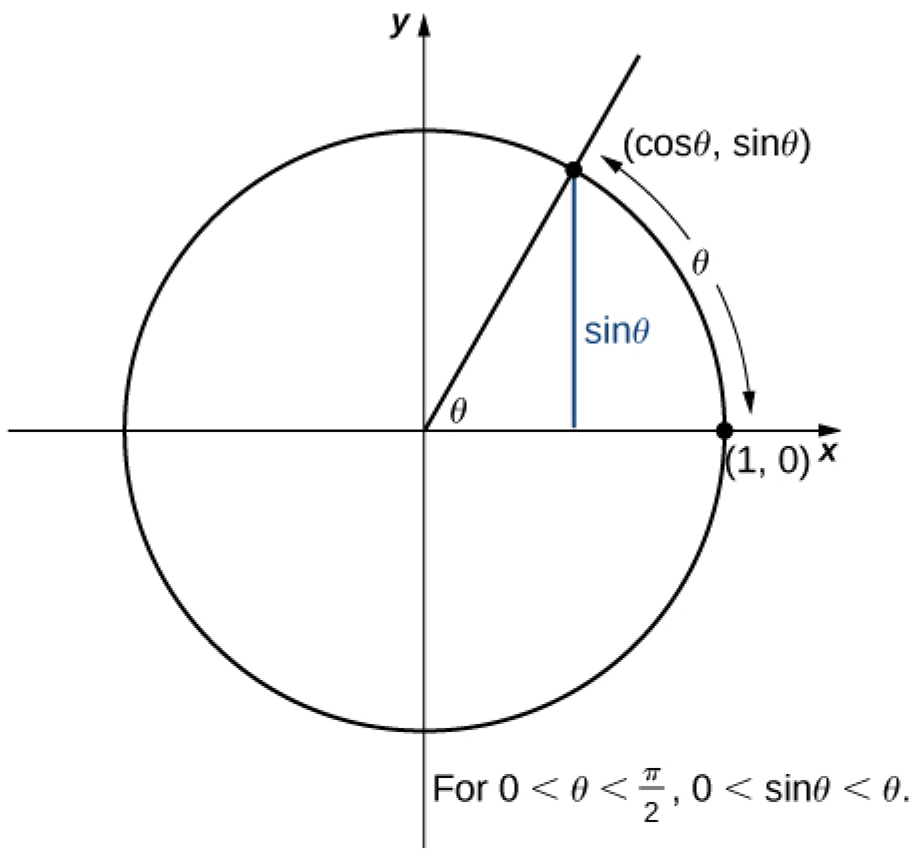


Figure 2.29 The sine function is shown as a line on the unit circle.

Because $\lim\limits_{\theta \to 0^+} 0 = 0$ and $\lim\limits_{\theta \to 0^+} \theta = 0,$ by using the squeeze theorem we conclude that

$$\lim_{\theta \to 0^+} \sin\theta = 0.$$

To see that $\lim\limits_{\theta \to 0^-} \sin\theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0, 0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta.$ Consequently, $0 < -\sin\theta < -\theta.$ It follows that $0 > \sin\theta > \theta.$ An application of the squeeze theorem produces the desired limit. Thus, since $\lim\limits_{\theta \to 0^+} \sin\theta = 0$ and $\lim\limits_{\theta \to 0^-} \sin\theta = 0,$

$$\lim_{\theta \to 0} \sin\theta = 0. \tag{2.16}$$

Next, using the identity $\cos\theta = \sqrt{1 - \sin^2\theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2},$ we see that

$$\lim_{\theta \to 0} \cos\theta = \lim_{\theta \to 0} \sqrt{1 - \sin^2\theta} = 1. \tag{2.17}$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$. To evaluate this limit, we use the unit circle in Figure 2.30. Notice that this figure adds one additional triangle to Figure 2.30. We see that the length of the side opposite angle θ in this new triangle is $\tan \theta$. Thus, we see that for $0 < \theta < \frac{\pi}{2}$, $\sin \theta < \theta < \tan \theta$.

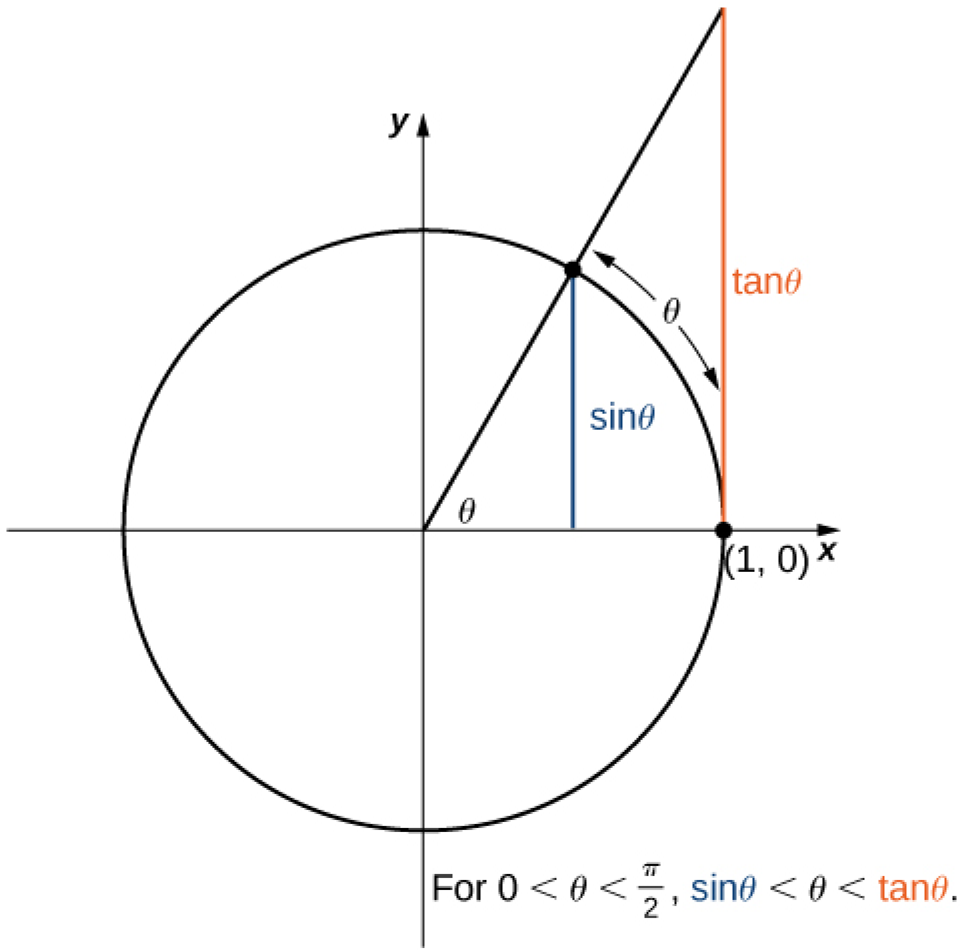


Figure 2.30 The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin \theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Equivalently, we have

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim\limits_{\theta\to 0^+} 1 = 1 = \lim\limits_{\theta\to 0^+} \cos\theta,$ we conclude that $\lim\limits_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$ By applying a manipulation similar to that used in demonstrating that $\lim\limits_{\theta\to 0^-} \sin\theta = 0,$ we can show that $\lim\limits_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$ Thus,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1. \tag{2.18}$$

In Example 2.25 we use this limit to establish $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.$ This limit also proves useful in later chapters.

EXAMPLE 2.25

Evaluating an Important Trigonometric Limit

Evaluate $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta}$.

[Show Solution]

CHECKPOINT 2.20

Evaluate $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta}$.

STUDENT PROJECT

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician Archimedes (ca. 287–212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of *n* triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height *h* and the base *b* of the isosceles triangle in Figure 2.31 in terms of θ and *r*.

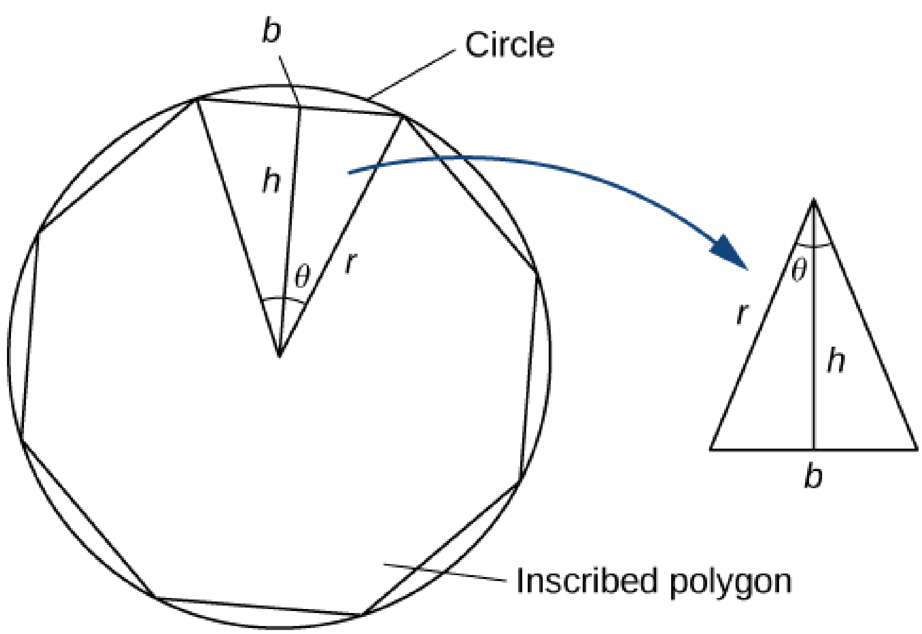


Figure 2.31

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and *r*.
(Substitute $(1/2)\sin\theta$ for $\sin(\theta/2)\cos(\theta/2)$ in your expression.)
3. If an *n*-sided regular polygon is inscribed in a circle of radius *r*, find a relationship between θ and *n*. Solve this for *n*. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)
4. Find an expression for the area of the *n*-sided polygon in terms of *r* and θ.
5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (*Hint:* $\lim\limits_{\theta \to 0} \frac{(\sin\theta)}{\theta} = 1$).

The technique of estimating areas of regions by using polygons is revisited in Introduction to Integration.

Section 2.3 Exercises

In the following exercises, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

83. $\lim_{x \to 0} \left(4x^2 - 2x + 3\right)$

84. $\lim_{x \to 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$

85. $\lim_{x \to -2} \sqrt{x^2 - 6x + 3}$

86. $\lim_{x \to -1} (9x + 1)^2$

In the following exercises, use direct substitution to evaluate each limit.

87. $\lim_{x \to 7} x^2$

88. $\lim_{x \to -2} \left(4x^2 - 1\right)$

89. $\lim_{x \to 0} \frac{1}{1 + \sin x}$

90. $\lim_{x \to 2} e^{2x - x^2}$

91. $\lim_{x \to 1} \frac{2 - 7x}{x + 6}$

92. $\lim_{x \to 3} \ln e^{3x}$

In the following exercises, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit.

93. $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$

94. $\lim_{x \to 2} \frac{x - 2}{x^2 - 2x}$

95. $\lim\limits_{x \to 6} \frac{3x-18}{2x-12}$

96. $\lim\limits_{h \to 0} \frac{(1+h)^2-1}{h}$

97. $\lim\limits_{t \to 9} \frac{t-9}{\sqrt{t}-3}$

98. $\lim\limits_{h \to 0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h}$, where *a* is a non-zero real-valued constant

99. $\lim\limits_{\theta \to \pi} \frac{\sin\theta}{\tan\theta}$

100. $\lim\limits_{x \to 1} \frac{x^3-1}{x^2-1}$

101. $\lim\limits_{x \to 1/2} \frac{2x^2+3x-2}{2x-1}$

102. $\lim\limits_{x \to -3} \frac{\sqrt{x+4}-1}{x+3}$

In the following exercises, use direct substitution to obtain an undefined expression. Then, use the method of Example 2.23 to simplify the function to help determine the limit.

103. $\lim\limits_{x \to -2^-} \frac{2x^2+7x-4}{x^2+x-2}$

104. $\lim\limits_{x \to -2^+} \frac{2x^2+7x-4}{x^2+x-2}$

105. $\lim\limits_{x \to 1^-} \frac{2x^2+7x-4}{x^2+x-2}$

106. $\lim\limits_{x \to 1^+} \frac{2x^2+7x-4}{x^2+x-2}$

In the following exercises, assume that $\lim\limits_{x \to 6} f(x) = 4$, $\lim\limits_{x \to 6} g(x) = 9$, and $\lim\limits_{x \to 6} h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.

107. $\lim\limits_{x \to 6} 2f(x)g(x)$

108. $\lim\limits_{x \to 6} \frac{g(x) - 1}{f(x)}$

109. $\lim\limits_{x \to 6} \left(f(x) + \frac{1}{3}g(x)\right)$

110. $\lim\limits_{x \to 6} \frac{(h(x))^3}{2}$

111. $\lim\limits_{x \to 6} \sqrt{g(x) - f(x)}$

112. $\lim\limits_{x \to 6} x \cdot h(x)$

113. $\lim\limits_{x \to 6} [(x + 1) \cdot f(x)]$

114. $\lim\limits_{x \to 6} (f(x) \cdot g(x) - h(x))$

[T] In the following exercises, use a calculator to draw the graph of each piecewise-defined function and study the graph to evaluate the given limits.

115. $f(x) = \begin{cases} x^2, & x \le 3 \\ x + 4, & x > 3 \end{cases}$

a. $\lim\limits_{x \to 3^-} f(x)$

b. $\lim\limits_{x \to 3^+} f(x)$

116. $g(x) = \begin{cases} x^3 - 1, & x \le 0 \\ 1, & x > 0 \end{cases}$

a. $\lim\limits_{x \to 0^-} g(x)$

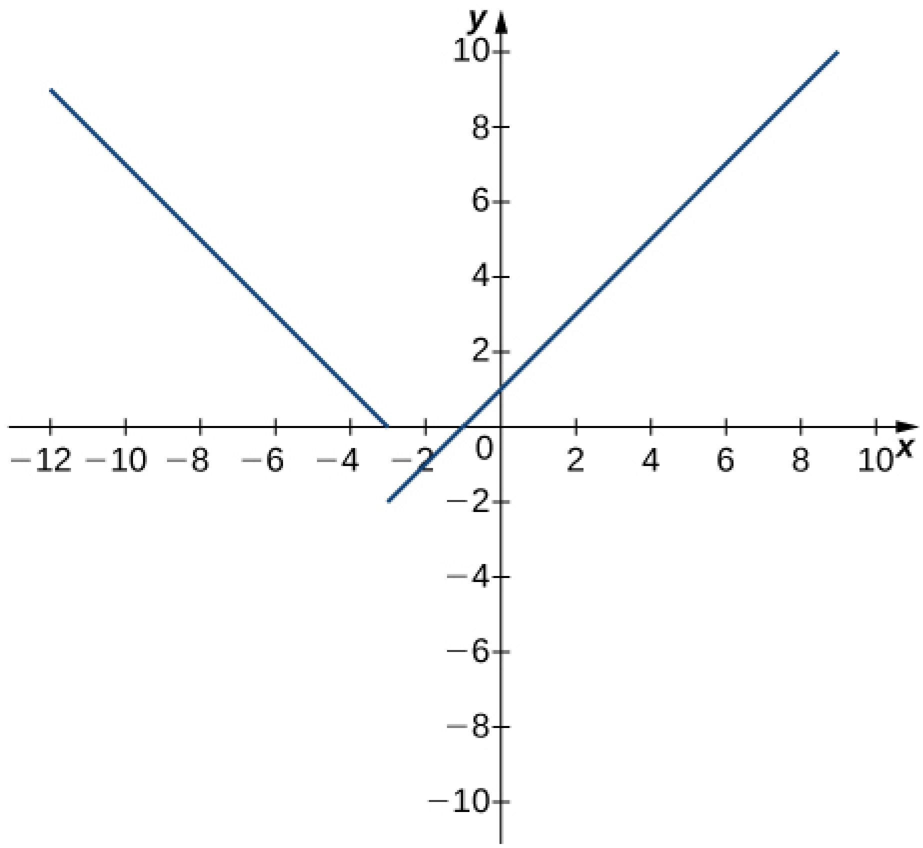
b. $\lim\limits_{x \to 0^+} g(x)$

117. $h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \ge 2 \end{cases}$

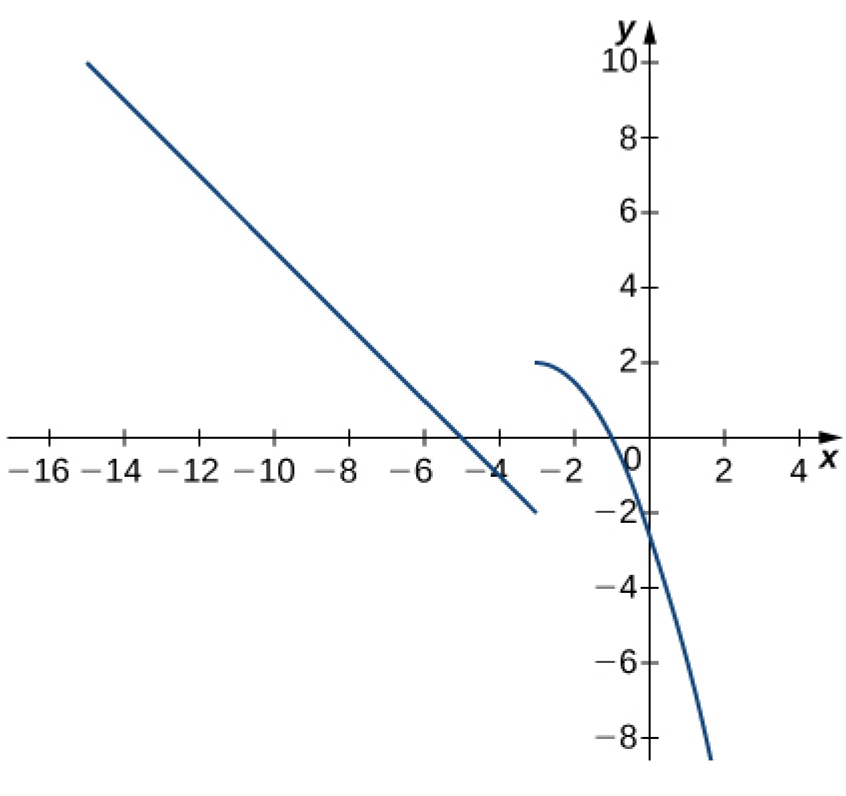
a. $\lim\limits_{x \to 2^-} h(x)$

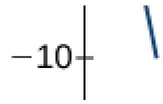
b. $\lim\limits_{x \to 2^+} h(x)$

In the following exercises, use the following graphs and the limit laws to evaluate each limit.



$y = f(x)$





$y = g(x)$

118. $\lim\limits_{x \to -3^+} (f(x) + g(x))$

119. $\lim\limits_{x \to -3^-} (f(x) - 3g(x))$

120. $\lim\limits_{x \to 0} \frac{f(x)g(x)}{3}$

121. $\lim\limits_{x \to -5} \frac{2 + g(x)}{f(x)}$

122. $\lim\limits_{x \to 1} (f(x))^2$

123. $\lim\limits_{x \to 1} \sqrt[3]{f(x) - g(x)}$

124. $\lim\limits_{x \to -7} (x \cdot g(x))$

125. $\lim\limits_{x \to -9} [x \cdot f(x) + 2 \cdot g(x)]$

126. **[T]** True or False? If $2x - 1 \le g(x) \le x^2 - 2x + 3,$ then $\lim\limits_{x \to 2} g(x) = 0.$

For the following problems, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f(x), g(x),$ and $h(x)$ when possible.

127. **[T]** $\lim\limits_{\theta \to 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$

128. $\lim\limits_{x \to 0} f(x),$ where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{cases}$

129. **[T]** In physics, the magnitude of an electric field generated by a point charge at a distance *r* in vacuum is governed by Coulomb's law: $E(r) = \frac{q}{4\pi\varepsilon_0 r^2},$ where *E* represents the magnitude of the electric field, *q* is the charge of the particle, *r* is the distance between the particle and where the strength of the field is measured, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant: $8.988 \times 10^9 \,\mathrm{N} \cdot \mathrm{m}^2/\mathrm{C}^2.$

a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}$.

b. Evaluate $\lim_{r \to 0^+} E(r)$. What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

130. **[T]** The density of an object is given by its mass divided by its volume: $\rho = m/V$.

a. Use a calculator to plot the volume as a function of density $(V = m/\rho)$, assuming you are examining something of mass 8 kg $(m = 8)$.

b. Evaluate $\lim_{\rho \to 0^+} V(\rho)$ and explain the physical meaning.

Learning Objectives

2.4.1. Explain the three conditions for continuity at a point.
2.4.2. Describe three kinds of discontinuities.
2.4.3. Define continuity on an interval.
2.4.4. State the theorem for limits of composite functions.
2.4.5. Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity at a point* where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure 2.32. We see that the graph of $f(x)$ has a hole at *a*. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at *a*, we need the following condition:

$$\text{i. } f(a) \text{ is defined.}$$

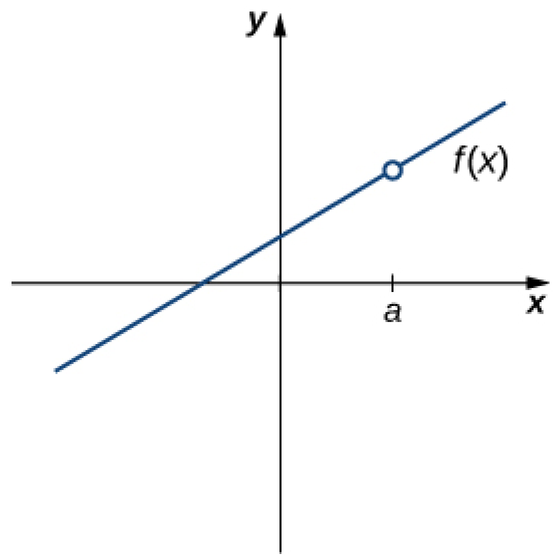


Figure 2.32 The function $f(x)$ is not continuous at *a* because $f(a)$ is undefined.

