## Chapter Outline



**Figure 4.1** As a rocket is being launched, at what rate should the angle of a video camera change to continue viewing the rocket? (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

A rocket is being launched from the ground and cameras are recording the event. A video camera is located on the ground a certain distance from the launch pad. At what rate should the angle of inclination (the angle the camera makes with the ground) change to allow the camera to record the flight of the rocket as it heads upward? (See Example 4.3.)

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

## Learning Objectives

4.3.1. Define absolute extrema.
4.3.2. Define local extrema.
4.3.3. Explain how to find the critical points of a function over a closed interval.
4.3.4. Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

## Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty$, $f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers $x$ and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (see the following figure).

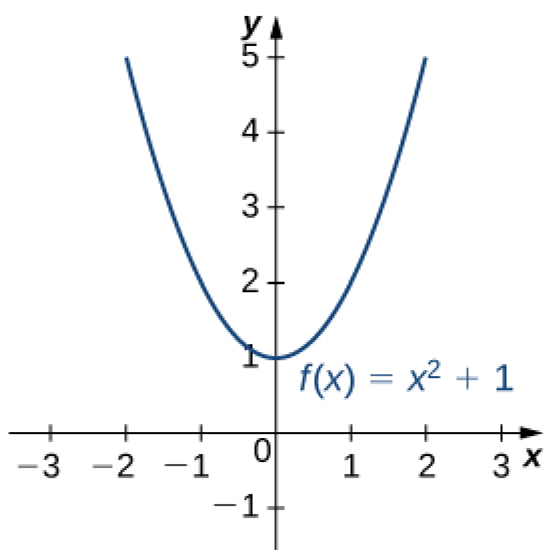



**Figure 4.12** The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

**DEFINITION**

Let $f$ be a function defined over an interval $I$ and let $c \in I$. We say $f$ has an **absolute maximum** on $I$ at $c$ if $f(c) \geq f(x)$ for all $x \in I$. We say $f$ has an **absolute minimum** on $I$ at $c$ if $f(c) \leq f(x)$ for all $x \in I$. If $f$ has an absolute maximum on $I$ at $c$ or an absolute minimum on $I$ at $c$, we say $f$ has an **absolute extremum** on $I$ at $c$.

Before proceeding, let's note two important issues regarding this definition. First, the term *absolute* here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function $f$ has an absolute extremum over an interval $I$ at $c$, the absolute extremum is $f(c)$. The real number $c$ is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers $x$, we say $f$ has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in Figure 4.13(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure 4.13 shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function $f$ over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

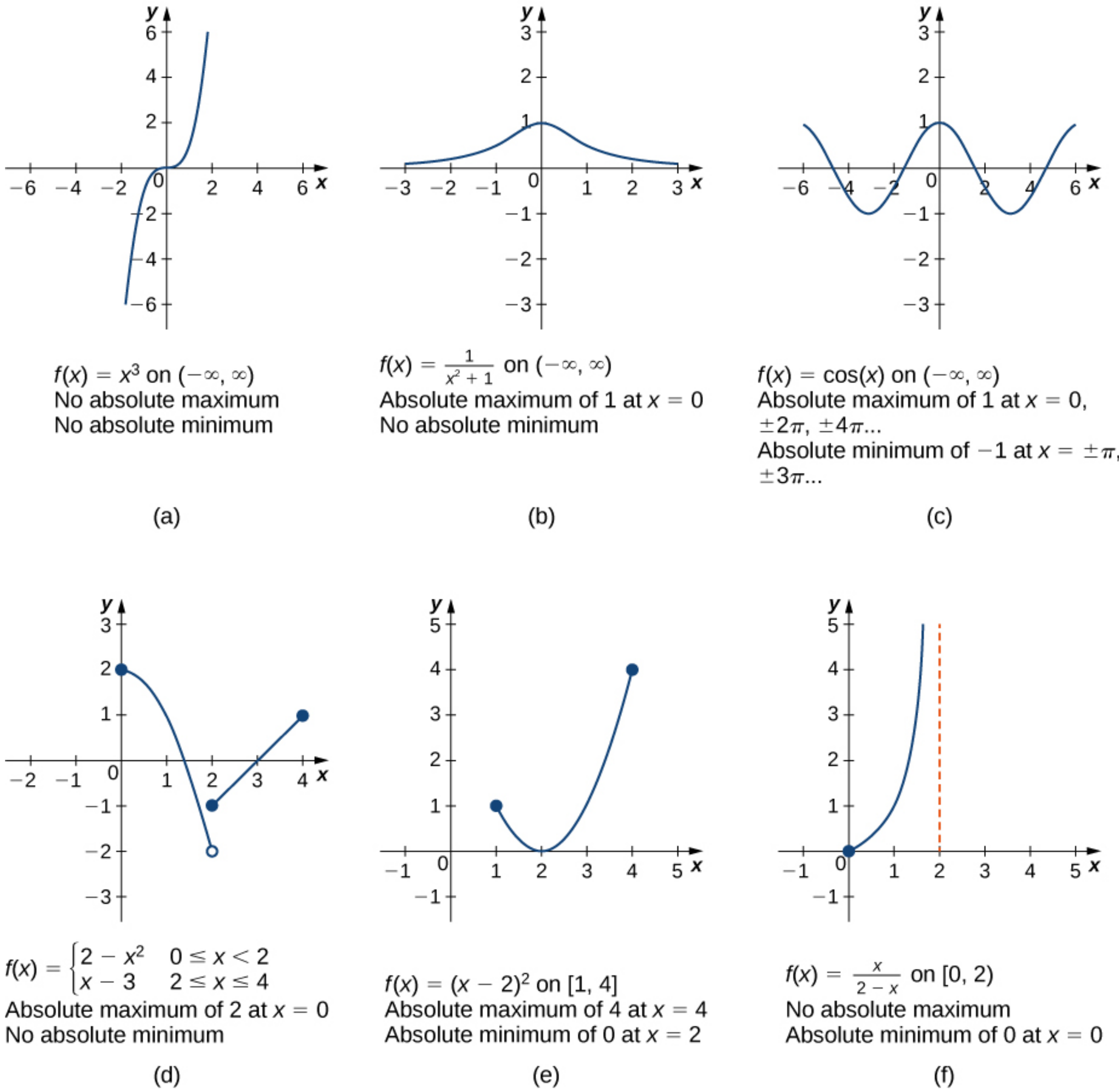



**Figure 4.13** Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

### THEOREM 4.1

#### Extreme Value Theorem

If $f$ is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which $f$ has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which $f$ has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval $I$ is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over $I.$ For example, consider the functions shown in Figure 4.13(d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4],$ the function is discontinuous at $x = 2.$ The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2),$ but is not defined at $x = 2,$ and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2),$ but does not have an absolute maximum over $[0, 2).$ These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let’s examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

## Local Extrema and Critical Points

Consider the function $f$ shown in Figure 4.14. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2.$ However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of $f,$ the value $f(0)$ is larger than $f(x)$ for all $x$ near 0. We say $f$ has a local maximum at $x = 0.$ Similarly, the function $f$ does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for $x$ near 1.

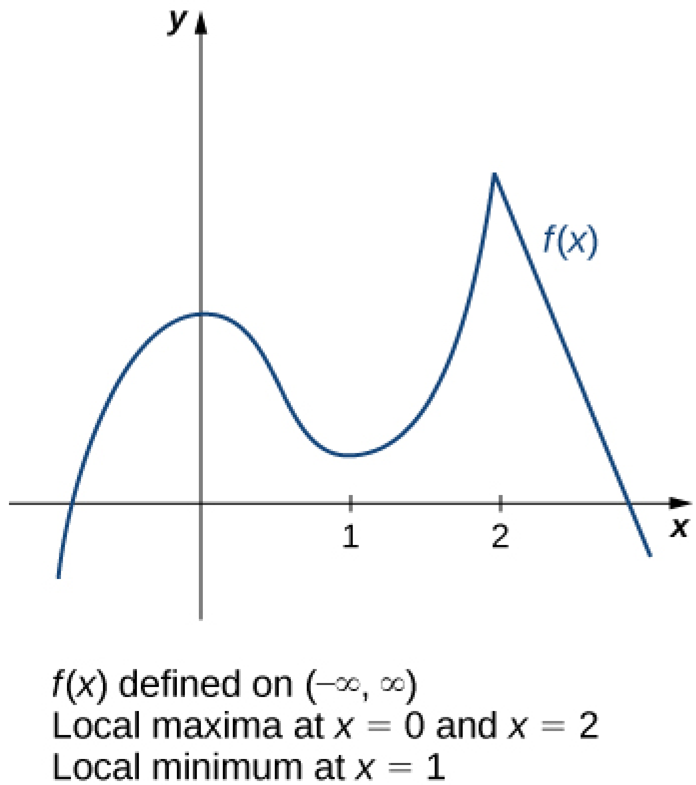



**Figure 4.14** This function $f$ has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

### DEFINITION

A function $f$ has a **local maximum** at $c$ if there exists an open interval $I$ containing $c$ such that $I$ is contained in the domain of $f$ and $f(c) \geq f(x)$ for all $x \in I.$ A function $f$ has a **local minimum** at $c$ if there exists an open interval $I$ containing $c$ such that $I$ is contained in the domain of $f$ and $f(c) \leq f(x)$ for all $x \in I.$ A function $f$ has a **local extremum** at $c$ if $f$ has a local maximum at $c$ or $f$ has a local minimum at $c.$

Note that if $f$ has an absolute extremum at $c$ and $f$ is defined over an interval containing $c,$ then $f(c)$ is also considered a local extremum. If an absolute extremum for a function $f$ occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function $f,$ it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure 4.14 again. The local extrema occur at $x = 0,$ $x = 1,$ and $x = 2.$ Notice that at $x = 0$ and $x = 1,$ the derivative $f'(x) = 0.$ At $x = 2,$ the derivative $f'(x)$ does not exist, since the function $f$ has a corner there. In fact, if $f$ has a local extremum at a point $x = c,$ the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value $c$ is known as a critical point and it is important in finding extreme values for functions.

**DEFINITION**

Let $c$ be an interior point in the domain of $f.$ We say that $c$ is a **critical point** of $f$ if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if $f$ has a local extremum at a point $x = c,$ then $c$ must be a critical point of $f.$ This fact is known as **Fermat's theorem.**

**THEOREM 4.2**

### Fermat's Theorem

If $f$ has a local extremum at $c$ and $f$ is differentiable at $c,$ then $f'(c) = 0.$

### Proof

Suppose $f$ has a local extremum at $c$ and $f$ is differentiable at $c.$ We need to show that $f'(c) = 0.$ To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0,$ and therefore $f'(c) = 0.$ Since $f$ has a local extremum at $c,$ $f$ has a local maximum or local minimum at $c.$ Suppose $f$ has a local maximum at $c.$ The case in which $f$ has a local minimum at $c$ can be handled similarly. There then exists an open interval $I$ such that $f(c) \geq f(x)$ for all $x \in I.$ Since $f$ is differentiable at $c,$ from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c).$ Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c},$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}. \tag{4.5}$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for $x$ near $c$. Therefore, for $x$ near $c$, but $x > c$, we have $\frac{f(x) - f(c)}{x - c} \leq 0$. From Equation 4.4 we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if $f$ has a local extremum at $c$, then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function $f$ must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus $f$ does not have a local extremum at $x = 0$. In Figure 4.15, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

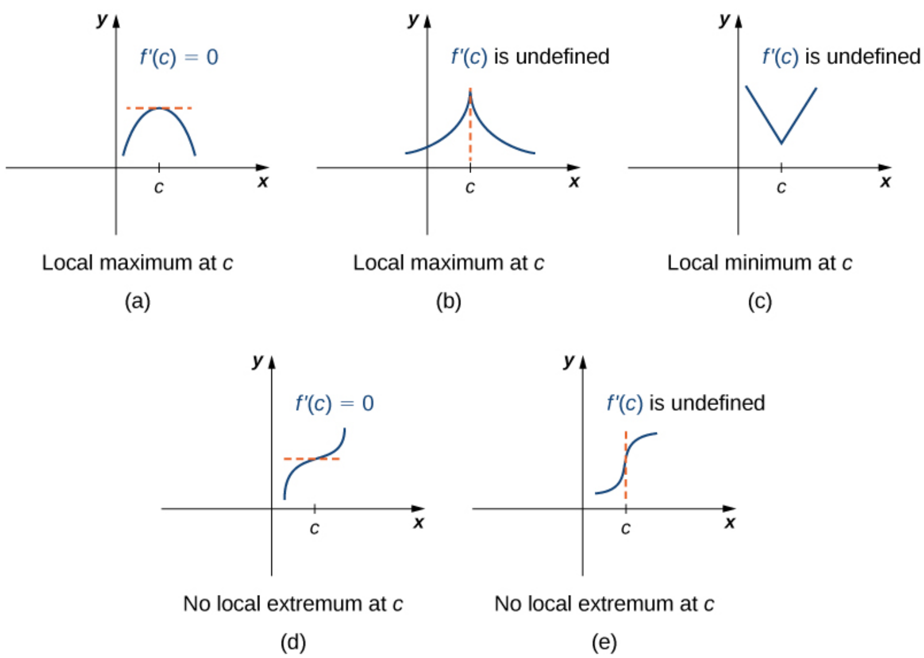


**Figure 4.15** (a–e) A function $f$ has a critical point at $c$ if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

### EXAMPLE 4.12

#### Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

b. $f(x) = \left(x^2 - 1\right)^3$

c. $f(x) = \frac{4x}{1+x^2}$

[Show Solution]

### CHECKPOINT 4.12

Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1.$

## Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in Figure 4.13, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by Fermat’s Theorem, the point $c$ at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

**THEOREM 4.3**

### Location of Absolute Extrema

Let $f$ be a continuous function over a closed, bounded interval $I$. The absolute maximum of $f$ over $I$ and the absolute minimum of $f$ over $I$ must occur at endpoints of $I$ or at critical points of $f$ in $I$.

With this idea in mind, let’s examine a procedure for locating absolute extrema.

**PROBLEM-SOLVING STRATEGY: LOCATING ABSOLUTE EXTREMA OVER A CLOSED INTERVAL**

Consider a continuous function $f$ defined over the closed interval $[a, b]$.

1. Evaluate $f$ at the endpoints $x = a$ and $x = b$.
2. Find all critical points of $f$ that lie over the interval $(a, b)$ and evaluate $f$ at those critical points.
3. Compare all values found in (1) and (2). From Location of Absolute Extrema, the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of $f$. The smallest of these values is the absolute minimum of $f$.

Now let’s look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

**EXAMPLE 4.13**

### Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

[Show Solution]

**CHECKPOINT 4.13**

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function $f$ has a local extremum at a point $c$, then $c$ must be a critical point of $f$. However, $c$ being a critical point is not a sufficient condition for $f$ to have a local extremum at $c$. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

## Section 4.3 Exercises

90. In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $h = -\frac{b}{(2a)}$. Prove this formula using calculus.

91. If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

92. If you are examining a function over an interval $(a, b)$, for $a$ and $b$ finite, is it possible not to have an absolute maximum or absolute minimum?

93. When you are checking for critical points, explain why you also need to determine points where $f'(x)$ is undefined. Draw a graph to support your explanation.

94. Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

95. Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming *a* is non-zero? Explain why or why not using graphical arguments.

96. Let $m$ be the number of local minima and $M$ be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

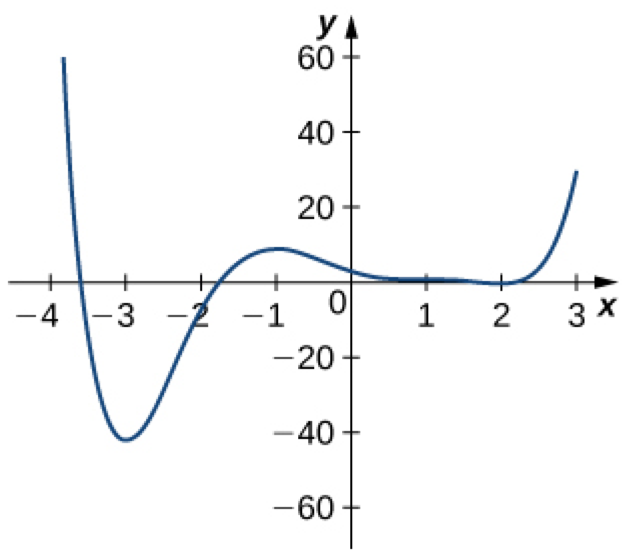
97. Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

98. Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

99. **[T]** Graph the function $y = e^{ax}$. For which values of $a$, on any infinite domain, will you have an absolute minimum and absolute maximum?

For the following exercises, determine where the local and absolute maxima and minima occur on the graph given. Assume the graph represents the entirety of each function.

100.

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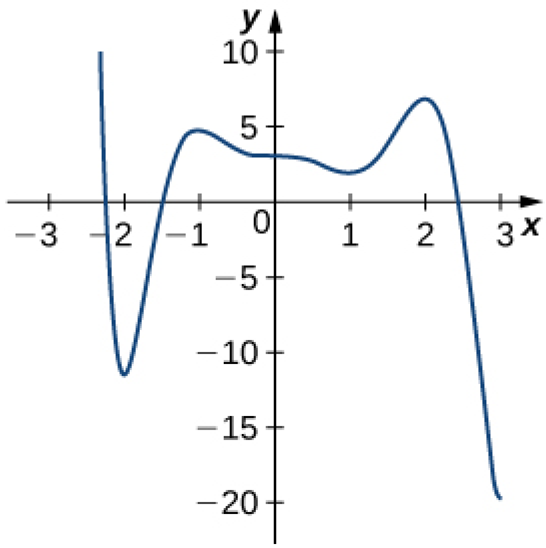
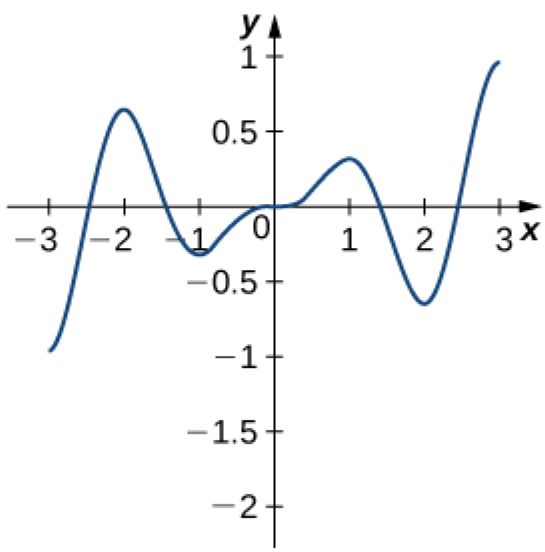



102.

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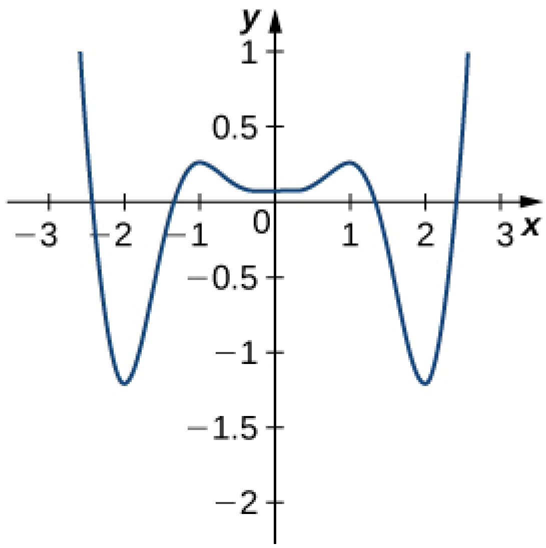



For the following problems, draw graphs of $f(x)$, which is continuous, over the interval $[-4, 4]$ with the following properties:

104. Absolute maximum at $x = 2$ and absolute minima at $x = \pm 3$

105. Absolute minimum at $x = 1$ and absolute maximum at $x = 2$

106. Absolute maximum at $x = 4$, absolute minimum at $x = -1$, local maximum at $x = -2$, and a critical point that is not a maximum or minimum at $x = 2$

107. Absolute maxima at $x = 2$ and $x = -3$, local minimum at $x = 1$, and absolute minimum at $x = 4$

For the following exercises, find the critical points in the domains of the following functions.

