Chapter Outline

Figure 4.1 Americans use (and lose) millions of golf balls a year, which keeps golf ball manufacturers in business. In this chapter, we study a profit model and learn methods for calculating optimal production levels for a typical golf ball manufacturing company. (credit: modification of work by oatsy40, Flickr)

In Introduction to Applications of Derivatives, we studied how to determine the maximum and minimum of a function of one variable over a closed interval. This function might represent the temperature over a given time interval, the position of a car as a function of time, or the altitude of a jet plane as it travels from New York to San Francisco. In each of these examples, the function has one independent variable.

Suppose, however, that we have a quantity that depends on more than one variable. For example, temperature can depend on location and the time of day, or a company's profit model might depend on the number of units sold and the amount of money spent on advertising. In this chapter, we look at a company that produces golf balls. We develop a profit model and, under various restrictions, we find that the optimal level of production and advertising dollars spent determines the maximum possible profit. Depending on the nature of the restrictions, both the method of solution and the solution itself changes (see Example 4.41).

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of *derivative* we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

Learning Objectives

4.1.1. Recognize a function of two variables and identify its domain and range.
4.1.2. Sketch a graph of a function of two variables.
4.1.3. Sketch several traces or level curves of a function of two variables.
4.1.4. Recognize a function of three or more variables and identify its level surfaces.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

> **DEFINITION**
>
> A **function of two variables** $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the *domain* of the function. The *range* of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in the following figure.

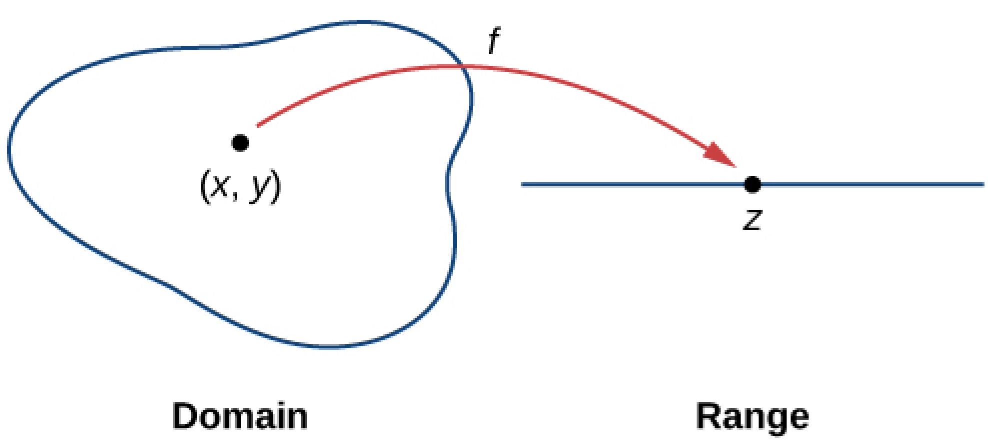


Figure 4.2 The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let’s take a look.

EXAMPLE 4.1

Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$
b. $g(x, y) = \sqrt{9 - x^2 - y^2}$

[Show Solution]

CHECKPOINT 4.1

Find the domain and range of the function $f(x, y) = \sqrt{36 - 9x^2 - 9y^2}$.

Graphing Functions of Two Variables

Suppose we wish to graph the function $z = (x, y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y = f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x, y) in the plane, and every point in the plane has an ordered pair (x, y) associated with it. With a function of two variables, each ordered pair (x, y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x, y, z). The graph of a function $z = (x, y)$ of two variables is called a **surface**.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x, y) coordinate system laying flat. Then, every point in the domain of the function f has a unique z-value associated with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

EXAMPLE 4.2

Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x, y) = \sqrt{9 - x^2 - y^2}$
b. $f(x, y) = x^2 + y^2$

[Show Solution]

EXAMPLE 4.3

Nuts and Bolts

A profit function for a hardware manufacturer is given by

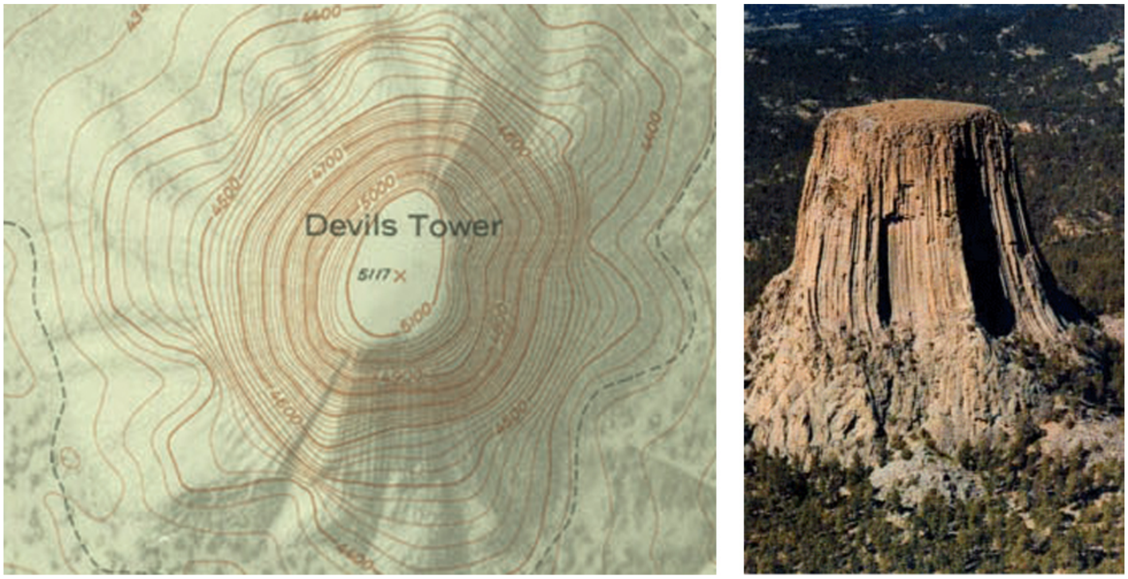
$$f(x, y) = 16 - (x - 3)^2 - (y - 2)^2,$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month (measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

[Show Solution]

Level Curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A topographical map contains curved lines called *contour lines*. Each contour line corresponds to the points on the map that have equal elevation (Figure 4.7). A level curve of a function of two variables $f(x, y)$ is completely analogous to a contour line on a topographical map.



(a) (b)

Figure 4.7 (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

DEFINITION

Given a function $f(x, y)$ and a number c in the range of $f,$ a **level curve of a function of two variables** for the value c is defined to be the set of points satisfying the equation $f(x, y) = c.$

Returning to the function $g(x, y) = \sqrt{9 - x^2 - y^2},$ we can determine the level curves of this function. The range of g is the closed interval $[0, 3].$ First, we choose any number in this closed interval—say, $c = 2.$ The level curve corresponding to $c = 2$ is described by the equation

$$\sqrt{9 - x^2 - y^2} = 2.$$

To simplify, square both sides of this equation:

$$9 - x^2 - y^2 = 4.$$

Now, multiply both sides of the equation by -1 and add 9 to each side:

$$x^2 + y^2 = 5.$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c = 3$, then the circle has radius 0, so it consists solely of the origin. Figure 4.8 is a graph of the level curves of this function corresponding to $c = 0, 1, 2,$ and 3. Note that in the previous derivation it may be possible that we introduced extra solutions by squaring both sides. This is not the case here because the range of the square root function is nonnegative.

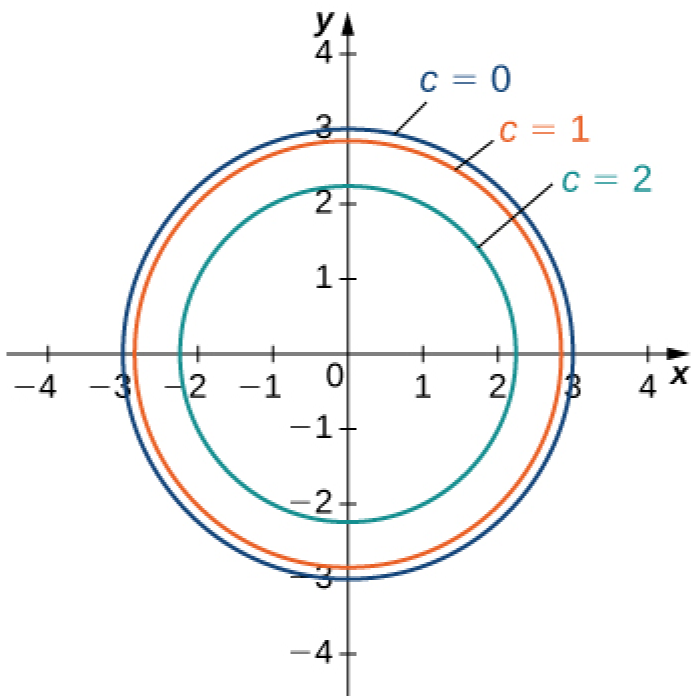


Figure 4.8 Level curves of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, using $c = 0, 1, 2,$ and 3 ($c = 3$ corresponds to the origin).

A graph of the various level curves of a function is called a **contour map**.

EXAMPLE 4.4

Making a Contour Map

Given the function $f(x, y) = \sqrt{8 + 8x - 4y - 4x^2 - y^2}$, find the level curve corresponding to $c = 0$. Then create a contour map for this function. What are the domain and range of f?

[Show Solution]

CHECKPOINT 4.2

Find and graph the level curve of the function $g(x, y) = x^2 + y^2 - 6x + 2y$ corresponding to $c = 15$.

Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the xy-plane, but as their name implies, vertical traces are graphed in the xz- or yz-planes.

DEFINITION

Consider a function $z = f(x, y)$ with domain $D \subseteq \mathbb{R}^2$. A **vertical trace** of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b$.

EXAMPLE 4.5

Finding Vertical Traces

Find vertical traces for the function $f(x, y) = \sin x \cos y$ corresponding to $x = -\frac{\pi}{4}, 0, \text{ and } \frac{\pi}{4}$, and $y = -\frac{\pi}{4}, 0, \text{ and } \frac{\pi}{4}$.

[Show Solution]

CHECKPOINT 4.3

Determine the equation of the vertical trace of the function $g(x, y) = -x^2 - y^2 + 2x + 4y - 1$ corresponding to $y = 3$, and describe its graph.

Functions of two variables can produce some striking-looking surfaces. The following figure shows two examples.