However, as we see in Figure 2.33, this condition alone is insufficient to guarantee continuity at the point *a*. Although $f(a)$ is defined, the function has a gap at *a*. In this example, the gap exists because $\lim\limits_{x \to a} f(x)$ does not exist. We must add another condition for continuity at *a*—namely,

$$\text{ii. } \lim_{x \to a} f(x) \text{ exists.}$$

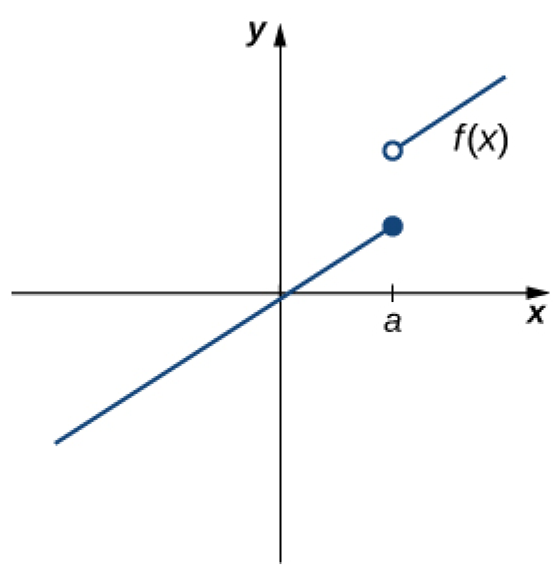


Figure 2.33 The function $f(x)$ is not continuous at *a* because $\lim\limits_{x \to a} f(x)$ does not exist.

However, as we see in Figure 2.34, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at *a*. We must add a third condition to our list:

$$\text{iii. } \lim_{x \to a} f(x) = f(a).$$

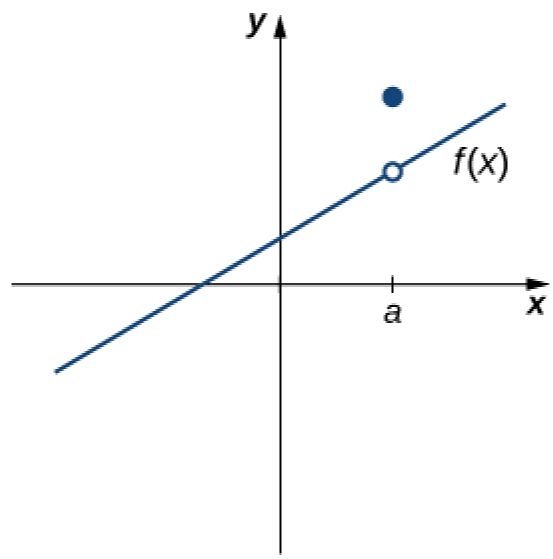


Figure 2.34 The function $f(x)$ is not continuous at *a* because $\lim_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

DEFINITION

A function $f(x)$ is **continuous at a point** *a* if and only if the following three conditions are satisfied:

i. $f(a)$ is defined
ii. $\lim_{x \to a} f(x)$ exists
iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** *a* if it fails to be continuous at *a*.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

PROBLEM-SOLVING STRATEGY: DETERMINING CONTINUITY AT A POINT

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at *a*. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim\limits_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$. If $\lim\limits_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at *a* and the problem is solved. If $\lim\limits_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim\limits_{x \to a} f(x)$. If $\lim\limits_{x \to a} f(x) \neq f(a)$, then the function is not continuous at *a*. If $\lim\limits_{x \to a} f(x) = f(a)$, then the function is continuous at *a*.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

EXAMPLE 2.26

Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = (x^2 - 4)/(x - 2)$ is continuous at $x = 2$. Justify the conclusion.

[Show Solution]

EXAMPLE 2.27

Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function

$f(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

[Show Solution]

EXAMPLE 2.28

Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$.

[Show Solution]

CHECKPOINT 2.21

Using the definition, determine whether the function

$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x + 4 & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

THEOREM 2.8

Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply Continuity of Polynomials and Rational Functions to determine the points at which a given rational function is continuous.

EXAMPLE 2.29

Continuity of a Rational Function

For what values of *x* is $f(x) = \frac{x+1}{x-5}$ continuous?

[Show Solution]

CHECKPOINT 2.22

For what values of *x* is $f(x) = 3x^4 - 4x^2$ continuous?

Types of Discontinuities

As we have seen in Example 2.26 and Example 2.27, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure 2.37 illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

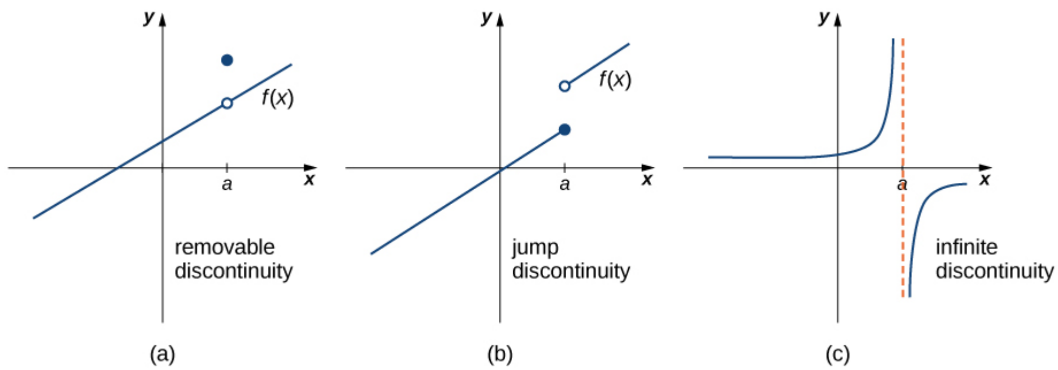


Figure 2.37 Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

DEFINITION

If $f(x)$ is discontinuous at *a*, then

1. f has a **removable discontinuity** at *a* if $\lim\limits_{x \to a} f(x)$ exists. (Note: When we state that $\lim\limits_{x \to a} f(x)$ exists, we mean that $\lim\limits_{x \to a} f(x) = L,$ where *L* is a real number.)
2. f has a **jump discontinuity** at *a* if $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist, but $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x).$ (Note: When we state that $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at *a* if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty.$

EXAMPLE 2.30

Classifying a Discontinuity

In Example 2.26, we showed that $f(x) = \frac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

[Show Solution]

EXAMPLE 2.31

Classifying a Discontinuity

In Example 2.27, we showed that $f(x) = \begin{cases} -x^2+4 & \text{if } x \le 3 \\ 4x-8 & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

[Show Solution]

EXAMPLE 2.32

Classifying a Discontinuity

Determine whether $f(x) = \frac{x+2}{x+1}$ is continuous at −1. If the function is discontinuous at −1, classify the discontinuity as removable, jump, or infinite.

[Show Solution]

CHECKPOINT 2.23

For $f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$, decide whether *f* is continuous at 1. If *f* is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

CONTINUITY FROM THE RIGHT AND FROM THE LEFT

A function $f(x)$ is said to be **continuous from the right** at *a* if $\lim\limits_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at *a* if $\lim\limits_{x \to a^-} f(x) = f(a)$.

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at *a* and is continuous from the left at *b*. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at *b*. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim\limits_{x \to a^+} f(x) = f(a)$ and $\lim\limits_{x \to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim\limits_{x \to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

EXAMPLE 2.33

Continuity on an Interval

State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

[Show Solution]

EXAMPLE 2.34

Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4 - x^2}$ is continuous.

[Show Solution]

CHECKPOINT 2.24

State the interval(s) over which the function $f(x) = \sqrt{x + 3}$ is continuous.

The Composite Function Theorem allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

THEOREM 2.9

Composite Function Theorem

If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L,$ then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

Before we move on to Example 2.35, recall that earlier, in the section on limit laws, we showed $\lim_{x \to 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In Example 2.35 we see how to combine this result with the composite function theorem.

EXAMPLE 2.35

Limit of a Composite Cosine Function

Evaluate $\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

[Show Solution]

CHECKPOINT 2.25

Evaluate $\lim_{x \to \pi} \sin(x - \pi)$.

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

THEOREM 2.10

Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim\limits_{x \to a} \cos x = \cos a$ for all values of *a*.

$$\begin{aligned}
\lim_{x \to a} \cos x &= \lim_{x \to a} \cos((x - a) + a) && \text{rewrite } x = x \\
&= \lim_{x \to a} (\cos(x - a)\cos a - \sin(x - a)\sin a) && \text{apply the identity for the cosine} \\
&= \cos\left(\lim_{x \to a} (x - a)\right)\cos a - \sin\left(\lim_{x \to a} (x - a)\right)\sin a && \lim_{x \to a} (x - a) = 0, \text{ and } \sin x \text{ and} \\
&= \cos(0)\cos a - \sin(0)\sin a && \text{evaluate } \cos(0) \text{ and si} \\
&= 1 \cdot \cos a - 0 \cdot \sin a = \cos a.
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where *a* and *b* are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem**.

THEOREM 2.11

The Intermediate Value Theorem

Let *f* be continuous over a closed, bounded interval $[a, b]$. If *z* is any real number between $f(a)$ and $f(b)$, then there is a number *c* in $[a, b]$ satisfying $f(c) = z$ in Figure 2.38.

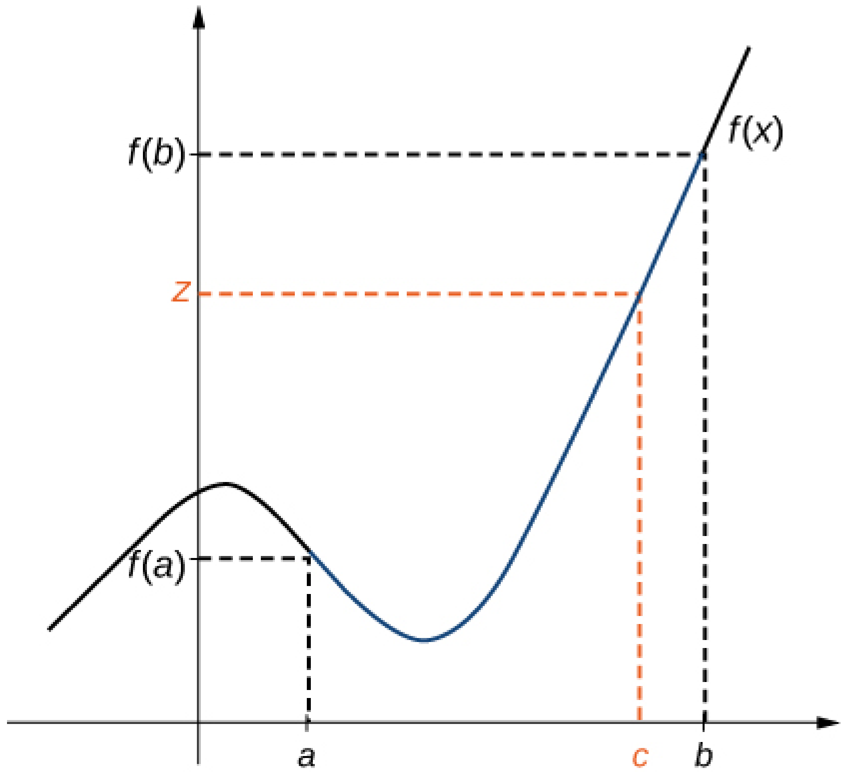


Figure 2.38 There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

EXAMPLE 2.36

Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

[Show Solution]

EXAMPLE 2.37

When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2], f(0) > 0$ and $f(2) > 0,$ can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

[Show Solution]

EXAMPLE 2.38

When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x, f(-1) = -1 < 0$ and $f(1) = 1 > 0.$ Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

[Show Solution]

CHECKPOINT 2.26

Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Section 2.4 Exercises

For the following exercises, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as jump, removable, infinite, or other.

131. $f(x) = \frac{1}{\sqrt{x}}$

132. $f(x) = \frac{2}{x^2+1}$

133. $f(x) = \frac{x}{x^2-x}$

134. $g(t) = t^{-1} + 1$

135. $f(x) = \frac{5}{e^x-2}$

136. $f(x) = \frac{|x-2|}{x-2}$

137. $H(x) = \tan 2x$

138. $f(t) = \frac{t+3}{t^2+5t+6}$

For the following exercises, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

139. $f(x)\frac{2x^2-5x+3}{x-1}$ at $x = 1$

140. $h(\theta) = \frac{\sin\theta-\cos\theta}{\tan\theta}$ at $\theta = \pi$

141. $g(u) = \begin{cases} \frac{6u^2+u-2}{2u-1} & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2} & \text{if } u = \frac{1}{2} \end{cases}$, at $u = \frac{1}{2}$

142. $f(y) = \frac{\sin(\pi y)}{\tan(\pi y)}$, at $y = 1$

143. $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$, at $x = 0$

144. $f(x) = \begin{cases} x\sin(x) \text{ if } x \leq \pi \\ x\tan(x) \text{ if } x > \pi \end{cases}$, at $x = \pi$

In the following exercises, find the value(s) of *k* that makes each function continuous over the given interval.

145. $f(x) = \begin{cases} 3x + 2, & x < k \\ 2x - 3, & k \leq x \leq 8 \end{cases}$

146. $f(\theta) = \begin{cases} \sin\theta, & 0 \leq \theta < \frac{\pi}{2} \\ \cos(\theta + k), & \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$

147. $f(x) = \begin{cases} \frac{x^2+3x+2}{x+2}, & x \neq -2 \\ k, & x = -2 \end{cases}$

148. $f(x) = \begin{cases} e^{kx}, & 0 \leq x < 4 \\ x+3, & 4 \leq x \leq 8 \end{cases}$

149. $f(x) = \begin{cases} \sqrt{kx}, & 0 \leq x \leq 3 \\ x+1, & 3 < x \leq 10 \end{cases}$

In the following exercises, use the Intermediate Value Theorem (IVT).

150. Let $h(x) = \begin{cases} 3x^2 - 4, & x \leq 2 \\ 5 + 4x, & x > 2 \end{cases}$ Over the interval $[0, 4]$, there is no value of *x* such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

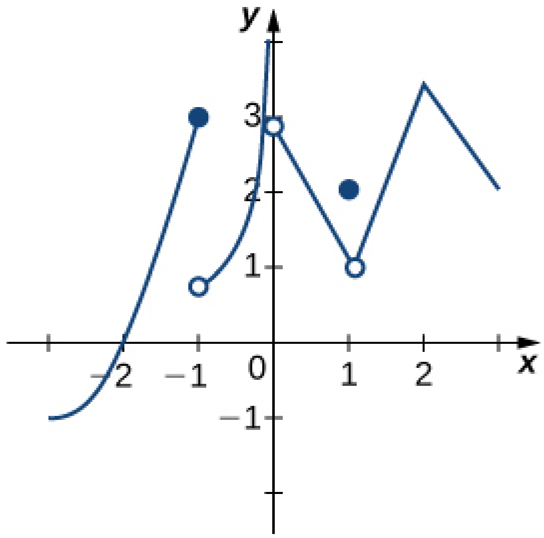
151. A particle moving along a line has at each time *t* a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value *c* for $2 < c < 5$ such that $h(c) = 0$.

152. **[T]** Use the statement "The cosine of *t* is equal to *t* cubed."

a. Write a mathematical equation of the statement.
b. Prove that the equation in part a. has at least one real solution.
c. Use a calculator to find an interval of length 0.01 that contains a solution.

153. Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

154. Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.
b. For each value in part a., state why the formal definition of continuity does not apply.
c. Classify each discontinuity as either jump, removable, or infinite.

155. Let $f(x) = \begin{cases} 3x, x > 1 \\ x^3, x < 1 \end{cases}$.

a. Sketch the graph of *f*.
b. Is it possible to find a value *k* such that $f(1) = k,$ which makes $f(x)$ continuous for all real numbers? Briefly explain.

156. Let $f(x) = \frac{x^4 - 1}{x^2 - 1}$ for $x \neq -1, 1.$

a. Sketch the graph of *f*.
b. Is it possible to find values k_1 and k_2 such that $f(-1) = k_1$ and $f(1) = k_2,$ and that makes $f(x)$ continuous for all real numbers? Briefly explain.

157. Sketch the graph of the function $y = f(x)$ with properties i. through vii.

i. The domain of *f* is $(-\infty, +\infty)$.
ii. *f* has an infinite discontinuity at $x = -6$.
iii. $f(-6) = 3$
iv. $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) = 2$
v. $f(-3) = 3$
vi. *f* is left continuous but not right continuous at $x = 3$.
vii. $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$

158. Sketch the graph of the function $y = f(x)$ with properties i. through iv.

i. The domain of *f* is $[0, 5]$.

ii. $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist and are equal.

iii. $f(x)$ is left continuous but not continuous at $x = 2,$ and right continuous but not continuous at $x = 3.$

iv. $f(x)$ has a removable discontinuity at $x = 1,$ a jump discontinuity at $x = 2,$ and the following limits hold: $\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = 2.$

In the following exercises, suppose $y = f(x)$ is defined for all *x*. For each description, sketch a graph with the indicated property.

159. Discontinuous at $x = 1$ with $\lim_{x \to -1} f(x) = -1$ and $\lim_{x \to 2} f(x) = 4$

160. Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \to 0} f(x) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

161. $f(t) = \frac{2}{e^t - e^{-t}}$ is continuous everywhere.

162. If the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal, then *f* cannot be discontinuous at $x = a.$

163. If a function is not continuous at a point, then it is not defined at that point.

164. According to the IVT, $\cos x - \sin x - x = 2$ has a solution over the interval $[-1, 1].$

165. If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b].$

166. The function $f(x) = \frac{x^2 - 4x + 3}{x^2 - 1}$ is continuous over the interval $[0, 3].$

167. If $f(x)$ is continuous everywhere and $f(a), f(b) > 0,$ then there is no root of $f(x)$ in the interval $[a, b].$

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \frac{|q_1 q_2|}{r^2},$ where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and *r* is the distance between the two particles.

168. To simplify the calculation of a model with many interacting particles, after some threshold value $r = R,$ we approximate *F* as zero.

a. Explain the physical reasoning behind this assumption.
b. What is the force equation?
c. Evaluate the force *F* using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} coulombs (C), and the Coulomb constant $k_e = 8.988 \times 10^9 \text{Nm}^2/\text{C}^2$ are 1 m apart. Also, assume $R < 1$ m. How much inaccuracy does our approximation generate? Is our approximation reasonable?
d. Is there any finite value of *R* for which this system remains continuous at *R*?

169. Instead of making the force 0 at *R*, instead we let the force be 10^{-20} for $r \geq R.$ Assume two protons, which have a magnitude of charge 1.6022×10^{-19} C, and the Coulomb constant $k_e = 8.988 \times 10^9 \text{Nm}^2/\text{C}^2.$ Is there a value *R* that can make this system continuous? If so, find it.

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2,$ where *m* is the mass of the rocket, *d* is the distance of the rocket from the center of Earth, and *k* is a constant.

170. **[T]** Determine the value and units of *k* given that the mass of the rocket is 3 million kg. (*Hint*: The distance from the center of Earth to its surface is 6378 km.)

171. **[T]** After a certain distance *D* has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by

$$F(d) = \begin{cases} -\dfrac{mk}{d^2} & \text{if } d < D \\ 10{,}000 & \text{if } d \geq D \end{cases}.$$

Using the value of k found in the previous exercise, find the necessary condition *D* such that the force function remains continuous.

172. As the rocket travels away from Earth's surface, there is a distance *D* where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as

$$F(d) = \begin{cases} -\dfrac{m_1 k}{d^2} \text{ if } d < D \\ -\dfrac{m_2 k}{d^2} \text{ if } d \geq D \end{cases}.$$

Is there a *D* value such that this function is continuous, assuming $m_1 \neq m_2$?

Prove the following functions are continuous everywhere

173. $f(\theta) = \sin\theta$

174. $g(x) = |x|$

175. Where is $f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x \text{ is rational} \end{cases}$ continuous?

Chapter Outline

Figure 3.1 The Hennessey Venom GT can go from 0 to 200 mph in 14.51 seconds. (credit: modification of work by Codex41, Flickr)

The Hennessey Venom GT is one of the fastest cars in the world. In 2014, it reached a record-setting speed of 270.49 mph. It can go from 0 to 200 mph in 14.51 seconds. The techniques in this chapter can be used to calculate the acceleration the Venom achieves in this feat (see Example 3.8.)