108. $y = 4x^3 - 3x$

109. $y = 4\sqrt{x} - x^2$

110. $y = \frac{1}{x-1}$

111. $y = \ln(x - 2)$

112. $y = \tan(x)$

113. $y = \sqrt{4 - x^2}$

114. $y = x^{3/2} - 3x^{5/2}$

115. $y = \frac{x^2 - 1}{x^2 + 2x - 3}$

116. $y = \sin^2(x)$

117. $y = x + \frac{1}{x}$

For the following exercises, find the local and/or absolute maxima for the functions over the specified domain.

118. $f(x) = x^2 + 3$ over $[-1, 4]$

119. $y = x^2 + \frac{2}{x}$ over $[1, 4]$

120. $y = \left(x - x^2\right)^2$ over $[-1, 1]$

121. $y = \frac{1}{\left(x - x^2\right)}$ over $(0, 1)$

122. $y = \sqrt{9 - x}$ over $[1, 9]$

123. $y = x + \sin(x)$ over $[0, 2\pi]$

124. $y = \frac{x}{1 + x}$ over $[0, 100]$

125. $y = |x + 1| + |x - 1|$ over $[-3, 2]$

126. $y = \sqrt{x} - \sqrt{x^3}$ over $[0, 4]$

127. $y = \sin x + \cos x$ over $[0, 2\pi]$

128. $y = 4\sin\theta - 3\cos\theta$ over $[0, 2\pi]$

For the following exercises, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

129. $y = x^2 + 4x + 5$

130. $y = x^3 - 12x$

131. $y = 3x^4 + 8x^3 - 18x^2$

132. $y = x^3(1 - x)^6$

133. $y = \frac{x^2 + x + 6}{x - 1}$

134. $y = \frac{x^2-1}{x-1}$

For the following functions, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

135. **[T]** $y = 3x\sqrt{1 - x^2}$

136. **[T]** $y = x + \sin(x)$

137. **[T]** $y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$

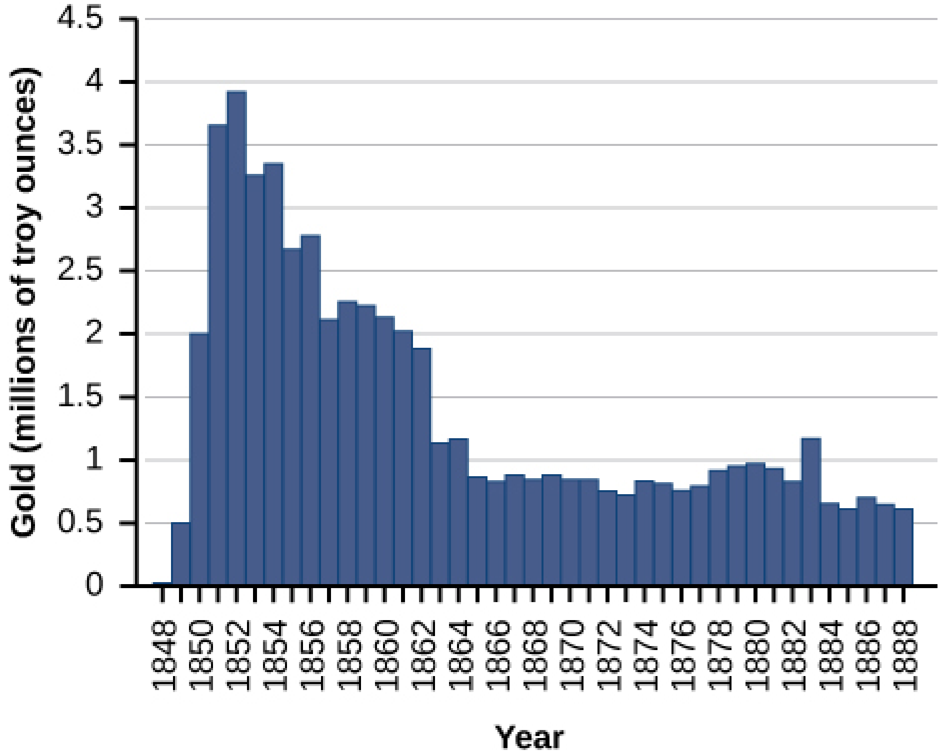
138. **[T]** $y = \frac{x^3+6x^2-x-30}{x-2}$

139. **[T]** $y = \frac{\sqrt{4-x^2}}{\sqrt{4+x^2}}$

140. A company that produces cell phones has a cost function of $C = x^2 - 1200x + 36{,}400,$ where $C$ is cost in dollars and $x$ is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

141. A ball is thrown into the air and its position is given by $h(t) = -4.9t^2 + 60t + 5\text{ m}.$ Find the height at which the ball stops ascending. How long after it is thrown does this happen?

For the following exercises, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G(t) = \frac{(25t)}{\left(t^2+16\right)},$ where $t$ is the number of years since the rush began $(0 \le t \le 40)$ and $G$ is ounces of gold produced (in millions). A summary of the data is shown in the following figure.




142. Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

143. Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Find the critical points, maxima, and minima for the following piecewise functions.

144. $y = \begin{cases} x^2 - 4x & 0 \le x \le 1 \\ x^2 - 4 & 1 < x \le 2 \end{cases}$

145. $y = \begin{cases} x^2 + 1 & x \le 1 \\ x^2 - 4x + 5 & x > 1 \end{cases}$

For the following exercises, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

146. $y = ax^2 + bx + c,$ given that $a > 0$

147. $y = (x - 1)^a,$ given that $a > 1$ and $a$ is an integer.

## Learning Objectives

4.4.1. Explain the meaning of Rolle's theorem.
4.4.2. Describe the significance of the Mean Value Theorem.
4.4.3. State three important consequences of the Mean Value Theorem.

The **Mean Value Theorem** is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

## Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function $f$ are equal at the endpoints of an interval, then there must be an interior point $c$ where $f'(c) = 0$. Figure 4.21 illustrates this theorem.

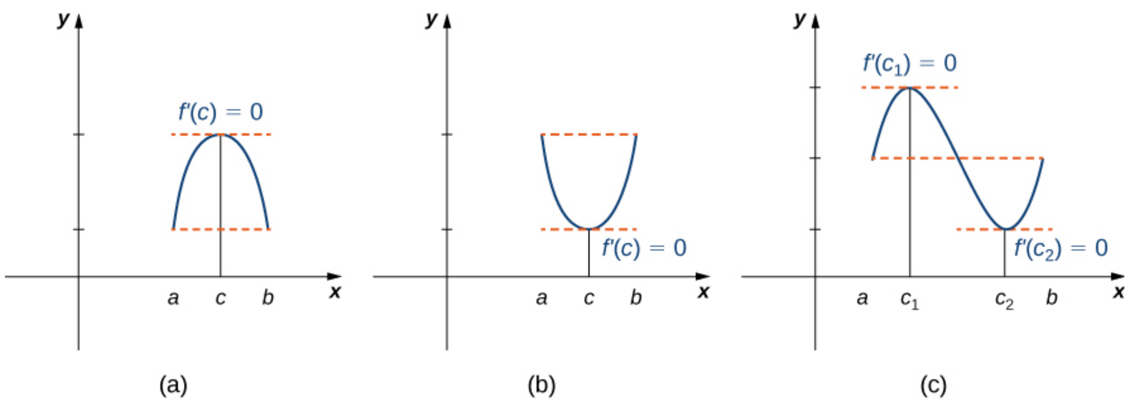



**Figure 4.21** If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between $a$ and $b$.

**THEOREM 4.4**

### Rolle's Theorem

Let $f$ be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval $(a, b)$ such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

### Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.

2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that $f(x) < k$.

Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since $f$ is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than $k$. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because $f$ has a maximum at an interior point $c$, and $f$ is differentiable at $c$, by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function $f$ is critical. If $f$ is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

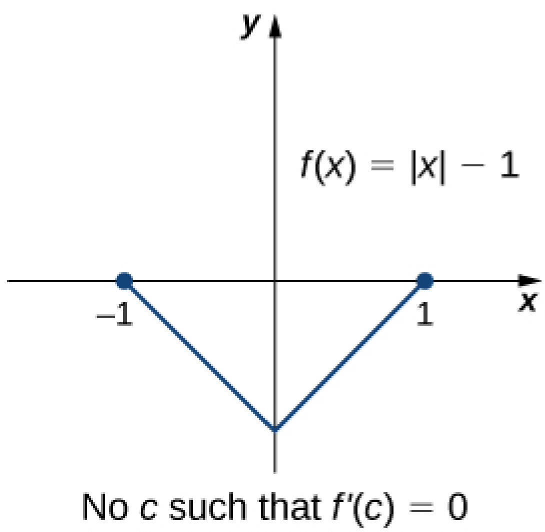



**Figure 4.22** Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points $c$ where $f'(c) = 0$.

### EXAMPLE 4.14

#### Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values $c$ in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$
b. $f(x) = x^3 - 4x$ over $[-2, 2]$

[Show Solution]

### CHECKPOINT 4.14

Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points $c$ guaranteed by Rolle's theorem.

## The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions $f$ defined on a closed interval $[a, b]$ with $f(a) = f(b)$. The Mean Value Theorem generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (Figure 4.25). The Mean Value Theorem states that if $f$ is continuous over the closed interval $[a, b]$ and differentiable over the open interval $(a, b)$, then there exists a point $c \in (a, b)$ such that the tangent line to the graph of $f$ at $c$ is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

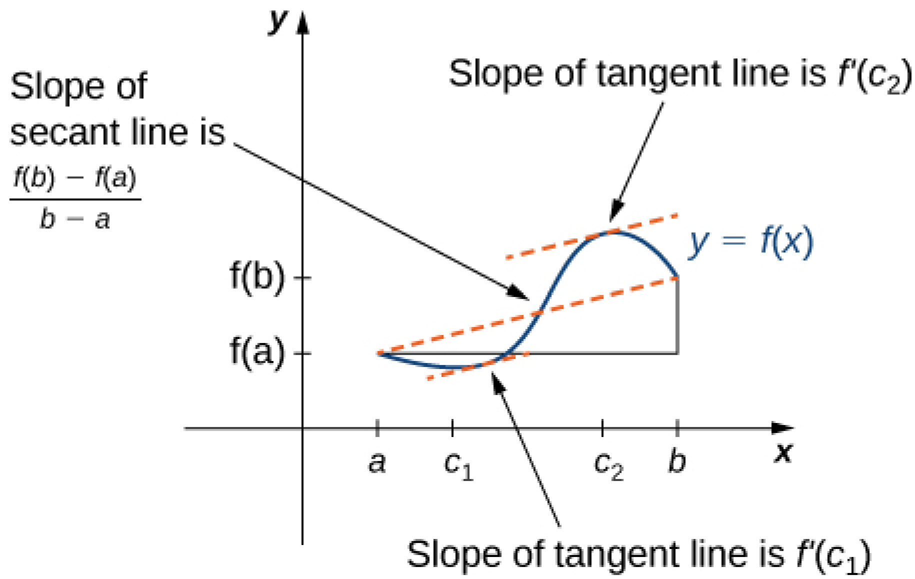



**Figure 4.25** The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values $c_1$ and $c_2$ such that the tangent line to $f$ at $c_1$ and $c_2$ has the same slope as the secant line.

**THEOREM 4.5**

### Mean Value Theorem

Let $f$ be continuous over the closed interval $[a, b]$ and differentiable over the open interval $(a, b)$. Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

## Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point $(x, y)$ on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

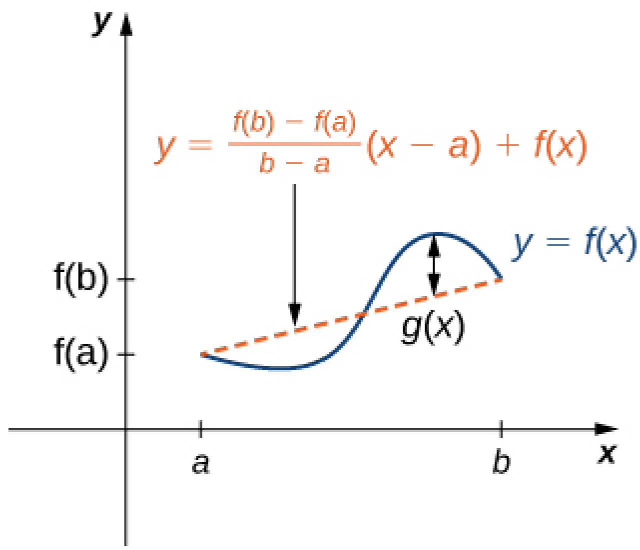



**Figure 4.26** The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point $(x, y)$ on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of $f$ intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since $f$ is a differentiable function over $(a, b)$, $g$ is also a differentiable function over $(a, b)$. Furthermore, since $f$ is continuous over $[a, b]$, $g$ is also continuous over $[a, b]$. Therefore, $g$ satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

**EXAMPLE 4.15**

### Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that $f$ satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values $c$ guaranteed by the Mean Value Theorem.

[Show Solution]

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \le t \le 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

**EXAMPLE 4.16**

### Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position $t$ seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100.$

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity $v_{\text{avg}}$ of the rock for when the rock is released and the rock hits the ground.
c. Find the time $t$ guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is $v_{\text{avg}}$.

**[Show Solution]**

**CHECKPOINT 4.15**

Suppose a ball is dropped from a height of 200 ft. Its position at time $t$ is $s(t) = -16t^2 + 200.$ Find the time $t$ when the instantaneous velocity of the ball equals its average velocity.

## Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all $x$ in some interval $I$, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

**THEOREM 4.6**

### Corollary 1: Functions with a Derivative of Zero

Let $f$ be differentiable over an interval $I$. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

## Proof

Since $f$ is differentiable over $I$, $f$ must be continuous over $I$. Suppose $f(x)$ is not constant for all $x$ in $I$. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since $f$ is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From Corollary 1: Functions with a Derivative of Zero, it follows that if two functions have the same derivative, they differ by, at most, a constant.

**THEOREM 4.7**

### Corollary 2: Constant Difference Theorem

If $f$ and $g$ are differentiable over an interval $I$ and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant $C$.

## Proof

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant $C$ such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function $f$ is increasing over $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas $f$ is decreasing over $I$ if $f(x)_1 > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (Figure 4.29). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function $f$, if there exists a function $F$ such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to $f$ are $F(x) + C$ for some constant $C$. We discuss this result in more detail later in the chapter.

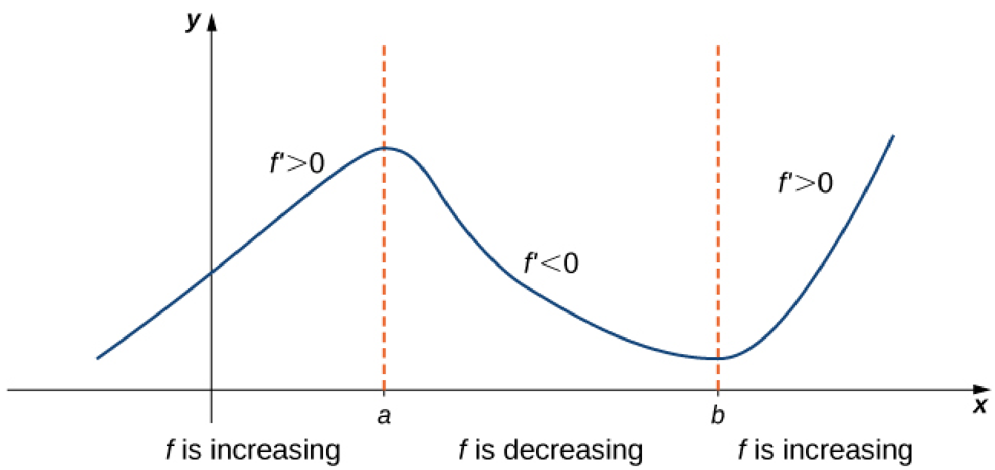



**Figure 4.29** If a function has a positive derivative over some interval $I$, then the function increases over that interval $I$; if the derivative is negative over some interval $I$, then the function decreases over that interval $I$.

**THEOREM 4.8**

### Corollary 3: Increasing and Decreasing Functions

Let $f$ be continuous over the closed interval $[a, b]$ and differentiable over the open interval $(a, b)$.

i. If $f'(x) > 0$ for all $x \in (a, b)$, then $f$ is an increasing function over $[a, b]$.
ii. If $f'(x) < 0$ for all $x \in (a, b)$, then $f$ is a decreasing function over $[a, b]$.

## Proof

We will prove i.; the proof of ii. is similar. Suppose $f$ is not an increasing function on $I$. Then there exist $a$ and $b$ in $I$ such that $a < b$, but $f(a) \geq f(b)$. Since $f$ is a differentiable function over $I$, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore $f$ must be an increasing function over $I$.

□

## Section 4.4 Exercises

148. Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

149. Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

150. When are Rolle's theorem and the Mean Value Theorem equivalent?

151. If you have a function with a discontinuity, is it still possible to have $f'(c)(b - a) = f(b) - f(a)$? Draw such an example or prove why not.

For the following exercises, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

152. $y = \sin(\pi x)$

153. $y = \frac{1}{x^3}$

154. $y = \sqrt{4 - x^2}$

155. $y = \sqrt{x^2 - 4}$

156. $y = \ln(3x - 5)$

For the following exercises, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points $c$ such that $f'(c)(b - a) = f(b) - f(a)$.

157. **[T]** $y = 3x^3 + 2x + 1$ over $[-1, 1]$

158. **[T]** $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

159. **[T]** $y = x^2\cos(\pi x)$ over $[-2, 2]$

160. **[T]** $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

For the following exercises, use the Mean Value Theorem and find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2 - 0)$.

161. $f(x) = x^3$

162. $f(x) = \sin(\pi x)$

163. $f(x) = \cos(2\pi x)$

164. $f(x) = 1 + x + x^2$

165. $f(x) = (x - 1)^{10}$

166. $f(x) = (x - 1)^9$

For the following exercises, show there is no $c$ such that $f(1) - f(-1) = f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

167. $f(x) = \left|x - \frac{1}{2}\right|$

168. $f(x) = \frac{1}{x^2}$

169. $f(x) = \sqrt{|x|}$

170. $f(x) = \lfloor x \rfloor$ (*Hint*: This is called the *floor function* and it is defined so that $f(x)$ is the largest integer less than or equal to $x$.)

For the following exercises, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

171. $y = e^x$ over $[0, 1]$

172. $y = \ln(2x + 3)$ over $\left[-\frac{3}{2}, 0\right]$

173. $f(x) = \tan(2\pi x)$ over $[0, 2]$

174. $y = \sqrt{9 - x^2}$ over $[-3, 3]$

175. $y = \frac{1}{|x+1|}$ over $[0, 3]$

176. $y = x^3 + 2x + 1$ over $[0, 6]$

177. $y = \frac{x^2+3x+2}{x}$ over $[-1, 1]$

178. $y = \frac{x}{\sin(\pi x)+1}$ over $[0, 1]$

179. $y = \ln(x + 1)$ over $[0, e - 1]$

180. $y = x\sin(\pi x)$ over $[0, 2]$

181. $y = 5 + |x|$ over $[-1, 1]$

For the following exercises, consider the roots of the equation.

182. Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root. What is it?

183. Find the conditions for exactly one root (double root) for the equation $y = x^2 + bx + c$

184. Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

For the following exercises, use a calculator to graph the function over the interval $[a, b]$ and graph the secant line from $a$ to $b$. Use the calculator to estimate all values of $c$ as guaranteed by the Mean Value Theorem. Then, find the exact value of $c$, if possible, or write the final equation and use a calculator to estimate to four digits.

185. **[T]** $y = \tan(\pi x)$ over $\left[-\frac{1}{4}, \frac{1}{4}\right]$

186. **[T]** $y = \frac{1}{\sqrt{x+1}}$ over $[0, 3]$

187. **[T]** $y = \left|x^2 + 2x - 4\right|$ over $[-4, 0]$

188. **[T]** $y = x + \frac{1}{x}$ over $\left[\frac{1}{2}, 4\right]$

189. **[T]** $y = \sqrt{x + 1} + \frac{1}{x^2}$ over $[3, 8]$

190. At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

191. Two cars drive from one spotlight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

192. Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

193. Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

## Learning Objectives

4.5.1. Explain how the sign of the first derivative affects the shape of a function's graph.
4.5.2. State the first derivative test for critical points.
4.5.3. Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
4.5.4. Explain the concavity test for a function over an open interval.
4.5.5. Explain the relationship between a function and its first and second derivatives.
4.5.6. State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function $f$ has a local extremum at a point $c$, then $c$ must be a critical point of $f$. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but $f$ does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

## The First Derivative Test

Corollary $3$ of the Mean Value Theorem showed that if the derivative of a function is positive over an interval $I$ then the function is increasing over $I$. On the other hand, if the derivative of the function is negative over an interval $I$, then the function is decreasing over $I$ as shown in the following figure.