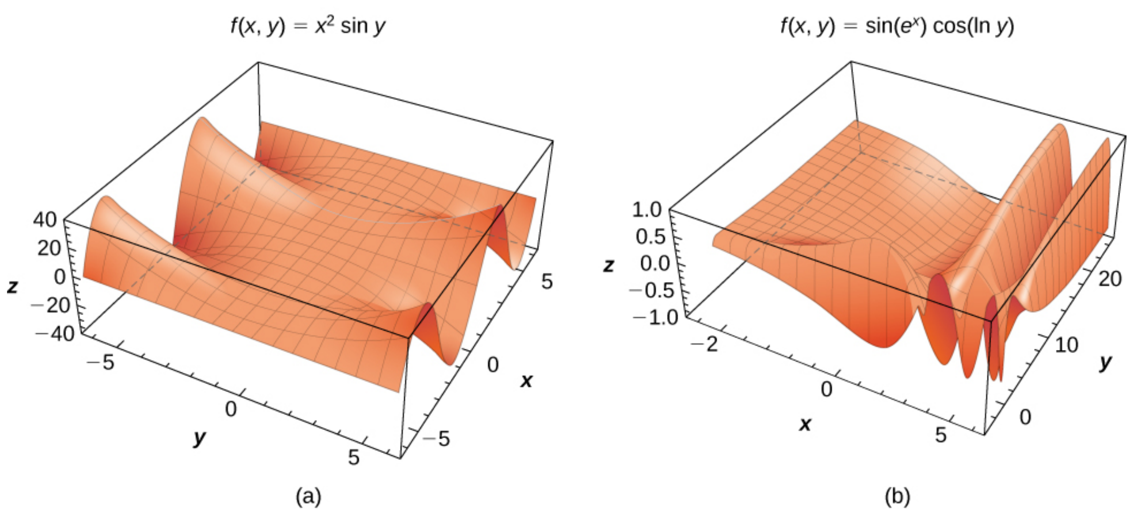


Figure 4.12 Examples of surfaces representing functions of two variables: (a) a combination of a power function and a sine function and (b) a combination of trigonometric, exponential, and logarithmic functions.

Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$f(x, y, z) = x^2 - 2xy + y^2 + 3yz - z^2 + 4x - 2y + 3x - 6 \text{ (a polynomial in three variables)}$$

and

$$g(x, y, t) = \left(x^2 - 4xy + y^2\right)\sin t - (3x + 5y)\cos t.$$

In the first function, (x, y, z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x, y) can represent a point in the plane, and t can represent time. The function might map a point in the plane to a third quantity (for example, pressure) at a given time

$t.$ The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

EXAMPLE 4.6

Domains for Functions of Three Variables

Find the domain of each of the following functions:

a. $f(x, y, z) = \frac{3x - 4y + 2z}{\sqrt{9 - x^2 - y^2 - z^2}}$

b. $g(x, y, t) = \frac{\sqrt{2t - 4}}{x^2 - y^2}$

[Show Solution]

CHECKPOINT 4.4

Find the domain of the function $h(x, y, t) = (3t - 6)\sqrt{y - 4x^2 + 4}.$

Functions of two variables have level curves, which are shown as curves in the $xy\text{-plane}.$ However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

DEFINITION

Given a function $f(x, y, z)$ and a number c in the range of $f,$ a **level surface of a function of three variables** is defined to be the set of points satisfying the equation $f(x, y, z) = c.$

EXAMPLE 4.7

Finding a Level Surface

Find the level surface for the function $f(x, y, z) = 4x^2 + 9y^2 - z^2$ corresponding to $c = 1$.

[Show Solution]

CHECKPOINT 4.5

Find the equation of the level surface of the function

$$g(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y - 6z$$

corresponding to $c = 2$, and describe the surface, if possible.

Section 4.1 Exercises

For the following exercises, evaluate each function at the indicated values.

1. $W(x, y) = 4x^2 + y^2$. Find $W(2, -1)$, $W(-3, 6)$.

2. $W(x, y) = 4x^2 + y^2$. Find $W(2 + h, 3 + h)$.

3. The volume of a right circular cylinder is calculated by a function of two variables, $V(x, y) = \pi x^2 y$, where x is the radius of the right circular cylinder and y represents the height of the cylinder. Evaluate $V(2, 5)$ and explain what this means.

4. An oxygen tank is constructed of a right cylinder of height y and radius x with two hemispheres of radius x mounted on the top and bottom of the cylinder. Express the volume of the tank as a function of two variables, x and y, find $V(10, 2)$, and explain what this means.

For the following exercises, find the domain of the function.

5. $V(x, y) = 4x^2 + y^2$

6. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

7. $f(x, y) = 4\ln(y^2 - x)$

8. $g(x, y) = \sqrt{16 - 4x^2 - y^2}$

9. $z(x, y) = y^2 - x^2$

10. $f(x, y) = \frac{y+2}{x^2}$

Find the range of the functions.

11. $g(x, y) = \sqrt{16 - 4x^2 - y^2}$

12. $V(x, y) = 4x^2 + y^2$

13. $z = y^2 - x^2$

For the following exercises, find the level curves of each function at the indicated value of c to visualize the given function.

14. $z(x, y) = y^2 - x^2, \;\; c = 1$

15. $z(x, y) = y^2 - x^2, \;\; c = 4$

16. $g(x, y) = x^2 + y^2; c = 4, c = 9$

17. $g(x, y) = 4 - x - y; c = 0, 4$

18. $f(x, y) = xy; c = 1; c = -1$

19. $h(x, y) = 2x - y; c = 0, -2, 2$

20. $f(x, y) = x^2 - y; c = 1, 2$

21. $g(x, y) = \frac{x}{x+y}; c = -1, 0, 2$

22. $g(x, y) = x^3 - y; c = -1, 0, 2$

23. $g(x, y) = e^{xy}; c = \frac{1}{2}, 3$

24. $f(x, y) = x^2; c = 4, 9$

25. $f(x, y) = xy - x; c = -2, 0, 2$

26. $h(x, y) = \ln(x^2 + y^2); c = -1, 0, 1$

27. $g(x, y) = \ln\left(\frac{y}{x^2}\right); c = -2, 0, 2$

28. $z = f(x, y) = \sqrt{x^2 + y^2},\ c = 3$

29. $f(x, y) = \frac{y+2}{x^2},\ c =$ any constant

For the following exercises, find the vertical traces of the functions at the indicated values of x and y, and plot the traces.

30. $z = 4 - x - y; x = 2$

31. $f(x, y) = 3x + y^3, x = 1$

32. $z = \cos\sqrt{x^2 + y^2}\ x = 1$

Find the domain of the following functions.

33. $z = \sqrt{100 - 4x^2 - 25y^2}$

34. $z = \ln\left(x - y^2\right)$

35. $f(x, y, z) = \frac{1}{\sqrt{36 - 4x^2 - 9y^2 - z^2}}$

36. $f(x, y, z) = \sqrt{49 - x^2 - y^2 - z^2}$

37. $f(x, y, z) = \sqrt[3]{16 - x^2 - y^2 - z^2}$

38. $f(x, y) = \cos\sqrt{x^2 + y^2}$

For the following exercises, plot a graph of the function.

39. $z = f(x, y) = \sqrt{x^2 + y^2}$

40. $z = x^2 + y^2$

41. Use technology to graph $z = x^2y$.

Sketch the following by finding the level curves. Verify the graph using technology.

42. $f(x, y) = \sqrt{4 - x^2 - y^2}$

43. $f(x, y) = 2 - \sqrt{x^2 + y^2}$

44. $z = 1 + e^{-x^2 - y^2}$

45. $z = \cos\sqrt{x^2 + y^2}$

46. $z = y^2 - x^2$

47. Describe the contour lines for several values of c for $z = x^2 + y^2 - 2x - 2y$.

Find the level surface for the functions of three variables and describe it.

48. $w(x, y, z) = x - 2y + z, c = 4$

49. $w(x, y, z) = x^2 + y^2 + z^2, c = 9$

50. $w(x, y, z) = x^2 + y^2 - z^2, c = -4$

51. $w(x, y, z) = x^2 + y^2 - z^2, c = 4$

52. $w(x, y, z) = 9x^2 - 4y^2 + 36z^2, c = 0$

For the following exercises, find an equation of the level curve of f that contains the point P.

53. $f(x, y) = 1 - 4x^2 - y^2, P(0, 1)$

54. $g(x, y) = y^2 \arctan x, P(1, 2)$

55. $g(x, y) = e^{xy}(x^2 + y^2), P(1, 0)$

56. The strength E of an electric field at point (x, y, z) resulting from an infinitely long charged wire lying along the y-axis is given by $E(x, y, z) = k/\sqrt{x^2 + y^2},$ where k is a positive constant. For simplicity, let $k = 1$ and find the equations of the level surfaces for $E = 10 \text{ and } E = 100.$

57. A thin plate made of iron is located in the $xy\text{-plane}.$ The temperature T in degrees Celsius at a point $P(x, y)$ is inversely proportional to the square of its distance from the origin. Express T as a function of $x \text{ and } y.$

58. Refer to the preceding problem. Using the temperature function found there, determine the proportionality constant if the temperature at point $P(1, 2)\ is\ 50°\text{C}.$ Use this constant to determine the temperature at point $Q(3, 4).$

59. Refer to the preceding problem. Find the level curves for $T = 40°\text{C} \text{ and } T = 100°\text{C},$ and describe what the level curves represent.

Learning Objectives

4.2.1. Calculate the limit of a function of two variables.
4.2.2. Learn how a function of two variables can approach different values at a boundary point, depending on the path of approach.
4.2.3. State the conditions for continuity of a function of two variables.
4.2.4. Verify the continuity of a function of two variables at a point.
4.2.5. Calculate the limit of a function of three or more variables and verify the continuity of the function at a point.

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

Limit of a Function of Two Variables

Recall from The Limit of a Function the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0,$ there exists a $\delta > 0,$ such that if $0 < |x - a| < \delta$ for all x in the domain of $f,$ then

$$|f(x) - L| < \varepsilon.$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

DEFINITION

Consider a point $(a, b) \in \mathbb{R}^2.$ A **δ disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b)—that is,

$$\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, (x - a)^2 + (y - b)^2 < \delta^2\right\}$$

as shown in the following graph.

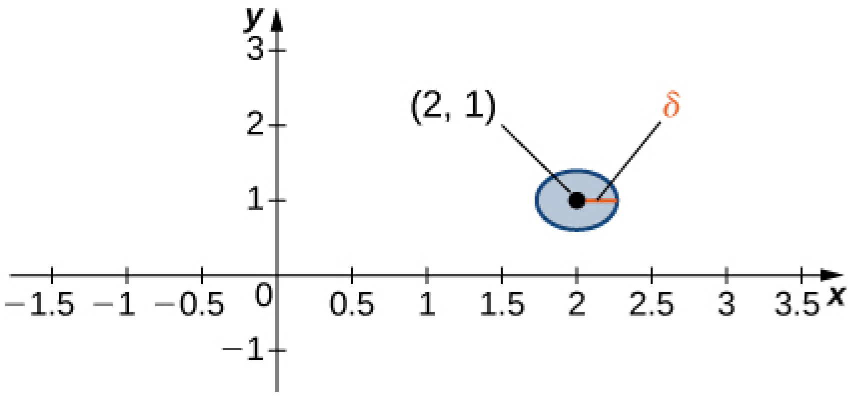


Figure 4.14 A δ disk centered around the point $(2, 1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x, y) in the δ disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - \delta < x < a + \delta.$$

In more than one dimension, we use a δ disk.