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

Learning Objectives

3.1.1. Recognize the meaning of the tangent to a curve at a point.
3.1.2. Calculate the slope of a tangent line.
3.1.3. Identify the derivative as the limit of a difference quotient.
3.1.4. Calculate the derivative of a given function at a point.
3.1.5. Describe the velocity as a rate of change.
3.1.6. Explain the difference between average velocity and instantaneous velocity.
3.1.7. Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac Newton $(1643–1727)$ and the German Gottfried Leibniz $(1646–1716)$, whose images appear in Figure 3.2. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

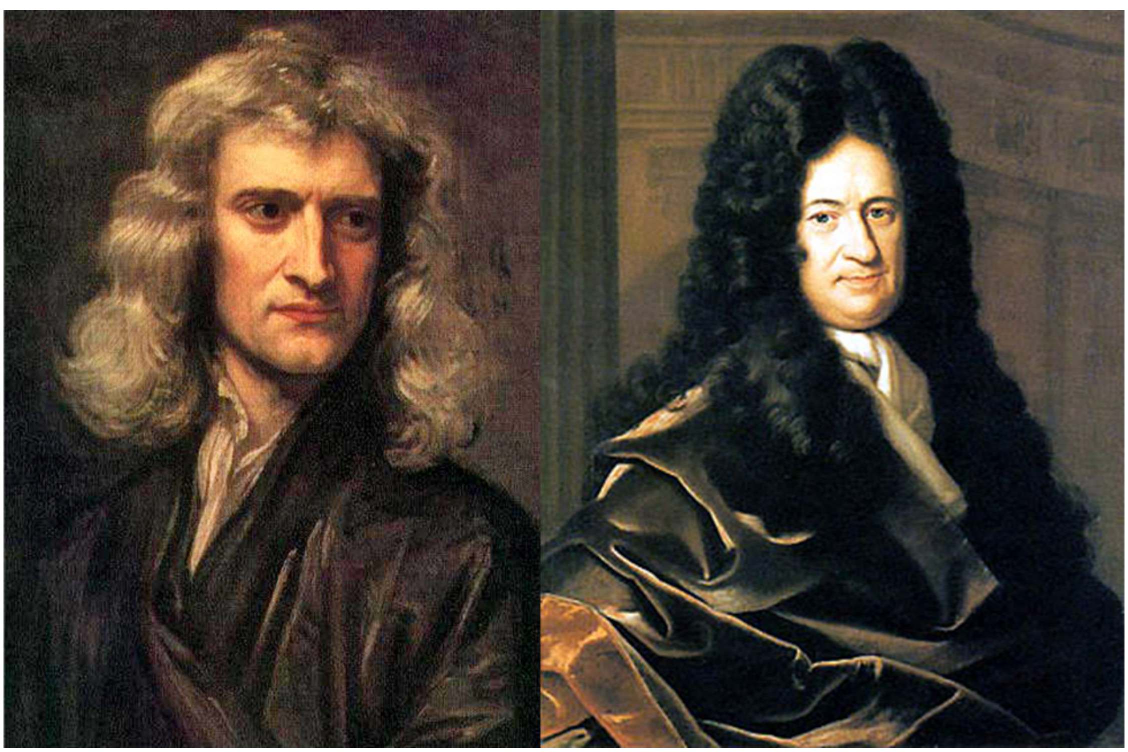


Figure 3.2 Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in Figure 3.3. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

We can also calculate the slope of a secant line to a function at a value *a* by using this equation and replacing x with $a + h$, where h is a value close to 0. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}.$$

DEFINITION

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a} \tag{3.1}$$

is a **difference quotient.**

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a + h) - f(a)}{h} \tag{3.2}$$

is a difference quotient with increment h.

MEDIA

View the development of the derivative with this applet.

These two expressions for calculating the slope of a secant line are illustrated in Figure 3.3. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

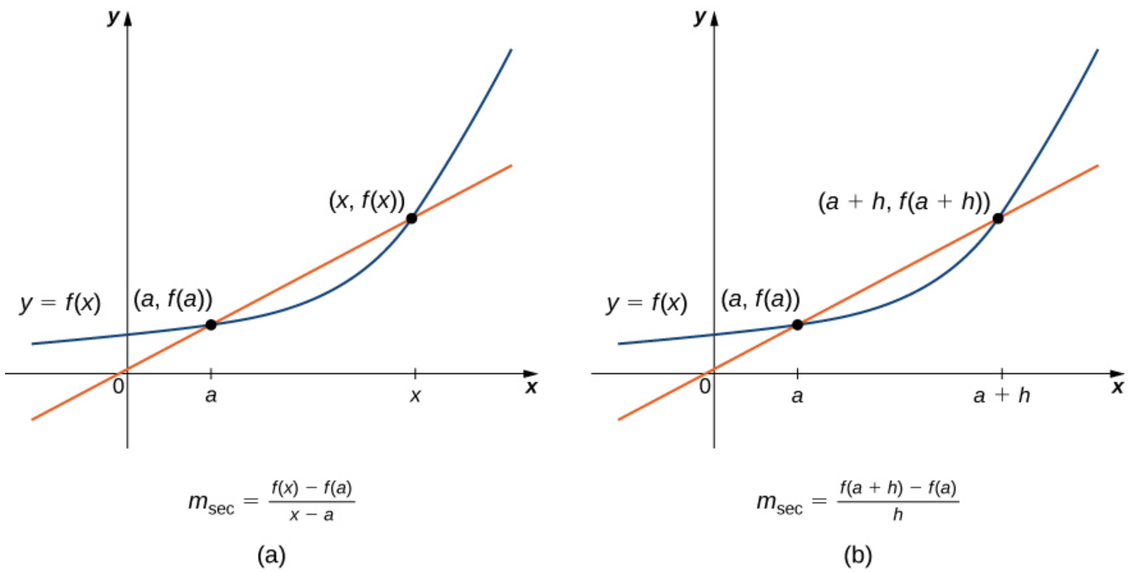
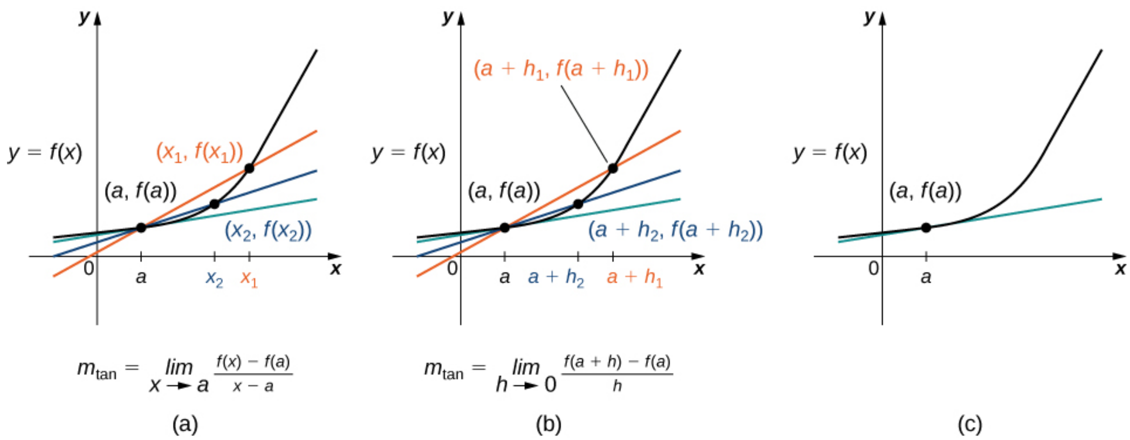


Figure 3.3 We can calculate the slope of a secant line in either of two ways.

In Figure 3.4(a) we see that, as the values of x approach $a,$ the slopes of the secant lines provide better estimates of the rate of change of the function at $a.$ Furthermore, the secant lines themselves approach the tangent line to the function at $a,$ which represents the limit of the secant lines. Similarly, Figure 3.4(b) shows that as the values of h get closer to $0,$ the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at $a,$ as shown in Figure 3.4(c).



$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Figure 3.4 The secant lines approach the tangent line (shown in green) as the second point approaches the first.

MEDIA

You can use this site to explore graphs to see if they have a tangent line at a point.

In Figure 3.5 we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be locally linear in the immediate vicinity of $x = 1$.

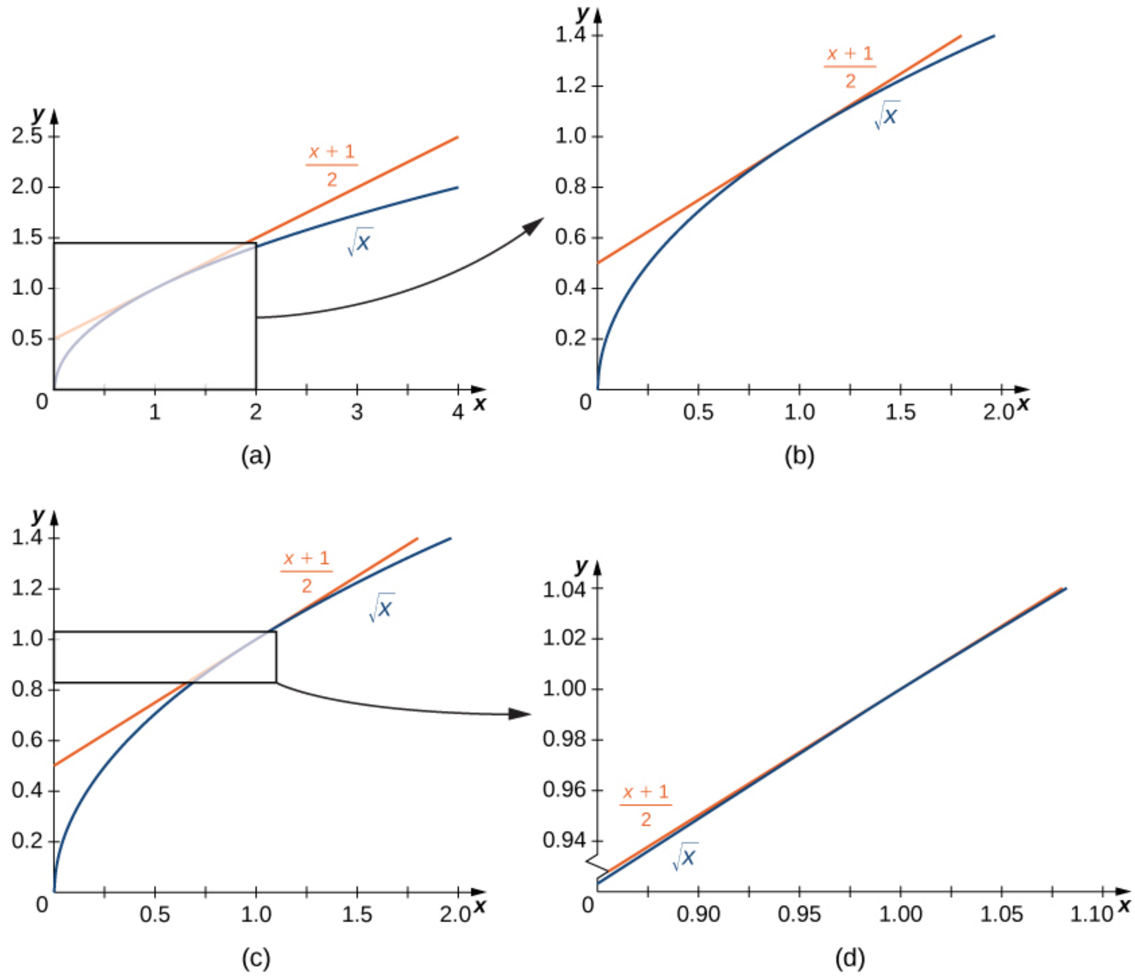


Figure 3.5 For values of x close to $1,$ the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

DEFINITION

Let $f(x)$ be a function defined in an open interval containing a. The *tangent line* to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.3}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{3.4}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

EXAMPLE 3.1

Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

[Show Solution]

EXAMPLE 3.2

The Slope of a Tangent Line Revisited

Use Equation 3.4 to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

[Show Solution]

EXAMPLE 3.3

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

[Show Solution]

CHECKPOINT 3.1

Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative**. The process of finding a derivative is called **differentiation**.

DEFINITION

Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.5}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{3.6}$$

EXAMPLE 3.4

Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 3.5.

[Show Solution]

CHECKPOINT 3.2

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 3.6.

EXAMPLE 3.5

Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 3.5.

[Show Solution]

EXAMPLE 3.6

Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 3.6.

[Show Solution]

CHECKPOINT 3.3

For $f(x) = x^2 + 3x + 2$, find $f'(1)$.

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the average velocity of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \tag{3.7}$$

As the values of t approach a, the values of v_{ave} approach the value we call the instantaneous velocity at a. That is, instantaneous velocity at a, denoted $v(a)$, is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \tag{3.8}$$

To better understand the relationship between average velocity and instantaneous velocity, see Figure 3.8. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t]$.

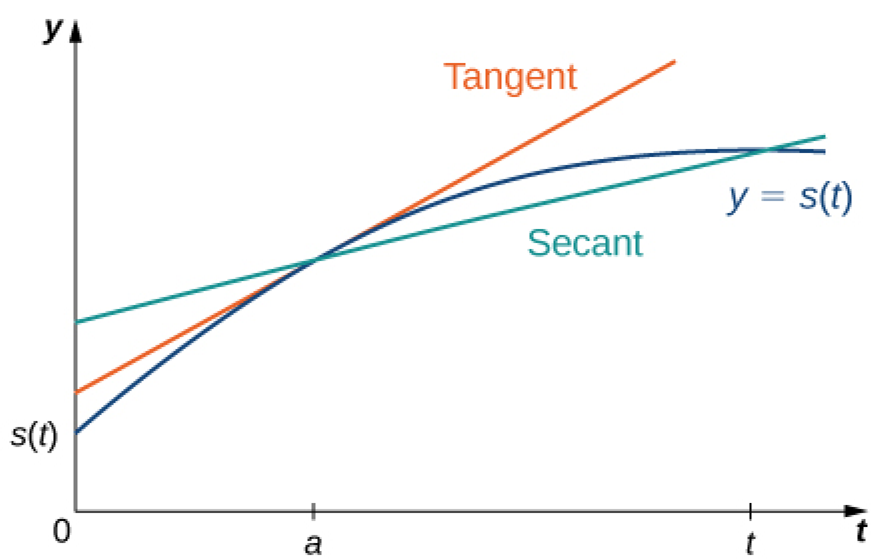


Figure 3.8 The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use Equation 3.5 to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using Equation 3.7.

EXAMPLE 3.7

Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ (Figure 3.9). Use a table of values to estimate $v(0)$. Check the estimate by using Equation 3.5.

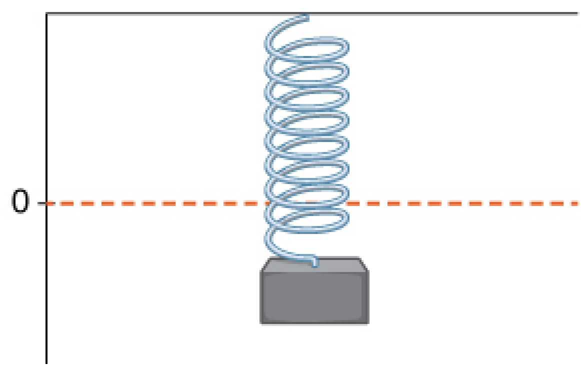


Figure 3.9 A lead weight suspended from a spring in vertical oscillatory motion.

[Show Solution]

CHECKPOINT 3.4

A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64, 0 \le t \le 2$. Find its instantaneous velocity 1 second after it is dropped, using Equation 3.5.

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

DEFINITION

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

EXAMPLE 3.8

Chapter Opener: Estimating Rate of Change of Velocity

Figure 3.10 (credit: modification of work by Codex41, Flickr)

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its acceleration) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?

[Show Solution]

EXAMPLE 3.9

Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70°\text{F}$ at 9 p.m., reaches a low of $60°$ during the night, and rises back to $70°$ by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10,$ where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

[Show Solution]

EXAMPLE 3.10

Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10{,}000$ dollars. Find the rate of change of profit when $10{,}000$ games are produced. Should the toy company increase or decrease production?

[Show Solution]

CHECKPOINT 3.5

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10.$ The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25),$ the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Section 3.1 Exercises

For the following exercises, use Equation 3.1 to find the slope of the secant line between the values x_1 and x_2 for each function $y = f(x)$.

1. $f(x) = 4x + 7; x_1 = 2, x_2 = 5$

2. $f(x) = 8x - 3; x_1 = -1, x_2 = 3$

3. $f(x) = x^2 + 2x + 1; x_1 = 3, x_2 = 3.5$

4. $f(x) = -x^2 + x + 2; x_1 = 0.5, x_2 = 1.5$

5. $f(x) = \frac{4}{3x-1}; x_1 = 1, x_2 = 3$

6. $f(x) = \frac{x-7}{2x+1}; x_1 = 0, x_2 = 2$

7. $f(x) = \sqrt{x}; x_1 = 1, x_2 = 16$

8. $f(x) = \sqrt{x-9}; x_1 = 10, x_2 = 13$

9. $f(x) = x^{1/3} + 1; x_1 = 0, x_2 = 8$

10. $f(x) = 6x^{2/3} + 2x^{1/3}; x_1 = 1, x_2 = 27$

For the following functions,

a. use Equation 3.4 to find the slope of the tangent line $m_{\tan} = f'(a),$ and
b. find the equation of the tangent line to f at $x = a.$

11. $f(x) = 3 - 4x, a = 2$

12. $f(x) = \frac{x}{5} + 6, a = -1$

13. $f(x) = x^2 + x, a = 1$

14. $f(x) = 1 - x - x^2, a = 0$

15. $f(x) = \frac{7}{x}, a = 3$

16. $f(x) = \sqrt{x+8}, a = 1$

17. $f(x) = 2 - 3x^2, a = -2$

18. $f(x) = \frac{-3}{x-1}, a = 4$

19. $f(x) = \frac{2}{x+3}, a = -4$

20. $f(x) = \frac{3}{x^2}, a = 3$

For the following functions $y = f(x)$, find $f'(a)$ using Equation 3.1.

21. $f(x) = 5x + 4, a = -1$

22. $f(x) = -7x + 1, a = 3$

23. $f(x) = x^2 + 9x, a = 2$

24. $f(x) = 3x^2 - x + 2, a = 1$

25. $f(x) = \sqrt{x}, a = 4$

26. $f(x) = \sqrt{x-2}, a = 6$

27. $f(x) = \frac{1}{x}, a = 2$

28. $f(x) = \frac{1}{x-3}, a = -1$

29. $f(x) = \frac{1}{x^3}, a = 1$

30. $f(x) = \frac{1}{\sqrt{x}}, a = 4$

For the following exercises, given the function $y = f(x)$,

a. find the slope of the secant line PQ for each point $Q(x, f(x))$ with x value given in the table.
b. Use the answers from a. to estimate the value of the slope of the tangent line at P.
c. Use the answer from b. to find the equation of the tangent line to f at point P.

31. **[T]** $f(x) = x^2 + 3x + 4, P(1, 8)$ (Round to 6 decimal places.)

x	**Slope** m_{PQ}	*x*	**Slope** m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

32. **[T]** $f(x) = \frac{x+1}{x^2-1}, P(0, -1)$

x	**Slope** m_{PQ}	*x*	**Slope** m_{PQ}
0.1	(i)	-0.1	(vii)
0.01	(ii)	-0.01	(viii)
0.001	(iii)	-0.001	(ix)
0.0001	(iv)	-0.0001	(x)
0.00001	(v)	-0.00001	(xi)
0.000001	(vi)	-0.000001	(xii)

33. **[T]** $f(x) = 10e^{0.5x}, P(0, 10)$ (Round to 4 decimal places.)

x	**Slope** m_{PQ}
-0.1	(i)
-0.01	(ii)
-0.001	(iii)

x	Slope m_{PQ}
-0.0001	(iv)
-0.00001	(v)
–0.000001	(vi)

34. **[T]** $f(x) = \tan(x), P(\pi, 0)$

x	Slope m_{PQ}
3.1	(i)
3.14	(ii)
3.141	(iii)
3.1415	(iv)
3.14159	(v)
3.141592	(vi)

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;
b. the average velocity between $t = 2$ and $t = 2 + h$, where (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, and (iv) $h = 0.0001$; and
c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

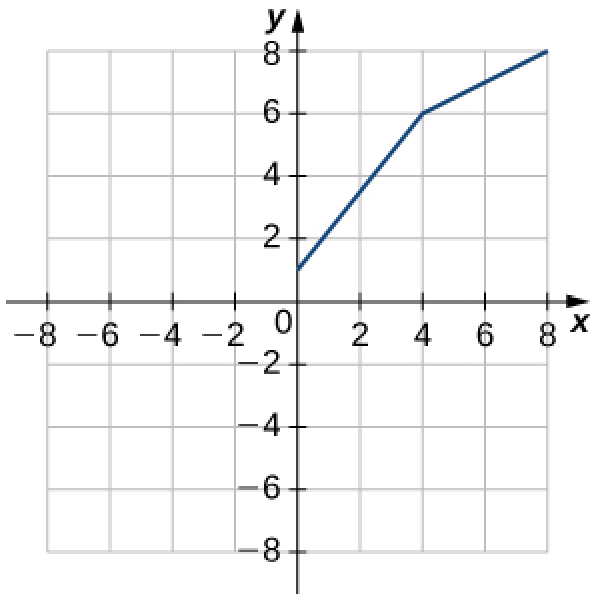
35. $s(t) = \frac{1}{3}t + 5$

36. $s(t) = t^2 - 2t$

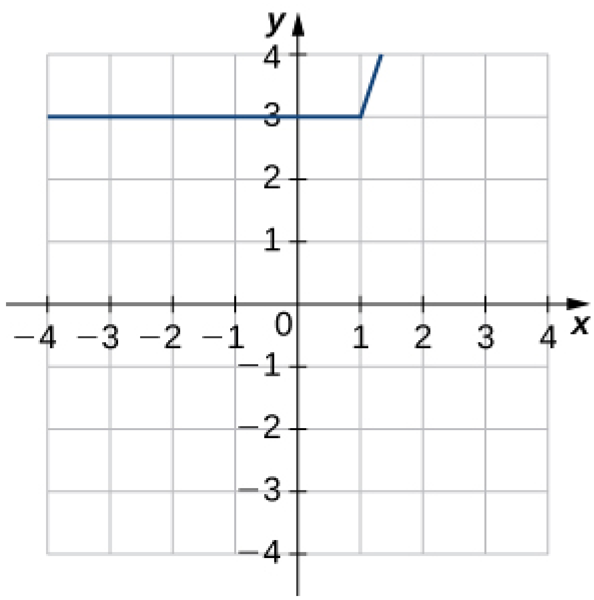
37. $s(t) = 2t^3 + 3$

38. $s(t) = \frac{16}{t^2} - \frac{4}{t}$

39. Use the following graph to evaluate a. $f'(1)$ and b. $f'(6)$.



40. Use the following graph to evaluate a. $f'(-3)$ and b. $f'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.

41. $f(x) = x^{1/3}, x = 0$

42. $f(x) = x^{2/3}, x = 0$

43. $f(x) = \begin{cases} 1, x < 1 \\ x, x \geq 1 \end{cases}, x = 1$

44. $f(x) = \frac{|x|}{x}, x = 0$

45. **[T]** The position in feet of a race car along a straight track after t seconds is modeled by the function $s(t) = 8t^2 - \frac{1}{16}t^3.$

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:

i. [4, 4.1]
ii. [4, 4.01]
iii. [4, 4.001]
iv. [4, 4.0001]

b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t = 4$ seconds.