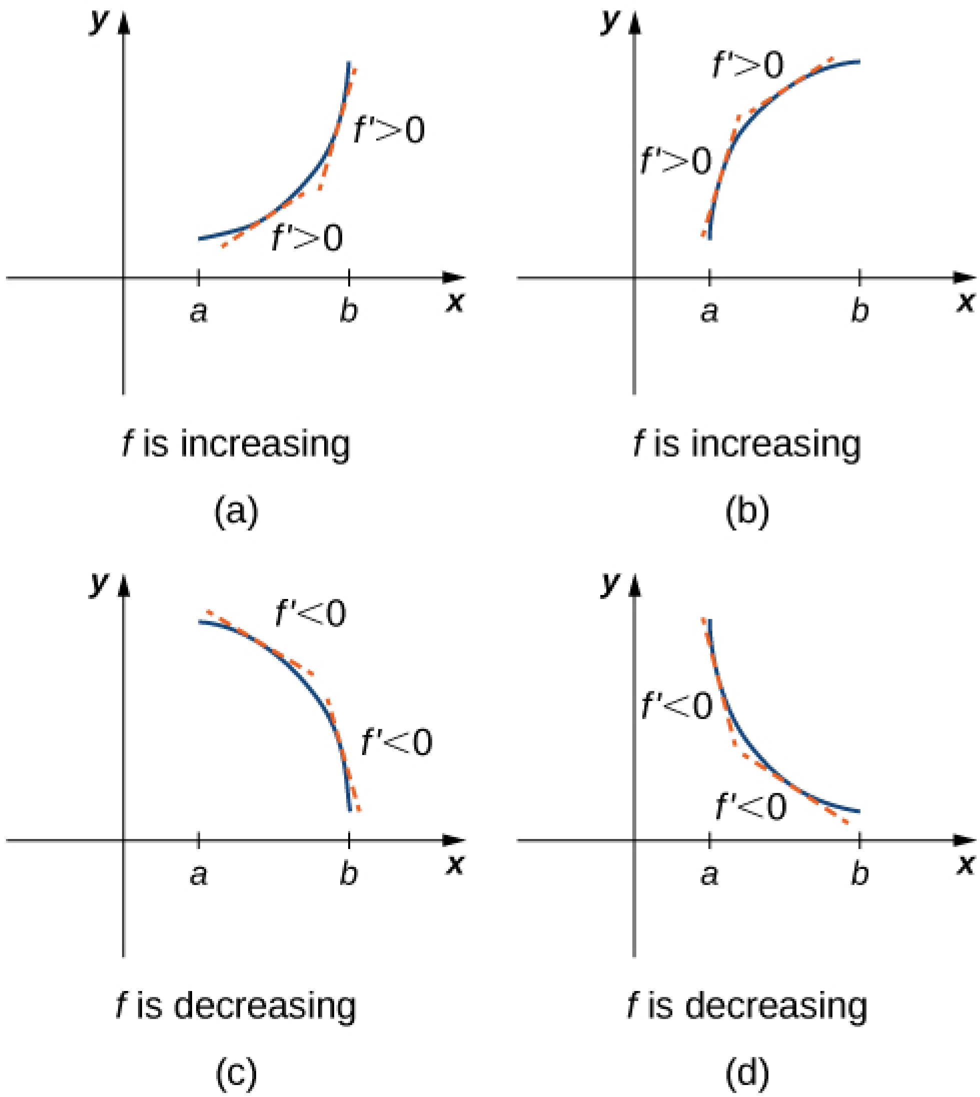



**Figure 4.30** Both functions are increasing over the interval $(a, b)$ . At each point $x$, the derivative $f'(x) > 0$. Both functions are decreasing over the interval $(a, b)$ . At each point $x$, the derivative $f'(x) < 0$.

A continuous function $f$ has a local maximum at point $c$ if and only if $f$ switches from increasing to decreasing at point $c$. Similarly, $f$ has a local minimum at $c$ if and only if $f$ switches from decreasing to increasing at $c$. If $f$ is a continuous function over an interval $I$ containing $c$ and differentiable over $I$, except possibly at $c$, the only way $f$ can switch from increasing to decreasing (or vice versa) at point $c$ is if $f'$ changes sign as $x$ increases through $c$. If $f$ is differentiable at $c$, the only way that $f'$. can change sign as $x$ increases through $c$ is if $f'(c) = 0$. Therefore, for a function $f$ that is continuous over an interval $I$ containing $c$ and differentiable over $I$, except possibly at $c$, the only way $f$ can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function $f$, we look for points $c$ in the domain of $f$ such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of $f$.

Note that $f$ need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure 4.31, we show that if a continuous function $f$ has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if $f$ has a local extremum at a critical point, then the sign of $f'$ switches as $x$ increases through that point.

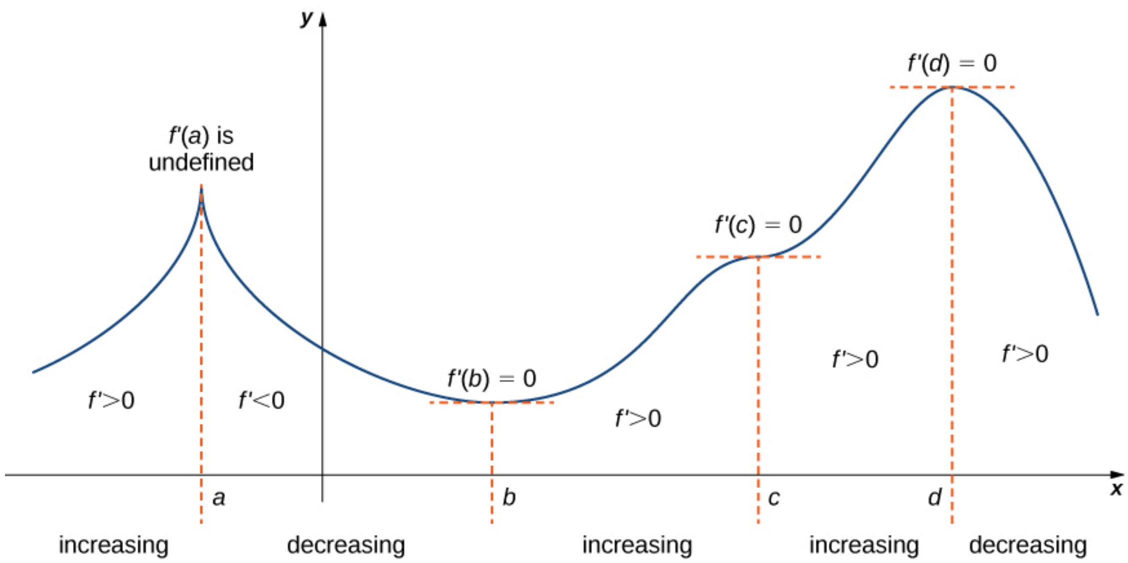



**Figure 4.31** The function $f$ has four critical points: $a, b, c,$ and $d$. The function $f$ has local maxima at $a$ and $d$, and a local minimum at $b$. The function $f$ does not have a local extremum at $c$. The sign of $f'$ changes at all local extrema.

Using Figure 4.31, we summarize the main results regarding local extrema.

- If a continuous function $f$ has a local extremum, it must occur at a critical point $c$.
- The function has a local extremum at the critical point $c$ if and only if the derivative $f'$ switches sign as $x$ increases through $c$.
- Therefore, to test whether a function has a local extremum at a critical point $c$, we must determine the sign of $f'(x)$ to the left and right of $c$.

This result is known as the **first derivative test**.

**THEOREM 4.9**

**First Derivative Test**

Suppose that $f$ is a continuous function over an interval $I$ containing a critical point $c$. If $f$ is differentiable over $I$, except possibly at point $c$, then $f(c)$ satisfies one of the following descriptions:

i. If $f'$ changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of $f$.
ii. If $f'$ changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of $f$.
iii. If $f'$ has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of $f$.

We can summarize the first derivative test as a strategy for locating local extrema.

**PROBLEM-SOLVING STRATEGY: USING THE FIRST DERIVATIVE TEST**

Consider a function $f$ that is continuous over an interval $I$.

1. Find all critical points of $f$ and divide the interval $I$ into smaller intervals using the critical points as endpoints.
2. Analyze the sign of $f'$ in each of the subintervals. If $f'$ is continuous over a given subinterval (which is typically the case), then the sign of $f'$ in that subinterval does not change and, therefore, can be determined by choosing an arbitrary test point $x$ in that subinterval and by evaluating the sign of $f'$ at that test point. Use the sign analysis to determine whether $f$ is increasing or decreasing over that interval.
3. Use First Derivative Test and the results of step $2$ to determine whether $f$ has a local maximum, a local minimum, or neither at each of the critical points.

Now let's look at how to use this strategy to locate all local extrema for particular functions.

**EXAMPLE 4.17**

### Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

[Show Solution]

**CHECKPOINT 4.16**

Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

**EXAMPLE 4.18**

### Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Use a graphing utility to confirm your results.

[Show Solution]

**CHECKPOINT 4.17**

Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x - 1}$.

## Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure 4.34(a) shows a function $f$ with a graph that curves upward. As $x$ increases, the slope of the tangent line increases. Thus, since the derivative increases as $x$ increases, $f'$ is an increasing function. We say this function $f$ is concave up. Figure 4.34(b) shows a function $f$ that curves downward. As $x$ increases, the slope of the tangent line decreases. Since the derivative decreases as $x$ increases, $f'$ is a decreasing function. We say this function $f$ is concave down.

### DEFINITION

Let $f$ be a function that is differentiable over an open interval $I$. If $f'$ is increasing over $I$, we say $f$ is **concave up** over $I$. If $f'$ is decreasing over $I$, we say $f$ is **concave down** over $I$.

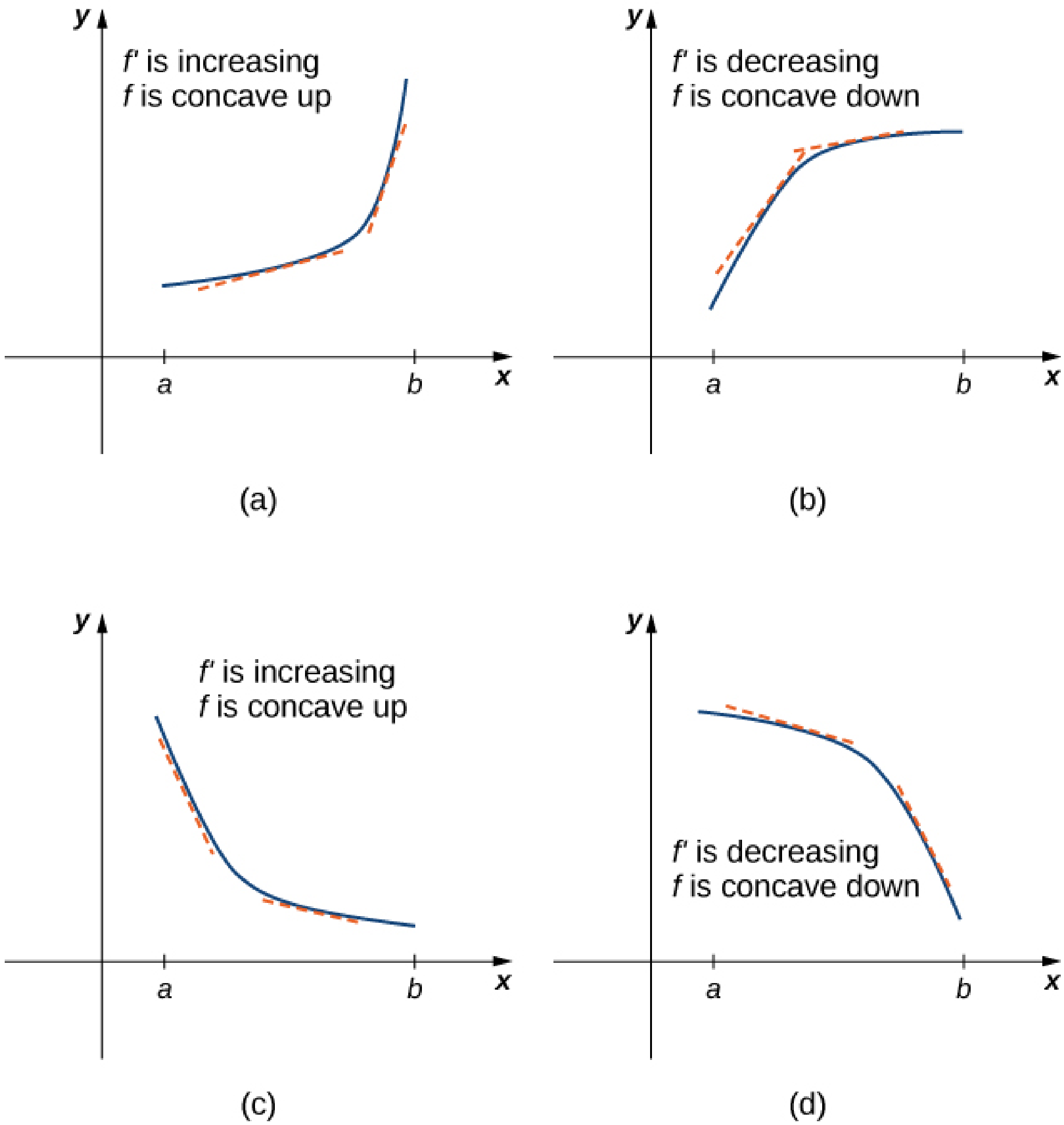



**Figure 4.34** (a), (c) Since $f'$ is increasing over the interval $(a, b)$, we say $f$ is concave up over $(a, b)$. (b), (d) Since $f'$ is decreasing over the interval $(a, b)$, we say $f$ is concave down over $(a, b)$.

In general, without having the graph of a function $f$, how can we determine its concavity? By definition, a function $f$ is concave up if $f'$ is increasing. From Corollary $3$, we know that if $f'$ is a differentiable function, then $f'$ is increasing if its derivative $f''(x) > 0$. Therefore, a function $f$ that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function $f$ is concave down if $f'$ is decreasing. We know that a differentiable function $f'$ is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function $f$ is concave down when $f''(x) < 0$. Applying this logic is known as the **concavity test**.

### THEOREM 4.10

#### Test for Concavity

Let $f$ be a function that is twice differentiable over an interval $I$.

i. If $f''(x) > 0$ for all $x \in I$, then $f$ is concave up over $I$.
ii. If $f''(x) < 0$ for all $x \in I$, then $f$ is concave down over $I$.

We conclude that we can determine the concavity of a function $f$ by looking at the second derivative of $f$. In addition, we observe that a function $f$ can switch concavity (Figure 4.35). However, a continuous function can switch concavity only at a point $x$ if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function $f$ is concave up and concave down, we look for those values of $x$ where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of $f$ into smaller intervals and determine the sign of $f''$ over each of these smaller intervals. If $f''$ changes sign as we pass through a point $x$, then $f$ changes concavity. It is important to remember that a function $f$ may not change concavity at a point $x$ even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, $f$ does change concavity at a point $a$ and $f$ is continuous at $a$, we say the point $(a, f(a))$ is an inflection point of $f$.

### DEFINITION

If $f$ is continuous at $a$ and $f$ changes concavity at $a$, the point $(a, f(a))$ is an **inflection point** of $f$.

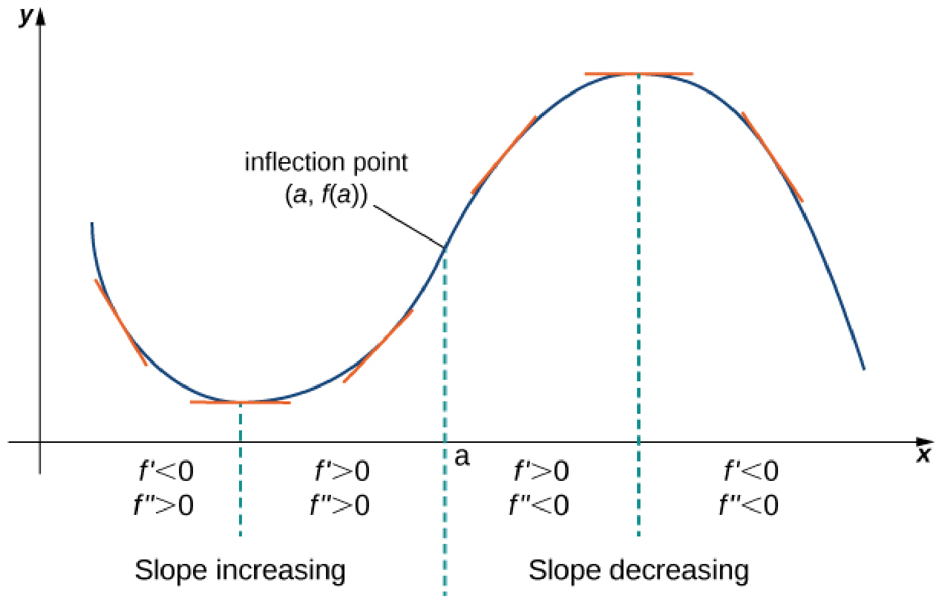



**Figure 4.35** Since $f''(x) > 0$ for $x < a$, the function $f$ is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function $f$ is concave down over the interval $(a, \infty)$. The point $(a, f(a))$ is an inflection point of $f$.

**EXAMPLE 4.19**

### Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where $f$ is concave up and all intervals where $f$ is concave down. List all inflection points for $f$. Use a graphing utility to confirm your results.

[Show Solution]

**CHECKPOINT 4.18**

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where $f$ is concave up and all intervals where $f$ is concave down.

We now summarize, in Table 4.1, the information that the first and second derivatives of a function $f$ provide about the graph of $f$, and illustrate this information in Figure 4.37.

| Sign of $f'$ | Sign of $f''$ | Is $f$ increasing or decreasing? | Concavity |
|---|---|---|---|
| Positive | Positive | Increasing | Concave up |
| Positive | Negative | Increasing | Concave down |
| Negative | Positive | Decreasing | Concave up |
| Negative | Negative | Decreasing | Concave down |

**Table 4.1** What Derivatives Tell Us about Graphs

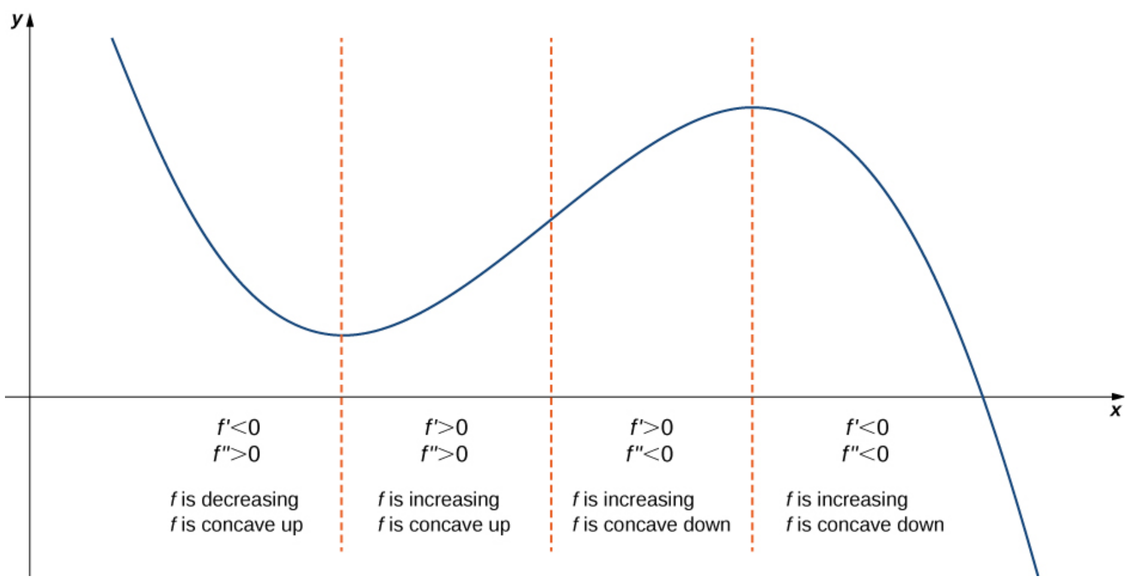



**Figure 4.37** Consider a twice-differentiable function $f$ over an open interval $I$. If $f'(x) > 0$ for all $x \in I$, the function is increasing over $I$. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over $I$. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on $I$.