DEFINITION

Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is $L,$ written

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x, y) in a δ disk around $(a, b),$ except possibly for (a, b) itself, the value of $f(x, y)$ is no more than ε away from L (Figure 4.15). Using symbols, we write the following: For any $\varepsilon > 0,$ there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

Figure 4.15 The limit of a function involving two variables requires that $f(x, y)$ be within ε of L whenever (x, y) is within δ of (a, b). The smaller the value of ε, the smaller the value of δ.

Proving that a limit exists using the definition of a limit of a function of two variables can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem in The Limit Laws.

THEOREM 4.1

Limit laws for functions of two variables

Let $f(x, y)$ and $g(x, y)$ be defined for all $(x, y) \neq (a, b)$ in a neighborhood around (a, b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$\lim_{(x,y) \to (a,b)} f(x, y) = L$ and $\lim_{(x,y) \to (a,b)} g(x, y) = M,$ and let c be a constant.
Then each of the following statements holds:

Constant Law:

$$\lim_{(x,y) \to (a,b)} c = c \tag{4.2}$$

Identity Laws:

$$\lim_{(x,y) \to (a,b)} x = a \tag{4.3}$$

$$\lim_{(x,y) \to (a,b)} y = b \tag{4.4}$$

Sum Law:

$$\lim_{(x,y) \to (a,b)} (f(x, y) + g(x, y)) = L + M \tag{4.5}$$

Difference Law:

$$\lim_{(x,y) \to (a,b)} (f(x, y) - g(x, y)) = L - M \tag{4.6}$$

Constant Multiple Law:

$$\lim_{(x,y) \to (a,b)} (cf(x, y)) = cL \tag{4.7}$$

Product Law:

$$\lim_{(x,y) \to (a,b)} (f(x, y)g(x, y)) = LM \tag{4.8}$$

Quotient Law:

$$\lim_{(x,y) \to (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \text{ for } M \neq 0 \tag{4.9}$$

Power Law:

$$\lim_{(x,y) \to (a,b)} (f(x, y))^n = L^n \tag{4.10}$$

for any positive integer n.

Root Law:

$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} \qquad \boxed{4.11}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

EXAMPLE 4.8

Finding the Limit of a Function of Two Variables

Find each of the following limits:

a. $\lim_{(x,y)\to(2,-1)} \left(x^2 - 2xy + 3y^2 - 4x + 3y - 6\right)$

b. $\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$

[Show Solution]

CHECKPOINT 4.6

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2 - y}{y^2 + x - 1}}.$$

Since we are taking the limit of a function of two variables, the point (a, b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a, b). If this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

EXAMPLE 4.9

Limits That Fail to Exist

Show that neither of the following limits exist:

a. $\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2}$

b. $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$

[Show Solution]

CHECKPOINT 4.7

Show that

$$\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$$

does not exist.

Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

DEFINITION

Let S be a subset of $\mathbb{R}^2$ (Figure 4.17).

1. A point P_0 is called an **interior point** of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a **boundary point** of S if every δ disk centered around P_0 contains points both inside and outside S.

Figure 4.17 In the set S shown, $(-1, 1)$ is an interior point and $(2, 3)$ is a boundary point.

DEFINITION

Let S be a subset of $\mathbb{R}^2$ (Figure 4.17).

1. S is called an **open set** if every point of S is an interior point.
2. S is called a **closed set** if it contains all its boundary points.

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

DEFINITION

Let S be a subset of $\mathbb{R}^2$ (Figure 4.17).

1. An open set S is a **connected set** if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a **region** if it is open, connected, and nonempty.

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the $\delta\,\text{disk}$ is not contained inside the domain. By definition, some of the points of the $\delta\,\text{disk}$ are inside the domain and some are outside. Therefore, we need only consider points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

DEFINITION

Let f be a function of two variables, x and $y,$ and suppose (a, b) is on the boundary of the domain of $f.$ Then, the limit of $f(x, y)$ as (x, y) approaches (a, b) is $L,$ written

$$\lim_{(x,y) \to (a,b)} f(x, y) = L,$$

if for any $\varepsilon > 0,$ there exists a number $\delta > 0$ such that for any point (x, y) inside the domain of f and within a suitably small distance positive δ of $(a, b),$ the value of $f(x, y)$ is no more than ε away from L (Figure 4.15). Using symbols, we can write: For any $\varepsilon > 0,$ there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

EXAMPLE 4.10

Limit of a Function at a Boundary Point

Prove $\lim_{(x,y) \to (4,3)} \sqrt{25 - x^2 - y^2} = 0.$

[Show Solution]

CHECKPOINT 4.8

Evaluate the following limit:

$$\lim_{(x,y) \to (5,-2)} \sqrt{29 - x^2 - y^2}.$$

Continuity of Functions of Two Variables

In Continuity, we defined the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x = a$:

1. $f(a)$ exists.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

DEFINITION

A function $f(x, y)$ is continuous at a point (a, b) in its domain if the following conditions are satisfied:

1. $f(a, b)$ exists.

2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

EXAMPLE 4.11

Demonstrating Continuity for a Function of Two Variables

Show that the function $f(x,y) = \frac{3x+2y}{x+y+1}$ is continuous at point $(5,-3)$.

[Show Solution]

CHECKPOINT 4.9

Show that the function $f(x,y) = \sqrt{26 - 2x^2 - y^2}$ is continuous at point $(2,-3)$.

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point $\left(x_0, y_0\right)$ in its domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $\sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2} < \delta$ it is true, $|f(x,y) - f(a,b)| < \varepsilon$. This definition can be combined with the formal definition (that is, the *epsilon–delta definition*) of continuity of a function of one variable to prove the following theorems:

THEOREM 4.2

The Sum of Continuous Functions Is Continuous

If $f(x, y)$ is continuous at (x_0, y_0), and $g(x, y)$ is continuous at (x_0, y_0), then $f(x, y) + g(x, y)$ is continuous at (x_0, y_0).

THEOREM 4.3

The Product of Continuous Functions Is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).

THEOREM 4.4

The Composition of Continuous Functions Is Continuous

Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq \mathbb{R}$. Suppose g is continuous at some point $(x_0, y_0) \in D$ and define $z_0 = g(x_0, y_0)$. Let f be a function that maps $\mathbb{R}$ to $\mathbb{R}$ such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0, y_0) as shown in the following figure.

Figure 4.20 The composition of two continuous functions is continuous.

Let's now use the previous theorems to show continuity of functions in the following examples.

EXAMPLE 4.12

More Examples of Continuity of a Function of Two Variables

Show that the functions $f(x, y) = 4x^3y^2$ and $g(x, y) = \cos\left(4x^3y^2\right)$ are continuous everywhere.

[Show Solution]

CHECKPOINT 4.10

Show that the functions $f(x, y) = 2x^2y^3 + 3$ and $g(x, y) = \left(2x^2y^3 + 3\right)^4$ are continuous everywhere.

Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x, y, z)$ that gives the temperature at a physical location (x, y, z) in three dimensions. Or perhaps a function $g(x, y, z, t)$ can indicate air pressure at a location (x, y, z) at time $t.$ How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

DEFINITION

Let $\left(x_0, y_0, z_0\right)$ be a point in $\mathbb{R}^3.$ Then, a ***δ* ball** in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from $\left(x_0, y_0, z_0\right)$—that is,

$$\left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, \sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2 + \left(z - z_0\right)^2} < \delta\right\}.$$

To define a δ ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P = \left(w_0, x_0, y_0, z_0\right)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$\left\{(w,x,y,z) \in \mathbb{R}^4 \,\middle|\, \sqrt{\left(w - w_0\right)^2 + \left(x - x_0\right)^2 + \left(y - y_0\right)^2 + \left(z - z_0\right)^2} < \delta\right\}.$$

To show that a limit of a function of three variables exists at a point $\left(x_0, y_0, z_0\right)$, it suffices to show that for any point in a δ ball centered at $\left(x_0, y_0, z_0\right)$, the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

EXAMPLE 4.13

Finding the Limit of a Function of Three Variables

Find $\lim\limits_{(x,y,z) \to (4,1,-3)} \dfrac{x^2y - 3z}{2x + 5y - z}$.

[Show Solution]

CHECKPOINT 4.11

Find $\lim\limits_{(x,y,z) \to (4,-1,3)} \sqrt{13 - x^2 - 2y^2 + z^2}$.

Section 4.2 Exercises

For the following exercises, find the limit of the function.

60. $\lim_{(x,y)\to(1,2)} x$

61. $\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$

62. Show that the limit $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$ exists and is the same along the paths: y-axis and x-axis, and along $y = x$.

For the following exercises, evaluate the limits at the indicated values of x and y. If the limit does not exist, state this and explain why the limit does not exist.

63. $\lim_{(x,y)\to(0,0)} \frac{4x^2+10y^2+4}{4x^2-10y^2+6}$

64. $\lim_{(x,y)\to(11,13)} \sqrt{\frac{1}{xy}}$

65. $\lim_{(x,y)\to(0,1)} \frac{y^2\sin x}{x}$

66. $\lim_{(x,y)\to(0,0)} \sin\left(\frac{x^8+y^7}{x-y+10}\right)$

67. $\lim_{(x,y)\to(\pi/4,1)} \frac{y\tan x}{y+1}$

68. $\lim_{(x,y)\to(0,\pi/4)} \frac{\sec x+2}{3x-\tan y}$

69. $\lim_{(x,y)\to(2,5)} \left(\frac{1}{x}-\frac{5}{y}\right)$

70. $\lim_{(x,y)\to(4,4)} x\ln y$

71. $\lim_{(x,y)\to(4,4)} e^{-x^2-y^2}$

72. $\lim_{(x,y)\to(0,0)} \sqrt{9-x^2-y^2}$

73. $\lim_{(x,y)\to(1,2)} \left(x^2y^3-x^3y^2+3x+2y\right)$

74. $\lim\limits_{(x,y) \to (\pi,\pi)} x \sin\left(\frac{x+y}{4}\right)$

75. $\lim\limits_{(x,y) \to (0,0)} \frac{xy+1}{x^2+y^2+1}$

76. $\lim\limits_{(x,y) \to (0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

77. $\lim\limits_{(x,y) \to (0,0)} \ln\left(x^2+y^2\right)$

For the following exercises, complete the statement.

78. A point $\left(x_0, y_0\right)$ in a plane region R is an interior point of R if _______________.

79. A point $\left(x_0, y_0\right)$ in a plane region R is called a boundary point of R if __________.

For the following exercises, use algebraic techniques to evaluate the limit.

80. $\lim\limits_{(x,y) \to (2,1)} \frac{x-y-1}{\sqrt{x-y}-1}$

81. $\lim\limits_{(x,y) \to (0,0)} \frac{x^4-4y^4}{x^2+2y^2}$

82. $\lim\limits_{(x,y) \to (0,0)} \frac{x^3-y^3}{x-y}$

83. $\lim\limits_{(x,y) \to (0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

For the following exercises, evaluate the limits of the functions of three variables.

84. $\lim\limits_{(x,y,z) \to (1,2,3)} \frac{xz^2-y^2z}{xyz-1}$

85. $\lim\limits_{(x,y,z) \to (0,0,0)} \frac{x^2-y^2-z^2}{x^2+y^2-z^2}$

For the following exercises, evaluate the limit of the function by determining the value the function approaches along the indicated paths. If the limit does not exist, explain why not.