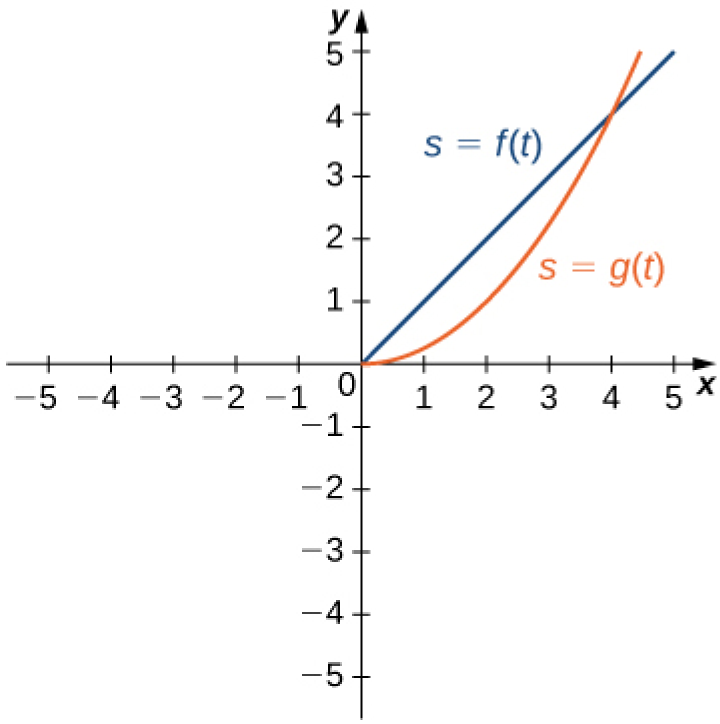
46. **[T]** The distance in feet that a ball rolls down an incline is modeled by the function $s(t) = 14t^2,$ where *t* is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:

i. [5, 5.1]
ii. [5, 5.01]
iii. [5, 5.001]
iv. [5, 5.0001]

b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t = 5$ seconds.

47. Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s = f(t)$ and $s = g(t),$ where s is measured in feet and t is measured in seconds.



a. Which vehicle has traveled farther at $t = 2$ seconds?
b. What is the approximate velocity of each vehicle at $t = 3$ seconds?
c. Which vehicle is traveling faster at $t = 4$ seconds?
d. What is true about the positions of the vehicles at $t = 4$ seconds?

48. **[T]** The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300.$

a. Calculate the average cost per jar over the following intervals:

   i. [100, 100.1]
   ii. [100, 100.01]
   iii. [100, 100.001]
   iv. [100, 100.0001]

b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

49. **[T]** For the function $f(x) = x^3 - 2x^2 - 11x + 12,$ do the following.

a. Use a graphing calculator to graph *f* in an appropriate viewing window.
b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{\tan} = f'(a) = 0.$

50. **[T]** For the function $f(x) = \frac{x}{1+x^2}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\tan} = f'(a) = 0$.

51. Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for $N(x)$.
b. What is $N(100)$? Explain the physical meaning.
c. What is $N'(100)$? Explain the physical meaning.

52. **[T]** For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the nDeriv function, which numerically finds the derivative, on a graphing calculator to estimate $f'(-2), f'(-0.5), f'(1.7)$, and $f'(2.718)$.

53. **[T]** For the function $f(x) = \frac{x^2}{x^2+1}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the nDeriv function on a graphing calculator to find $f'(-4), f'(-2), f'(2)$, and $f'(4)$.

Learning Objectives

3.2.1. Define the derivative function of a given function.
3.2.2. Graph a derivative function from the graph of a given function.
3.2.3. State the connection between derivatives and continuity.
3.2.4. Describe three conditions for when a function does not have a derivative.
3.2.5. Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

DEFINITION

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.9}$$

A function $f(x)$ is said to be **differentiable at a** if $f'(a)$ exists. More generally, a function is said to be **differentiable on S** if it is differentiable at every point in an open set S, and a **differentiable function** is one in which $f'(x)$ exists on its domain.

In the next few examples we use Equation 3.9 to find the derivative of a function.

EXAMPLE 3.11

Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

[Show Solution]

EXAMPLE 3.12

Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

[Show Solution]

CHECKPOINT 3.6

Find the derivative of $f(x) = x^2$.

We use a variety of different notations to express the derivative of a function. In Example 3.12 we showed that if $f(x) = x^2 - 2x,$ then $f'(x) = 2x - 2.$ If we had expressed this function in the form $y = x^2 - 2x,$ we could have expressed the derivative as $y' = 2x - 2$ or $\frac{dy}{dx} = 2x - 2.$ We could have conveyed the same information by writing $\frac{d}{dx}\left(x^2 - 2x\right) = 2x - 2.$ Thus, for the function $y = f(x),$ each of the following notations represents the derivative of $f(x)$:

$$f'(x), \ \frac{dy}{dx}, \ y', \ \frac{d}{dx}(f(x)).$$

In place of $f'(a)$ we may also use $\left.\frac{dy}{dx}\right|_{x=a}$ Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (Figure 3.11). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to $x,$ is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

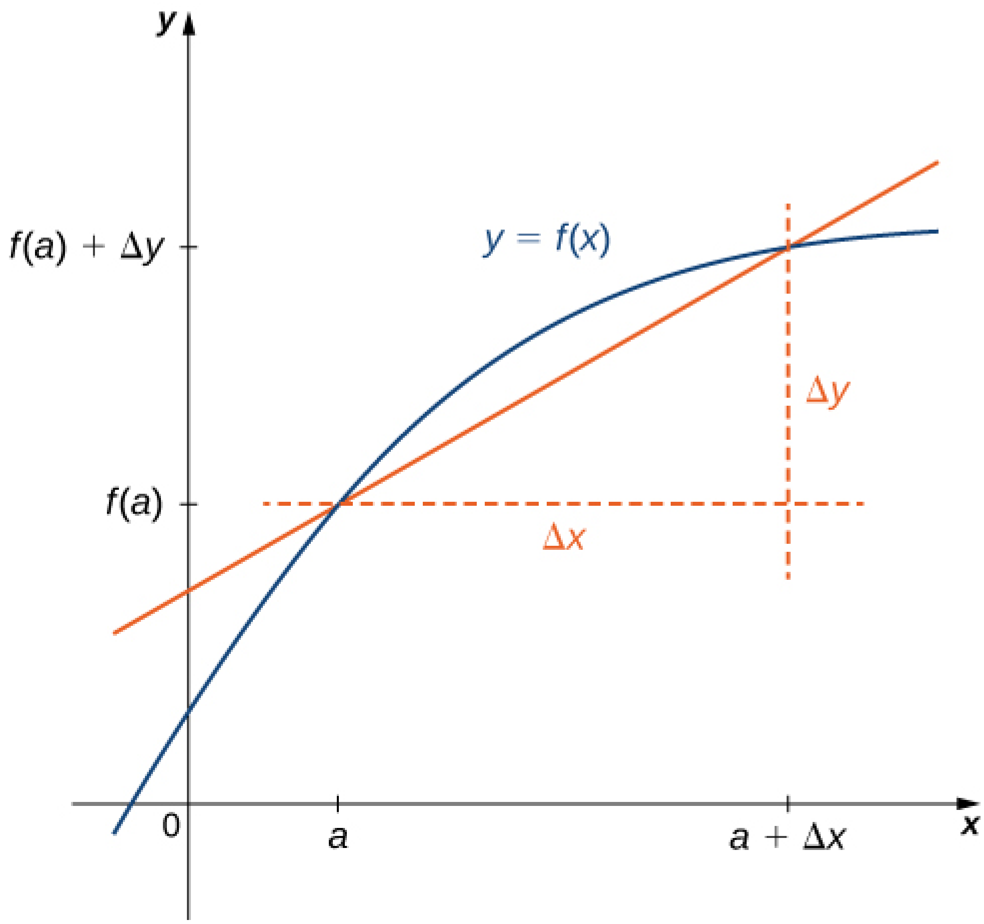


Figure 3.11 The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$

Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In Example 3.11 we found that for $f(x) = \sqrt{x}, f'(x) = 1/2\sqrt{x}$. If we graph these functions on the same axes, as in Figure 3.12, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

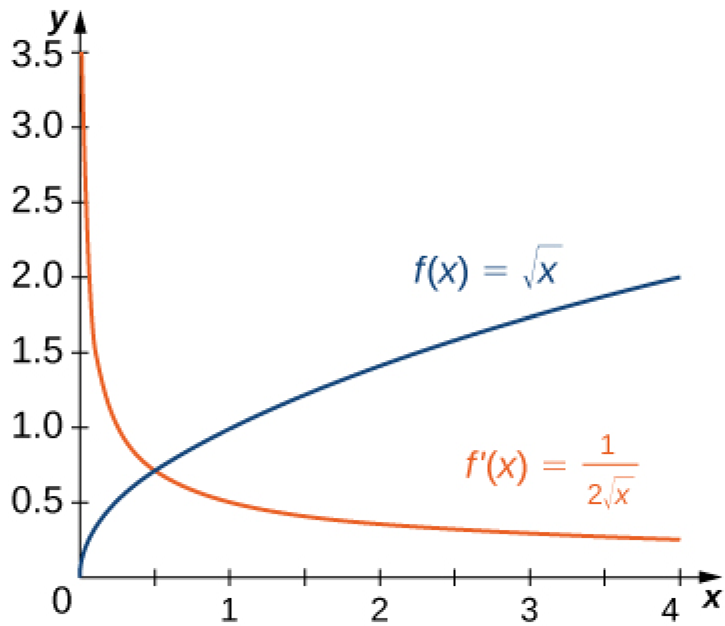


Figure 3.12 The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In Example 3.12 we found that for $f(x) = x^2 - 2x, f'(x) = 2x - 2$. The graphs of these functions are shown in Figure 3.13. Observe that $f(x)$ is decreasing for $x < 1$. For these same values of $x, f'(x) < 0$. For values of $x > 1, f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

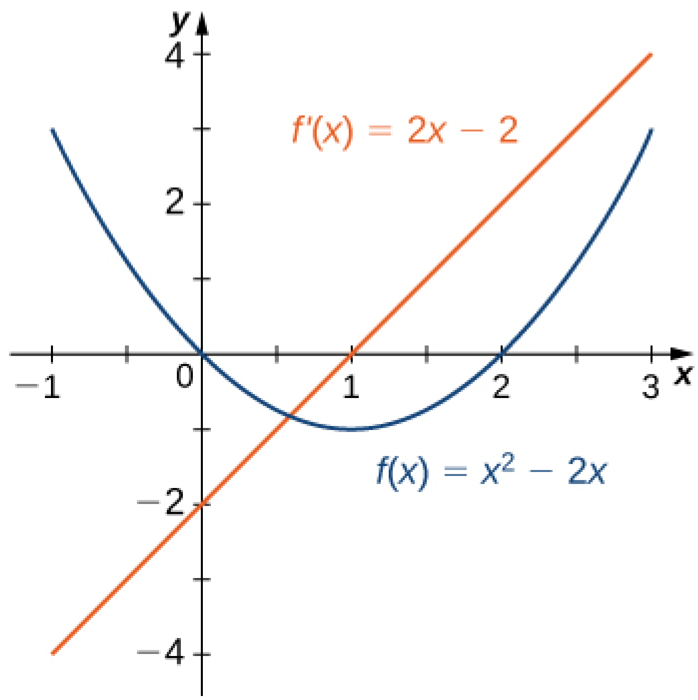
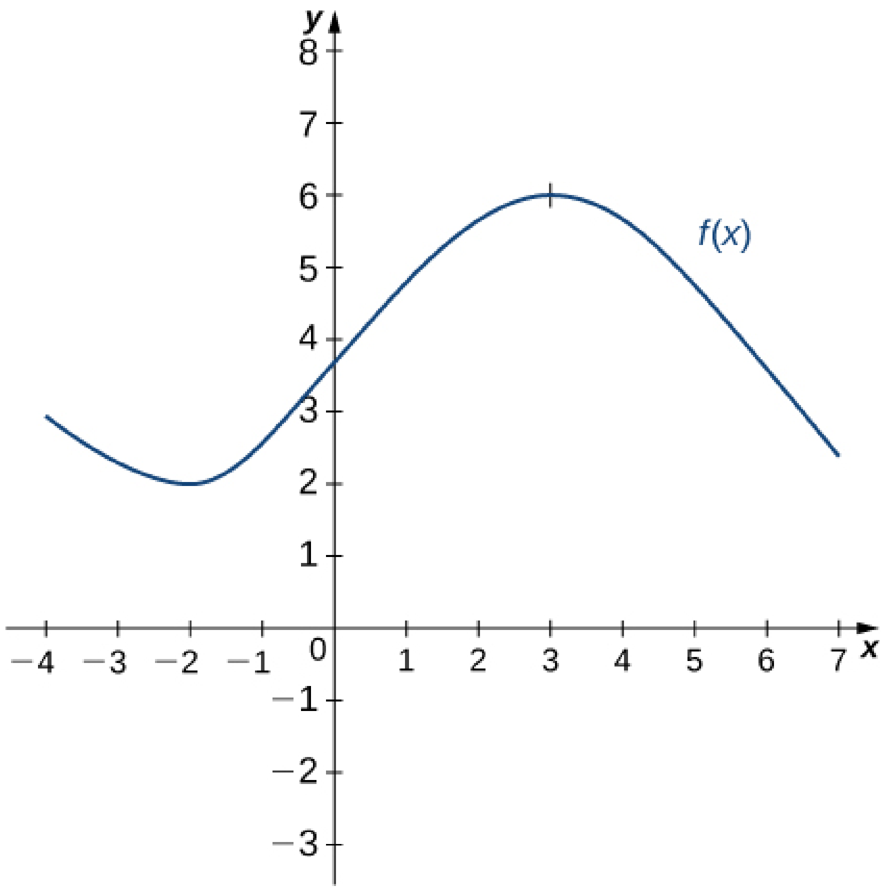


Figure 3.13 The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent.

EXAMPLE 3.13

Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



[Show Solution]

CHECKPOINT 3.7

Sketch the graph of $f(x) = x^2 - 4.$ On what interval is the graph of $f'(x)$ above the x-axis?

Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there; however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

THEOREM 3.1

Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at $a,$ then f is continuous at $a.$

Proof

If $f(x)$ is differentiable at $a,$ then $f'(a)$ exists and

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \to a} f(x) = f(a).$ Thus,

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} (f(x) - f(a) + f(a)) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right) && \text{Multiply and divide } f(x) - f(a) \text{ by } x \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \cdot \left(\lim_{x \to a} (x - a) \right) + \lim_{x \to a} f(a) \\
&= f'(a) \cdot 0 + f(a) \\
&= f(a).
\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a),$ we conclude that f is continuous at $a.$

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x) = |x|.$ This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x) = |x|,$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \text{ and } \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

See Figure 3.14.

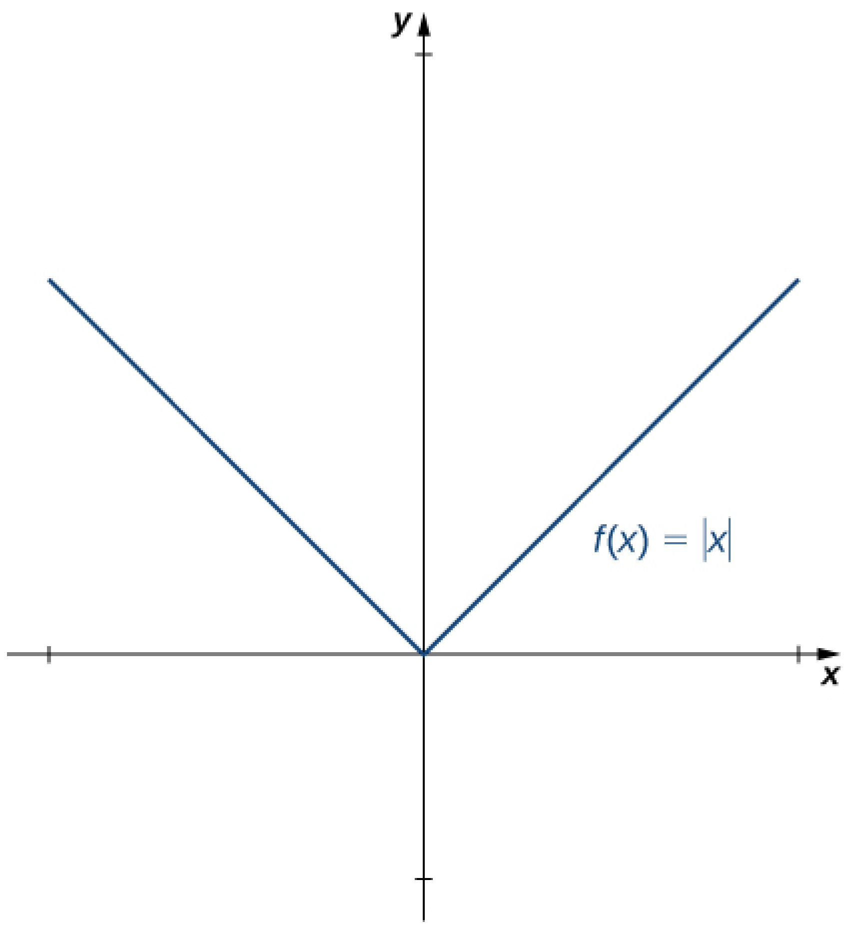


Figure 3.14 The function $f(x) = |x|$ is continuous at 0 but is not differentiable at 0.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x) = \sqrt[3]{x}$:

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x) = \sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 (Figure 3.15).

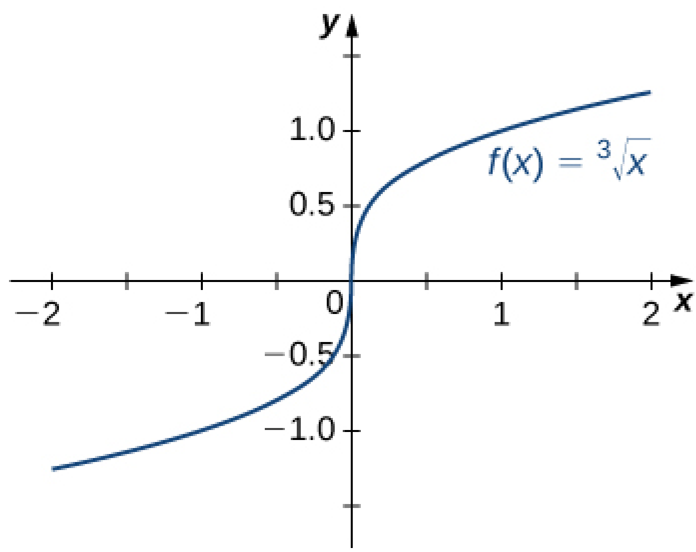


Figure 3.15 The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0.$ It is continuous at 0 but is not differentiable at $0.$

The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at $0.$ We see that

$$f'(0) = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero (Figure 3.16).

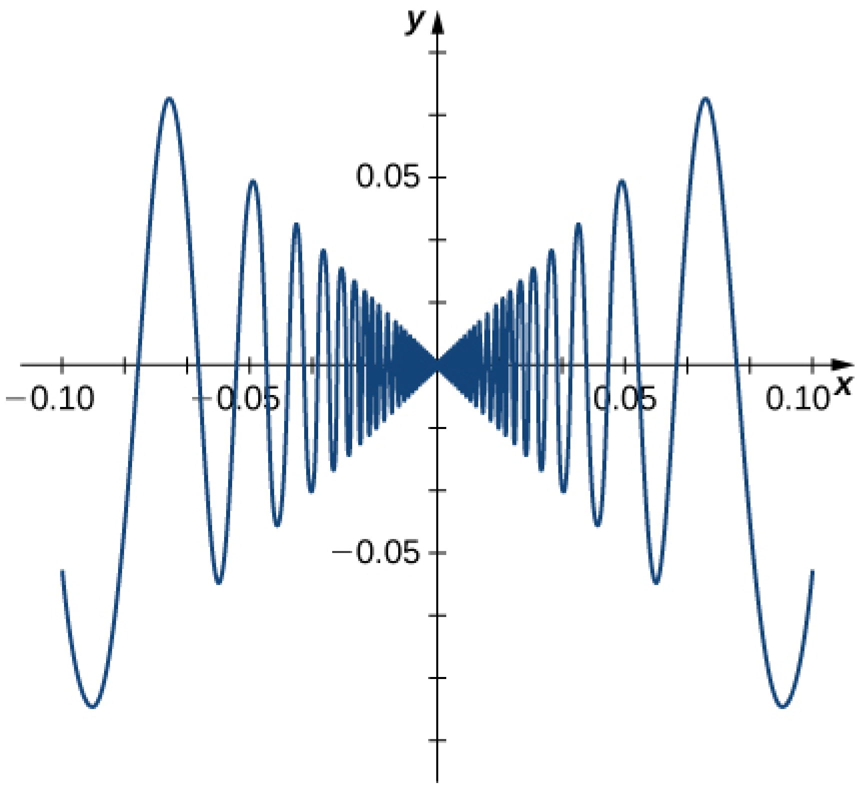


Figure 3.16 The function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be "smooth" at that point.
3. As we saw in the example of $f(x) = \sqrt[3]{x},$ a function fails to be differentiable at a point where there is a vertical tangent line.
4. As we saw with $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

EXAMPLE 3.14

A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (Figure 3.17). The function that describes the track is to have the form

$$f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c \text{ if} x < -10 \\ -\frac{1}{4}x + \frac{5}{2} \text{ if} x \geq -10 \end{cases}$$

where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at $-10.$ Find values of b and c that make $f(x)$ both continuous and differentiable.