## The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let $f$ be a twice-differentiable function such that $f'(a) = 0$ and $f''$ is continuous over an open interval $I$ containing $a$. Suppose $f''(a) < 0$. Since $f''$ is continuous over $I$, $f''(x) < 0$ for all $x \in I$ (Figure 4.38). Then, by Corollary $3$, $f'$ is a decreasing function over $I$. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, $f$ has a local maximum at $x = a$. On the other hand, suppose there exists a point $b$ such that $f'(b) = 0$ but $f''(b) > 0$. Since $f''$ is continuous over an open interval $I$ containing $b$, then $f''(x) > 0$ for all $x \in I$ (Figure 4.38). Then, by Corollary $3$, $f'$ is an increasing function over $I$. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, $f$ has a local minimum at $x = b$.

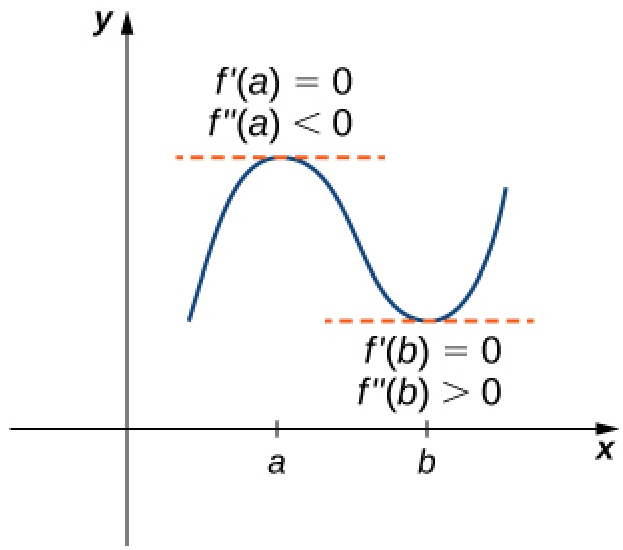



**Figure 4.38** Consider a twice-differentiable function $f$ such that $f''$ is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval $I$ containing $a$ such that for all $x$ in $I$, $f$ is increasing if $x < a$ and $f$ is decreasing if $x > a$. As a result, $f$ has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval $I$ containing $b$ such that for all $x$ in $I$, $f$ is decreasing if $x < b$ and $f$ is increasing if $x > b$. As a result, $f$ has a local minimum at $x = b$.

**THEOREM 4.11**

### Second Derivative Test

Suppose $f'(c) = 0$, $f''$ is continuous over an interval containing $c$.

i. If $f''(c) > 0$, then $f$ has a local minimum at $c$.
ii. If $f''(c) < 0$, then $f$ has a local maximum at $c$.
iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then $f$ may have a local maximum, local minimum, or neither at $c$. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x$, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether $f$ has a local maximum or local minimum at a critical point $c$ where $f'(c) = 0$.

**EXAMPLE 4.20**

### Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

[Show Solution]

**CHECKPOINT 4.19**

Consider the function $f(x) = x^3 - \left(\frac{3}{2}\right)x^2 - 18x$. The points $c = 3, -2$ satisfy $f'(c) = 0$. Use the second derivative test to determine whether $f$ has a local maximum or local minimum at those points.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \to \pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

## Section 4.5 Exercises

194. If $c$ is a critical point of $f(x)$, when is there no local maximum or minimum at $c$? Explain.

195. For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

196. For the function $y = x^3$, is $x = 0$ an inflection point?

197. Is it possible for a point $c$ to be both an inflection point and a local extrema of a twice differentiable function?

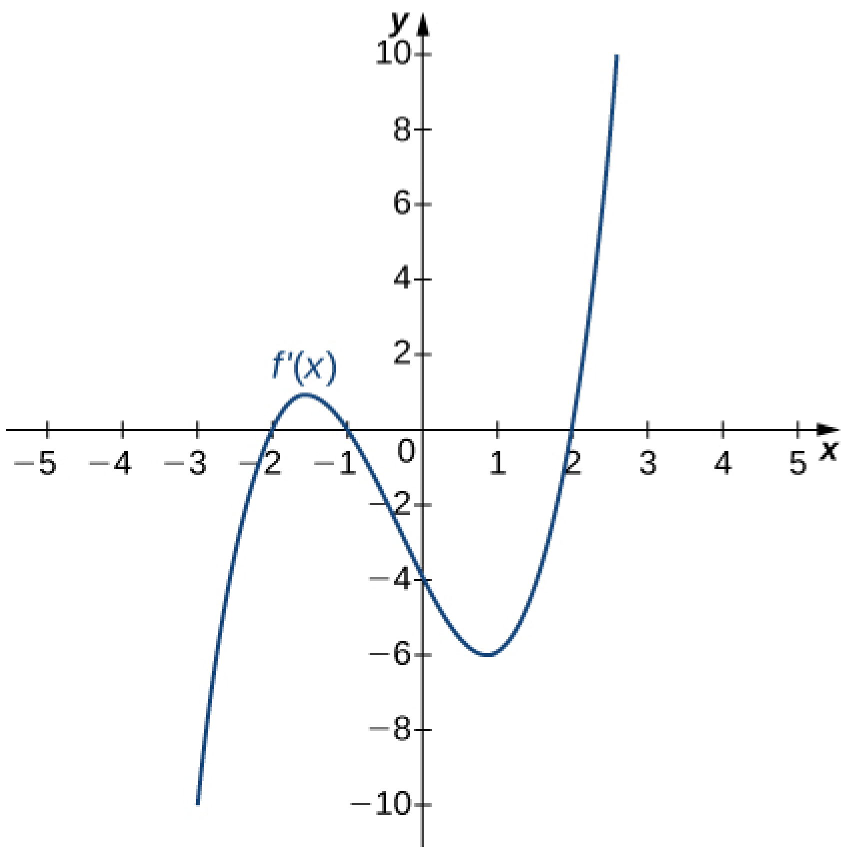
198. Why do you need continuity for the first derivative test? Come up with an example.

199. Explain whether a concave-down function has to cross $y = 0$ for some value of $x$.

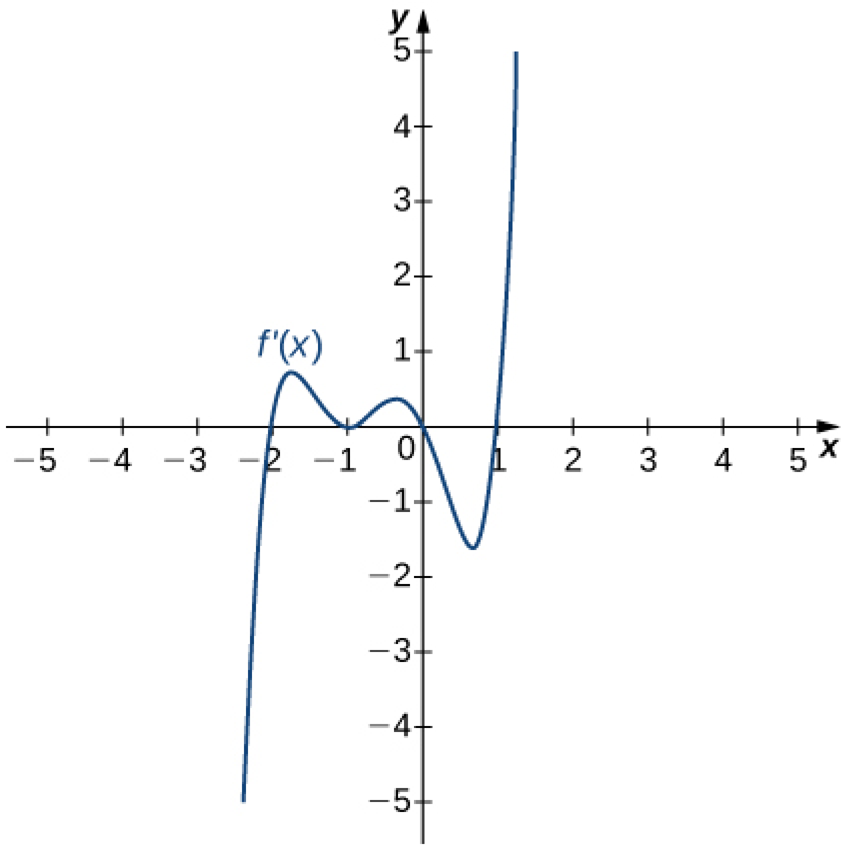
200. Explain whether a polynomial of degree $2$ can have an inflection point.

For the following exercises, analyze the graphs of $f'$, then list all intervals where $f$ is increasing or decreasing.

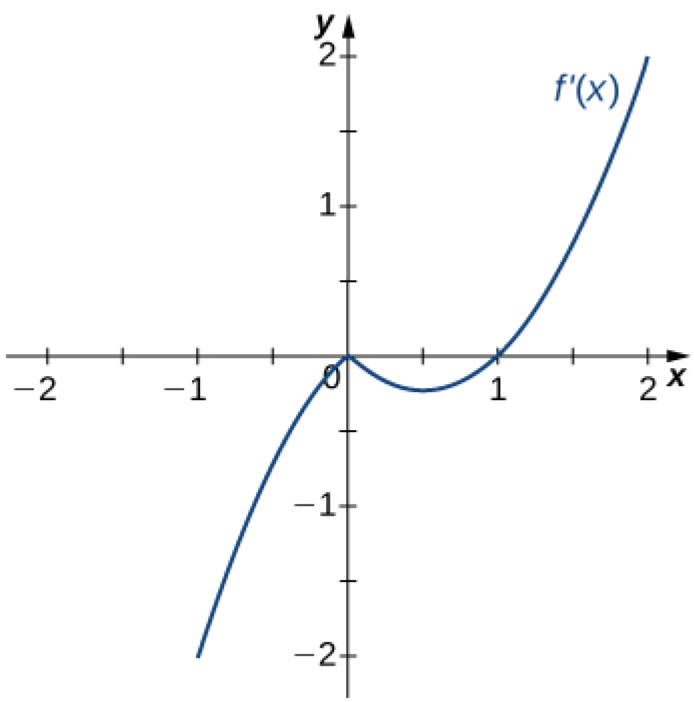
201.



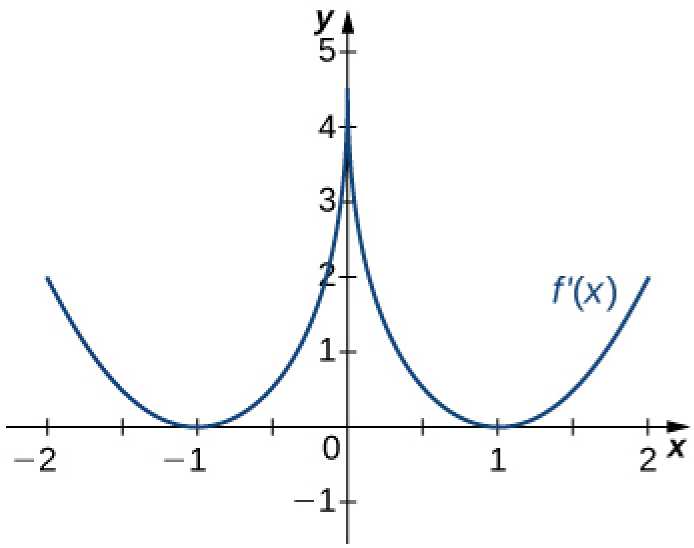

202.



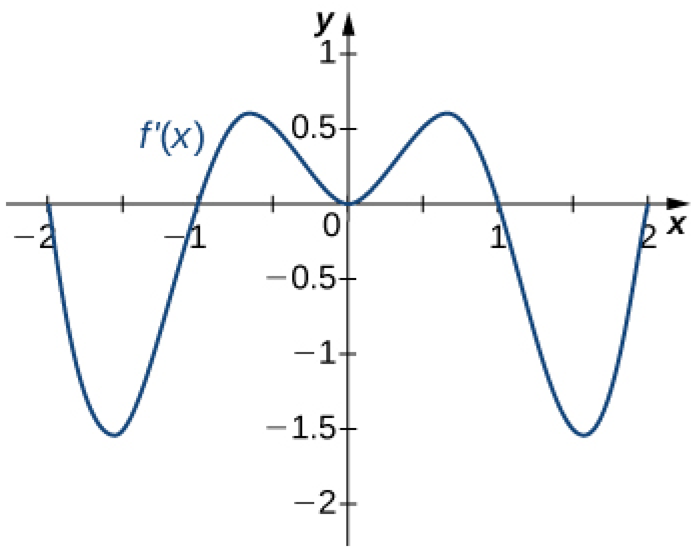

203.



204.



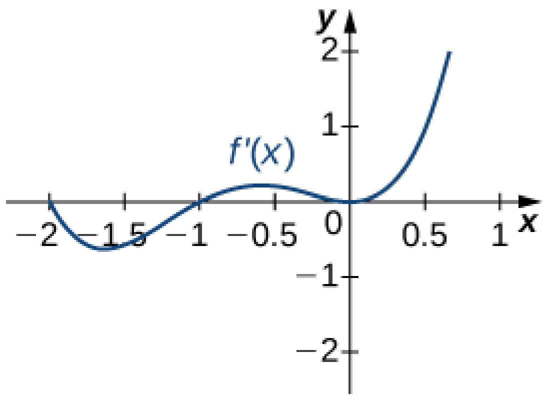

205.



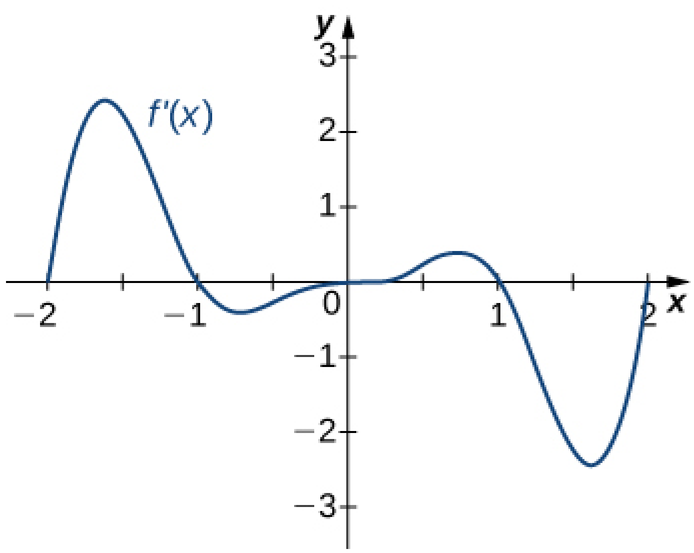

For the following exercises, analyze the graphs of $f'$, then list all intervals where

a. $f$ is increasing and decreasing and
b. the minima and maxima are located.

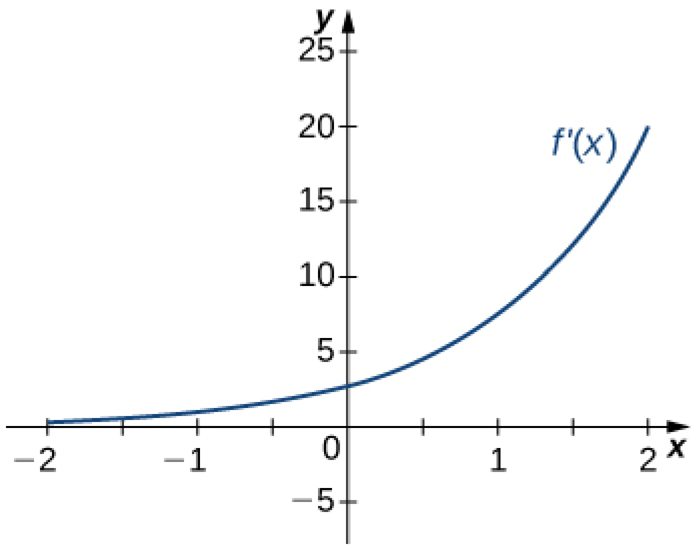
206.



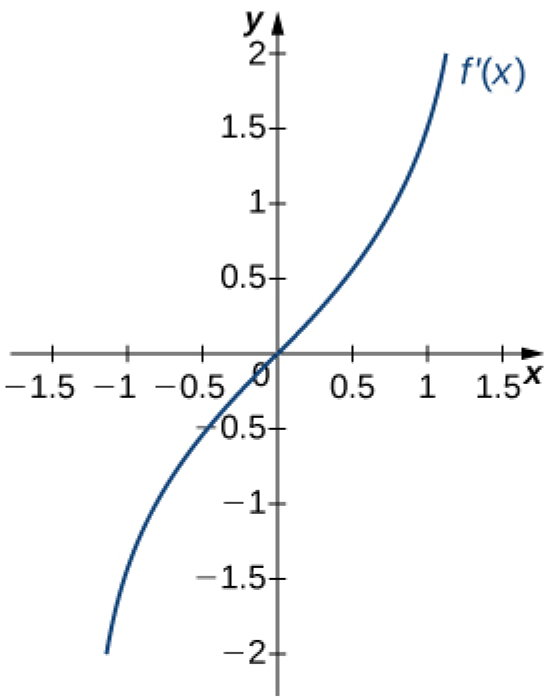

207.



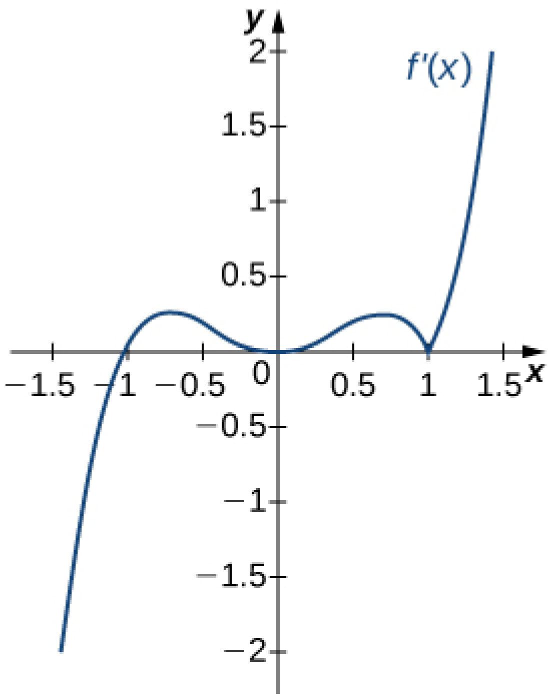

208.




209.

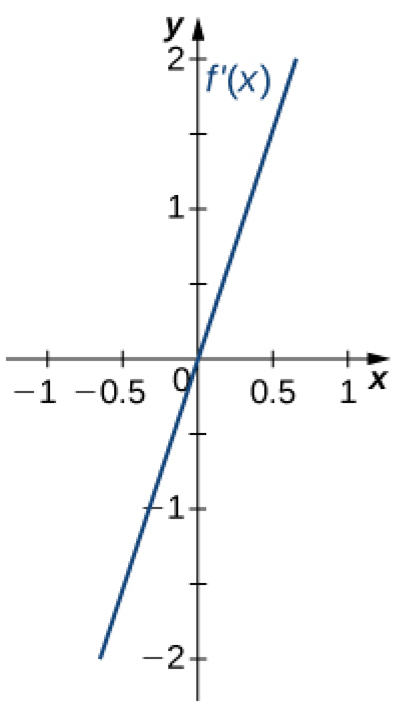



210.

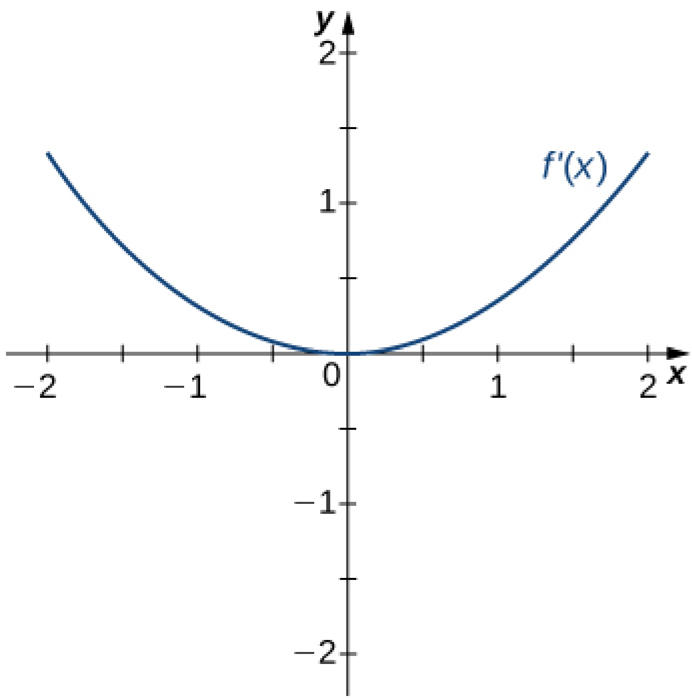



For the following exercises, analyze the graphs of $f'$, then list all inflection points and intervals $f$ that are concave up and concave down.