86. $\lim\limits_{(x,y) \to (0,0)} \frac{xy+y^3}{x^2+y^2}$

a. Along the x-axis ($y = 0$)
b. Along the y-axis ($x = 0$)
c. Along the path $y = 2x$

87. Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$ using the results of previous problem.

88. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$
a. Along the *x*-axis ($y = 0$)
b. Along the *y*-axis ($x = 0$)
c. Along the path $y = x^2$

89. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ using the results of previous problem.

Discuss the continuity of the following functions. Find the largest region in the xy-plane in which the following functions are continuous.

90. $f(x, y) = \sin(xy)$

91. $f(x, y) = \ln(x + y)$

92. $f(x, y) = e^{3xy}$

93. $f(x, y) = \frac{1}{xy}$

For the following exercises, determine the region in which the function is continuous. Explain your answer.

94. $f(x, y) = \frac{x^2y}{x^2+y^2}$

95. $f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

(*Hint*: Show that the function approaches different values along two different paths.)

96. $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$

97. Determine whether $g(x, y) = \frac{x^2-y^2}{x^2+y^2}$ is continuous at $(0, 0)$.

98. Create a plot using graphing software to determine where the limit does not exist. Determine the region of the coordinate plane in which $f(x, y) = \frac{1}{x^2 - y}$ is continuous.

99. Determine the region of the xy-plane in which the composite function $g(x, y) = \arctan\left(\frac{xy^2}{x+y}\right)$ is continuous. Use technology to support your conclusion.

100. Determine the region of the xy-plane in which $f(x, y) = \ln(x^2 + y^2 - 1)$ is continuous. Use technology to support your conclusion. (*Hint*: Choose the range of values for x and y carefully!)

101. At what points in space is $g(x, y, z) = x^2 + y^2 - 2z^2$ continuous?

102. At what points in space is $g(x, y, z) = \frac{1}{x^2 + z^2 - 1}$ continuous?

103. Show that $\lim_{(x,y) \to (0,0)} \frac{1}{x^2 + y^2}$ does not exist at $(0, 0)$ by plotting the graph of the function.

104. **[T]** Evaluate $\lim_{(x,y) \to (0,0)} \frac{-xy^2}{x^2 + y^4}$ by plotting the function using a CAS. Determine analytically the limit along the path $x = y^2$.

105. **[T]**

a. Use a CAS to draw a contour map of $z = \sqrt{9 - x^2 - y^2}$.
b. What is the name of the geometric shape of the level curves?
c. Give the general equation of the level curves.
d. What is the maximum value of z?
e. What is the domain of the function?
f. What is the range of the function?

106. *True or False*: If we evaluate $\lim_{(x,y) \to (0,0)} f(x)$ along several paths and each time the limit is 1, we can conclude that $\lim_{(x,y) \to (0,0)} f(x) = 1$.

107. Use polar coordinates to find $\lim_{(x,y) \to (0,0)} \frac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$. You can also find the limit using L'Hôpital's rule.

108. Use polar coordinates to find $\lim\limits_{(x,y) \to (0,0)} \cos\left(x^2 + y^2\right)$.

109. Discuss the continuity of $f(g(x, y))$ where $f(t) = 1/t$ and $g(x, y) = 2x - 5y$.

110. Given $f(x, y) = x^2 - 4y$, find $\lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$.

111. Given $f(x, y) = x^2 - 4y$, find $\lim\limits_{h \to 0} \dfrac{f(1+h, y) - f(1, y)}{h}$.

Learning Objectives

4.3.1. Calculate the partial derivatives of a function of two variables.
4.3.2. Calculate the partial derivatives of a function of more than two variables.
4.3.3. Determine the higher-order derivatives of a function of two variables.
4.3.4. Explain the meaning of a partial differential equation and give an example.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

DEFINITION

Let $f(x, y)$ be a function of two variables. Then the **partial derivative** of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}. \tag{4.12}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}. \tag{4.13}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

EXAMPLE 4.14

Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

[Show Solution]

CHECKPOINT 4.12

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = 4x^2 + 2xy - y^2 + 3x - 2y + 5.$$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x) = f(x, y)$ as a function of x. Then

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x}.$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y) = f(x, y)$ as a function of y. Then

$$h'(x) = \lim_{k \to 0} \frac{h(x+k) - h(x)}{k} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} = \frac{\partial f}{\partial y}.$$

All differentiation rules from Introduction to Derivatives apply.

EXAMPLE 4.15

Calculating Partial Derivatives

Calculate $\partial f / \partial x$ and $\partial f / \partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$

b. $g(x, y) = \sin\left(x^2 y - 2x + 4\right)$

[Show Solution]

CHECKPOINT 4.13

Calculate $\partial f / \partial x$ and $\partial f / \partial y$ for the function $f(x, y) = \tan\left(x^3 - 3x^2 y^2 + 2y^4\right)$ by holding the opposite variable constant, then differentiating.

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in $\mathbb{R}^3$. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x+h, y) - f(x, y)}{h}.$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure 4.21 illustrates a surface described by an arbitrary function $z = f(x, y)$.

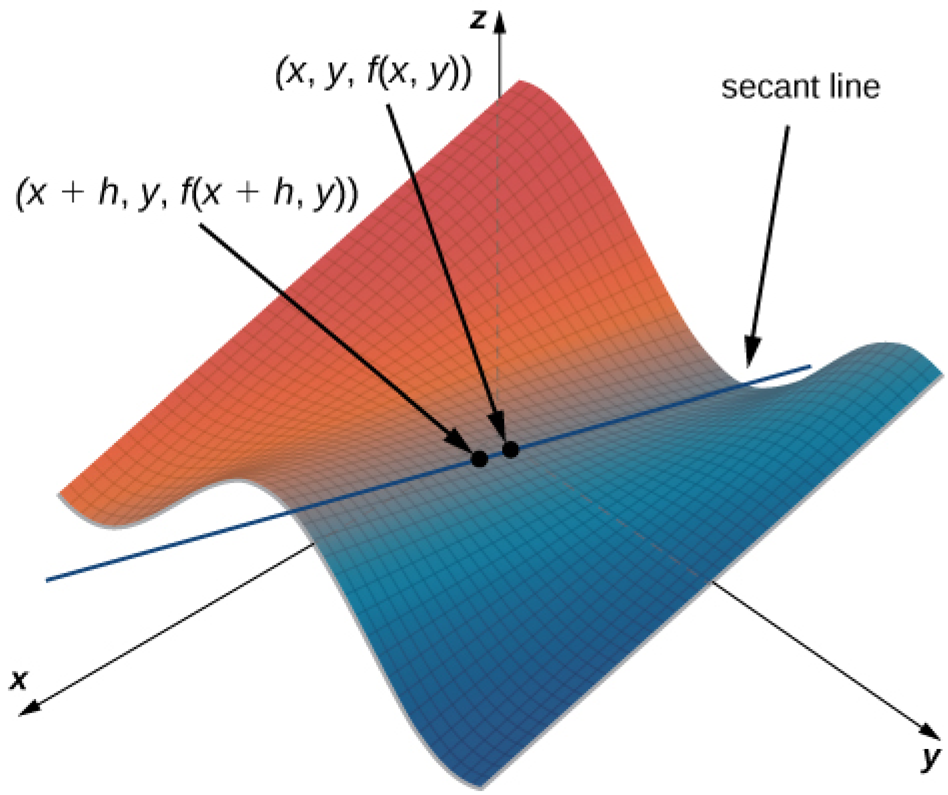


Figure 4.21 Secant line passing through the points $(x, y, f(x, y))$ and $(x + h, y, f(x + h, y))$.

In Figure 4.21, the value of h is positive. If we graph $f(x, y)$ and $f(x + h, y)$ for an arbitrary point (x, y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x + h, y) - f(x, y)}{h}.$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f / \partial x$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the x-axis and $\partial f / \partial y$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called *directional derivatives*, which we discuss in Directional Derivatives and the Gradient.

We now return to the idea of contour maps, which we introduced in Functions of Several Variables. We can use a contour map to estimate partial derivatives of a function $g(x, y)$.

EXAMPLE 4.16

Partial Derivatives from a Contour Map

Use a contour map to estimate $\partial g/\partial x$ at the point $\left(\sqrt{5}, 0\right)$ for the function $g(x, y) = \sqrt{9 - x^2 - y^2}$.

[Show Solution]

CHECKPOINT 4.14

Use a contour map to estimate $\partial f/\partial y$ at point $\left(0, \sqrt{2}\right)$ for the function

$$f(x, y) = x^2 - y^2.$$

Compare this with the exact answer.

Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w = f(x, y, z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

DEFINITION

Let $f(x, y, z)$ be a function of three variables. Then, the *partial derivative of f with respect to x,* written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}. \qquad (4.14)$$

The *partial derivative of f with respect to y*, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k, z) - f(x, y, z)}{k}. \tag{4.15}$$

The *partial derivative of f with respect to* z, written as $\partial f / \partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = \lim_{m \to 0} \frac{f(x, y, z + m) - f(x, y, z)}{m}. \tag{4.16}$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of $x, y,$ and $z,$ and we wish to calculate $\partial f / \partial x,$ then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

EXAMPLE 4.17

Calculating Partial Derivatives for a Function of Three Variables

Use the limit definition of partial derivatives to calculate $\partial f / \partial x$ for the function

$$f(x, y, z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z.$$

Then, find $\partial f / \partial y$ and $\partial f / \partial z$ by setting the other two variables constant and differentiating accordingly.

[Show Solution]

CHECKPOINT 4.15

Use the limit definition of partial derivatives to calculate $\partial f / \partial x$ for the function

$$f(x, y, z) = 2x^2 - 4x^2y + 2y^2 + 5xz^2 - 6x + 3z - 8.$$

Then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

EXAMPLE 4.18

Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

a. $f(x, y, z) = \frac{x^2y - 4xz + y^2}{x - 3yz}$

b. $g(x, y, z) = \sin\left(x^2y - z\right) + \cos\left(x^2 - yz\right)$

[Show Solution]

CHECKPOINT 4.16

Calculate $\partial f/\partial x,$ $\partial f/\partial y,$ and $\partial f/\partial z$ for the function $f(x, y, z) = \sec\left(x^2y\right) - \tan\left(x^3yz^2\right).$

Higher-Order Partial Derivatives

Consider the function

$$f(x, y) = 2x^3 - 4xy^2 + 5y^3 - 6xy + 5x - 4y + 12.$$

Its partial derivatives are

$$\frac{\partial f}{\partial x} = 6x^2 - 4y^2 - 6y + 5 \text{ and } \frac{\partial f}{\partial y} = -8xy + 15y^2 - 6x - 4.$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives, third-order derivatives*, and so on. In general, they are referred to as **higher-order partial derivatives**. There are four second-order partial derivatives for any function (provided they all exist):

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right], \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right], \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right], \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].$$

An alternative notation for each is $f_{xx}, f_{yx}, f_{xy},$ and $f_{yy},$ respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and $f_{yx},$ are commonly called **mixed partial derivatives**.