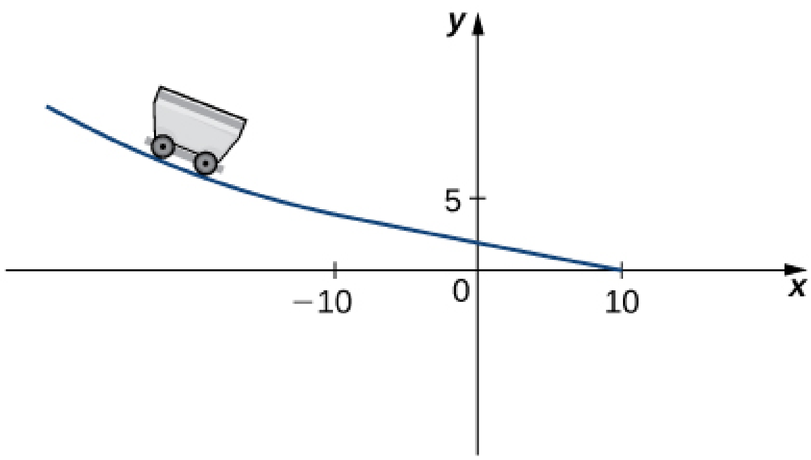


Figure 3.17 For the car to move smoothly along the track, the function must be both continuous and differentiable.

[Show Solution]

CHECKPOINT 3.8

Find values of a and b that make $f(x) = \begin{cases} ax + b \text{ if } x < 3 \\ x^2 \text{ if } x \geq 3 \end{cases}$ both continuous and differentiable at 3.

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

$$f''(x),\ f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x)$$

$$y''(x),\ y'''(x),\ y^{(4)}(x), \ldots, y^{(n)}(x)$$

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly. Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.

EXAMPLE 3.15

Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

[Show Solution]

CHECKPOINT 3.9

Find $f''(x)$ for $f(x) = x^2$.

EXAMPLE 3.16

Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

[Show Solution]

CHECKPOINT 3.10

For $s(t) = t^3$, find $a(t)$.

Section 3.2 Exercises

For the following exercises, use the definition of a derivative to find $f'(x)$.

54. $f(x) = 6$

55. $f(x) = 2 - 3x$

56. $f(x) = \frac{2x}{7} + 1$

57. $f(x) = 4x^2$

58. $f(x) = 5x - x^2$

59. $f(x) = \sqrt{2x}$

60. $f(x) = \sqrt{x-6}$

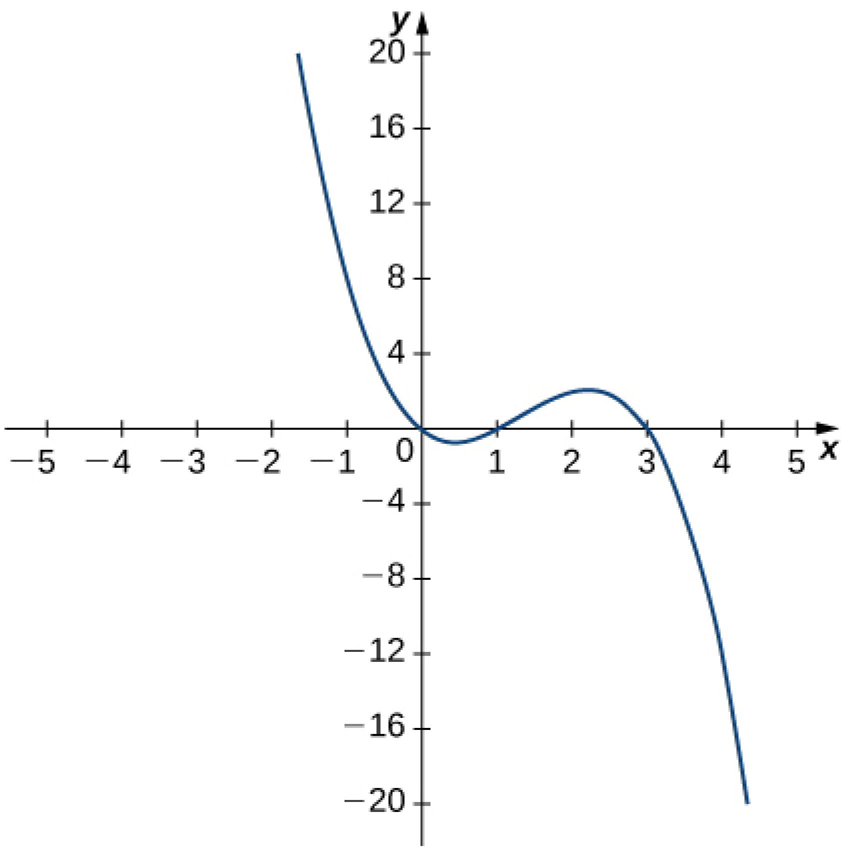
61. $f(x) = \frac{9}{x}$

62. $f(x) = x + \frac{1}{x}$

63. $f(x) = \frac{1}{\sqrt{x}}$

For the following exercises, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.

64.



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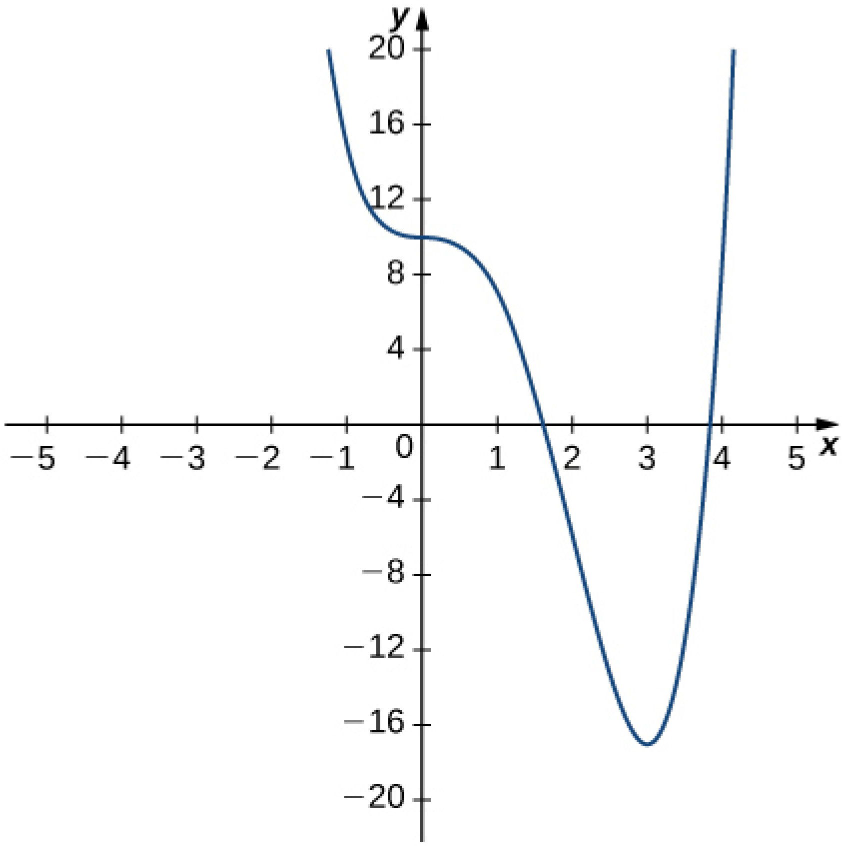

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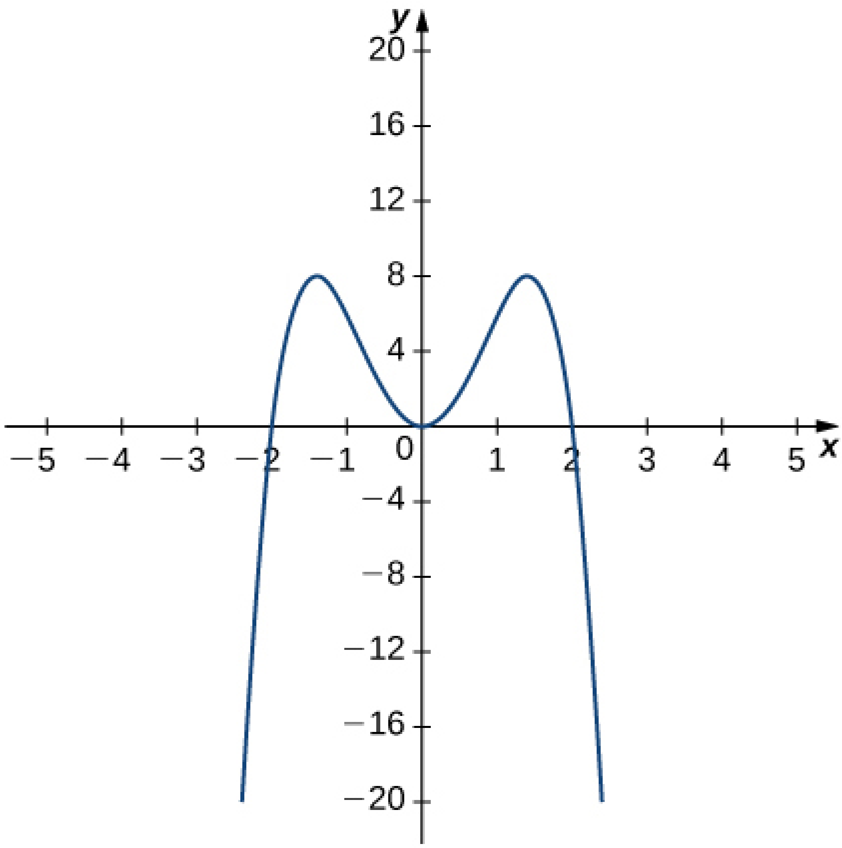

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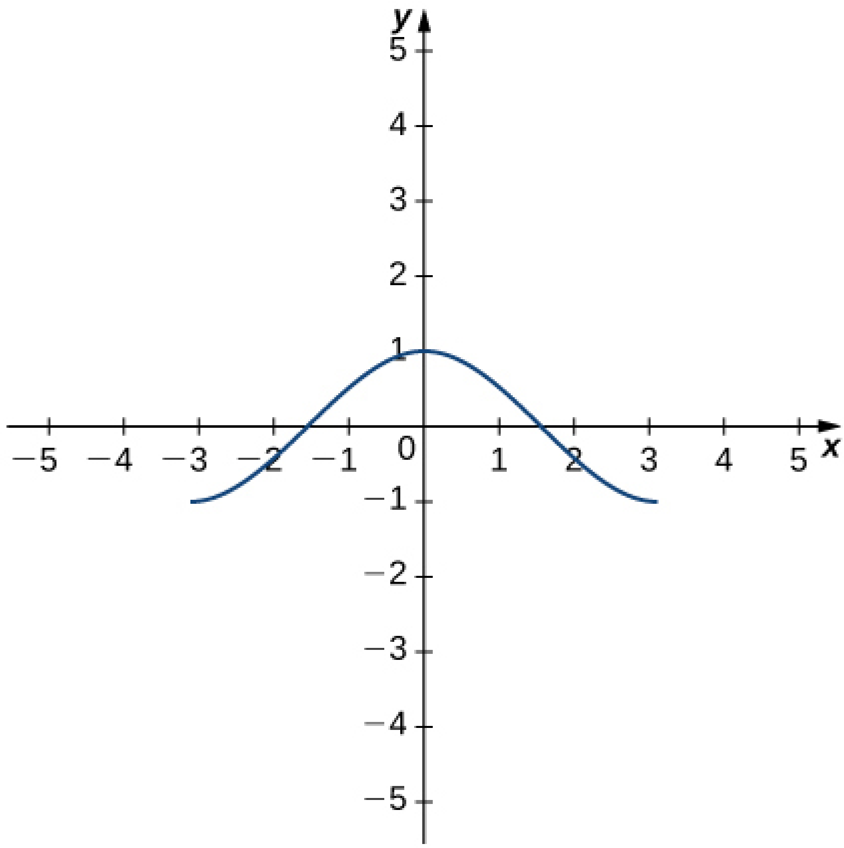


For the following exercises, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a.

68. $\lim_{h \to 0} \frac{(1+h)^{2/3}-1}{h}$

69. $\lim_{h \to 0} \frac{\left[3(2+h)^2+2\right]-14}{h}$

70. $\lim_{h \to 0} \frac{\cos(\pi+h)+1}{h}$

71. $\lim_{h \to 0} \frac{(2+h)^4-16}{h}$

72. $\lim_{h \to 0} \frac{[2(3+h)^2-(3+h)]-15}{h}$

73. $\lim_{h \to 0} \frac{e^h-1}{h}$

For the following functions,

a. sketch the graph and
b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

74. $f(x) = \begin{cases} 2\sqrt{x}, 0 \le x \le 1 \\ 3x - 1, x > 1 \end{cases}$

75. $f(x) = \begin{cases} 3, x < 1 \\ 3x, x \ge 1 \end{cases}$

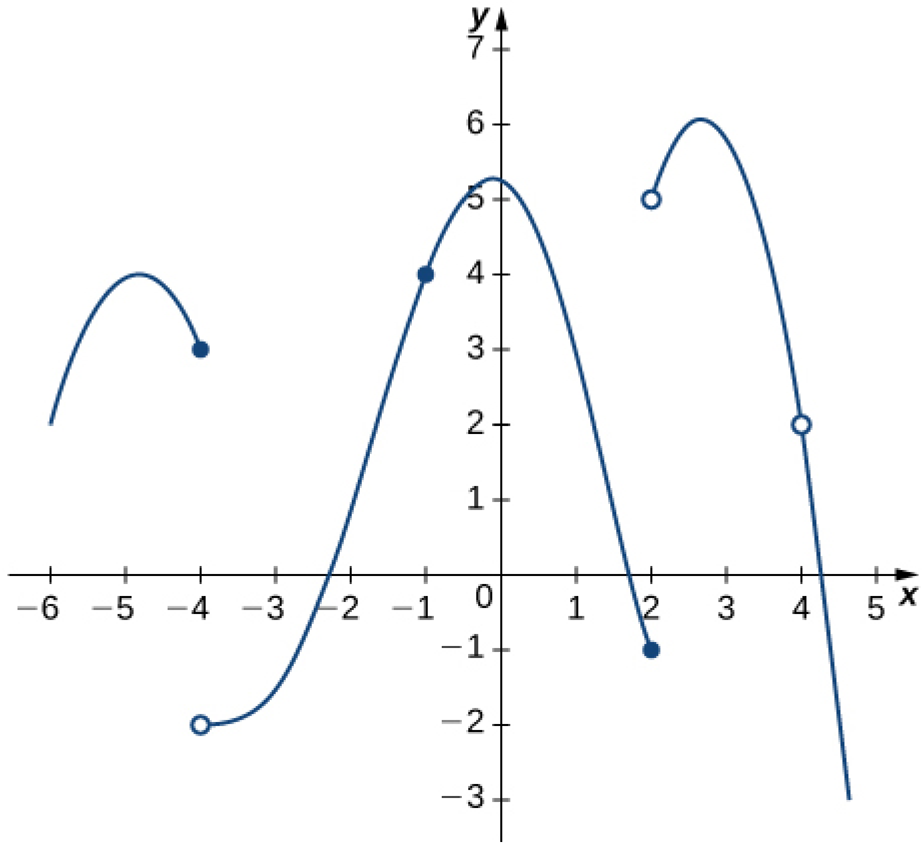
76. $f(x) = \begin{cases} -x^2 + 2, x \le 1 \\ x, x > 1 \end{cases}$

77. $f(x) = \begin{cases} 2x, x \le 1 \\ \frac{2}{x}, x > 1 \end{cases}$

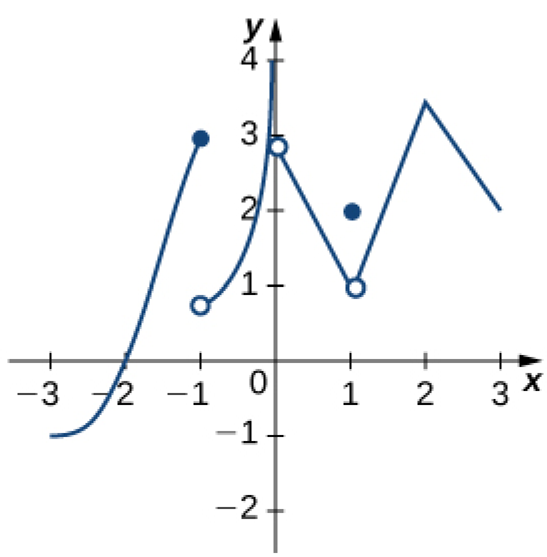
For the following graphs,

a. determine for which values of $x = a$ the $\lim\limits_{x \to a} f(x)$ exists but f is not continuous at $x = a,$ and
b. determine for which values of $x = a$ the function is continuous but not differentiable at $x = a.$

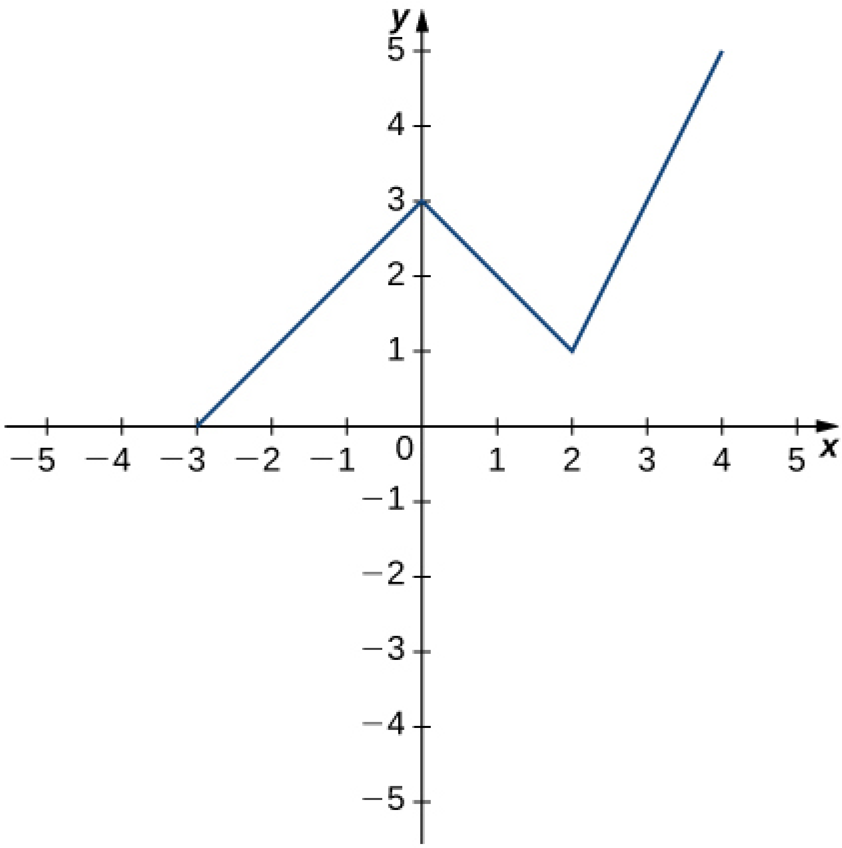
78.



79.



80. Use the graph to evaluate a. $f'(-0.5)$, b. $f'(0)$, c. $f'(1)$, d. $f'(2)$, and e. $f'(3)$, if it exists.



For the following functions, use $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ to find $f''(x)$.

81. $f(x) = 2 - 3x$

82. $f(x) = 4x^2$

83. $f(x) = x + \frac{1}{x}$

For the following exercises, use a calculator to graph $f(x)$. Determine the function $f'(x)$, then use a calculator to graph $f'(x)$.

84. **[T]** $f(x) = -\frac{5}{x}$

85. **[T]** $f(x) = 3x^2 + 2x + 4.$

86. **[T]** $f(x) = \sqrt{x} + 3x$

87. **[T]** $f(x) = \frac{1}{\sqrt{2x}}$

88. **[T]** $f(x) = 1 + x + \frac{1}{x}$

89. **[T]** $f(x) = x^3 + 1$

For the following exercises, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\frac{f(x+h) - f(x)}{h}$
b. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

90. $P(x)$ denotes the population of a city at time x in years.

91. $C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

92. $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios.

93. $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

94. $B(x)$ denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since $1990.$

95. $p(x)$ denotes atmospheric pressure at an altitude of x feet.

96. Sketch the graph of a function $y = f(x)$ with all of the following properties:
a. $f'(x) > 0$ for $-2 \le x < 1$
b. $f'(2) = 0$
c. $f'(x) > 0$ for $x > 2$
d. $f(2) = 2$ and $f(0) = 1$
e. $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$
f. $f'(1)$ does not exist.

97. Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y = T(x).$
a. Give a physical interpretation, with units, of $T'(x).$
b. If we know that $T'(1000) = -0.1,$ explain the physical meaning.

98. Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.
a. What does $\frac{P(b) - P(a)}{b - a}$ for $0 < a < b$ measure, and what are the units?

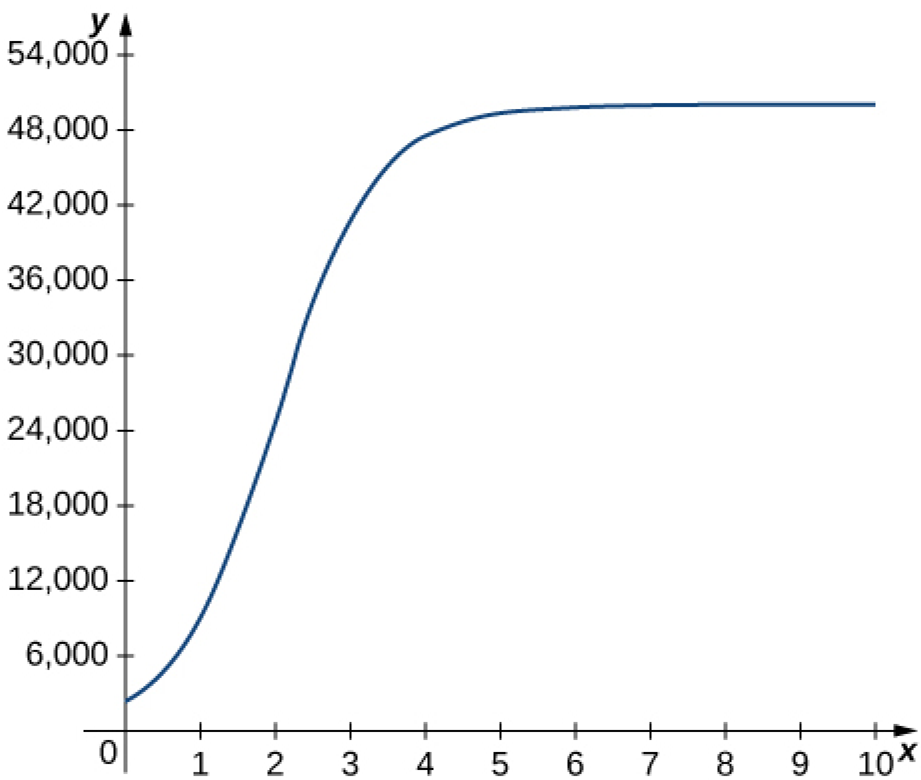
b. What does $P'(x)$ measure, and what are the units?

c. Suppose that $P'(30) = 5,$ what is the approximate change in profit if the number of items sold increases from 30 to 31?