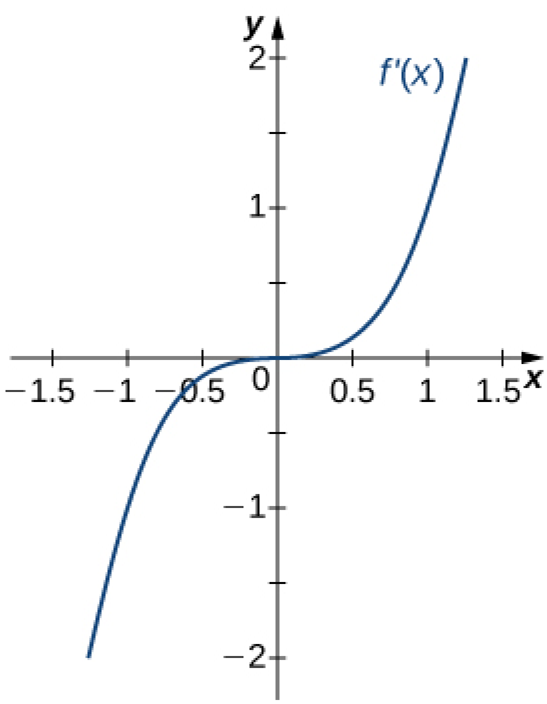
211.



212.




213.

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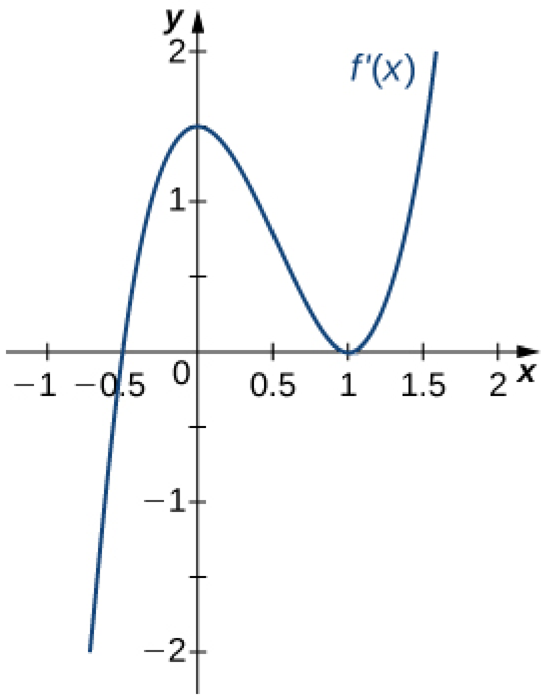
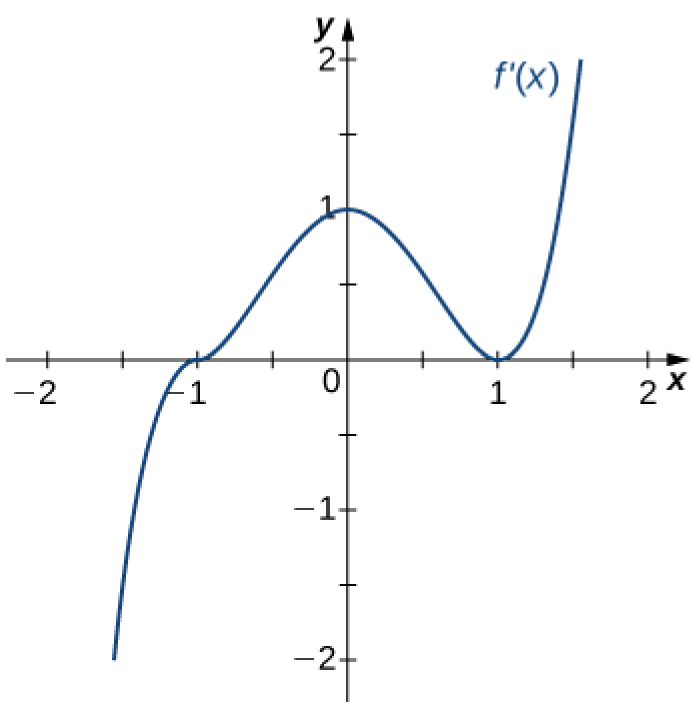



215.




For the following exercises, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

216. $f(x) > 0, f'(x) > 0$ over $x > 1, -3 < x < 0, f'(x) = 0$ over $0 < x < 1$

217. $f'(x) > 0$ over $x > 2, -3 < x < -1, f'(x) < 0$ over $-1 < x < 2, f''(x) < 0$ for all $x$

218. $f''(x) < 0$ over $-1 < x < 1, f''(x) > 0, -3 < x < -1, 1 < x < 3$, local maximum at $x = 0$, local minima at $x = \pm 2$

219. There is a local maximum at $x = 2$, local minimum at $x = 1$, and the graph is neither concave up nor concave down.

220. There are local maxima at $x = \pm 1$, the function is concave up for all $x$, and the function remains positive for all $x$.

For the following exercises, determine

a. intervals where $f$ is increasing or decreasing and
b. local minima and maxima of $f$.

221. $f(x) = \sin x + \sin^3 x$ over $-\pi < x < \pi$

222. $f(x) = x^2 + \cos x$

For the following exercises, determine a. intervals where $f$ is concave up or concave down, and b. the inflection points of $f$.

223. $f(x) = x^3 - 4x^2 + x + 2$

For the following exercises, determine

a. intervals where $f$ is increasing or decreasing,
b. local minima and maxima of $f$,
c. intervals where $f$ is concave up and concave down, and
d. the inflection points of $f$.

224. $f(x) = x^2 - 6x$

225. $f(x) = x^3 - 6x^2$

226. $f(x) = x^4 - 6x^3$

227. $f(x) = x^{11} - 6x^{10}$

228. $f(x) = x + x^2 - x^3$

229. $f(x) = x^2 + x + 1$

230. $f(x) = x^3 + x^4$

For the following exercises, determine

a. intervals where $f$ is increasing or decreasing,
b. local minima and maxima of $f$,
c. intervals where $f$ is concave up and concave down, and
d. the inflection points of $f$. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

231. **[T]** $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$

232. **[T]** $f(x) = x + \sin(2x)$ over $x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

233. **[T]** $f(x) = \sin x + \tan x$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

234. **[T]** $f(x) = (x-2)^2(x-4)^2$

235. **[T]** $f(x) = \frac{1}{1-x}, x \neq 1$

236. **[T]** $f(x) = \frac{\sin x}{x}$ over $x = [2\pi, 0) \cup (0, 2\pi]$

237. $f(x) = \sin(x) e^x$ over $x = [-\pi, \pi]$

238. $f(x) = \ln x \sqrt{x}, x > 0$

239. $f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}, x > 0$

240. $f(x) = \frac{e^x}{x}, x \neq 0$

For the following exercises, interpret the sentences in terms of $f, f', \text{ and } f''$.

241. The population is growing more slowly. Here $f$ is the population.

242. A bike accelerates faster, but a car goes faster. Here $f =$ Bike's position minus Car's position.

243. The airplane lands smoothly. Here $f$ is the plane's altitude.

244. Stock prices are at their peak. Here $f$ is the stock price.

245. The economy is picking up speed. Here $f$ is a measure of the economy, such as GDP.

For the following exercises, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0, f'(3) = 0$. Determine whether the following statements are *true or false*. Justify your answer.

246. $f(x) = 0$ for some $1 \leq x \leq 3$

247. $f''(x) = 0$ for some $1 \leq x \leq 3$

248. There is no absolute maximum at $x = 3$

249. If $f(x)$ has three roots, then it has $1$ inflection point.

250. If $f(x)$ has one inflection point, then it has three real roots.

## Learning Objectives

4.6.1. Calculate the limit of a function as $x$ increases or decreases without bound.
4.6.2. Recognize a horizontal asymptote on the graph of a function.
4.6.3. Estimate the end behavior of a function as $x$ increases or decreases without bound.
4.6.4. Recognize an oblique asymptote on the graph of a function.
4.6.5. Analyze a function and its derivatives to draw its graph.

We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function $f$ defined on an unbounded domain, we also need to know the behavior of $f$ as $x \to \pm\infty$. In this section, we define limits at infinity and show how these limits affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function $f$.

## Limits at Infinity

We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity. Back in Introduction to Functions and Graphs, we looked at vertical asymptotes; in this section we deal with horizontal and oblique asymptotes.

### Limits at Infinity and Horizontal Asymptotes

Recall that $\lim\limits_{x \to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to $L$ as long as $x$ is sufficiently close to $a$. We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in Figure 4.40 and numerically in Table 4.2, as the values of $x$ get larger, the values of $f(x)$ approach $2$. We say the limit as $x$ approaches $\infty$ of $f(x)$ is $2$ and write $\lim\limits_{x \to \infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approaches $2$. We say the limit as $x$ approaches $-\infty$ of $f(x)$ is $2$ and write $\lim\limits_{x \to -\infty} f(x) = 2$.

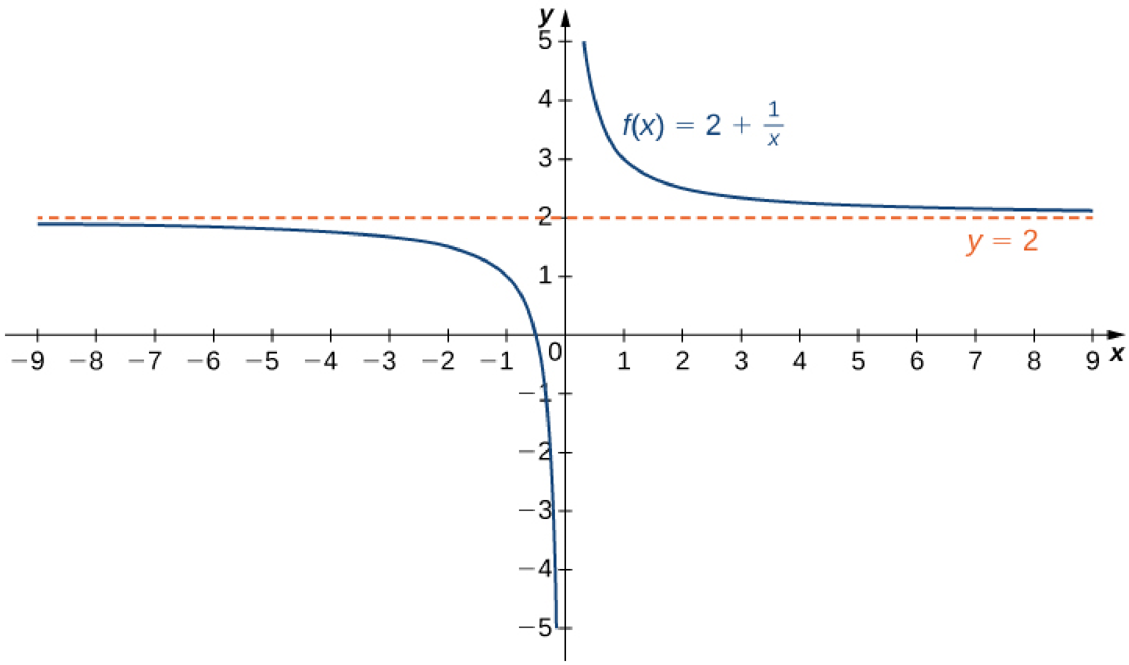



**Figure 4.40** The function approaches the asymptote $y = 2$ as $x$ approaches $\pm\infty$.

| | | | | |
|---|---|---|---|---|
| $x$ | 10 | 100 | 1,000 | 10,000 |
| $2 + \frac{1}{x}$ | 2.1 | 2.01 | 2.001 | 2.0001 |
| $x$ | −10 | −100 | −1000 | −10,000 |
| $2 + \frac{1}{x}$ | 1.9 | 1.99 | 1.999 | 1.9999 |

**Table 4.2** Values of a function $f$ as $x \to \pm\infty$

More generally, for any function $f$, we say the limit as $x \to \infty$ of $f(x)$ is $L$ if $f(x)$ becomes arbitrarily close to $L$ as long as $x$ is sufficiently large. In that case, we write $\lim\limits_{x\to\infty} f(x) = L$. Similarly, we say the limit as $x \to -\infty$ of $f(x)$ is $L$ if $f(x)$ becomes arbitrarily close to $L$ as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim\limits_{x\to-\infty} f(x) = L$. We now look at the definition of a function having a limit at infinity.

### DEFINITION

(Informal) If the values of $f(x)$ become arbitrarily close to $L$ as $x$ becomes sufficiently large, we say the function $f$ has a **limit at infinity** and write

$$\lim_{x \to \infty} f(x) = L.$$

If the values of $f(x)$ becomes arbitrarily close to $L$ for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function $f$ has a limit at negative infinity and write

$$\lim_{x \to -\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value $L$ as $x \to \infty$ or $x \to -\infty$, the graph of $f$ approaches the line $y = L$. In that case, the line $y = L$ is a horizontal asymptote of $f$ (Figure 4.41). For example, for the function $f(x) = \frac{1}{x}$, since $\lim_{x \to \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \frac{1}{x}$.

### DEFINITION

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of $f$.

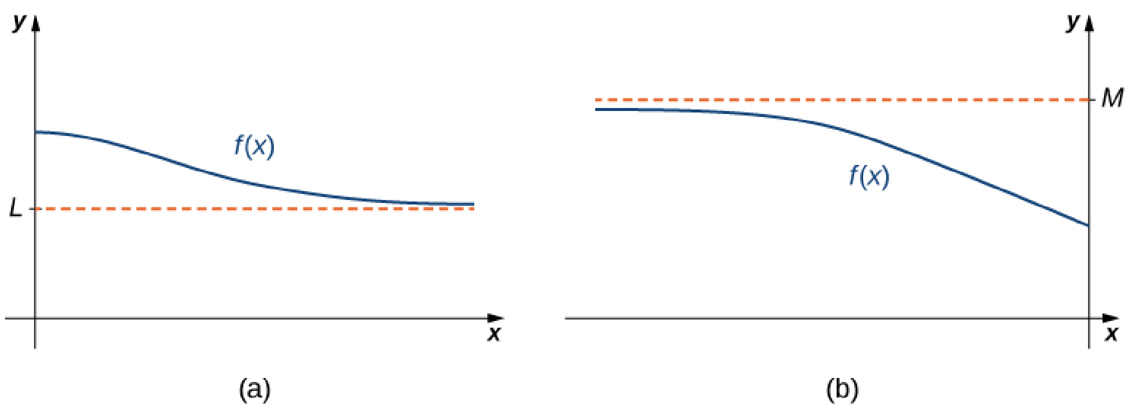



**Figure 4.41** (a) As $x \to \infty$, the values of $f$ are getting arbitrarily close to $L$. The line $y = L$ is a horizontal asymptote of $f$. (b) As $x \to -\infty$, the values of $f$ are getting arbitrarily close to $M$. The line $y = M$ is a horizontal asymptote of $f$.

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as $x$ approaches the vertical asymptote. However, a function may cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function $f(x) = \frac{(\cos x)}{x} + 1$ shown in Figure 4.42 intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.

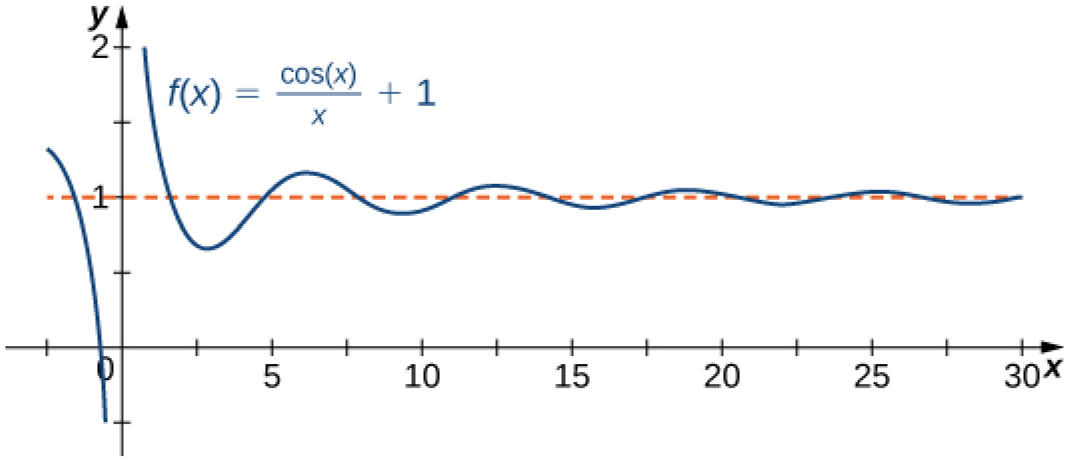



**Figure 4.42** The graph of $f(x) = (\cos x)/x + 1$ crosses its horizontal asymptote $y = 1$ an infinite number of times.

The algebraic limit laws and squeeze theorem we introduced in Introduction to Limits also apply to limits at infinity. We illustrate how to use these laws to compute several limits at infinity.

### EXAMPLE 4.21

#### Computing Limits at Infinity

For each of the following functions $f$, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Determine the horizontal asymptote(s) for $f$.

a. $f(x) = 5 - \frac{2}{x^2}$

b. $f(x) = \frac{\sin x}{x}$

c. $f(x) = \tan^{-1}(x)$

[Show Solution]

## CHECKPOINT 4.20

Evaluate $\lim\limits_{x \to -\infty} \left(3 + \frac{4}{x}\right)$ and $\lim\limits_{x \to \infty} \left(3 + \frac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x) = 3 + \frac{4}{x}$, if any.

### Infinite Limits at Infinity

Sometimes the values of a function $f$ become arbitrarily large as $x \to \infty$ (or as $x \to -\infty$). In this case, we write $\lim\limits_{x \to \infty} f(x) = \infty$ (or $\lim\limits_{x \to -\infty} f(x) = \infty$). On the other hand, if the values of $f$ are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim\limits_{x \to \infty} f(x) = -\infty$ (or $\lim\limits_{x \to -\infty} f(x) = -\infty$).

For example, consider the function $f(x) = x^3$. As seen in Table 4.3 and Figure 4.47, as $x \to \infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim\limits_{x \to \infty} x^3 = \infty$. On the other hand, as $x \to -\infty$, the values of $f(x) = x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim\limits_{x \to -\infty} x^3 = -\infty$.

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 10 | 20 | 50 | 100 | 1000 |
| $x^3$ | 1000 | 8000 | 125,000 | 1,000,000 | 1,000,000,000 |
| $x$ | −10 | −20 | −50 | −100 | −1000 |
| $x^3$ | −1000 | −8000 | −125,000 | −1,000,000 | −1,000,000,000 |

**Table 4.3** Values of a power function as $x \to \pm\infty$

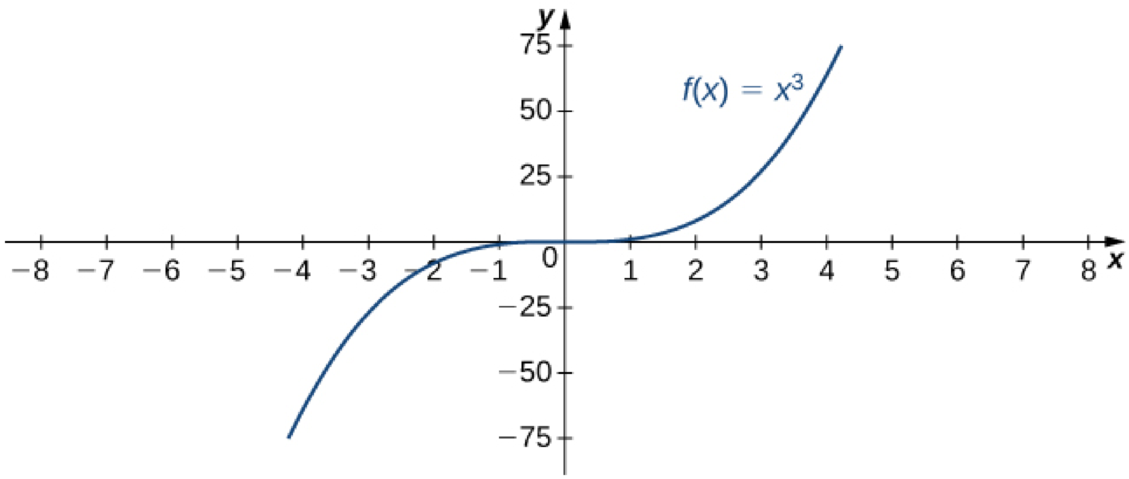



**Figure 4.47** For this function, the functional values approach infinity as $x \to \pm\infty$.

### DEFINITION

(Informal) We say a function $f$ has an infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for $x$ sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for $x$ sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

## Formal Definitions

Earlier, we used the terms *arbitrarily close*, *arbitrarily large*, and *sufficiently large* to define limits at infinity informally. Although these terms provide accurate descriptions of limits at infinity, they are not precise mathematically. Here are more formal definitions of limits at infinity. We then look at how to use these definitions to prove results involving limits at infinity.