EXAMPLE 4.19

Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function

$$f(x, y) = xe^{-3y} + \sin(2x - 5y).$$

[Show Solution]

CHECKPOINT 4.17

Calculate all four second partial derivatives for the function

$$f(x, y) = \sin(3x - 2y) + \cos(x + 4y).$$

At this point we should notice that, in both Example 4.19 and the checkpoint, it was true that $\partial^2 f / \partial x \partial y = \partial^2 f / \partial y \partial x.$ Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

THEOREM 4.5

Equality of Mixed Partial Derivatives (Clairaut's Theorem)

Suppose that $f(x, y)$ is defined on an open disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most advanced calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x, y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Partial Differential Equations

In Introduction to Differential Equations, we studied differential equations in which the unknown function had one independent variable. A **partial differential equation** is an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives. Examples of partial differential equations are

$$u_t = c^2\left(u_{xx} + u_{yy}\right) \tag{4.17}$$

(heat equation in two dimensions)

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right) \tag{4.18}$$

(wave equation in two dimensions)

$$u_{xx} + u_{yy} = 0 \tag{4.19}$$

(Laplace's equation in two dimensions)

In the first two equations, the unknown function u has three independent variables—$t, x,$ and y—and c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time. In Laplace's equation, the unknown function u has two independent variables x and y.

EXAMPLE 4.20

A Solution to the Wave Equation

Verify that

$$u(x, y, t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$$

is a solution to the wave equation

$$u_{tt} = 4\left(u_{xx} + u_{yy}\right). \tag{4.20}$$

[Show Solution]

CHECKPOINT 4.18

Verify that $u(x, y, t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$ is a solution to the heat equation

$$u_t = 9\left(u_{xx} + u_{yy}\right). \tag{4.21}$$

Since the solution to the two-dimensional heat equation is a function of three variables, it is not easy to create a visual representation of the solution. We can graph the solution for fixed values of *t*, which amounts to snapshots of the heat distributions at fixed times. These snapshots show how the heat is distributed over a two-dimensional surface as time progresses. The graph of the preceding solution at time $t = 0$ appears in the following figure. As time progresses, the extremes level out, approaching zero as *t* approaches infinity.

Figure 4.23

If we consider the heat equation in one dimension, then it is possible to graph the solution over time. The heat equation in one dimension becomes

$$u_t = c^2 u_{xx},$$

where c^2 represents the thermal diffusivity of the material in question. A solution of this differential equation can be written in the form

$$u_m(x, t) = e^{-\pi^2 m^2 c^2 t} \sin(m\pi x) \tag{4.22}$$

where m is any positive integer. A graph of this solution using $m = 1$ appears in Figure 4.24, where the initial temperature distribution over a wire of length 1 is given by $u(x, 0) = \sin \pi x$. Notice that as time progresses, the wire cools off. This is seen because, from left to right, the highest temperature (which occurs in the middle of the wire) decreases and changes color from red to blue.

Figure 4.24 Graph of a solution of the heat equation in one dimension over time.

STUDENT PROJECT

Lord Kelvin and the Age of Earth

Figure 4.25 (a) William Thomson (Lord Kelvin), 1824-1907, was a British physicist and electrical engineer; (b) Kelvin used the heat diffusion equation to estimate the age of Earth (credit: modification of work by NASA).

During the late 1800s, the scientists of the new field of geology were coming to the conclusion that Earth must be “millions and millions” of years old. At about the same time, Charles Darwin had published his treatise on evolution. Darwin’s view was that evolution needed many millions of years to take place, and he made a bold claim that the Weald chalk fields, where important fossils were found, were the result of 300 million years of erosion.

At that time, eminent physicist William Thomson (Lord Kelvin) used an important partial differential equation, known as the *heat diffusion equation*, to estimate the age of Earth by determining how long it would take Earth to cool from molten rock to what we had at that time. His conclusion was a range of 20 to 400 million years, but most likely about 50 million years. For many decades, the proclamations of this irrefutable icon of science did not sit well with geologists or with Darwin.

MEDIA

Read Kelvin’s paper on estimating the age of the Earth.

Kelvin made reasonable assumptions based on what was known in his time, but he also made several assumptions that turned out to be wrong. One incorrect assumption was that Earth is solid and that the cooling was therefore via conduction only, hence justifying the use of the diffusion equation. But the most serious error was a forgivable one—omission of the fact that Earth contains radioactive elements that continually supply heat beneath Earth's mantle. The discovery of radioactivity came near the end of Kelvin's life and he acknowledged that his calculation would have to be modified.

Kelvin used the simple one-dimensional model applied only to Earth's outer shell, and derived the age from graphs and the roughly known temperature gradient near Earth's surface. Let's take a look at a more appropriate version of the diffusion equation in radial coordinates, which has the form

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial^2 r} + \frac{2}{r}\frac{\partial T}{\partial r}\right]. \qquad \boxed{4.23}$$

Here, $T(r, t)$ is temperature as a function of r (measured from the center of Earth) and time t. K is the heat conductivity—for molten rock, in this case. The standard method of solving such a partial differential equation is by separation of variables, where we express the solution as the product of functions containing each variable separately. In this case, we would write the temperature as

$$T(r, t) = R(r)f(t).$$

1. Substitute this form into Equation 4.13 and, noting that $f(t)$ is constant with respect to distance (r) and $R(r)$ is constant with respect to time (t), show that

$$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right].$$

2. This equation represents the separation of variables we want. The left-hand side is only a function of t and the right-hand side is only a function of r, and they must be equal for all values of r and t. Therefore, they both must be equal to a constant. Let's call that constant $-\lambda^2$. (The convenience of this choice is seen on substitution.) So, we have

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\lambda^2 \quad \text{and} \quad \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right] = -\lambda^2.$$

Now, we can verify through direct substitution for each equation that the solutions are $f(t) = Ae^{-\lambda^2 t}$ and $R(r) = B\left(\frac{\sin \alpha r}{r}\right) + C\left(\frac{\cos \alpha r}{r}\right)$, where $\alpha = \lambda/\sqrt{K}$. Note that $f(t) = Ae^{+\lambda n^2 t}$ is also a valid solution, so we could have chosen $+\lambda^2$ for our constant. Can you see why it would not be valid for this case as time increases?

3. Let's now apply boundary conditions.
    a. The temperature must be finite at the center of Earth, $r = 0$. Which of the two constants, B or C, must therefore be zero to keep R finite at $r = 0$? (Recall that $\sin(\alpha r)/r \to \alpha =$ as $r \to 0$, but $\cos(\alpha r)/r$ behaves very differently.)
    b. Kelvin argued that when magma reaches Earth's surface, it cools very rapidly. A person can often touch the surface within weeks of the flow. Therefore, the surface reached a moderate temperature very early and remained nearly constant at a surface temperature T_s. For simplicity, let's set $T = 0$ at $r = R_E$ and find α such that this is the temperature there for all time t. (Kelvin took the value to be $300\text{ K} \approx 80°\text{F}$. We can add this 300 K constant to our solution later.) For this to be true, the sine argument must be zero at $r = R_E$. Note that α has an infinite series of values that satisfies this condition. Each value of α represents a valid solution (each with its own value for A). The total or general solution is the sum of all these solutions.
    c. At $t = 0$, we assume that all of Earth was at an initial hot temperature T_0 (Kelvin took this to be about 7000 K.) The application of this boundary condition involves the more advanced application of Fourier coefficients. As noted in part b. each value of α_n represents a valid solution, and the general solution is a sum of all these solutions. This results in a series solution:

$$T(r, t) = \left(\frac{T_0 R_{\mathrm{E}}}{\pi}\right) \sum_n \frac{(-1)^{n-1}}{n} e^{-\lambda n^2 t} \frac{\sin(\alpha_n r)}{r}, \text{ where } \alpha_n = n\pi/R_{\mathrm{E}}.$$

Note how the values of α_n come from the boundary condition applied in part b. The term $\frac{-1^{n-1}}{n}$ is the constant A_n for each term in the series, determined from applying the Fourier method. Letting $\beta = \frac{\pi}{R_{\mathrm{E}}}$, examine the first few terms of this solution shown here and note how λ^2 in the exponential causes the higher terms to decrease quickly as time progresses:

$$T(r,t) = \frac{T_0 R_E}{\pi r}\left(\begin{array}{c} e^{-K\beta^2 t}(\sin \beta r) - \frac{1}{2}e^{-4K\beta^2 t}(\sin 2\beta r) + \frac{1}{3}e^{-9K\beta^2 t}(\sin 3\beta r) \\ -\frac{1}{4}e^{-16K\beta^2 t}(\sin 4\beta r) + \frac{1}{5}e^{-25K\beta^2 t}(\sin 5\beta r)... \end{array}\right).$$

Near time $t = 0,$ many terms of the solution are needed for accuracy. Inserting values for the conductivity K and $\beta = \pi/R_E$ for time approaching merely thousands of years, only the first few terms make a significant contribution. Kelvin only needed to look at the solution near Earth's surface (Figure 4.26) and, after a long time, determine what time best yielded the estimated temperature gradient known during his era ($1°\text{F}$ increase per $50\,\text{ft}$). He simply chose a range of times with a gradient close to this value. In Figure 4.26, the solutions are plotted and scaled, with the $300 - \text{K}$ surface temperature added. Note that the center of Earth would be relatively cool. At the time, it was thought Earth must be solid.

Figure 4.26 Temperature versus radial distance from the center of Earth. (a) Kelvin's results, plotted to scale. (b) A close-up of the results at a depth of $4.0\,\text{mi}$ below Earth's surface.

Epilog

On May $20, 1904,$ physicist Ernest Rutherford spoke at the Royal Institution to announce a revised calculation that included the contribution of radioactivity as a source of Earth's heat. In Rutherford's own words:

"I came into the room, which was half-dark, and presently spotted Lord Kelvin in the audience, and realised that I was in for trouble at the last part of my speech dealing with the age of the Earth, where my views conflicted with his. To my relief, Kelvin fell fast asleep, but as I came to the important point, I saw the old bird sit up, open an eye and cock a baleful glance at me.

Then a sudden inspiration came, and I said Lord Kelvin had limited the age of the Earth, *provided no new source* [*of heat*] *was discovered*. That prophetic utterance referred to what we are now considering tonight, radium! Behold! The old boy beamed upon me."

Rutherford calculated an age for Earth of about 500 million years. Today's accepted value of Earth's age is about 4.6 billion years.

Section 4.3 Exercises

For the following exercises, calculate the partial derivative using the limit definitions only.

112. $\frac{\partial z}{\partial x}$ for $z = x^2 - 3xy + y^2$

113. $\frac{\partial z}{\partial y}$ for $z = x^2 - 3xy + y^2$

For the following exercises, calculate the sign of the partial derivative using the graph of the surface.

114. $f_x(1, 1)$

115. $f_x(-1, 1)$

116. $f_y(1, 1)$

117. $f_x(0, 0)$

For the following exercises, calculate the partial derivatives.