99. The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of $50{,}000$ citizens.

a. Describe what $N'(t)$ represents and how it behaves as t increases.

b. What does the derivative tell us about how this town is affected by the flu outbreak?



For the following exercises, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	Height (meters)
0	0
1	2
2	4

Time (seconds)	Height (meters)
3	13
4	25
5	32

100. What is the physical meaning of $h'(t)$? What are the units?

101. **[T]** Construct a table of values for $h'(t)$ and graph both $h(t)$ and $h'(t)$ on the same graph. (*Hint:* for **interior points**, estimate both the left limit and right limit and average them. An interior point of an interval I is an element of I which is not an endpoint of I.)

102. **[T]** The best linear fit to the data is given by $H(t) = 7.229t - 4.905,$ where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t).$ Graph $H(t)$ with the given data and, on a separate coordinate plane, graph $H'(t).$

103. **[T]** The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429,$ where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t).$ Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t).$

104. **[T]** The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683,$ where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t).$ Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t).$ Does the linear, quadratic, or cubic function fit the data best?

105. Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t), G''(t)$ and $F''(t)$ are. What are the physical meanings of $H''(t), G''(t)$ and $F''(t),$ and what are their units?

Learning Objectives

3.3.1. State the constant, constant multiple, and power rules.
3.3.2. Apply the sum and difference rules to combine derivatives.
3.3.3. Use the product rule for finding the derivative of a product of functions.
3.3.4. Use the quotient rule for finding the derivative of a quotient of functions.
3.3.5. Extend the power rule to functions with negative exponents.
3.3.6. Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$ by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit. The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated. In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c.$ For this function, both $f(x) = c$ and $f(x + h) = c,$ so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c-c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

THEOREM 3.2

The Constant Rule

Let c be a constant.

If $f(x) = c$, then $f'(c) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

EXAMPLE 3.17

Applying the Constant Rule

Find the derivative of $f(x) = 8$.

[Show Solution]

CHECKPOINT 3.11

Find the derivative of $g(x) = -3$.

The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right) = 2x \text{ and } \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}\left(x^n\right)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, note that the technique used in this case is essentially the same as the technique used to prove the general case.

EXAMPLE 3.18

Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

[Show Solution]

CHECKPOINT 3.12

Find $\frac{d}{dx}\left(x^4\right)$.

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

THEOREM 3.3

The Power Rule

Let n be a positive integer. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

$$\text{Since } (x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by *h*:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \left(nx^{n-1} + \binom{n}{2} x^{n-2}h + \binom{n}{3} x^{n-3}h^2 + \ldots + nxh^{n-1} + h^n \right) \\ &= nx^{n-1}. \end{aligned}$$

□

EXAMPLE 3.19

Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

[Show Solution]

CHECKPOINT 3.13

Find the derivative of $f(x) = x^7$.

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

THEOREM 3.4

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) + g(x), j'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of g:

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) - g(x), j'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant *k* multiplied by a function *f* is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x));$$

that is,

$$\text{for } j(x) = kf(x), j'(x) = kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $j(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$j'(x) = \lim_{h \to 0} \frac{j(x+h) - j(x)}{h}.$$

By substituting $j(x+h) = f(x+h) + g(x+h)$ and $j(x) = f(x) + g(x)$, we obtain

$$j'(x) = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}.$$

Rearranging and regrouping the terms, we have

$$j'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$j'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right) = f'(x) + g'(x).$$

□

EXAMPLE 3.20

Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

[Show Solution]

EXAMPLE 3.21

Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

[Show Solution]

CHECKPOINT 3.14

Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

EXAMPLE 3.22

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

[Show Solution]

CHECKPOINT 3.15

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

THEOREM 3.5

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

$$\text{if } j(x) = f(x)g(x), \text{ then } j'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \to 0} g(x+h) = g(x)$.

By applying the limit definition of the derivative to $j(x) = f(x)g(x)$, we obtain

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$j'(x) = \lim_{h \to 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} \right) + \lim_{h \to 0} \left(\frac{f(x)g(x+h) - f(x)g(x)}{h} \right).$$

Rearranging, we obtain

$$j'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x) \right).$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$j'(x) = f'(x)g(x) + g'(x)f(x).$$

□

EXAMPLE 3.23

Applying the Product Rule to Functions at a Point

For $j(x) = f(x)g(x),$ use the product rule to find $j'(2)$ if $f(2) = 3, f'(2) = -4, g(2) = 1,$ and $g'(2) = 6.$

[Show Solution]

EXAMPLE 3.24

Applying the Product Rule to Binomials

For $j(x) = (x^2 + 2)(3x^3 - 5x),$ find $j'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

[Show Solution]

CHECKPOINT 3.16

Use the product rule to obtain the derivative of $j(x) = 2x^5\left(4x^2 + x\right).$

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}\left(x^2\right) = 2x, \text{ not } \frac{\frac{d}{dx}\left(x^3\right)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

THEOREM 3.6

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.$$

That is,

$$\text{if } j(x) = \frac{f(x)}{g(x)}, \text{ then } j'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

The proof of the **quotient rule** is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

EXAMPLE 3.25

Applying the Quotient Rule

Use the quotient rule to find the derivative of $k(x) = \frac{5x^2}{4x+3}$.

[Show Solution]

CHECKPOINT 3.17

Find the derivative of $h(x) = \frac{3x+1}{4x-3}$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

THEOREM 3.7

Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k,$ so that n is a positive integer with $k = -n.$ Since for each positive integer $n, x^{-n} = \frac{1}{x^n},$ we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n.$ In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}.$ Thus,

$$\frac{d}{\mathrm{dx}}\left(x^{-n}\right) = \frac{0\left(x^n\right) - 1\left(nx^{n-1}\right)}{\left(x^n\right)^2}.$$

Simplifying, we see that

$$\frac{d}{\mathrm{dx}}\left(x^{-n}\right) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{(n-1)-2n} = -nx^{-n-1}.$$

Finally, observe that since $k = -n,$ by substituting we have

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

□

EXAMPLE 3.26

Using the Extended Power Rule

Find $\frac{d}{dx}\left(x^{-4}\right)$.

[Show Solution]

EXAMPLE 3.27

Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find the derivative of $f(x) = \frac{6}{x^2}$.

[Show Solution]

CHECKPOINT 3.18

Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended power rule.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

EXAMPLE 3.28

Combining Differentiation Rules

For $k(x) = 3h(x) + x^2 g(x),$ find $k'(x).$

[Show Solution]

EXAMPLE 3.29

Extending the Product Rule

For $k(x) = f(x)g(x)h(x),$ express $k'(x)$ in terms of $f(x), g(x), h(x),$ and their derivatives.

[Show Solution]

EXAMPLE 3.30

Combining the Quotient Rule and the Product Rule

For $h(x) = \frac{2x^3 k(x)}{3x+2},$ find $h'(x).$

[Show Solution]

CHECKPOINT 3.19

Find $\frac{d}{dx}(3f(x) - 2g(x))$.

EXAMPLE 3.31

Determining Where a Function Has a Horizontal Tangent

Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

[Show Solution]

EXAMPLE 3.32

Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t) = \frac{t}{t^2+1}$. What is the initial velocity of the object?

[Show Solution]

CHECKPOINT 3.20

Find the values of x for which the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

STUDENT PROJECT

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure 3.20).

Figure 3.20 The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

****

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (Figure 3.21). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

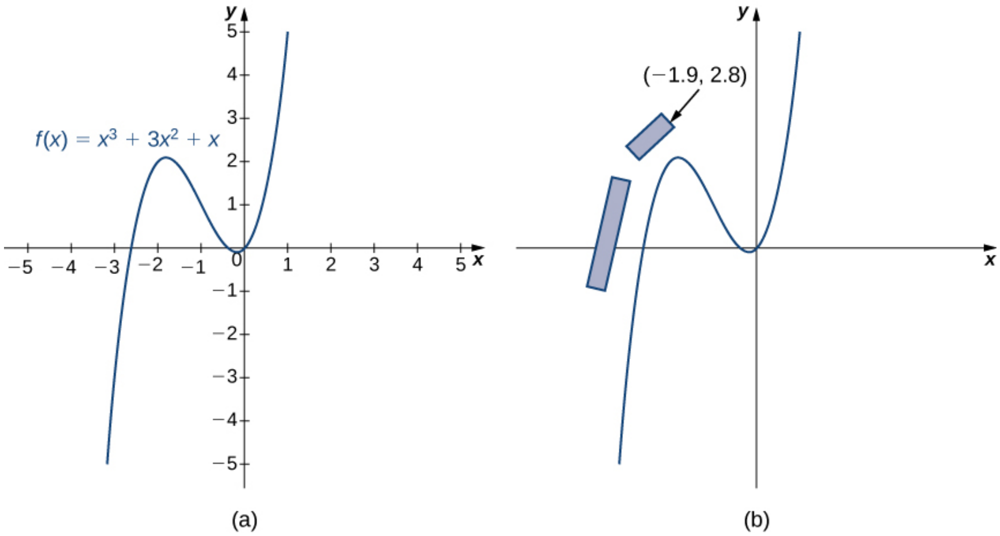


Figure 3.21 (a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the *x*-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Section 3.3 Exercises

For the following exercises, find $f'(x)$ for each function.

106. $f(x) = x^7 + 10$

107. $f(x) = 5x^3 - x + 1$

108. $f(x) = 4x^2 - 7x$

109. $f(x) = 8x^4 + 9x^2 - 1$

110. $f(x) = x^4 + \frac{2}{x}$

111. $f(x) = 3x\left(18x^4 + \frac{13}{x+1}\right)$

112. $f(x) = (x + 2)\left(2x^2 - 3\right)$

113. $f(x) = x^2\left(\frac{2}{x^2} + \frac{5}{x^3}\right)$

114. $f(x) = \frac{x^3+2x^2-4}{3}$

115. $f(x) = \frac{4x^3-2x+1}{x^2}$

116. $f(x) = \frac{x^2+4}{x^2-4}$

117. $f(x) = \frac{x+9}{x^2-7x+1}$

For the following exercises, find the equation of the tangent line $T(x)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

118. **[T]** $y = 3x^2 + 4x + 1$ at $(0, 1)$

119. **[T]** $y = 2\sqrt{x} + 1$ at $(4, 5)$

120. **[T]** $y = \frac{2x}{x-1}$ at $(-1, 1)$

121. **[T]** $y = \frac{2}{x} - \frac{3}{x^2}$ at $(1, -1)$

For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x. Find the derivative of each of the functions $h(x)$.

122. $h(x) = 4f(x) + \frac{g(x)}{7}$

123. $h(x) = x^3 f(x)$

124. $h(x) = \frac{f(x)g(x)}{2}$

125. $h(x) = \frac{3f(x)}{g(x)+2}$

For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	−2	0
$g(x)$	2	3	−4	6
$f'(x)$	−1	7	8	−3
$g'(x)$	4	1	2	9

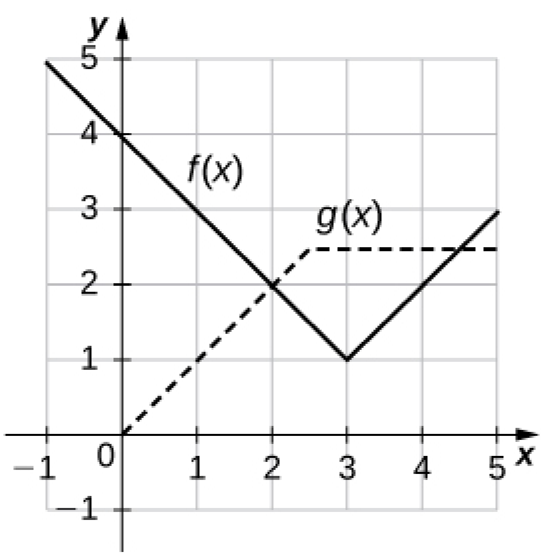
126. Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

127. Find $h'(2)$ if $h(x) = \frac{f(x)}{g(x)}$.

128. Find $h'(3)$ if $h(x) = 2x + f(x)g(x)$.

129. Find $h'(4)$ if $h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$.

For the following exercises, use the following figure to find the indicated derivatives, if they exist.



130. Let $h(x) = f(x) + g(x)$. Find

a. $h'(1)$,
b. $h'(3)$, and
c. $h'(4)$.

131. Let $h(x) = f(x)g(x)$. Find

a. $h'(1)$,
b. $h'(3)$, and
c. $h'(4)$.

132. Let $h(x) = \frac{f(x)}{g(x)}$. Find

a. $h'(1)$,
b. $h'(3)$, and
c. $h'(4)$.

For the following exercises,

a. evaluate $f'(a)$, and
b. graph the function $f(x)$ and the tangent line at $x = a$.

133. **[T]** $f(x) = 2x^3 + 3x - x^2, a = 2$

134. **[T]** $f(x) = \frac{1}{x} - x^2, a = 1$

135. **[T]** $f(x) = x^2 - x^{12} + 3x + 2, a = 0$

136. **[T]** $f(x) = \frac{1}{x} - x^{2/3}, a = -1$

137. Find the equation of the tangent line to the graph of $f(x) = 2x^3 + 4x^2 - 5x - 3$ at $x = -1$.

138. Find the equation of the tangent line to the graph of $f(x) = x^2 + \frac{4}{x} - 10$ at $x = 8$.

139. Find the equation of the tangent line to the graph of $f(x) = (3x - x^2)(3 - x - x^2)$ at $x = 1$.

140. Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x intercept of 6.

141. Find the equation of the line passing through the point $P(3, 3)$ and tangent to the graph of $f(x) = \frac{6}{x-1}$.

142. Determine all points on the graph of $f(x) = x^3 + x^2 - x - 1$ for which

a. the tangent line is horizontal
b. the tangent line has a slope of -1.

143. Find a quadratic polynomial such that $f(1) = 5, f'(1) = 3$ and $f''(1) = -6$.

144. A car driving along a freeway with traffic has traveled $s(t) = t^3 - 6t^2 + 9t$ meters in t seconds.

a. Determine the time in seconds when the velocity of the car is 0.
b. Determine the acceleration of the car when the velocity is 0.

145. **[T]** A herring swimming along a straight line has traveled $s(t) = \frac{t^2}{t^2+2}$ feet in t seconds.

Determine the velocity of the herring when it has traveled 3 seconds.

146. The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P(t) = \frac{8t+3}{0.2t^2+1}$, where t is measured in years.

a. Determine the initial flounder population.
b. Determine $P'(10)$ and briefly interpret the result.

147. **[T]** The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t) = \frac{2t^2+t}{t^3+50}$, where C is measured in milligrams per liter of blood.

a. Find the rate of change of $C(t)$.
b. Determine the rate of change for $t = 8, 12, 24,$ and 36.
c. Briefly describe what seems to be occurring as the number of hours increases.

148. A book publisher has a cost function given by $C(x) = \frac{x^3+2x+3}{x^2}$, where x is the number of copies of a book in thousands and C is the cost, per book, measured in dollars. Evaluate $C'(2)$ and explain its meaning.

149. **[T]** According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \frac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

a. Suppose that $G, m_1,$ and m_2 are constants. Find the rate of change of force F with respect to distance d.
b. Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11}$ Nm^2/kg^2, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

Learning Objectives

3.4.1. Determine a new value of a quantity from the old value and the amount of change.
3.4.2. Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
3.4.3. Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
3.4.4. Predict the future population from the present value and the population growth rate.
3.4.5. Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h) - f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

$$\frac{f(a+h) - f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For small enough values of h, $f'(a) \approx \frac{f(a+h)-f(a)}{h}$. We can then solve for $f(a+h)$ to get the amount of change formula:

$$f(a+h) \approx f(a) + f'(a)h. \tag{3.10}$$

We can use this formula if we know only $f(a)$ and $f'(a)$ and wish to estimate the value of $f\,(a+h)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in Figure 3.22, we are approximating $f(a+h)$ by the y coordinate at $a+h$ on the line tangent to $f(x)$ at $x=a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f'\,(a)$.

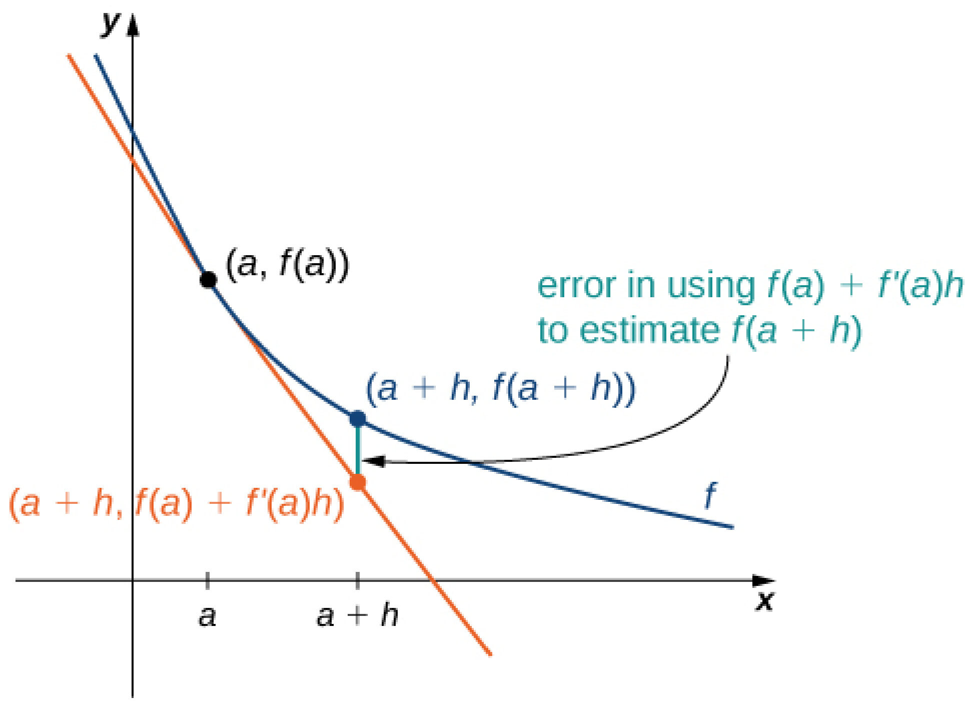


Figure 3.22 The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f\,(a+h) \approx f\,(a) + f'\,(a)\,\mathrm{h}$.

MEDIA

Here is an interesting demonstration of rate of change.

EXAMPLE 3.33

Estimating the Value of a Function

If $f\,(3)=2$ and $f'\,(3)=5$, estimate $f(3.2)$.

[Show Solution]

CHECKPOINT 3.21

Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

DEFINITION

Let $s(t)$ be a function giving the position of an object at time t.

The velocity of the object at time t is given by $v(t) = s'(t)$.

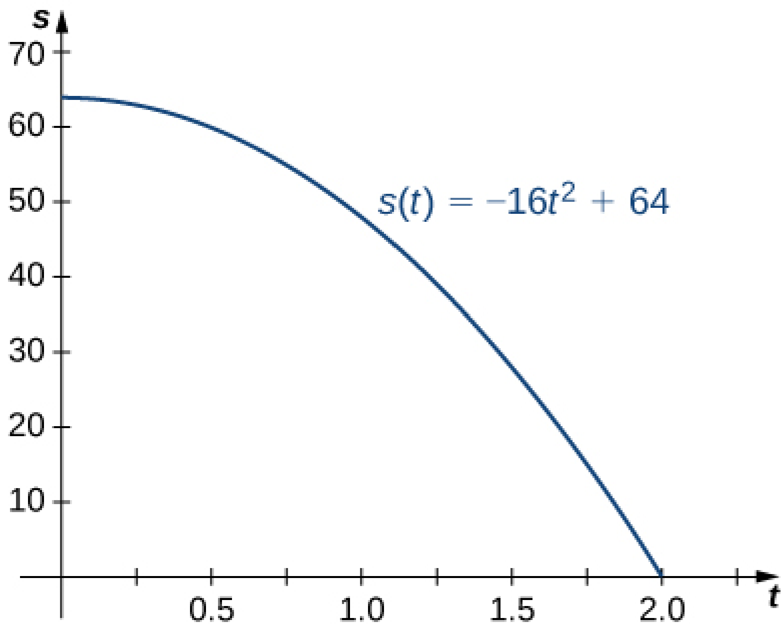
The speed of the object at time t is given by $|v(t)|$.

The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

EXAMPLE 3.34

Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



a. What is the instantaneous velocity of the ball when it hits the ground?
b. What is the average velocity during its fall?

[Show Solution]

EXAMPLE 3.35

Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?
b. Is the particle speeding up or slowing down at time $t = 1$?

[Show Solution]

EXAMPLE 3.36

Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4, t \geq 0.$

a. Find $v(t)$.
b. At what time(s) is the particle at rest?
c. On what time intervals is the particle moving from left to right? From right to left?
d. Use the information obtained to sketch the path of the particle along a coordinate axis.

[Show Solution]

CHECKPOINT 3.22

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1.$ Is the particle moving from right to left or from left to right at time $t = 3$?

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

DEFINITION

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

EXAMPLE 3.37

Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

[Show Solution]

CHECKPOINT 3.23

The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3{,}000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

DEFINITION

If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.

If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.

If $P(x) = R(x) - C(x)$ is the profit obtained from selling *x* items, then the **marginal profit** $MP(x)$ is defined to be
$MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since *x* represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

EXAMPLE 3.38

Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation
$p(x) = 9 - 0.03x, 0 \leq x \leq 300.$

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \text{ for } 0 \leq x \leq 300.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

[Show Solution]

CHECKPOINT 3.24

Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

Section 3.4 Exercises

For the following exercises, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.
b. Determine the time intervals when the object is slowing down or speeding up.