### DEFINITION

(Formal) We say a function $f$ has a **limit at infinity**, if there exists a real number $L$ such that for all $\varepsilon > 0,$ there exists $N > 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x > N.$ In that case, we write

$$\lim_{x \to \infty} f(x) = L$$

(see Figure 4.48).

We say a function $f$ has a limit at negative infinity if there exists a real number $L$ such that for all $\varepsilon > 0,$ there exists $N < 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x < N.$ In that case, we write

$$\lim_{x \to -\infty} f(x) = L.$$

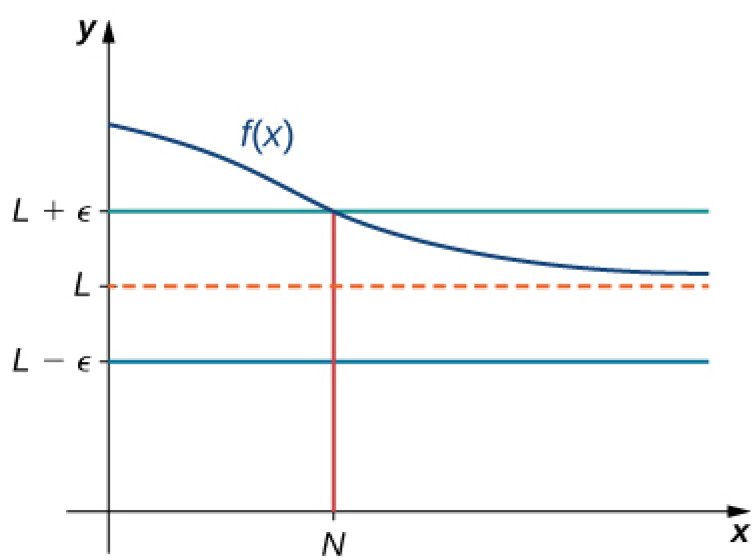



**Figure 4.48** For a function with a limit at infinity, for all $x > N,$ $|f(x) - L| < \varepsilon.$

Earlier in this section, we used graphical evidence in Figure 4.40 and numerical evidence in Table 4.2 to conclude that $\lim_{x \to \infty} \left(2 + \frac{1}{x}\right) = 2.$ Here we use the formal definition of limit at infinity to prove this result rigorously.

### EXAMPLE 4.22 A FINITE LIMIT AT INFINITY EXAMPLE

Use the formal definition of limit at infinity to prove that $\lim_{x \to \infty} \left(2 + \frac{1}{x}\right) = 2.$

**[Show Solution]**

### CHECKPOINT 4.21

Use the formal definition of limit at infinity to prove that $\lim_{x \to \infty} \left(3 - \frac{1}{x^2}\right) = 3.$

We now turn our attention to a more precise definition for an infinite limit at infinity.

### DEFINITION

(Formal) We say a function $f$ has an **infinite limit at infinity** and write

$$\lim_{x \to \infty} f(x) = \infty$$

if for all $M > 0,$ there exists an $N > 0$ such that

$$f(x) > M$$

for all $x > N$ (see Figure 4.49).

We say a function has a negative infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = -\infty$$

if for all $M < 0,$ there exists an $N > 0$ such that

$$f(x) < M$$

for all $x > N.$

Similarly we can define limits as $x \to -\infty.$

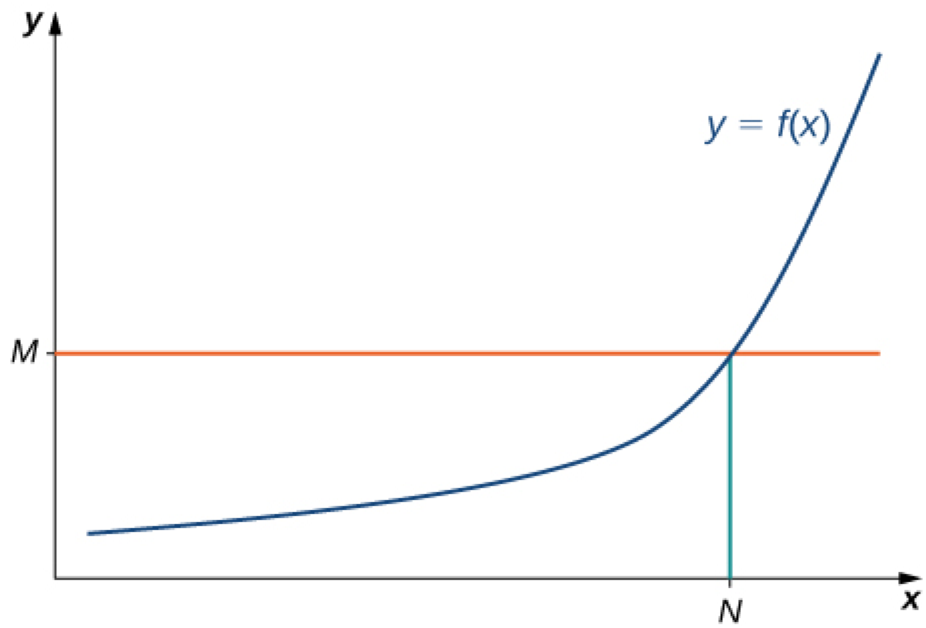



**Figure 4.49** For a function with an infinite limit at infinity, for all $x > N,\ f(x) > M.$

Earlier, we used graphical evidence (Figure 4.47) and numerical evidence (Table 4.3) to conclude that $\lim\limits_{x\to\infty} x^3 = \infty.$ Here we use the formal definition of infinite limit at infinity to prove that result.

## EXAMPLE 4.23 AN INFINITE LIMIT AT INFINITY

Use the formal definition of infinite limit at infinity to prove that $\lim\limits_{x\to\infty} x^3 = \infty.$

**[Show Solution]**

### CHECKPOINT 4.22

Use the formal definition of infinite limit at infinity to prove that $\lim_{x \to \infty} 3x^2 = \infty$.

## End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach $\infty$ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

### End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where $n$ is a positive integer. From Figure 4.50 and Figure 4.51, we see that

$$\lim_{x \to \infty} x^n = \infty;\ n = 1, 2, 3, \ldots$$

and

$$\lim_{x \to -\infty} x^n = \begin{cases} \infty;\ n = 2, 4, 6, \ldots \\ -\infty;\ n = 1, 3, 5, \ldots \end{cases}.$$

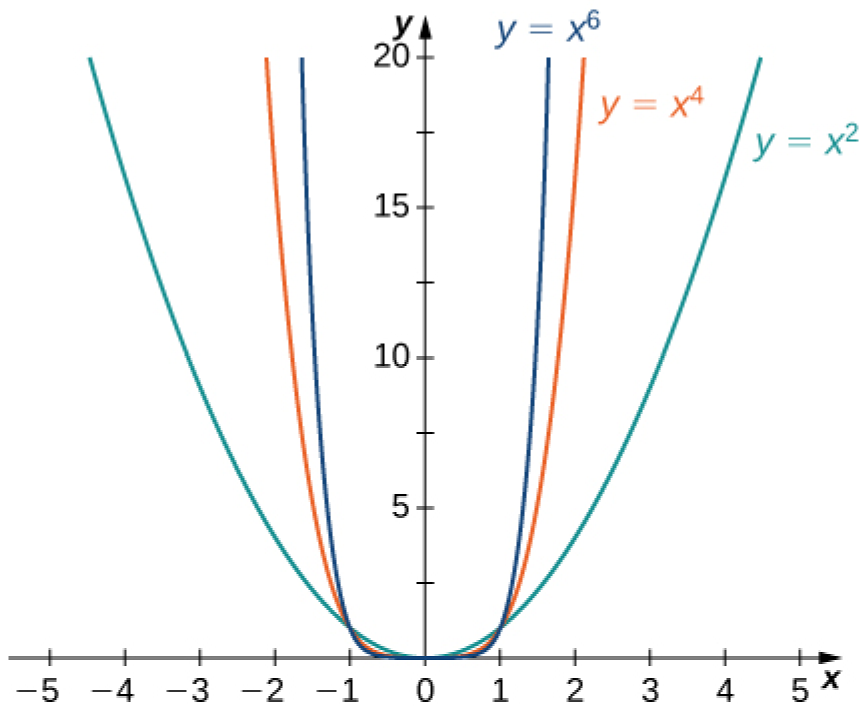



**Figure 4.50** For power functions with an even power of $n$, $\lim_{x \to \infty} x^n = \infty = \lim_{x \to -\infty} x^n$.

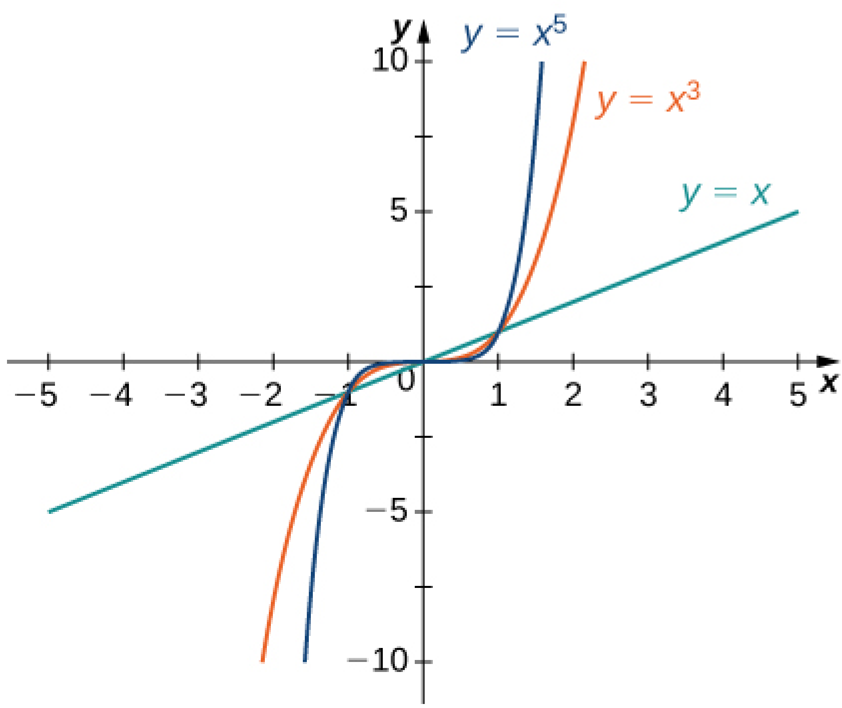



**Figure 4.51** For power functions with an odd power of $n$, $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$.

Using these facts, it is not difficult to evaluate $\lim_{x \to \infty} cx^n$ and $\lim_{x \to -\infty} cx^n$, where $c$ is any constant and $n$ is a positive integer. If $c > 0$, the graph of $y = cx^n$ is a vertical stretch or compression of $y = x^n$, and therefore

$$\lim_{x \to \infty} cx^n = \lim_{x \to \infty} x^n \text{ and } \lim_{x \to -\infty} cx^n = \lim_{x \to -\infty} x^n \text{ if } c > 0.$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the $x$-axis, and therefore

$$\lim_{x \to \infty} cx^n = -\lim_{x \to \infty} x^n \text{ and } \lim_{x \to -\infty} cx^n = -\lim_{x \to -\infty} x^n \text{ if } c < 0.$$

If $c = 0, y = cx^n = 0$, in which case $\lim_{x \to \infty} cx^n = 0 = \lim_{x \to -\infty} cx^n$.

**EXAMPLE 4.24**

### Limits at Infinity for Power Functions

For each function $f$, evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

a. $f(x) = -5x^3$
b. $f(x) = 2x^4$

[Show Solution]

**CHECKPOINT 4.23**

Let $f(x) = -3x^4$. Find $\lim_{x \to \infty} f(x)$.

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x \to \pm\infty} f(x)$ for any polynomial function $f$. Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

of degree $n \geq 1$ so that $a_n \neq 0.$ Factoring, we see that

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n} \frac{1}{x} + \ldots + \frac{a_1}{a_n} \frac{1}{x^{n-1}} + \frac{a_0}{a_n} \frac{1}{x^n}\right).$$

As $x \to \pm\infty,$ all the terms inside the parentheses approach zero except the first term. We conclude that

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n.$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$ as shown in Figure 4.52 and Table 4.4.

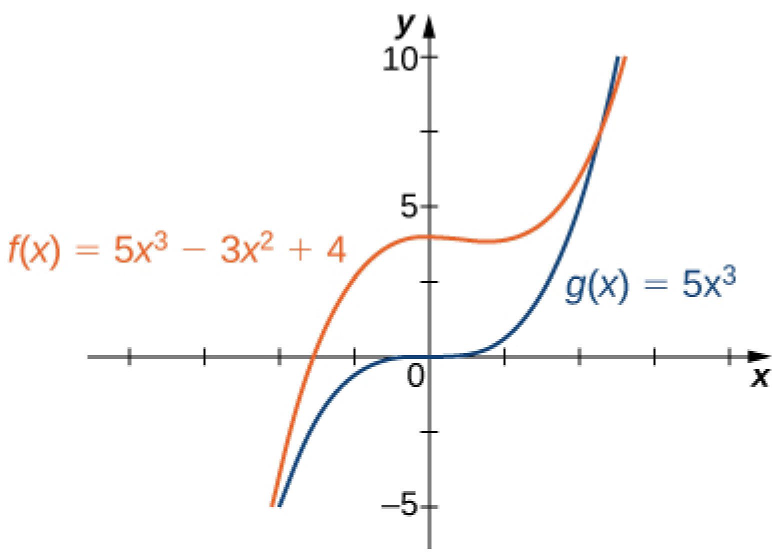



**Figure 4.52** The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

| | | | |
|---|---|---|---|
| $x$ | 10 | 100 | 1000 |
| $f(x) = 5x^3 - 3x^2 + 4$ | 4704 | 4,970,004 | 4,997,000,004 |
| $g(x) = 5x^3$ | 5000 | 5,000,000 | 5,000,000,000 |
| $x$ | −10 | −100 | −1000 |
| | | | |

| | | | |
|---|---|---|---|
| $f(x) = 5x^3 - 3x^2 + 4$ | −5296 | −5,029,996 | −5,002,999,996 |
| $g(x) = 5x^3$ | −5000 | −5,000,000 | −5,000,000,000 |

**Table 4.4** A polynomial's end behavior is determined by the term with the largest exponent.

### End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In Example 4.25, we show that the limits at infinity of a rational function $f(x) = \frac{p(x)}{q(x)}$ depend on the relationship between the degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we divide the numerator and denominator by the highest power of $x$ appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of $x$.

**EXAMPLE 4.25**

### Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \to \infty$ and $x \to -\infty$. Then, use this information to describe the end behavior of the function.

a. $f(x) = \frac{3x-1}{2x+5}$ (*Note:* The degree of the numerator and the denominator are the same.)
b. $f(x) = \frac{3x^2+2x}{4x^3-5x+7}$ (*Note:* The degree of numerator is less than the degree of the denominator.)
c. $f(x) = \frac{3x^2+4x}{x+2}$ (*Note:* The degree of numerator is greater than the degree of the denominator.)

[Show Solution]

**CHECKPOINT 4.24**

Evaluate $\lim_{x \to \pm\infty} \frac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$ and use these limits to determine the end behavior of $f(x) = \frac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$.

Before proceeding, consider the graph of $f(x) = \frac{(3x^2 + 4x)}{(x + 2)}$ shown in Figure 4.56. As $x \to \infty$ and $x \to -\infty$, the graph of $f$ appears almost linear. Although $f$ is certainly not a linear function, we now investigate why the graph of $f$ seems to be approaching a linear function. First, using long division of polynomials, we can write

$$f(x) = \frac{3x^2 + 4x}{x + 2} = 3x - 2 + \frac{4}{x + 2}.$$

Since $\frac{4}{(x + 2)} \to 0$ as $x \to \pm\infty$, we conclude that

$$\lim_{x \to \pm\infty} (f(x) - (3x - 2)) = \lim_{x \to \pm\infty} \frac{4}{x + 2} = 0.$$

Therefore, the graph of $f$ approaches the line $y = 3x - 2$ as $x \to \pm\infty$. This line is known as an **oblique asymptote** for $f$ (Figure 4.56).

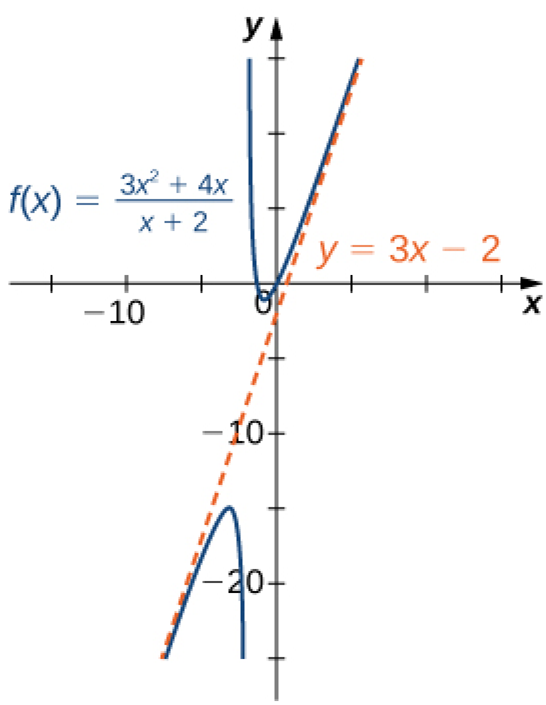


**Figure 4.56** The graph of the rational function $f(x) = \left(3x^2 + 4x\right)/(x + 2)$ approaches the oblique asymptote $y = 3x - 2 \text{ as } x \to \pm\infty.$

We can summarize the results of Example 4.25 to make the following conclusion regarding end behavior for rational functions. Consider a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

where $a_n \neq 0 \text{ and } b_m \neq 0.$

1. If the degree of the numerator is the same as the degree of the denominator $(n = m),$ then $f$ has a horizontal asymptote of $y = a_n/b_m$ as $x \to \pm\infty.$
2. If the degree of the numerator is less than the degree of the denominator $(n < m),$ then $f$ has a horizontal asymptote of $y = 0$ as $x \to \pm\infty.$
3. If the degree of the numerator is greater than the degree of the denominator $(n > m),$ then $f$ does not have a horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

where the degree of $r(x)$ is less than the degree of $q(x).$ As a result, $\lim\limits_{x \to \pm\infty} r(x)/q(x) = 0.$ Therefore, the values of $[f(x) - g(x)]$ approach zero as $x \to \pm\infty.$ If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ $(n = m + 1),$ the function $g(x)$ is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

## EXAMPLE 4.26

### Determining End Behavior for a Function Involving a Radical

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \frac{3x-2}{\sqrt{4x^2+5}}$ and describe the end behavior of $f.$

[Show Solution]

## CHECKPOINT 4.25

Evaluate $\lim\limits_{x\to\infty}\frac{\sqrt{3x^2+4}}{x+6}$.

## Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as $x \to \pm\infty$. For example, $\sin x$ oscillates between $1 \text{ and } -1$ (Figure 4.58). The tangent function $x$ has an infinite number of vertical asymptotes as $x \to \pm\infty$; therefore, it does not approach a finite limit nor does it approach $\pm\infty$ as $x \to \pm\infty$ as shown in Figure 4.59.