118. $\frac{\partial z}{\partial x}$ for $z = \sin(3x)\cos(3y)$

119. $\frac{\partial z}{\partial y}$ for $z = \sin(3x)\cos(3y)$

120. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = x^8 e^{3y}$

121. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = \ln\left(x^6 + y^4\right)$

122. Find $f_y(x, y)$ for $f(x, y) = e^{xy}\cos(x)\sin(y)$.

123. Let $z = e^{xy}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

124. Let $z = \ln\left(\frac{x}{y}\right)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

125. Let $z = \tan(2x - y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

126. Let $z = \sinh(2x + 3y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

127. Let $f(x, y) = \arctan\left(\frac{y}{x}\right)$. Evaluate $f_x(2, -2)$ and $f_y(2, -2)$.

128. Let $f(x, y) = \frac{xy}{x-y}$. Find $f_x(2, -2)$ and $f_y(2, -2)$.

Evaluate the partial derivatives at point $P(0, 1)$.

129. Find $\frac{\partial z}{\partial x}$ at $(0, 1)$ for $z = e^{-x}\cos(y)$.

130. Given $f(x, y, z) = x^3yz^2$, find $\frac{\partial^2 f}{\partial x \partial y}$ and $f_z(1, 1, 1)$.

131. Given $f(x, y, z) = 2\sin(x + y)$, find $f_x\left(0, \frac{\pi}{2}, -4\right)$, $f_y\left(0, \frac{\pi}{2}, -4\right)$, and $f_z\left(0, \frac{\pi}{2}, -4\right)$.

132. The area of a parallelogram with adjacent side lengths that are a and b, and in which the angle between these two sides is θ, is given by the function $A(a, b, \theta) = ba\sin(\theta)$. Find the rate of change of the area of the parallelogram with respect to the following:

a. Side *a*
b. Side *b*
c. Angle θ

133. Express the volume of a right circular cylinder as a function of two variables:

a. its radius r and its height h.
b. Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.
c. Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

134. Calculate $\frac{\partial w}{\partial z}$ for $w = z\sin(xy^2 + 2z)$.

Find the indicated higher-order partial derivatives.

135. f_{xy} for $z = \ln(x - y)$

136. f_{yx} for $z = \ln(x - y)$

137. Let $z = x^2 + 3xy + 2y^2$. Find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$.

138. Given $z = e^x \tan y$, find $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$.

139. Given $f(x, y, z) = xyz$, find f_{xyy}, f_{yxy}, and f_{yyx}.

140. Given $f(x, y, z) = e^{-2x}\sin\left(z^2 y\right)$, show that $f_{xyy} = f_{yxy}$.

141. Show that $z = \frac{1}{2}\left(e^y - e^{-y}\right)\sin x$ is a solution of the differential equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

142. Find $f_{xx}(x, y)$ for $f(x, y) = \frac{4x^2}{y} + \frac{y^2}{2x}$.

143. Let $f(x, y, z) = x^2y^3z - 3xy^2z^3 + 5x^2z - y^3z$. Find f_{xyz}.

144. Let $F(x, y, z) = x^3yz^2 - 2x^2yz + 3xz - 2y^3z$. Find F_{xyz}.

145. Given $f(x, y) = x^2 + x - 3xy + y^3 - 5$, find all points at which $f_x = f_y = 0$ simultaneously.

146. Given $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$, find all points at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

147. Given $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$, find all points on f at which $f_x = f_y = 0$ simultaneously.

148. Given $f(x, y) = 15x^3 - 3xy + 15y^3$, find all points at which $f_x(x, y) = f_y(x, y) = 0$ simultaneously.

149. Show that $z = e^x \sin y$ satisfies the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

150. Show that $f(x, y) = \ln\left(x^2 + y^2\right)$ solves Laplace's equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

151. Show that $z = e^{-t}\cos\left(\frac{x}{c}\right)$ satisfies the heat equation $\frac{\partial z}{\partial t} = -e^{-t}\cos\left(\frac{x}{c}\right).$

152. Find $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x,y)}{\Delta x}$ for $f(x, y) = -7x - 2xy + 7y$.

153. Find $\lim\limits_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}$ for $f(x, y) = -7x - 2xy + 7y$.

154. Find $\lim\limits_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x}$ for $f(x, y) = x^2y^2 + xy + y$.

155. Find $\lim\limits_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x}$ for $f(x, y) = \sin(xy)$.

156. The function $P(T, V) = \frac{nRT}{V}$ gives the pressure at a point in a gas as a function of temperature T and volume V. The letters n and R are constants. Find $\frac{\partial P}{\partial V}$ and $\frac{\partial P}{\partial T}$, and explain what these quantities represent.

157. The equation for heat flow in the xy-plane is $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. Show that $f(x, y, t) = e^{-2t}\sin x \sin y$ is a solution.

158. The basic wave equation is $f_{tt} = f_{xx}$. Verify that $f(x, t) = \sin(x + t)$ and $f(x, t) = \sin(x - t)$ are solutions.

159. The law of cosines can be thought of as a function of three variables. Let $x, y,$ and θ be two sides of any triangle where the angle θ is the included angle between the two sides. Then, $F(x, y, \theta) = x^2 + y^2 - 2xy\cos\theta$ gives the square of the third side of the triangle. Find $\frac{\partial F}{\partial \theta}$ and $\frac{\partial F}{\partial x}$ when $x = 2, y = 3,$ and $\theta = \frac{\pi}{6}.$

160. Suppose the sides of a rectangle are changing with respect to time. The first side is changing at a rate of 2 in./sec whereas the second side is changing at the rate of 4 in/sec. How fast is the diagonal of the rectangle changing when the first side measures 16 in. and the second side measures 20 in.? (Round answer to three decimal places.)

161. A Cobb-Douglas production function is $f(x, y) = 200x^{0.7}y^{0.3},$ where x and y represent the amount of labor and capital available. Let $x = 500$ and $y = 1000.$ Find $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$ at these values, which represent the marginal productivity of labor and capital, respectively.

162. The apparent temperature index is a measure of how the temperature feels, and it is based on two variables: $h,$ which is relative humidity, and $t,$ which is the air temperature.

$A = 0.885t - 22.4h + 1.20th - 0.544.$ Find $\frac{\partial A}{\partial t}$ and $\frac{\partial A}{\partial h}$ when $t = 20°\text{F}$ and $h = 0.90.$

Learning Objectives

4.4.1. Determine the equation of a plane tangent to a given surface at a point.
4.4.2. Use the tangent plane to approximate a function of two variables at a point.
4.4.3. Explain when a function of two variables is differentiable.
4.4.4. Use the total differential to approximate the change in a function of two variables.

In this section, we consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y = f(x)$. The slope of the tangent line at the point $x = a$ is given by $m = f'(a)$; what is the slope of a tangent plane? We learned about the equation of a plane in Equations of Lines and Planes in Space; in this section, we see how it can be applied to the problem at hand.

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A tangent plane at a regular point contains all of the lines tangent to that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly.

DEFINITION

Let $P_0 = \left(x_0, y_0, z_0\right)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the **tangent plane** to S at P_0 (Figure 4.27).

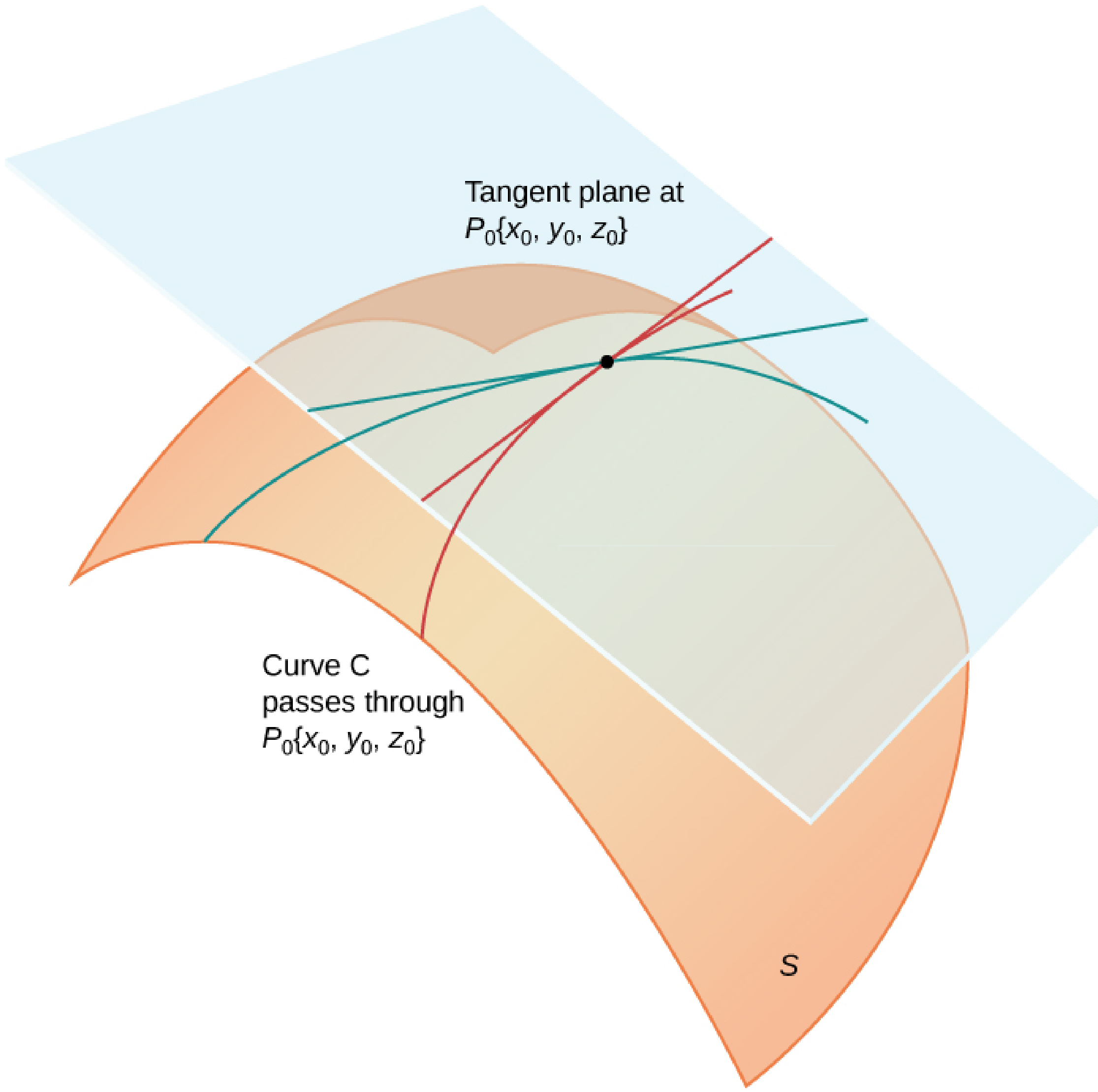


Figure 4.27 The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S that pass through P_0.

For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point, defined later in this section. We define the term tangent plane here and then explore the idea intuitively.