150. $s(t) = 2t^3 - 3t^2 - 12t + 8$

151. $s(t) = 2t^3 - 15t^2 + 36t - 10$

152. $s(t) = \frac{t}{1+t^2}$

153. A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.
a. Find the velocity of the rocket 3 seconds after being fired.
b. Find the acceleration of the rocket 3 seconds after being fired.

154. A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.
a. Determine how long it takes for the ball to hit the ground.
b. Determine the velocity of the ball when it hits the ground.

155. The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.
a. Determine the velocity of the car when $s(t) = 0$.
b. Determine the velocity of the car when $s(t) = 14$.

156. The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.
a. Determine the velocity of the bird at $t = 1$ sec.
b. Determine the acceleration of the bird at $t = 1$ sec.

c. Determine the acceleration of the bird when the velocity equals 0.

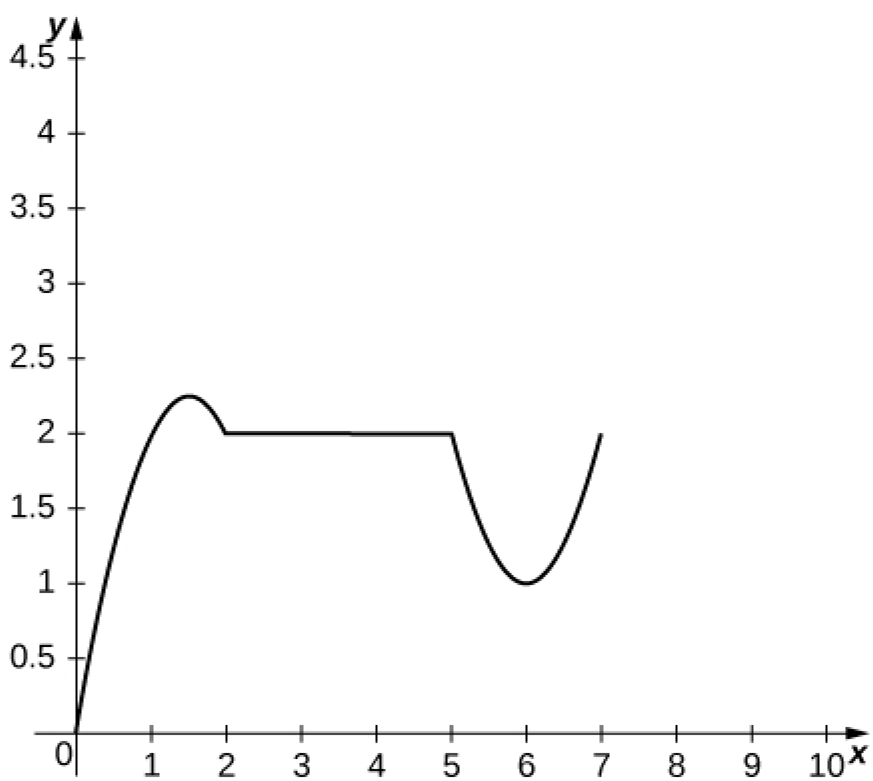
157. A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

a. Find the velocity of the potato after 0.5 s and 5.75 s.
b. Find the speed of the potato at 0.5 s and 5.75 s.
c. Determine when the potato reaches its maximum height.
d. Find the acceleration of the potato at 0.5 s and 1.5 s.
e. Determine how long the potato is in the air.
f. Determine the velocity of the potato upon hitting the ground.

158. The position function $s(t) = t^3 - 8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

a. Determine the direction the train is traveling when $s(t) = 0$.
b. Determine the direction the train is traveling when $a(t) = 0$.
c. Determine the time intervals when the train is slowing down or speeding up.

159. The following graph shows the position $y = s(t)$ of an object moving along a straight line.



a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.
b. Sketch the graph of the velocity function.
c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.
d. Determine the time intervals when the object is speeding up or slowing down.

160. The cost function, in dollars, of a company that manufactures food processors is given by $C(x) = 200 + \frac{7}{x} + \frac{x^2}{7}$, where x is the number of food processors manufactured.

a. Find the marginal cost function.
b. Use the marginal cost function to estimate the cost of manufacturing the thirteenth food processor.
c. Find the actual cost of manufacturing the thirteenth food processor.

161. The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p = 10 - 0.001x$.

a. Find the revenue function $R(x)$.
b. Find the marginal revenue function.
c. Find the marginal revenue at $x = 2000$ and 5000.

162. **[T]** A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.
b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

163. **[T]** In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.

Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75{,}000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

a. Find the marginal cost function.
b. Find the revenue and marginal revenue functions.
c. Find $R'(1000)$ and $R'(4000)$. Interpret the results.
d. Find the profit and marginal profit functions.
e. Find $P'(1000)$ and $P'(4000)$. Interpret the results.

164. A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

a. Find the rate of change function $P'(t)$ of the population function.
b. Find $P'(1), P'(2), P'(3)$, and $P'(4)$. Interpret what the results mean for the town.
c. Find $P''(1), P''(2), P''(3)$, and $P''(4)$. Interpret what the results mean for the town's population.

165. **[T]** A culture of bacteria grows in number according to the function $N(t) = 3000\left(1 + \frac{4t}{t^2+100}\right)$, where t is measured in hours.

a. Find the rate of change of the number of bacteria.
b. Find $N'(0), N'(10), N'(20)$, and $N'(30)$.
c. Interpret the results in (b).
d. Find $N''(0), N''(10), N''(20)$, and $N''(30)$. Interpret what the answers imply about the bacteria population growth.

166. The centripetal force of an object of mass m is given by $F(r) = \frac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.
b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

Years since 1800	**Population (millions)**
1	0.8795
11	1.040
21	1.264
31	1.516
41	1.661

Years since 1800	Population (millions)
51	2.000
61	2.634
71	3.272
81	3.911
91	4.422

Table 3.4 Population of London Source: http://en.wikipedia.org/wiki/Demographics_of_London.

167. **[T]**

a. Using a calculator or a computer program, find the best-fit linear function to measure the population.
b. Find the derivative of the equation in a. and explain its physical meaning.
c. Find the second derivative of the equation and explain its physical meaning.

168. **[T]**

a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.
b. Find the derivative of the equation and explain its physical meaning.
c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut's position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	Position (m)
0	0
1	–1
2	–2
3	–5
4	–7

Time after dropping (s)	Position (m)
5	–14

169. **[T]**

a. Using a calculator or computer program, find the best-fit quadratic curve to the data.
b. Find the derivative of the position function and explain its physical meaning.
c. Find the second derivative of the position function and explain its physical meaning.

170. **[T]**

a. Using a calculator or computer program, find the best-fit cubic curve to the data.
b. Find the derivative of the position function and explain its physical meaning.
c. Find the second derivative of the position function and explain its physical meaning.
d. Using the result from c. explain why a cubic function is not a good choice for this problem.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

171. **[T]** The Holling type I equation is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.

a. Graph the Holling type I equation, given $a = 0.5$.
b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.
c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.
d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

172. **[T]** The Holling type II equation is described by $f(x) = \frac{ax}{n+x}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type I and II equations?
b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.

c. Show that $f(n) = \frac{1}{2}a$ and interpret the meaning of the parameter n.
d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

173. **[T]** The Holling type III equation is described by $f(x) = \frac{ax^2}{n^2+x^2}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type II and III equations?
b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.
c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).
d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

174. **[T]** The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	**Population of lynx (hundreds)**
20	10
55	15
65	55
95	60

Table 3.5 Snowshoe Hare and Lynx Populations Source: http://www.biotopics.co.uk/newgcse/predatorprey.html.

a. Graph the data points and determine which Holling-type function fits the data best.
b. Using the meanings of the parameters a and n, determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.
c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

Learning Objectives

3.5.1. Find the derivatives of the sine and cosine function.
3.5.2. Find the derivatives of the standard trigonometric functions.
3.5.3. Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

$$f'\,(x) = \lim_{h \to 0} \frac{f\,(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0, $f'\,(x) \approx \frac{f(x+h) - f(x)}{h}$. We see that by using $h = 0.01$,

$$\frac{d}{dx}(\sin x) \approx \frac{\sin\,(x + 0.01) - \sin x}{0.01}$$

By setting $D\,(x) = \frac{\sin\,(x+0.01) - \sin x}{0.01}$ and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ (Figure 3.25).

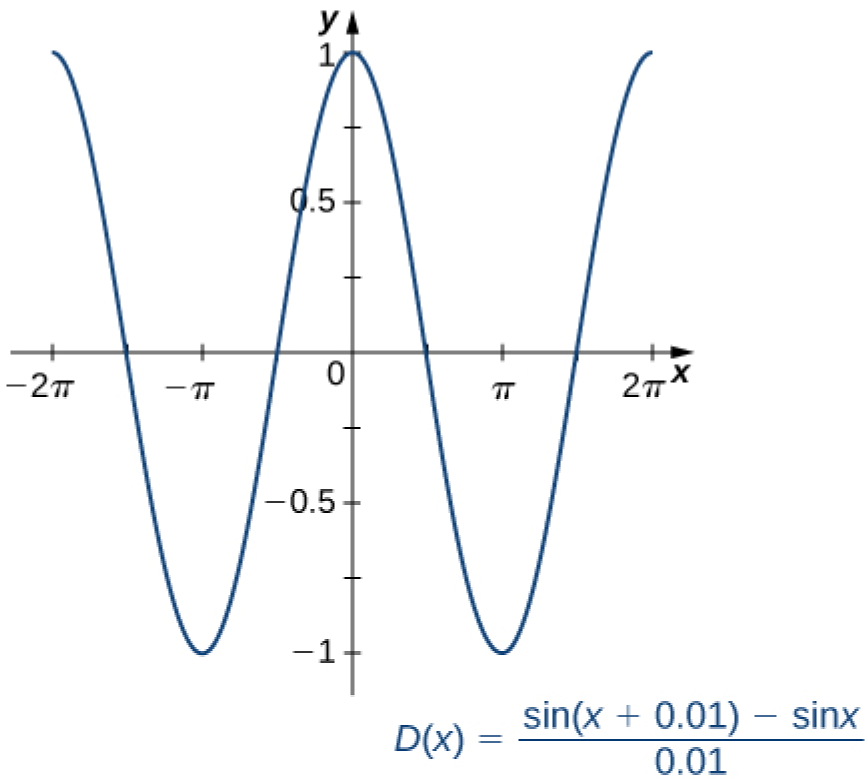


Figure 3.25 The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

THEOREM 3.8

The Derivatives of sin *x* and cos *x*

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

3.11

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x \tag{3.12}$$

Proof

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits we learned in Introduction to Limits:

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \text{ and } \lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{(\sin h)}{h}$ and $y = \frac{(\cos h - 1)}{h}$ are shown in Figure 3.26.

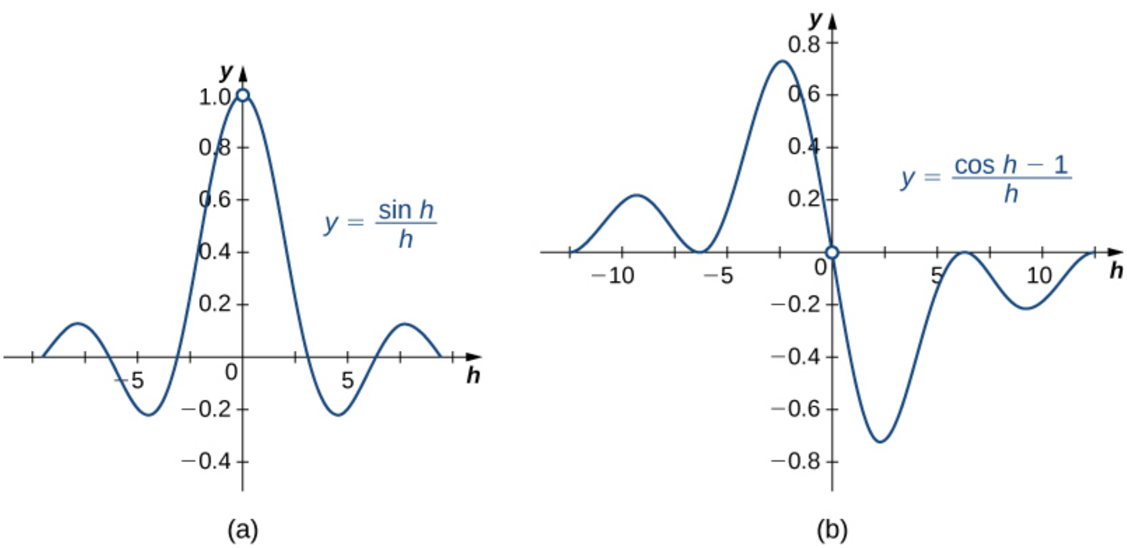


Figure 3.26 These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin\,(x + h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} && \text{Apply the definition of the d}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} && \text{Use trig identity for the sine}\\ &= \lim_{h\to 0}\left(\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right) && \text{Regroup.}\\ &= \lim_{h\to 0}\left(\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right) && \text{Factor out } \sin x \text{ and } \cos x.\\ &= \sin x\cdot 0+\cos x\cdot 1 && \text{Apply trig limit formulas.}\\ &= \cos x && \text{Simplify.}\end{aligned}$$

□

Figure 3.27 shows the relationship between the graph of $f(x) = \sin x$ and its derivative $f'(x) = \cos x$. Notice that at the points where $f(x) = \sin x$ has a horizontal tangent, its derivative $f'(x) = \cos x$ takes on the value zero. We also see that where $f(x) = \sin x$ is increasing, $f'(x) = \cos x > 0$ and where $f(x) = \sin x$ is decreasing, $f'(x) = \cos x < 0$.

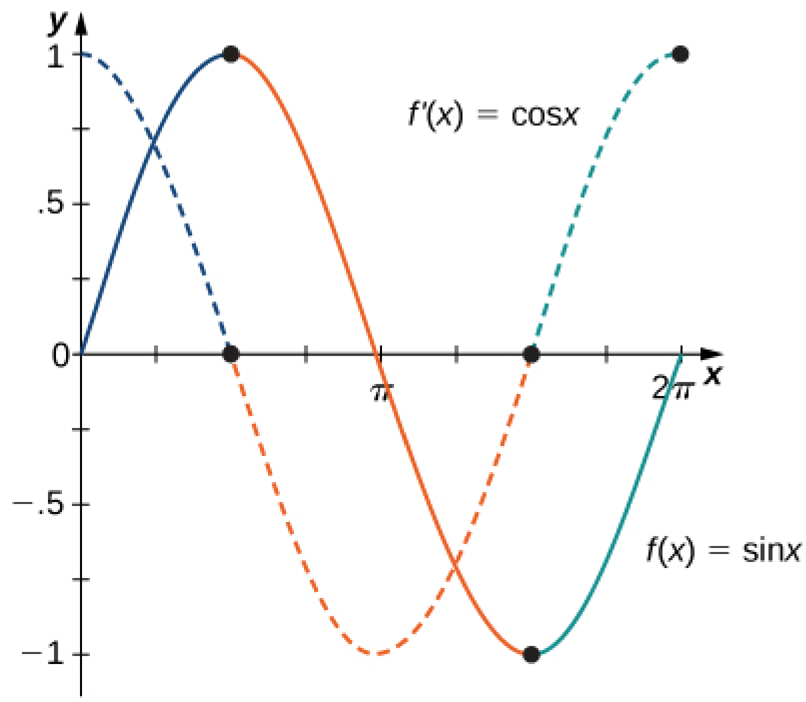


Figure 3.27 Where $f(x)$ has a maximum or a minimum, $f'(x) = 0$ that is, $f'(x) = 0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs.

EXAMPLE 3.39

Differentiating a Function Containing sin *x*

Find the derivative of $f(x) = 5x^3 \sin x$.

[Show Solution]

CHECKPOINT 3.25

Find the derivative of $f(x) = \sin x \cos x$.

EXAMPLE 3.40

Finding the Derivative of a Function Containing cos *x*

Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

[Show Solution]

CHECKPOINT 3.26

Find the derivative of $f(x) = \frac{x}{\cos x}$.

EXAMPLE 3.41

An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2\sin t - t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

[Show Solution]

CHECKPOINT 3.27

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

EXAMPLE 3.42

The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

[Show Solution]

CHECKPOINT 3.28

Find the derivative of $f(x) = \cot x$.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

THEOREM 3.9

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{3.13}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{3.14}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{3.15}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot \mathrm{x}. \tag{3.16}$$

EXAMPLE 3.43

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

[Show Solution]

EXAMPLE 3.44

Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x \tan x$.

[Show Solution]

CHECKPOINT 3.29

Find the derivative of $f(x) = 2\tan x - 3\cot x$.

CHECKPOINT 3.30

Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

EXAMPLE 3.45

Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

[Show Solution]

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of sin *x* equals sin *x*, so

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \ldots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$
$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \ldots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.$$

CHECKPOINT 3.31

For $y = \cos x$, find $\frac{d^4 y}{dx^4}$.

EXAMPLE 3.46

Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

[Show Solution]

CHECKPOINT 3.32

For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

EXAMPLE 3.47

An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

[Show Solution]

CHECKPOINT 3.33

A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Section 3.5 Exercises

For the following exercises, find $\frac{dy}{dx}$ for the given functions.

175. $y = x^2 - \sec x + 1$

176. $y = 3\csc x + \frac{5}{x}$

177. $y = x^2 \cot x$

178. $y = x - x^3 \sin x$

179. $y = \frac{\sec x}{x}$

180. $y = \sin x \tan x$

181. $y = (x + \cos x)(1 - \sin x)$

182. $y = \frac{\tan x}{1 - \sec x}$

183. $y = \frac{1 - \cot x}{1 + \cot x}$

184. $y = \cos x (1 + \csc x)$

For the following exercises, find the equation of the tangent line to each of the given functions at the indicated values of x. Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

185. **[T]** $f(x) = -\sin x, x = 0$

186. **[T]** $f(x) = \csc x, x = \frac{\pi}{2}$

187. **[T]** $f(x) = 1 + \cos x, x = \frac{3\pi}{2}$

188. **[T]** $f(x) = \sec x, x = \frac{\pi}{4}$

189. **[T]** $f(x) = x^2 - \tan x,\ x = 0$

190. **[T]** $f(x) = 5\cot x, \ x = \frac{\pi}{4}$

For the following exercises, find $\frac{d^2y}{dx^2}$ for the given functions.

191. $y = x\sin x - \cos x$

192. $y = \sin x\cos x$

193. $y = x - \frac{1}{2}\sin x$

194. $y = \frac{1}{x} + \tan x$

195. $y = 2\csc x$

196. $y = \sec^2 x$

197. Find all x values on the graph of $f(x) = -3\sin x\cos x$ where the tangent line is horizontal.

198. Find all x values on the graph of $f(x) = x - 2\cos x$ for $0 < x < 2\pi$ where the tangent line has slope 2.

199. Let $f(x) = \cot x$. Determine the points on the graph of f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

200. **[T]** A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6\cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

201. Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a\cos t + b\sin t$ where a and b are constants, t measures time in seconds, and s measures position in centimeters. If the position is 0 cm and the velocity is 3 cm/s when $t = 0$, find the values of a and b.

202. After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t) = -5\cos t$ cm at t seconds after the jump.

a. Sketch one period of the position function for $t \geq 0$.
b. Find the velocity function.
c. Sketch one period of the velocity function for $t \geq 0$.
d. Determine the times when the velocity is 0 over one period.
e. Find the acceleration function.
f. Sketch one period of the acceleration function for $t \geq 0$.

203. The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y = 10 + 5\sin x$ where y is the number of hamburgers sold

and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find y' and determine the intervals where the number of burgers being sold is increasing.

204. **[T]** The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y(t) = 0.5 + 0.3\cos t,$ where t is months since January. Find y' and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For the following exercises, use the quotient rule to derive the given equations.

205. $\frac{d}{dx}(\cot x) = -\csc^2 x$

206. $\frac{d}{dx}(\sec x) = \sec x \tan x$

207. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

208. Use the definition of derivative and the identity

$\cos\,(x + h) = \cos x \cos h - \sin x \sin h$ to prove that $\frac{d(\cos x)}{dx} = -\sin x.$

For the following exercises, find the requested higher-order derivative for the given functions.

209. $\frac{d^3 y}{dx^3}$ of $y = 3\cos x$

210. $\frac{d^2 y}{dx^2}$ of $y = 3\sin x + x^2 \cos x$

211. $\frac{d^4 y}{dx^4}$ of $y = 5\cos x$

212. $\frac{d^2 y}{dx^2}$ of $y = \sec x + \cot x$

213. $\frac{d^3 y}{dx^3}$ of $y = x^{10} - \sec x$

Learning Objectives

3.6.1. State the chain rule for the composition of two functions.
3.6.2. Apply the chain rule together with the power rule.
3.6.3. Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
3.6.4. Recognize the chain rule for a composition of three or more functions.
3.6.5. Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions $(x^n, \sin x, \cos x, \text{etc.})$ as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin\left(x^3\right)$ or $k(x) = \sqrt{3x^2 + 1}.$ In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the **chain rule**, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin\left(x^3\right).$ We can think of the derivative of this function with respect to *x* as the rate of change of $\sin\left(x^3\right)$ relative to the change in $x.$ Consequently, we want to know how $\sin\left(x^3\right)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin\left(x^3\right).$ This chain reaction gives us hints as to what is involved in computing the derivative of $\sin\left(x^3\right).$ First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin\left(x^3\right)$ suggests that the derivative of $\sin(u)$ with respect to $u,$ where $u = x^3,$ is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin\left(x^3\right)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin\left(x^3\right).$

$$h'(a) = \lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x - a}.$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

$$h'(a) = \lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a.$ That is,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}\left(x^3\right)_{x=a} = 3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^3$ and observing that as $x \to a, u \to a^3$:

$$\begin{aligned} \lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x^3 - a^3} &= \lim_{u \to a^3} \frac{\sin u - \sin\left(a^3\right)}{u - a^3} \\ &= \frac{d}{du}(\sin u)_{u=a^3} \\ &= \cos\left(a^3\right). \end{aligned}$$

Thus, $h'(a) = \cos\left(a^3\right) \cdot 3a^2.$

In other words, if $h(x) = \sin\left(x^3\right),$ then $h'(x) = \cos\left(x^3\right) \cdot 3x^2.$ Thus, if we think of $h(x) = \sin\left(x^3\right)$ as the composition $(f \circ g)(x) = f(g(x))$ where $f(x) =$ sin x and $g(x) = x^3,$ then the derivative of $h(x) = \sin\left(x^3\right)$ is the product of the derivative of $g(x) = x^3$ and the derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3.$ At this point, we anticipate that for $h(x) = \sin(g(x)),$ it is quite likely that $h'(x) = \cos(g(x))g'(x).$ As we determined above, this is the case for $h(x) = \sin\left(x^3\right).$

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

RULE: THE CHAIN RULE

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

$$h'(x) = f'(g(x))g'(x). \tag{3.17}$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

MEDIA

Watch an animation of the chain rule.