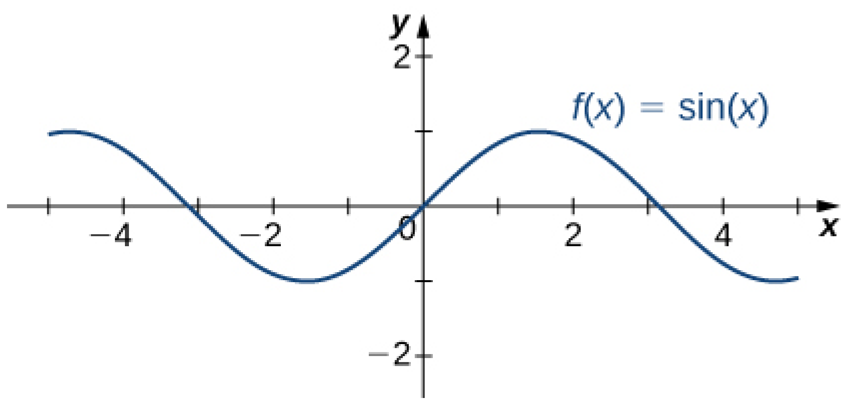



**Figure 4.58** The function $f(x) = \sin x$ oscillates between $1 \text{ and } -1$ as $x \to \pm\infty$

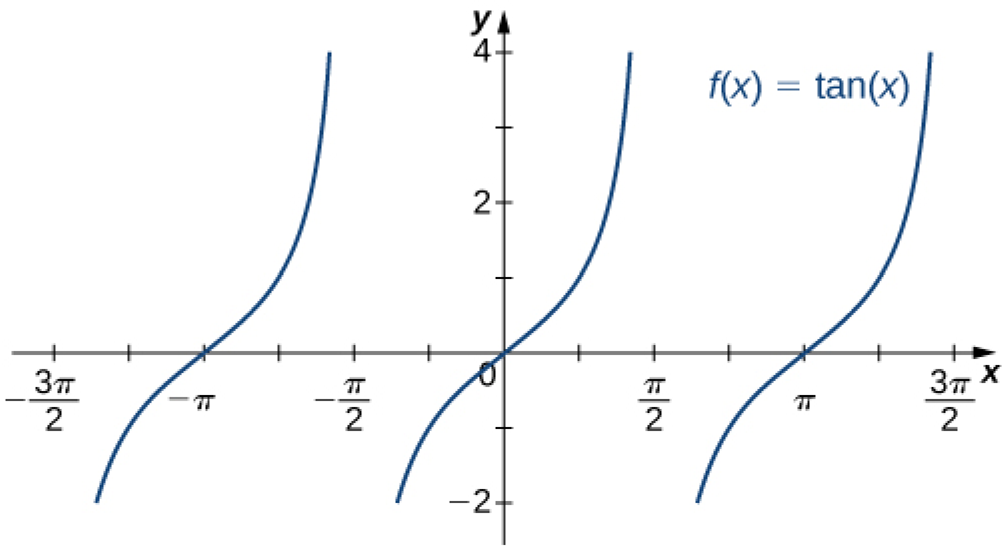



**Figure 4.59** The function $f(x) = \tan x$ does not approach a limit and does not approach $\pm\infty$ as $x \to \pm\infty$

Recall that for any base $b > 0, b \neq 1,$ the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1, y = b^x$ is increasing over `$(-\infty, \infty)$. If $0 < b < 1,$ $y = b^x$ is decreasing over $(-\infty, \infty)$. For the natural exponential function $f(x) = e^x,$ $e \approx 2.718 > 1$. Therefore, $f(x) = e^x$ is increasing on `$(-\infty, \infty)$ and the range is `$(0, \infty)$. The exponential function $f(x) = e^x$ approaches $\infty$ as $x \to \infty$ and approaches $0$ as $x \to -\infty$ as shown in Table 4.5 and Figure 4.60.

| $x$ | $-5$ | $-2$ | $0$ | $2$ | $5$ |
|---|---|---|---|---|---|
| $e^x$ | 0.00674 | 0.135 | 1 | 7.389 | 148.413 |

**Table 4.5** End behavior of the natural exponential function

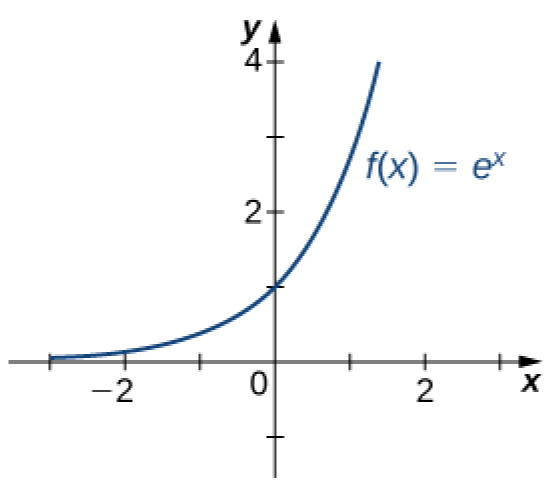


**Figure 4.60** The exponential function approaches zero as $x \to -\infty$ and approaches $\infty$ as $x \to \infty$.

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \to -\infty$ as $x \to 0^+$ and $\ln(x) \to \infty$ as $x \to \infty$ as shown in Figure 4.61 and Table 4.6.

| $x$ | 0.01 | 0.1 | 1 | 10 | 100 |
|---|---|---|---|---|---|
| $\ln(x)$ | −4.605 | −2.303 | 0 | 2.303 | 4.605 |

**Table 4.6** End behavior of the natural logarithm function

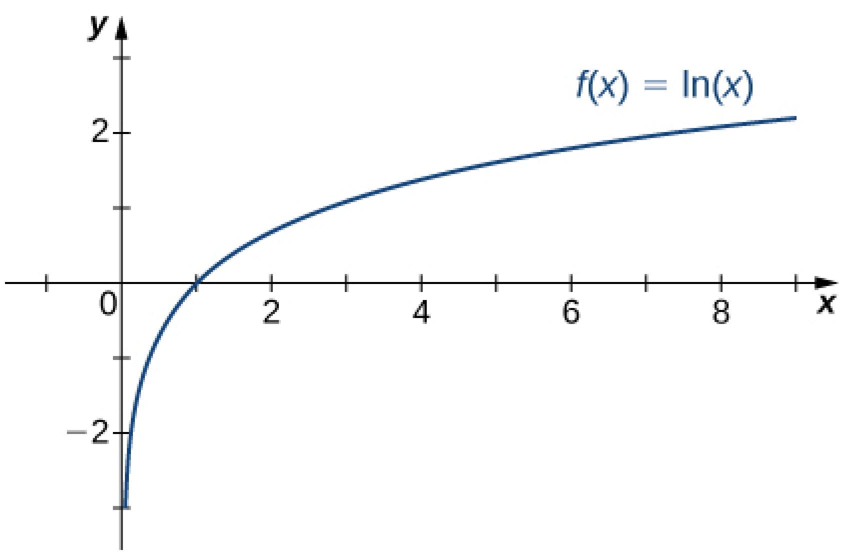



**Figure 4.61** The natural logarithm function approaches $\infty$ as $x \to \infty$.

## EXAMPLE 4.27

### Determining End Behavior for a Transcendental Function

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{(2+3e^x)}{(7-5e^x)}$ and describe the end behavior of $f$.

[Show Solution]

**CHECKPOINT 4.26**

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \frac{\left(3e^{x} - 4\right)}{\left(5e^{x} + 2\right)}$.

## Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

**PROBLEM-SOLVING STRATEGY: DRAWING THE GRAPH OF A FUNCTION**

Given a function $f,$ use the following steps to sketch a graph of $f$:

1. Determine the domain of the function.
2. Locate the $x$- and $y$-intercepts.
3. Evaluate $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number $L,$ then $y = L$ is a horizontal asymptote. If either of these limits is $\infty$ or $-\infty,$ determine whether $f$ has an oblique asymptote. If $f$ is a rational function such that $f(x) = \frac{p(x)}{q(x)},$ where the degree of the numerator is greater than the degree of the denominator, then $f$ can be written as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

   where the degree of $r(x)$ is less than the degree of $q(x).$ The values of $f(x)$ approach the values of $g(x)$ as $x \to \pm\infty.$ If $g(x)$ is a linear function, it is known as an *oblique asymptote*.
4. Determine whether $f$ has any vertical asymptotes.

5. Calculate $f'$. Find all critical points and determine the intervals where $f$ is increasing and where $f$ is decreasing. Determine whether $f$ has any local extrema.
6. Calculate $f''$. Determine the intervals where $f$ is concave up and where $f$ is concave down. Use this information to determine whether $f$ has any inflection points. The second derivative can also be used as an alternate means to determine or verify that $f$ has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

**EXAMPLE 4.28**

### Sketching a Graph of a Polynomial

Sketch a graph of $f(x) = (x - 1)^2(x + 2)$.

[Show Solution]

**CHECKPOINT 4.27**

Sketch a graph of $f(x) = (x - 1)^3(x + 2)$.

**EXAMPLE 4.29**

### Sketching a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2}{\left(1-x^2\right)}$.

[Show Solution]

### CHECKPOINT 4.28

Sketch a graph of $f(x) = \frac{(3x+5)}{(8+4x)}$.

### EXAMPLE 4.30

#### Sketching a Rational Function with an Oblique Asymptote

Sketch the graph of $f(x) = \frac{x^2}{(x-1)}$

[Show Solution]

### CHECKPOINT 4.29

Find the oblique asymptote for $f(x) = \frac{\left(3x^3-2x+1\right)}{\left(2x^2-4\right)}$.

### EXAMPLE 4.31

#### Sketching the Graph of a Function with a Cusp

Sketch a graph of $f(x) = (x-1)^{2/3}$.

[Show Solution]

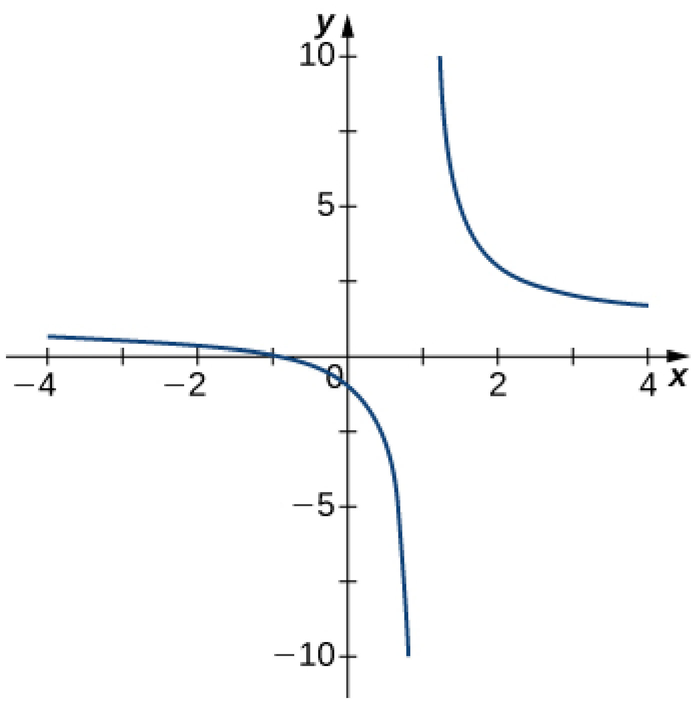
CHECKPOINT 4.30

Consider the function $f(x) = 5 - x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of $f$.

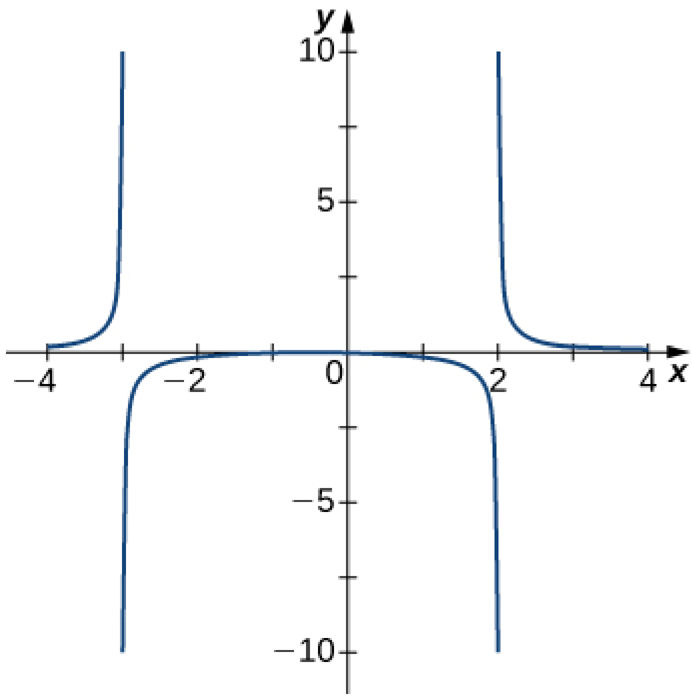
## Section 4.6 Exercises

For the following exercises, examine the graphs. Identify where the vertical asymptotes are located.

251.




252.

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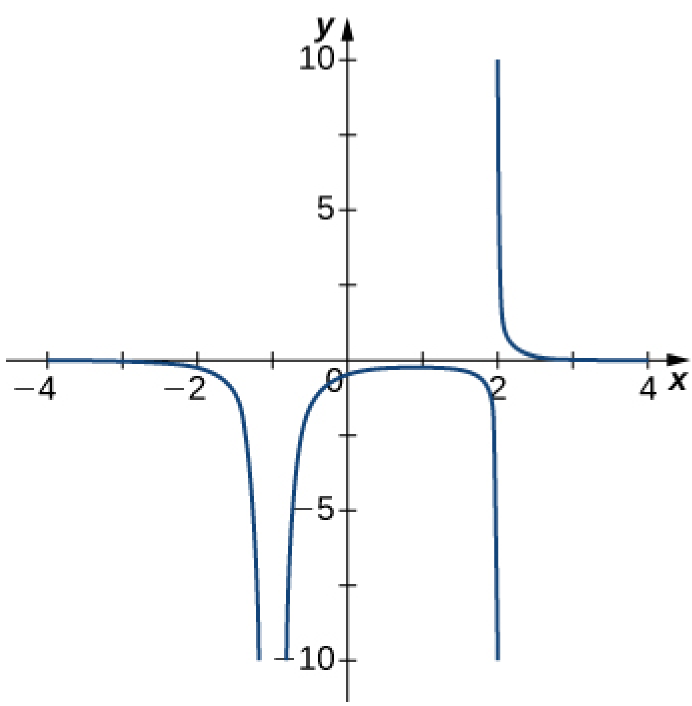


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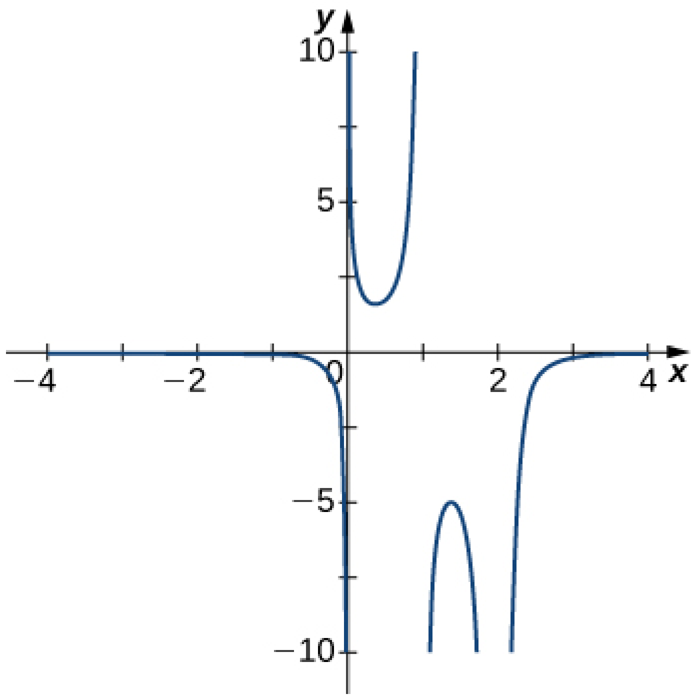


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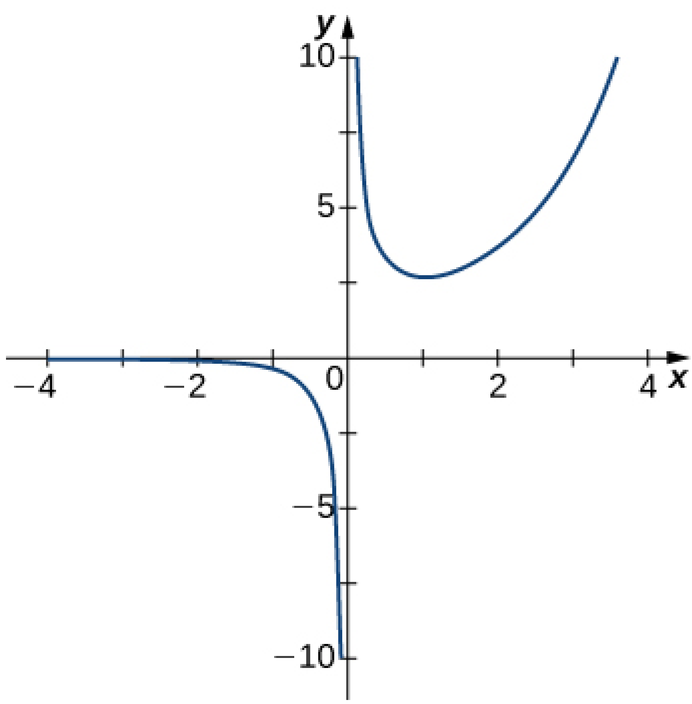



For the following functions $f(x)$, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

256. $f(x) = \frac{x+1}{x^2+5x+4}$, $a = -1$

257. $f(x) = \frac{x}{x-2}$, $a = 2$

258. $f(x) = (x+2)^{3/2}$, $a = -2$

259. $f(x) = (x-1)^{-1/3}$, $a = 1$

260. $f(x) = 1 + x^{-2/5}$, $a = 1$

For the following exercises, evaluate the limit.

261. $\lim\limits_{x\to\infty} \frac{1}{3x+6}$

262. $\lim\limits_{x\to\infty} \frac{2x-5}{4x}$

263. $\lim\limits_{x\to\infty} \frac{x^2-2x+5}{x+2}$

264. $\lim\limits_{x\to-\infty} \frac{3x^3-2x}{x^2+2x+8}$

265. $\lim\limits_{x\to-\infty} \frac{x^4-4x^3+1}{2-2x^2-7x^4}$

266. $\lim\limits_{x\to\infty} \frac{3x}{\sqrt{x^2+1}}$

267. $\lim\limits_{x\to-\infty} \frac{\sqrt{4x^2-1}}{x+2}$

268. $\lim\limits_{x\to\infty} \frac{4x}{\sqrt{x^2-1}}$

269. $\lim\limits_{x\to-\infty} \frac{4x}{\sqrt{x^2-1}}$

270. $\lim\limits_{x\to\infty} \frac{2\sqrt{x}}{x-\sqrt{x}+1}$

For the following exercises, find the horizontal and vertical asymptotes.

271. $f(x) = x - \frac{9}{x}$

272. $f(x) = \frac{1}{1-x^2}$

273. $f(x) = \frac{x^3}{4-x^2}$

274. $f(x) = \frac{x^2+3}{x^2+1}$

275. $f(x) = \sin(x)\sin(2x)$

276. $f(x) = \cos x + \cos(3x) + \cos(5x)$

277. $f(x) = \frac{x\sin(x)}{x^2-1}$

278. $f(x) = \frac{x}{\sin(x)}$

279. $f(x) = \frac{1}{x^3+x^2}$

280. $f(x) = \frac{1}{x-1} - 2x$

281. $f(x) = \frac{x^3+1}{x^3-1}$

282. $f(x) = \frac{\sin x+\cos x}{\sin x-\cos x}$

283. $f(x) = x - \sin x$

284. $f(x) = \frac{1}{x} - \sqrt{x}$

For the following exercises, construct a function $f(x)$ that has the given asymptotes.