DEFINITION

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = \left(x_0, y_0\right)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \quad (4.24)$$

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is $z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. A parallel vector to the first tangent line is $\mathbf{a} = \mathbf{j} + f_y(x_0, y_0)\mathbf{k}$; a parallel vector to the second tangent line is $\mathbf{b} = \mathbf{i} + f_x(x_0, y_0)\mathbf{k}$. We can take the cross product of these two vectors:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \left(\mathbf{j} + f_y(x_0, y_0)\mathbf{k}\right) \times \left(\mathbf{i} + f_x(x_0, y_0)\mathbf{k}\right) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} \\
&= f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}.
\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = \left(x_0, y_0, f(x_0, y_0)\right)$ in the equation for a plane:

$$\begin{aligned}
\mathbf{n} \cdot \left((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + \left(z - f(x_0, y_0)\right)\mathbf{k}\right) &= 0 \\
\left(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}\text{-}\mathbf{k}\right) \cdot \left((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + \left(z - f(x_0, y_0)\right)\mathbf{k}\right) &= 0 \\
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - \left(z - f(x_0, y_0)\right) &= 0.
\end{aligned}$$

Solving this equation for z gives Equation 4.24.

EXAMPLE 4.21

Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

[Show Solution]

CHECKPOINT 4.19

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = x^3 - x^2y + y^2 - 2x + 3y - 2$ at point $(-1, 3)$.

EXAMPLE 4.22

Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = \sin(2x)\cos(3y)$ at the point $(\pi/3, \pi/4)$.

[Show Solution]

A tangent plane to a surface does not always exist at every point on the surface. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

The graph of this function follows.

Figure 4.29 Graph of a function that does not have a tangent plane at the origin.

If either $x = 0$ or $y = 0,$ then $f(x, y) = 0,$ so the value of the function does not change on either the *x*- or *y*-axis. Therefore, $f_x(x, 0) = f_y(0, y) = 0,$ so as either $x \; or \; y$ approach zero, these partial derivatives stay equal to zero. Substituting them into Equation 4.24 gives $z = 0$ as the equation of the tangent line. However, if we approach the origin from a different direction, we get a different story. For example, suppose we approach the origin along the line $y = x.$ If we put $y = x$ into the original function, it becomes

$$f(x, x) = \frac{x(x)}{\sqrt{x^2 + (x)^2}} = \frac{x^2}{\sqrt{2x^2}} = \frac{|x|}{\sqrt{2}}.$$

When $x > 0,$ the slope of this curve is equal to $\sqrt{2}/2;$ when $x < 0,$ the slope of this curve is equal to $-\left(\sqrt{2}/2\right).$ This presents a problem. In the definition of *tangent plane*, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here. When we study differentiable functions, we will see that this function is not differentiable at the origin.

Linear Approximations

Recall from Linear Approximations and Differentials that the formula for the linear approximation of a function $f(x)$ at the point $x = a$ is given by

$$y \approx f(a) + f'(a)(x - a).$$

The diagram for the linear approximation of a function of one variable appears in the following graph.

Figure 4.30 Linear approximation of a function in one variable.

The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

DEFINITION

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the **linear approximation** of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{4.25}$$

Notice that this equation also represents the tangent plane to the surface defined by $z = f(x, y)$ at the point $\left(x_0, y_0\right)$. The idea behind using a linear approximation is that, if there is a point $\left(x_0, y_0\right)$ at which the precise value of $f(x, y)$ is known, then for values of (x, y) reasonably close to $\left(x_0, y_0\right)$, the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x, y)$ (Figure 4.31). Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point $\left(x_0, y_0\right)$.

Figure 4.31 Using a tangent plane for linear approximation at a point.

EXAMPLE 4.23

Using a Tangent Plane Approximation

Given the function $f(x, y) = \sqrt{41 - 4x^2 - y^2}$, approximate $f(2.1, 2.9)$ using point $(2, 3)$ for (x_0, y_0). What is the approximate value of $f(2.1, 2.9)$ to four decimal places?

[Show Solution]

CHECKPOINT 4.20

Given the function $f(x, y) = e^{5 - 2x + 3y}$, approximate $f(4.1, 0.9)$ using point $(4, 1)$ for (x_0, y_0). What is the approximate value of $f(4.1, 0.9)$ to four decimal places?

Differentiability

When working with a function $y = f(x)$ of one variable, the function is said to be differentiable at a point $x = a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is "smooth" at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

For a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in Figure 4.29. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

DEFINITION

A function $f(x, y)$ is **differentiable** at a point $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around $P,$ we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y), \tag{4.26}$$

where the error term E satisfies

$$\lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

The last term in Equation 4.26 is referred to as the *error term* and it represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point $P.$ For the function f to be differentiable at $P,$ the function must be smooth—that is, the graph of f must be close to the tangent plane for points near $P.$

EXAMPLE 4.24

Demonstrating Differentiability

Show that the function $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3).$

[Show Solution]

CHECKPOINT 4.21

Show that the function $f(x, y) = 3x - 4y^2$ is differentiable at point $(-1, 2).$

The function $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ is not differentiable at the origin. We can see this by calculating the partial derivatives. This function appeared earlier in the section, where we showed that $f_x(0, 0) = f_y(0, 0) = 0.$ Substituting this information into Equation 4.26 using $x_0 = 0$ and $y_0 = 0,$ we get

$$\begin{aligned} f(x, y) &= f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + E(x, y) \\ E(x, y) &= \frac{xy}{\sqrt{x^2+y^2}}. \end{aligned}$$

Calculating $\lim\limits_{(x,y) \to (x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$ gives

$$\begin{aligned} \lim_{(x,y) \to (x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y) \to (0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \\ &= \lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}. \end{aligned}$$

Depending on the path taken toward the origin, this limit takes different values. Therefore, the limit does not exist and the function f is not differentiable at the origin as shown in the following figure.

Figure 4.32 This function $f(x, y)$ is not differentiable at the origin.

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

THEOREM 4.6

Differentiability Implies Continuity

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y)$ is continuous at (x_0, y_0).

Differentiability Implies Continuity shows that if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example,

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous at the origin, but it is not differentiable at the origin. This observation is also similar to the situation in single-variable calculus.

Continuity of First Partials Implies Differentiability further explores the connection between continuity and differentiability at a point. This theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

THEOREM 4.7

Continuity of First Partials Implies Differentiability

Let $z = f(x, y)$ be a function of two variables with $\left(x_0, y_0\right)$ in the domain of f. If $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ all exist in a neighborhood of $\left(x_0, y_0\right)$ and are continuous at $\left(x_0, y_0\right)$, then $f(x, y)$ is differentiable there.

Recall that earlier we showed that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

was not differentiable at the origin. Let’s calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x} = \frac{y^3}{\left(x^2+y^2\right)^{3/2}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x^3}{\left(x^2+y^2\right)^{3/2}}.$$

The contrapositive of the preceding theorem states that if a function is not differentiable, then at least one of the hypotheses must be false. Let's explore the condition that $f_x(0, 0)$ must be continuous. For this to be true, it must be true that $\lim_{(x,y)\to(0,0)} f_x(0, 0) = f_x(0, 0)$:

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{\left(x^2 + y^2\right)^{3/2}}.$$

Let $x = ky$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{y^3}{\left(x^2+y^2\right)^{3/2}} &= \lim_{y\to 0} \frac{y^3}{\left((ky)^2+y^2\right)^{3/2}} \\
&= \lim_{y\to 0} \frac{y^3}{\left(k^2y^2+y^2\right)^{3/2}} \\
&= \lim_{y\to 0} \frac{y^3}{|y|^3\left(k^2+1\right)^{3/2}} \\
&= \frac{1}{\left(k^2+1\right)^{3/2}} \lim_{y\to 0} \frac{|y|}{y}.
\end{aligned}$$

If $y > 0$, then this expression equals $1/\left(k^2 + 1\right)^{3/2}$; if $y < 0$, then it equals $-\left(1/\left(k^2 + 1\right)^{3/2}\right)$. In either case, the value depends on k, so the limit fails to exist.

Differentials

In Linear Approximations and Differentials we first studied the concept of differentials. The differential of y, written dy, is defined as $f'(x)dx$. The differential is used to approximate $\Delta y = f(x + \Delta x) - f(x)$, where $\Delta x = dx$. Extending this idea to the linear approximation of a function of two variables at the point $\left(x_0, y_0\right)$ yields the formula for the total differential for a function of two variables.

DEFINITION

Let $z = f(x, y)$ be a function of two variables with $\left(x_0, y_0\right)$ in the domain of f, and let Δx and Δy be chosen so that $\left(x_0 + \Delta x, y_0 + \Delta y\right)$ is also in the domain of f. If f

is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x \text{ and } dy = \Delta y.$$

The differential dz, also called the **total differential** of $z = f(x, y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{4.27}$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of z. Now, let's define $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$. We use dz to approximate Δz, so

$$\Delta z \approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy. Since $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, this can be used further to approximate $f(x + \Delta x, y + \Delta y)$:

$$\begin{aligned} f(x + \Delta x, y + \Delta y) &= f(x, y) + \Delta z \\ &\approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y. \end{aligned}$$

See the following figure.

Figure 4.33 The linear approximation is calculated via the formula $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$

One such application of this idea is to determine error propagation. For example, if we are manufacturing a gadget and are off by a certain amount in measuring a given quantity, the differential can be used to estimate the error in the total volume of the gadget.

EXAMPLE 4.25

Approximation by Differentials

Find the differential dz of the function $f(x, y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2, -3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz?

[Show Solution]

CHECKPOINT 4.22

Find the differential dz of the function $f(x, y) = 4y^2 + x^2y - 2xy$ and use it to approximate Δz at point $(1, -1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

Differentiability of a Function of Three Variables

All of the preceding results for differentiability of functions of two variables can be generalized to functions of three variables. First, the definition:

DEFINITION

A function $f(x, y, z)$ is differentiable at a point $P\left(x_0, y_0, z_0\right)$ if for all points (x, y, z) in a δ disk around P we can write

$$\begin{aligned} f(x, y) \quad &= f\left(x_0, y_0, z_0\right) + f_x\left(x_0, y_0, z_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0, z_0\right)\left(y - y_0\right) \\ &+ f_z\left(x_0, y_0, z_0\right)\left(z - z_0\right) + E(x, y, z), \end{aligned} \tag{4.28}$$

where the error term *E* satisfies

$$\lim_{(x, y, z) \to \left(x_0, y_0, z_0\right)} \frac{E(x, y, z)}{\sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2 + \left(z - z_0\right)^2}} = 0.$$

If a function of three variables is differentiable at a point $\left(x_0, y_0, z_0\right)$, then it is continuous there. Furthermore, continuity of first partial derivatives at that point guarantees differentiability.

Section 4.4 Exercises

For the following exercises, find a unit normal vector to the surface at the indicated point.

163. $f(x, y) = x^3, (2, -1, 8)$

164. $\ln\left(\frac{x}{y-z}\right) = 0$ when $x = y = 1$

For the following exercises, as a useful review for techniques used in this section, find a normal vector and a tangent vector at point P.