PROBLEM-SOLVING STRATEGY: APPLYING THE CHAIN RULE

1. To differentiate $h(x) = f(g(x))$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x)) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = (g(x))^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = (g(x))^n$ as $f(g(x))$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'(g(x)) = n(g(x))^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x) = n(g(x))^{n-1}g'(x)$$

RULE: POWER RULE FOR COMPOSITION OF FUNCTIONS

For all values of *x* for which the derivative is defined, if

$$h(x) = (g(x))^n.$$

Then

$$h'(x) = n(g(x))^{n-1}g'(x). \tag{3.18}$$

EXAMPLE 3.48

Using the Chain and Power Rules

Find the derivative of $h(x) = \dfrac{1}{\left(3x^2+1\right)^2}$.

[Show Solution]

CHECKPOINT 3.34

Find the derivative of $h(x) = \left(2x^3 + 2x - 1\right)^4$.

EXAMPLE 3.49

Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

[Show Solution]

EXAMPLE 3.50

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

[Show Solution]

CHECKPOINT 3.35

Find the equation of the line tangent to the graph of $f(x) = \left(x^2 - 2\right)^3$ at $x = -2$.

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

EXAMPLE 3.51

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

[Show Solution]

In the following example we apply the rule that we have just derived.

EXAMPLE 3.52

Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos\left(5x^2\right)$.

[Show Solution]

EXAMPLE 3.53

Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec\left(4x^5 + 2x\right)$.

[Show Solution]

CHECKPOINT 3.36

Find the derivative of $h(x) = \sin(7x + 2)$.

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in Example 3.51 and Example 3.53. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

THEOREM 3.10

Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$$\frac{d}{dx}(\sin(g(x)) = \cos(g(x))g'(x) \qquad \frac{d}{dx}\sin u = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}(\cos(g(x)) = -\sin(g(x))g'(x) \qquad \frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}(\tan(g(x)) = \sec^2(g(x))g'(x) \qquad \frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\cot(g(x)) = -\csc^2(g(x))g'(x) \qquad \frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\sec(g(x)) = \sec(g(x)\tan(g(x))g'(x) \qquad \frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx}$$

$$\frac{d}{dx}(\csc(g(x)) = -\csc(g(x))\cot(g(x))g'(x) \qquad \frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx}.$$

EXAMPLE 3.54

Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x + 1)^5(3x - 2)^7$.

[Show Solution]

CHECKPOINT 3.37

Find the derivative of $h(x) = \frac{x}{(2x+3)^3}$.

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

$$k(x) = h(f(g(x))).$$

Then, applying the chain rule once we obtain

$$k'(x) = \frac{d}{dx}(h(f(g(x))) = h'(f(g(x))) \cdot \frac{d}{dx}f((g(x))).$$

Applying the chain rule again, we obtain

$$k'(x) = h'\Big(f(g(x))f'(g(x))g'(x)\Big).$$

RULE: CHAIN RULE FOR A COMPOSITION OF THREE FUNCTIONS

For all values of *x* for which the function is differentiable, if

$$k(x) = h(f(g(x))),$$

then

$$k'(x) = h'(f(g(x)))f'(g(x))g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

EXAMPLE 3.55

Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4\left(7x^2 + 1\right)$.

[Show Solution]

CHECKPOINT 3.38

Find the derivative of $h(x) = \sin^6\left(x^3\right)$.

EXAMPLE 3.56

Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time *t* is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

[Show Solution]

CHECKPOINT 3.39

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(4t)$. Find its acceleration at time t.

Proof

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Rewriting, we obtain

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

Although it is clear that

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a),$$

it is not obvious that

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a)).$$

To see that this is true, first recall that since *g* is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x \to a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$ and $b = g(a)$ and use change of variables in the limit to obtain

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'(g(a)).$$

Finally,

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'(g(a))g'(a).$$

□

EXAMPLE 3.57

Using the Chain Rule with Functional Values

Let $h(x) = f(g(x))$. If $g(1) = 4, g'(1) = 3,$ and $f'(4) = 7,$ find $h'(1)$.

[Show Solution]

CHECKPOINT 3.40

Given $h(x) = f(g(x))$. If $g(2) = -3, g'(2) = 4,$ and $f'(-3) = 7,$ find $h'(2)$.

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

$$h'(x) = \frac{dy}{dx}, f'(g(x)) = f'(u) = \frac{dy}{du} \text{ and } g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'(g(x))g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

RULE: CHAIN RULE USING LEIBNIZ'S NOTATION

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

EXAMPLE 3.58

Taking a Derivative Using Leibniz's Notation, Example 1

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

[Show Solution]

EXAMPLE 3.59

Taking a Derivative Using Leibniz's Notation, Example 2

Find the derivative of $y = \tan\left(4x^2 - 3x + 1\right)$.

[Show Solution]

CHECKPOINT 3.41

Use Leibniz's notation to find the derivative of $y = \cos\left(x^3\right)$. Make sure that the final answer is expressed entirely in terms of the variable x.

Section 3.6 Exercises

For the following exercises, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

214. $y = 3u - 6, u = 2x^2$

215. $y = 6u^3, u = 7x - 4$

216. $y = \sin u, u = 5x - 1$

217. $y = \cos u, u = \frac{-x}{8}$

218. $y = \tan u, u = 9x + 2$

219. $y = \sqrt{4u + 3}, u = x^2 - 6x$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and
b. find $\frac{dy}{dx}$ as a function of x.

220. $y = (3x - 2)^6$

221. $y = \left(3x^2 + 1\right)^3$

222. $y = \sin^5(x)$

223. $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

224. $y = \tan(\sec x)$

225. $y = \csc(\pi x + 1)$

226. $y = \cot^2 x$

227. $y = -6\sin^{-3} x$

For the following exercises, find $\frac{dy}{dx}$ for each function.

228. $y = \left(3x^2 + 3x - 1\right)^4$

229. $y = (5 - 2x)^{-2}$

230. $y = \cos^3(\pi x)$

231. $y = \left(2x^3 - x^2 + 6x + 1\right)^3$

232. $y = \frac{1}{\sin^2(x)}$

233. $y = (\tan x + \sin x)^{-3}$

234. $y = x^2 \cos^4 x$

235. $y = \sin(\cos 7x)$

236. $y = \sqrt{6 + \sec \pi x^2}$

237. $y = \cot^3(4x + 1)$

238. Let $y = [f(x)]^3$ and suppose that $f'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1.$ Find $f(1).$

239. Let $y = \left(f(x) + 5x^2\right)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1.$ Find $f'(-1)$

240. Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x.$ If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2,$ find $f'(4).$

241. **[T]** Find the equation of the tangent line to $y = -\sin\left(\frac{x}{2}\right)$ at the origin. Use a calculator to graph the function and the tangent line together.

242. **[T]** Find the equation of the tangent line to $y = \left(3x + \frac{1}{x}\right)^2$ at the point $(1, 16).$ Use a calculator to graph the function and the tangent line together.

243. Find the x-coordinates at which the tangent line to $y = \left(x - \frac{6}{x}\right)^8$ is horizontal.

244. **[T]** Find an equation of the line that is normal to $g(\theta) = \sin^2(\pi\theta)$ at the point $\left(\frac{1}{4}, \frac{1}{2}\right).$ Use a calculator to graph the function and the normal line together.

For the following exercises, use the information in the following table to find $h'(a)$ at the given value for $a.$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	1	–2	3	0
2	4	4	1	–1
3	3	–3	2	3

245. $h(x) = f(g(x)); a = 0$

246. $h(x) = g(f(x)); a = 0$

247. $h(x) = \left(x^4 + g(x)\right)^{-2}; a = 1$

248. $h(x) = \left(\frac{f(x)}{g(x)}\right)^2; a = 3$

249. $h(x) = f(x + f(x)); a = 1$

250. $h(x) = (1 + g(x))^3; a = 2$

251. $h(x) = g\left(2 + f\left(x^2\right)\right); a = 1$

252. $h(x) = f(g(\sin x)); a = 0$

253. **[T]** The position function of a freight train is given by $s(t) = 100(t + 1)^{-2},$ with s in meters and t in seconds. At time $t = 6$ s, find the train's

a. velocity and
b. acceleration.
c. Using a. and b. is the train speeding up or slowing down?

254. **[T]** A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches:

$$s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right).$$

a. Determine the position of the spring at $t = 1.5$ s.
b. Find the velocity of the spring at $t = 1.5$ s.

255. **[T]** The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300.$ In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

a. Find the marginal cost $C'(x)$.

b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.

c. Use b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

256. **[T]** The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r = 2 - \frac{100}{(t+7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.

b. Use a. to find the rate at which the area is expanding at $t = 4$ s.

257. **[T]** The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r = \frac{1}{(t+1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.

b. Use a. to find the rate at which the volume is changing at $t = 1$ min.

258. **[T]** The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10\cos\left[\frac{\pi}{12}(x-2)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

259. **[T]** The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Learning Objectives

3.9.1. Find the derivative of exponential functions.
3.9.2. Find the derivative of logarithmic functions.
3.9.3. Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in Introduction to Functions and Graphs, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x, b > 0,$ is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n, where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1, b^{-n} = \frac{1}{b^n},$ for a positive integer $n,$ and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and $t.$ These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the *continuity* of $B(x) = b^x, b > 0,$ we may interpret b^r as $\lim\limits_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4, 4^{3.1} < 4^\pi < 4^{3.2}, 4^{3.14} < 4^\pi < 4^{3.15},$$
$$4^{3.141} < 4^\pi < 4^{3.142}, 4^{3.1415} < 4^\pi < 4^{3.1416}, \ldots.$$

As we see in the following table, $4^\pi \approx 77.88.$

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944

x	4^x	x	4^x
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^{4}	256

Table 3.6 Approximating a Value of 4^{π}

We also assume that for $B(x) = b^x, b > 0,$ the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in Introduction to Functions and Graphs. Figure 3.33 provides graphs of the functions $y = 2^x, y = 3^x, y = 2.7^x,$ and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.

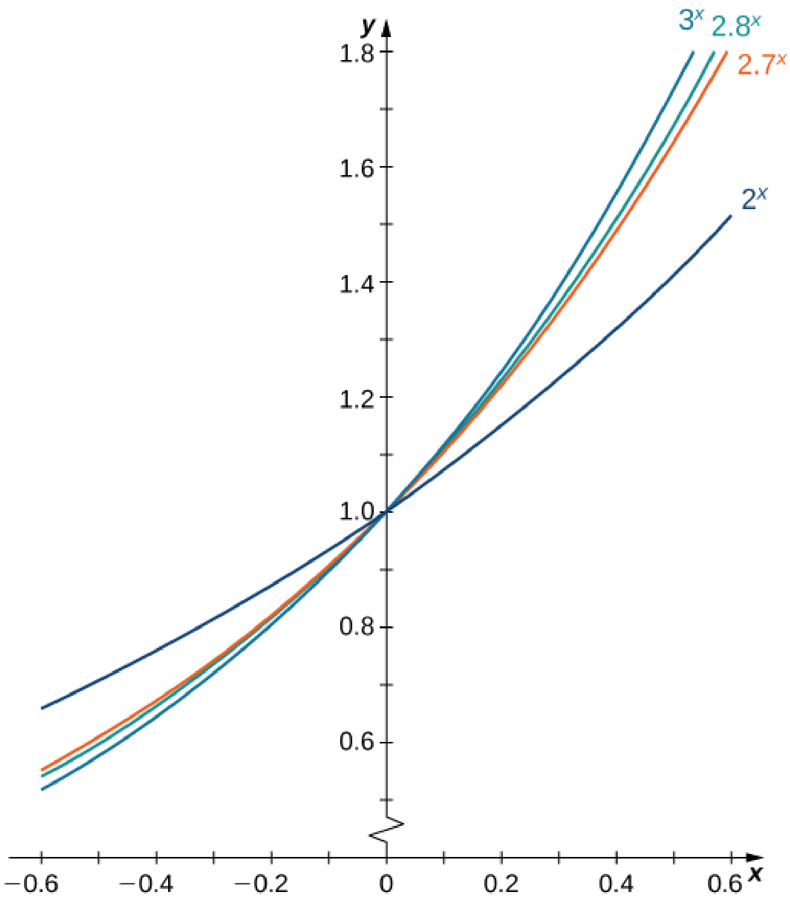


Figure 3.33 The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of $e,$ we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$ to obtain the estimate

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

See the following table.

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

Table 3.7 Estimating a Value of e

The evidence from the table suggests that $2.7182 < e < 2.7183.$

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in Figure 3.34. This line is tangent to the graph of $E(x) = e^x$ at $x = 0.$

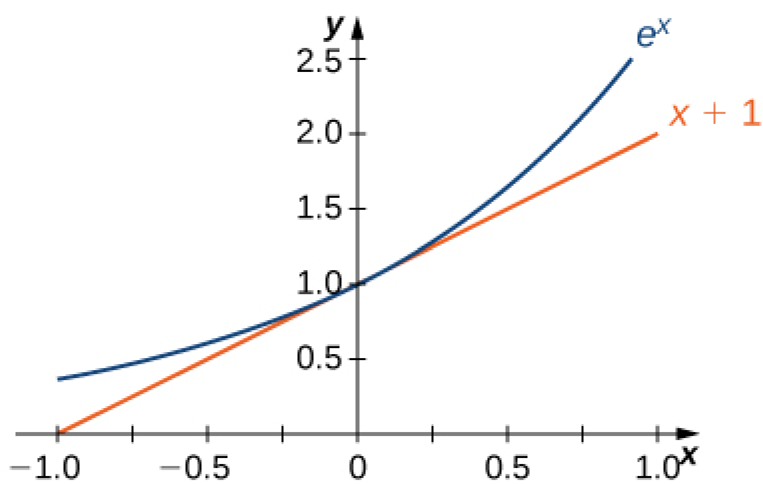


Figure 3.34 The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x, b > 0.$ Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}. \tag{3.28}$$

Turning to $B'(x),$ we obtain the following.

$$\begin{aligned}
B'(x) &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.} \\
&= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\
&= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} && \text{Factor out } b^x. \\
&= b^x \lim_{h \to 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\
&= b^x B'(0) && \text{Use } B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.
\end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0). \tag{3.29}$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

THEOREM 3.14

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x).$$

EXAMPLE 3.74

Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

[Show Solution]

EXAMPLE 3.75

Combining Differentiation Rules

Find the derivative of $y = \frac{e^{x^2}}{x}$.

[Show Solution]

CHECKPOINT 3.50

Find the derivative of $h(x) = xe^{2x}$.

EXAMPLE 3.76

Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

[Show Solution]

CHECKPOINT 3.51

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

THEOREM 3.15

The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x,$ then

$$\frac{dy}{dx} = \frac{1}{x}. \tag{3.30}$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0,$ the derivative of $h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)}g'(x). \tag{3.31}$$

Proof

If $x > 0$ and $y = \ln x,$ then $e^y = x.$ Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

$$h'(x) = \frac{1}{g(x)} g'(x).$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in Figure 3.35.

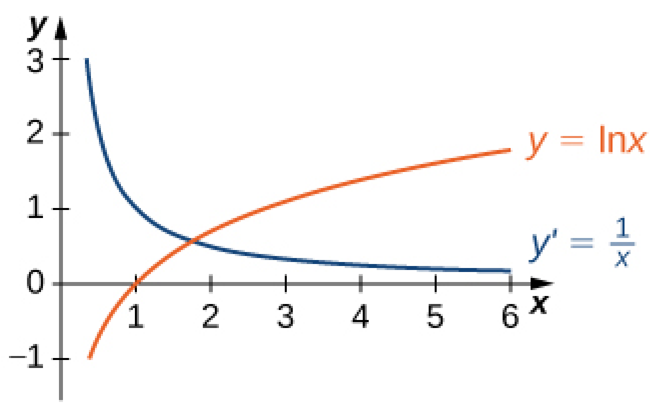


Figure 3.35 The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$.

EXAMPLE 3.77

Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln\left(x^3 + 3x - 4\right)$.

[Show Solution]

EXAMPLE 3.78

Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right)$.

[Show Solution]

CHECKPOINT 3.52

Differentiate: $f(x) = \ln(3x + 2)^5$.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = log_b x$ and $y = b^x$ for $b > 0, b \neq 1$.

THEOREM 3.16

Derivatives of General Exponential and Logarithmic Functions

Let $b > 0, b \neq 1,$ and let $g(x)$ be a differentiable function.

i. If, $y = \log_b x,$ then

$$\frac{dy}{dx} = \frac{1}{x \ln b}. \tag{3.32}$$

More generally, if $h(x) = \log_b(g(x)),$ then for all values of *x* for which $g(x) > 0,$

3.33

$$h'(x) = \frac{g'(x)}{g(x)\ln b}.$$

ii. If $y = b^x$, then

$$\frac{dy}{dx} = b^x \ln b. \tag{3.34}$$

More generally, if $h(x) = b^{g(x)}$, then

$$h'(x) = b^{g(x)} g'(x) \ln b. \tag{3.35}$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln\left(b^y\right) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in Equation 3.33 now follows from the chain rule.

If $y = b^x$, then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y}\frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative (Equation 3.35) follows from the chain rule.

□

EXAMPLE 3.79

Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x + 2}$.

[Show Solution]

EXAMPLE 3.80

Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

[Show Solution]

CHECKPOINT 3.53

Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

PROBLEM-SOLVING STRATEGY: USING LOGARITHMIC DIFFERENTIATION

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.

3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

EXAMPLE 3.81

Using Logarithmic Differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}$.

[Show Solution]

EXAMPLE 3.82

Using Logarithmic Differentiation

Find the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

[Show Solution]

EXAMPLE 3.83

Extending the Power Rule

Find the derivative of $y = x^r$ where r is an arbitrary real number.

[Show Solution]

CHECKPOINT 3.54

Use logarithmic differentiation to find the derivative of $y = x^x$.

CHECKPOINT 3.55

Find the derivative of $y = (\tan x)^{\pi}$.

Section 3.9 Exercises

For the following exercises, find $f'(x)$ for each function.

331. $f(x) = x^2 e^x$

332. $f(x) = \frac{e^{-x}}{x}$

333. $f(x) = e^{x^3 \ln x}$

334. $f(x) = \sqrt{e^{2x} + 2x}$

335. $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

336. $f(x) = \frac{10^x}{\ln 10}$

337. $f(x) = 2^{4x} + 4x^2$

338. $f(x) = 3^{\sin 3x}$

339. $f(x) = x^{\pi} \cdot \pi^x$

340. $f(x) = \ln\left(4x^3 + x\right)$

341. $f(x) = \ln\sqrt{5x - 7}$

342. $f(x) = x^2 \ln 9x$

343. $f(x) = \log(\sec x)$

344. $f(x) = \log_7 \left(6x^4 + 3\right)^5$

345. $f(x) = 2^x \cdot \log_3 7^{x^2 - 4}$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

346. $y = x^{\sqrt{x}}$

347. $y = (\sin 2x)^{4x}$

348. $y = (\ln x)^{\ln x}$

349. $y = x^{\log_2 x}$

350. $y = \left(x^2 - 1\right)^{\ln x}$

351. $y = x^{\cot x}$

352. $y = \frac{x + 11}{\sqrt[3]{x^2 - 4}}$

353. $y = x^{-1/2}\left(x^2 + 3\right)^{2/3}(3x - 4)^4$

354. **[T]** Find an equation of the tangent line to the graph of $f(x) = 4xe^{\left(x^2 - 1\right)}$ at the point where

$x = -1.$ Graph both the function and the tangent line.

355. **[T]** Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1.$ Graph both the function and the normal line.

356. **[T]** Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2.$ (*Hint*: Use implicit differentiation to find $\frac{dy}{dx}.$) Graph both the curve and the tangent line.

357. Consider the function $y = x^{1/x}$ for $x > 0.$
  a. Determine the points on the graph where the tangent line is horizontal.
  b. Determine the points on the graph where $y' > 0$ and those where $y' < 0.$

358. The formula $I(t) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

  a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\frac{\pi}{2}$	(ii)
π	(iii)
$\frac{3\pi}{2}$	(iv)
2π	(v)
$\frac{5\pi}{2}$	(vi)
3π	(vii)
$\frac{7\pi}{2}$	(viii)
4π	(ix)

b. Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

359. **[T]** The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.
b. Use a. to determine the rate at which the population is increasing in t years.
c. Use b. to determine the rate at which the population is increasing in 10 years.

360. **[T]** An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.
b. Use a. to determine the rate at which the substance is decaying in t hours.
c. Use b. to determine the rate of decay at $t = 4$ hours.

361. **[T]** The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function

$N(t) = 5.3e^{0.093t^2 - 0.87t}, (0 \le t \le 4),$ where $N(t)$ gives the number of cases (in thousands) and *t* is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

362. **[T]** The *relative rate of change* of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where $a, b,$ and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.
b. Use a. to find the relative rate of change of a population in $x = 20$ months when $a = 204, b = 0.0198,$ and $c = 0.15.$
c. Briefly interpret what the result of b. means.

For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

Years since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

Table 3.8 New York City Population Over Time Source: http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade.

363. **[T]** Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

364. **[T]** Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

365. **[T]** Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

366. **[T]** Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?
b. Estimate the population in 2010. Was the prediction correct from a.?