285. $x = 1$ and $y = 2$

286. $x = 1$ and $y = 0$

287. $y = 4, \ x = -1$

288. $x = 0$

For the following exercises, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

289. **[T]** $f(x) = \frac{1}{x+10}$

290. **[T]** $f(x) = \frac{x+1}{x^2+7x+6}$

291. **[T]** $\lim_{x \to -\infty} x^2 + 10x + 25$

292. **[T]** $\lim_{x \to -\infty} \frac{x+2}{x^2+7x+6}$

293. **[T]** $\lim_{x \to \infty} \frac{3x+2}{x+5}$

For the following exercises, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

294. $y = 3x^2 + 2x + 4$

295. $y = x^3 - 3x^2 + 4$

296. $y = \frac{2x+1}{x^2+6x+5}$

297. $y = \frac{x^3+4x^2+3x}{3x+9}$

298. $y = \frac{x^2+x-2}{x^2-3x-4}$

299. $y = \sqrt{x^2 - 5x + 4}$

300. $y = 2x\sqrt{16 - x^2}$

301. $y = \frac{\cos x}{x}$, on $x = [-2\pi, 2\pi]$

302. $y = e^x - x^3$

303. $y = x\tan x, x = [-\pi, \pi]$

304. $y = x\ln(x), x > 0$

305. $y = x^2\sin(x), x = [-2\pi, 2\pi]$

306. For $f(x) = \frac{P(x)}{Q(x)}$ to have an asymptote at $y = 2$ then the polynomials $P(x)$ and $Q(x)$ must have what relation?

307. For $f(x) = \frac{P(x)}{Q(x)}$ to have an asymptote at $x = 0,$ then the polynomials $P(x)$ and $Q(x).$ must have what relation?

308. If $f'(x)$ has asymptotes at $y = 3$ and $x = 1,$ then $f(x)$ has what asymptotes?

309. Both $f(x) = \frac{1}{(x-1)}$ and $g(x) = \frac{1}{(x-1)^2}$ have asymptotes at $x = 1$ and $y = 0.$ What is the most obvious difference between these two functions?

310. True or false: Every ratio of polynomials has vertical asymptotes.

## Learning Objectives

4.7.1. Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

## Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the **optimization problems** that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 4.32, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let’s look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

**EXAMPLE 4.32**

### Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 4.62). Given $100$ ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

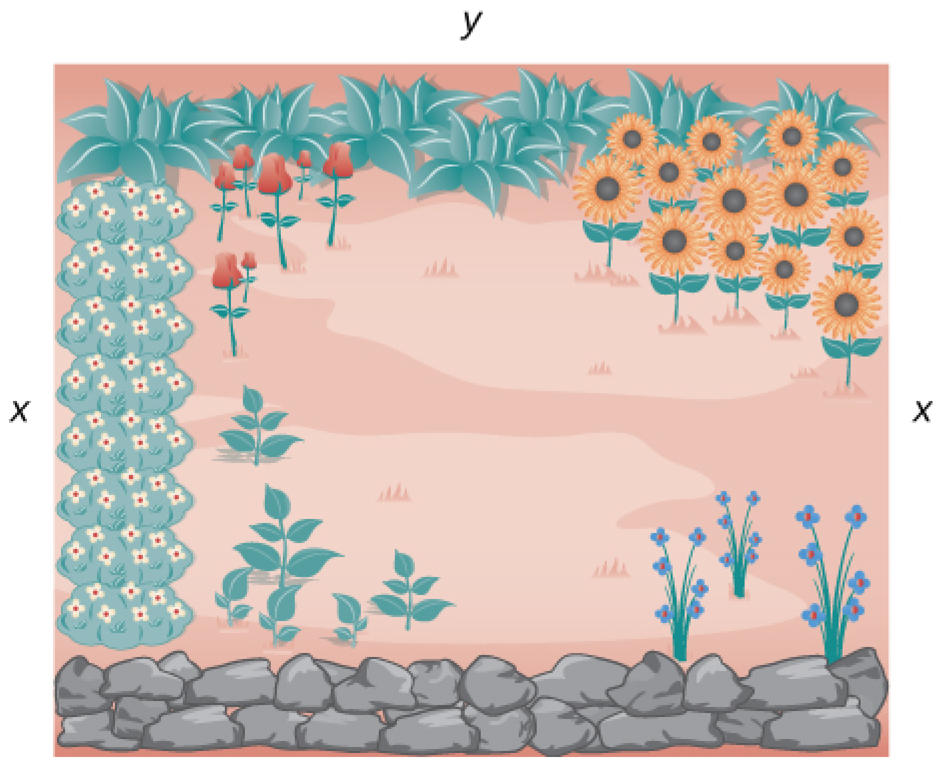



**Figure 4.62** We want to determine the measurements $x$ and $y$ that will create a garden with a maximum area using $100$ ft of fencing.

[Show Solution]

### CHECKPOINT 4.31

Determine the maximum area if we want to make the same rectangular garden as in Figure 4.63, but we have $200$ ft of fencing.

Now let's look at a general strategy for solving optimization problems similar to Example 4.32.

### PROBLEM-SOLVING STRATEGY: SOLVING OPTIMIZATION PROBLEMS

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step $3$. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step $4$ based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step $4$. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

**EXAMPLE 4.33**

### Maximizing the Volume of a Box

An open-top box is to be made from a $24$ in. by $36$ in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

[Show Solution]

**MEDIA**

Watch a video about optimizing the volume of a box.

### CHECKPOINT 4.32

Suppose the dimensions of the cardboard in Example 4.33 are 20 in. by 30 in. Let $x$ be the side length of each square and write the volume of the open-top box as a function of $x$. Determine the domain of consideration for $x$.

### EXAMPLE 4.34

#### Minimizing Travel Time

An island is $2\ \text{mi}$ due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is $6\ \text{mi}$ west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of $8\ \text{mph}$ and swims at a rate of $3\ \text{mph}$. How far should the visitor run before swimming to minimize the time it takes to reach the island?

**[Show Solution]**

### CHECKPOINT 4.33

Suppose the island is $1$ mi from shore, and the distance from the cabin to the point on the shore closest to the island is $15\ \text{mi}$. Suppose a visitor swims at the rate of $2.5\ \text{mph}$ and runs at a rate of $6\ \text{mph}$. Let $x$ denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

In business, companies are interested in maximizing revenue. In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let’s use these data to determine the price the company should charge to maximize the amount of money it brings in.

### EXAMPLE 4.35

#### Maximizing Revenue

Owners of a car rental company have determined that if they charge customers $p$ dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars $n$ they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge $\$50$ per day or less, they will rent all their cars. If they charge $\$200$ per day or more, they will not rent any cars. Assuming the owners plan to charge customers between $50 per day and $\$200$ per day to rent a car, how much should they charge to maximize their revenue?

[Show Solution]

### CHECKPOINT 4.34

A car rental company charges its customers $p$ dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

### EXAMPLE 4.36

#### Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

[Show Solution]

**CHECKPOINT 4.35**

Modify the area function $A$ if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

## Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of $4$ at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

**EXAMPLE 4.37**

**Minimizing Surface Area**

A rectangular box with a square base, an open top, and a volume of $216$ in.$^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

[Show Solution]

**CHECKPOINT 4.36**

Consider the same open-top box, which is to have volume $216 \text{ in.}^3$. Suppose the cost of the material for the base is $20¢/\text{in.}^2$ and the cost of the material for the sides is $30¢/\text{in.}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let $x$ be the side length of the base and $y$ be the height of the box.)

## Section 4.7 Exercises

For the following exercises, answer by proof, counterexample, or explanation.

311. When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

312. Why do you need to check the endpoints for optimization problems?

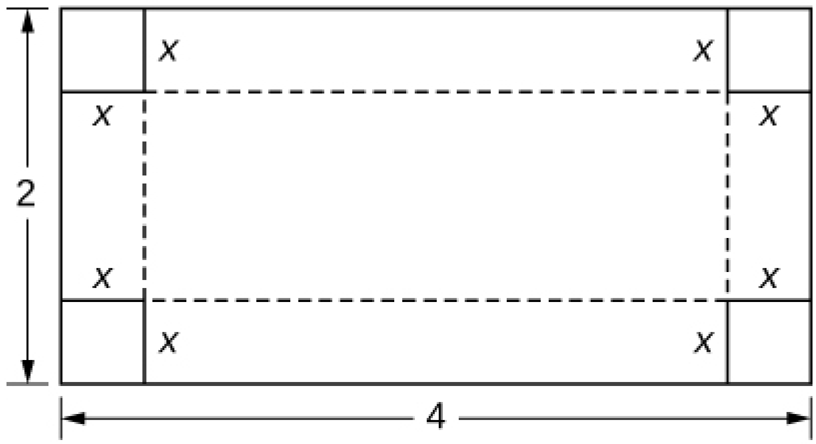
313. *True or False*. For every continuous nonlinear function, you can find the value $x$ that maximizes the function.

314. *True or False*. For every continuous nonconstant function on a closed, finite domain, there exists at least one $x$ that minimizes or maximizes the function.

For the following exercises, set up and evaluate each optimization problem.

315. To carry a suitcase on an airplane, the length $+\text{width}+$ height of the box must be less than or equal to $62 \text{ in}$. Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \left(\frac{1}{2}\right) h\right)^2$. What height allows you to have the largest volume?

316. You are constructing a cardboard box with the dimensions $2 \text{ m by } 4 \text{ m}$. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?




317. Find the positive integer that minimizes the sum of the number and its reciprocal.

318. Find two positive integers such that their sum is $10$, and minimize and maximize the sum of their squares.

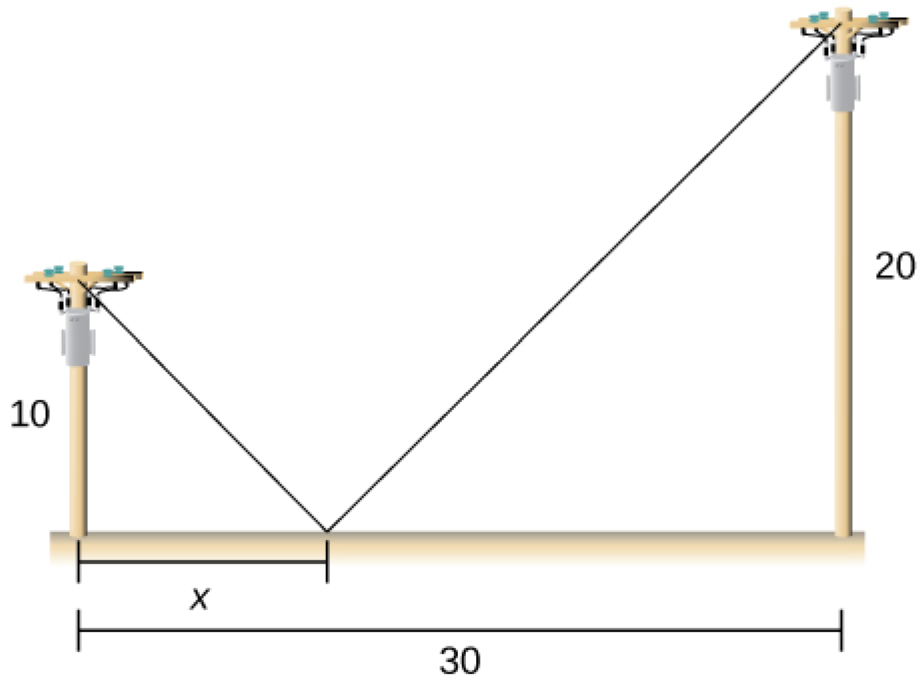
For the following exercises, consider the construction of a pen to enclose an area.

319. You have $400\text{ ft}$ of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

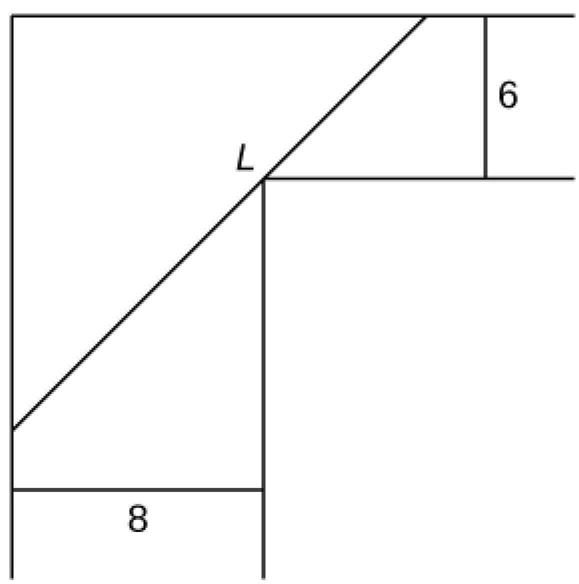
320. You have $800\text{ ft}$ of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

321. You need to construct a fence around an area of $1600\text{ ft}$. What are the dimensions of the rectangular pen to minimize the amount of material needed?

322. Two poles are connected by a wire that is also connected to the ground. The first pole is $20\text{ ft}$ tall and the second pole is $10\text{ ft}$ tall. There is a distance of $30\text{ ft}$ between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



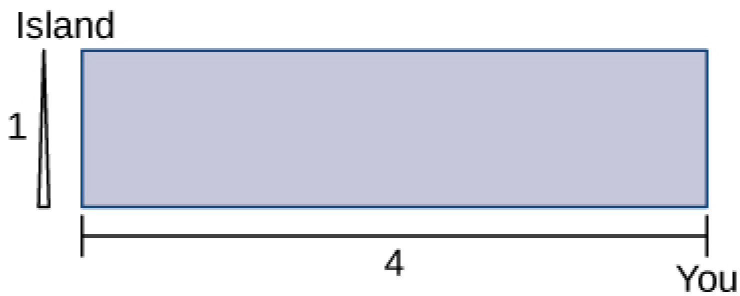

323. **[T]** You are moving into a new apartment and notice there is a corner where the hallway narrows from $8\text{ ft}$ to $6\text{ ft}$. What is the length of the longest item that can be carried horizontally around the corner?



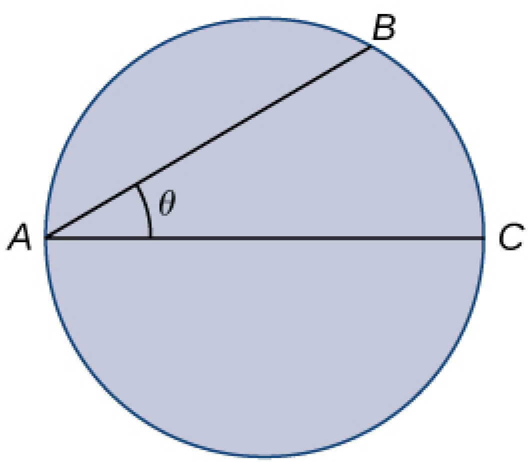

324. A patient's pulse measures $70\text{ bpm}$, $80\text{ bpm}$, then $120\text{ bpm}$. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2+(x-80)^2+(x-120)^2$? What value minimizes it?

325. In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

326. You can run at a speed of $6$ mph and swim at a speed of $3$ mph and are located on the shore, $4$ miles east of an island that is $1$ mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?




For the following problems, consider a lifeguard at a circular pool with diameter $40$ m. He must reach someone who is drowning on the exact opposite side of the pool, at position $C$. The lifeguard swims with a speed $v$ and runs around the pool at speed $w=3v$.




327. Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, $\theta$.

328. Find at what angle $\theta$ the lifeguard should swim to reach the drowning person in the least amount of time.

329. A truck uses gas as $g(v)=av+\frac{b}{v}$, where $v$ represents the speed of the truck and $g$ represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

For the following exercises, consider a limousine that gets $m(v) = \frac{(120-2v)}{5}$ mi/gal at speed $v$, the chauffeur costs $\$15/\text{h}$, and gas is $\$3.5/\text{gal}$.

330. Find the cost per mile at speed $v$.

331. Find the cheapest driving speed.

For the following exercises, consider a pizzeria that sell pizzas for a revenue of $R(x) = ax$ and costs $C(x) = b + cx + dx^2$, where $x$ represents the number of pizzas.

332. Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

333. Assume that $R(x) = 10x$ and $C(x) = 2x + x^2$. How many pizzas sold maximizes the profit?

334. Assume that $R(x) = 15x$, and $C(x) = 60 + 3x + \frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For the following exercises, consider a wire $4$ ft long cut into two pieces. One piece forms a circle with radius $r$ and the other forms a square of side $x$.

335. Choose $x$ to maximize the sum of their areas.

336. Choose $x$ to minimize the sum of their areas.

For the following exercises, consider two nonnegative numbers $x$ and $y$ such that $x + y = 10$. Maximize and minimize the quantities.

337. $xy$

338. $x^2y^2$

339. $y - \frac{1}{x}$

340. $x^2 - y$

For the following exercises, draw the given optimization problem and solve.

341. Find the volume of the largest right circular cylinder that fits in a sphere of radius $1$.

342. Find the volume of the largest right cone that fits in a sphere of radius $1$.

343. Find the area of the largest rectangle that fits into the triangle with sides $x = 0$, $y = 0$ and $\frac{x}{4} + \frac{y}{6} = 1$.

344. Find the largest volume of a cylinder that fits into a cone that has base radius $R$ and height $h$.

345. Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

346. Find the dimensions of a right cone with surface area $S = 4\pi$ that has the largest volume.

For the following exercises, consider the points on the given graphs. Use a calculator to graph the functions.

347. **[T]** Where is the line $y = 5 - 2x$ closest to the origin?

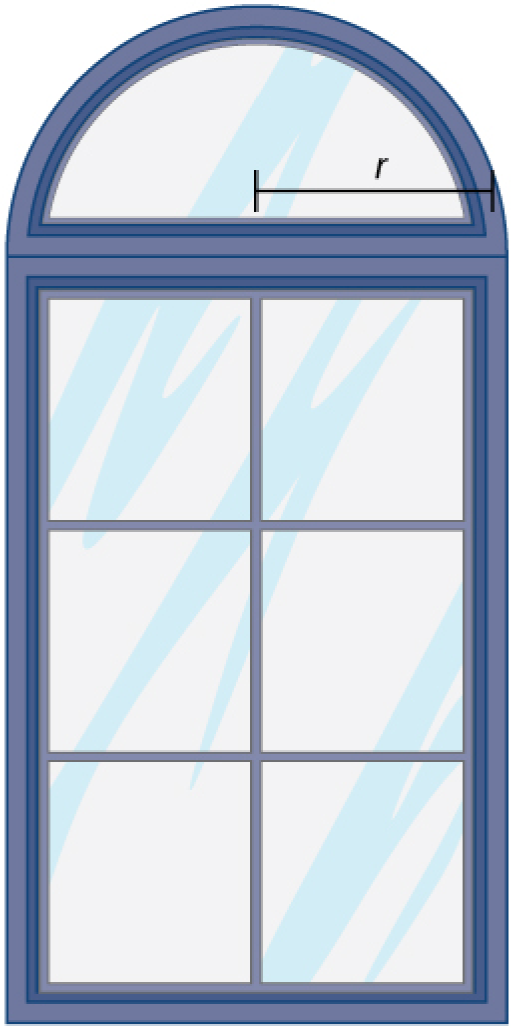
348. **[T]** Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

349. **[T]** Where is the parabola $y = x^2$ closest to point $(2, 0)$?

350. **[T]** Where is the parabola $y = x^2$ closest to point $(0, 3)$?

For the following exercises, set up, but do not evaluate, each optimization problem.

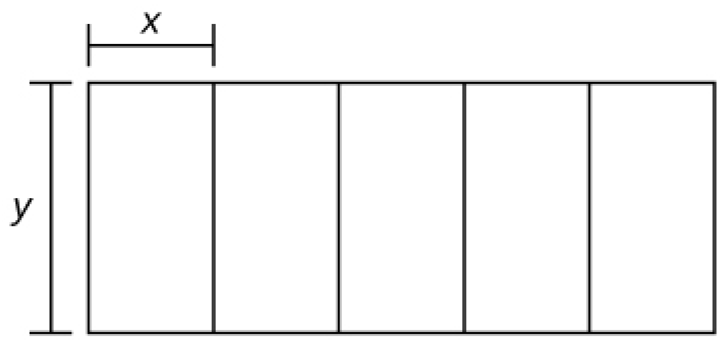
351. A window is composed of a semicircle placed on top of a rectangle. If you have $20 \text{ ft}$ of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use $r$ to represent the radius of the semicircle.




352. You have a garden row of $20$ watermelon plants that produce an average of $30$ watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

353. You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume $4\ \text{ft}^2$. Find the dimensions of the box that minimize cost. Use $x$ to represent the length of the side of the box.

354. You are building five identical pens adjacent to each other with a total area of $1000\ \text{m}^2$, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?




355. You are the manager of an apartment complex with $50$ units. When you set rent at $\$800/\text{month},$ all apartments are rented. As you increase rent by $\$25/\text{month},$ one fewer apartment is rented. Maintenance costs run $\$50/\text{month}$ for each occupied unit. What is the rent that maximizes the total amount of profit?