165. $x^2 + xy + y^2 = 3, P(-1, -1)$

166. $\left(x^2 + y^2\right)^2 = 9\left(x^2 - y^2\right), P(\sqrt{2}, 1)$

167. $xy^2 - 2x^2 + y + 5x = 6, P(4, 2)$

168. $2x^3 - x^2y^2 = 3x - y - 7, P(1, -2)$

169. $ze^{x^2 - y^2} - 3 = 0,\ P(2, 2, 3)$

For the following exercises, find the equation for the tangent plane to the surface at the indicated point. (*Hint:* Solve for z in terms of x and y.)

170. $-8x - 3y - 7z = -19, P(1, -1, 2)$

171. $z = -9x^2 - 3y^2, P(2, 1, -39)$

172. $x^2 + 10xyz + y^2 + 8z^2 = 0, P(-1, -1, -1)$

173. $z = \ln(10x^2 + 2y^2 + 1), P(0, 0, 0)$

174. $z = e^{7x^2 + 4y^2},\ P(0, 0, 1)$

175. $xy + yz + zx = 11, P(1, 2, 3)$

176. $x^2 + 4y^2 = z^2, P(3, 2, 5)$

177. $x^3 + y^3 = 3xyz, P\left(1, 2, \frac{3}{2}\right)$

178. $z = axy, P\left(1, \frac{1}{a}, 1\right)$

179. $z = \sin x + \sin y + \sin(x + y), P(0, 0, 0)$

180. $h(x, y) = \ln\sqrt{x^2 + y^2}, P(3, 4)$

181. $z = x^2 - 2xy + y^2, P(1, 2, 1)$

For the following exercises, find parametric equations for the normal line to the surface at the indicated point. (Recall that to find the equation of a line in space, you need a point line, $P_0(x_0, y_0, z_0)$, and a vector $\mathbf{n} = \langle a, b, c \rangle$ that is parallel to the line. Then the

n of the line is $x - x_0 = at, y - y_0 = bt, z - z_0 = ct.$)

$-3x + 9y + 4z = -4, P(1, -1, 2)$

$z = 5x^2 - 2y^2, P(2, 1, 18)$

184. $x^2 - 8xyz + y^2 + 6z^2 = 0, P(1, 1, 1)$

185. $z = \ln(3x^2 + 7y^2 + 1), P(0, 0, 0)$

186. $z = e^{4x^2 + 6y^2}, P(0, 0, 1)$

187. $z = x^2 - 2xy + y^2$ at point $P(1, 2, 1)$

For the following exercises, use the figure shown here.

188. The length of line segment AC is equal to what mathematical expression?

189. The length of line segment BC is equal to what mathematical expression?

190. Using the figure, explain what the length of line segment AB represents.

For the following exercises, complete each task.

191. Show that $f(x, y) = e^{xy}x$ is differentiable at point $(1, 0)$.

192. Find the total differential of the function $w = e^y \cos(x) + z^2$.

193. Show that $f(x, y) = x^2 + 3y$ is differentiable at every point. In other words, show that $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$ where both ε_1 and ε_2 approach zero as $(\Delta x, \Delta y)$ approaches $(0, 0)$.

194. Find the total differential of the function $z = \frac{xy}{y+x}$ where x changes from $10 \text{ to } 10.5$ and y changes from $15 \text{ to } 13$.

195. Let $z = f(x, y) = xe^y$. Compute Δz from $P(1, 2)$ to $Q(1.05, 2.1)$ and then find the approximate change in z from point P to point Q. Recall $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y),$ and dz and Δz are approximately equal.

196. The volume of a right circular cylinder is given by $V(r, h) = \pi r^2 h$. Find the differential dV. Interpret the formula geometrically.

197. See the preceding problem. Use differentials to estimate the volume of aluminum in an enclosed aluminum can with diameter 8.0 cm and height 12 cm if the aluminum is 0.04 cm thick.

198. Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from point $(1, 1)$ to point $(1.01, 0.97)$. Compare this approximation with the actual change in the function.

199. Let $z = f(x, y) = x^2 + 3xy - y^2$. Find the exact change in the function and the approximate change in the function as x changes from $2.00 \text{ to } 2.05$ and y changes from $3.00 \text{ to } 2.96$.

200. The centripetal acceleration of a particle moving in a circle is given by $a(r, v) = \frac{v^2}{r},$ where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration resulting from errors of 3% in v and 2% in r. (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in a is given by $\frac{da}{a}$.)

201. The radius r and height h of a right circular cylinder are measured with possible errors of $4\% \text{ and } 5\%,$ respectively. Approximate the maximum possible percentage error in measuring the volume (Recall that the percentage error is the ratio of the

amount of error over the original amount. So, in this case, the percentage error in V is given by $\frac{dV}{V}$.)

202. The base radius and height of a right circular cone are measured as 10 in. and 25 in., respectively, with a possible error in measurement of as much as 0.1 in. each. Use differentials to estimate the maximum error in the calculated volume of the cone.

203. The electrical resistance R produced by wiring resistors R_1 and R_2 in parallel can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 and R_2 are measured to be 7Ω and 6Ω, respectively, and if these measurements are accurate to within 0.05Ω, estimate the maximum possible error in computing R. (The symbol Ω represents an ohm, the unit of electrical resistance.)

204. The area of an ellipse with axes of length $2a$ and $2b$ is given by the formula $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

205. The period T of a simple pendulum with small oscillations is calculated from the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum and g is the acceleration resulting from gravity. Suppose that L and g have errors of, at most, 0.5% and 0.1%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

206. Electrical power P is given by $P = \frac{V^2}{R}$, where V is the voltage and R is the resistance. Approximate the maximum percentage error in calculating power if $120\ V$ is applied to a $2000 - \Omega$ resistor and the possible percent errors in measuring V and R are 3% and 4%, respectively.

For the following exercises, find the linear approximation of each function at the indicated point.

207. $f(x, y) = x\sqrt{y}, \quad P(1, 4)$

208. $f(x, y) = e^x \cos y; P(0, 0)$

209. $f(x, y) = \arctan(x + 2y), P(1, 0)$

210. $f(x, y) = \sqrt{20 - x^2 - 7y^2}, \quad P(2, 1)$

211. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad P(3, 2, 6)$

212. **[T]** Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2$ at point $(1, 2, 5),$ and graph the surface and the tangent plane at the point.

213. **[T]** Find the equation for the tangent plane to the surface at the indicated point, and graph the surface and the tangent plane: $z = \ln(10x^2 + 2y^2 + 1), P(0, 0, 0).$

214. **[T]** Find the equation of the tangent plane to the surface $z = f(x, y) = \sin(x + y^2)$ at point $\left(\frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}\right),$ and graph the surface and the tangent plane.

Learning Objectives

4.5.1. State the chain rules for one or two independent variables.
4.5.2. Use tree diagrams as an aid to understanding the chain rule for several independent and intermediate variables.
4.5.3. Perform implicit differentiation of a function of two or more variables.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

Chain Rules for One or Two Independent Variables

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x).$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

THEOREM 4.8

Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}, \tag{4.29}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that *f* is differentiable at the point $P\left(x_0, y_0\right)$, where $x_0 = g\left(t_0\right)$ and $y_0 = h\left(t_0\right)$ for a fixed value of t_0. We wish to prove that $z = f(x(t), y(t))$ is differentiable at $t = t_0$ and that Equation 4.29 holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f\left(x_0, y_0\right) + f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right) + E(x, y), \tag{4.30}$$

where $\lim\limits_{(x,y) \to \left(x_0, y_0\right)} \frac{E(x,y)}{\sqrt{\left(x-x_0\right)^2 + \left(y-y_0\right)^2}} = 0.$ We then subtract $z_0 = f\left(x_0, y_0\right)$ from both sides of this equation:

$$\begin{aligned} z(t) - z\left(t_0\right) &= f(x(t), y(t)) - f\left(x\left(t_0\right), y\left(t_0\right)\right) \\ &= f_x\left(x_0, y_0\right)\left(x(t) - x\left(t_0\right)\right) + f_y\left(x_0, y_0\right)\left(y(t) - y\left(t_0\right)\right) + E(x(t), y(t)). \end{aligned}$$

Next, we divide both sides by $t - t_0$:

$$\frac{z(t) - z\left(t_0\right)}{t - t_0} = f_x\left(x_0, y_0\right)\left(\frac{x(t) - x\left(t_0\right)}{t - t_0}\right) + f_y\left(x_0, y_0\right)\left(\frac{y(t) - y\left(t_0\right)}{t - t_0}\right) + \frac{E(x(t), y(t))}{t - t_0}.$$

Then we take the limit as t approaches t_0:

$$\begin{aligned} \lim_{t \to t_0} \frac{z(t) - z\left(t_0\right)}{t - t_0} &= f_x\left(x_0, y_0\right) \lim_{t \to t_0}\left(\frac{x(t) - x\left(t_0\right)}{t - t_0}\right) + f_y\left(x_0, y_0\right) \lim_{t \to t_0}\left(\frac{y(t) - y\left(t_0\right)}{t - t_0}\right) \\ &\quad + \lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0}. \end{aligned}$$

The left-hand side of this equation is equal to dz/dt, which leads to

$$\frac{dz}{dt} = f_x\left(x_0, y_0\right)\frac{dx}{dt} + f_y\left(x_0, y_0\right)\frac{dy}{dt} + \lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0}.$$

The last term can be rewritten as

$$\lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0} = \lim_{t \to t_0} \left(\frac{E(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{t - t_0} \right)$$

$$= \lim_{t \to t_0} \left(\frac{E(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right) \lim_{t \to t_0} \left(\frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{t - t_0} \right).$$

As t approaches t_0, $(x(t), y(t))$ approaches $\left(x\left(t_0\right), y\left(t_0\right)\right)$, so we can rewrite the last product as

$$\lim_{(x,y) \to \left(x_0, y_0\right)} \left(\frac{E(x, y)}{\sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}} \right) \lim_{(x,y) \to \left(x_0, y_0\right)} \left(\frac{\sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}}{t - t_0} \right).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\lim_{(x,y) \to \left(x_0, y_0\right)} \left(\frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{t - t_0} \right) = \lim_{(x,y) \to \left(x_0, y_0\right)} \left(\sqrt{\frac{(x-x_0)^2 + (y-y_0)^2}{(t-t_0)^2}} \right)$$

$$= \lim_{(x,y) \to \left(x_0, y_0\right)} \left(\sqrt{\left(\frac{x-x_0}{t-t_0}\right)^2 + \left(\frac{y-y_0}{t-t_0}\right)^2} \right)$$

$$= \sqrt{\left(\lim_{(x,y) \to \left(x_0, y_0\right)} \left(\frac{x-x_0}{t-t_0} \right) \right)^2 + \left(\lim_{(x,y) \to \left(x_0, y_0\right)} \left(\right. \right.}$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t = t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation 4.29 reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt.$ The variables x and y that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function $f,$ but are dependent variables for the variable $t.$ Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

EXAMPLE 4.26

Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z = f(x, y) = 4x^2 + 3y^2, x = x(t) = \sin t, y = y(t) = \cos t$
b. $z = f(x, y) = \sqrt{x^2 - y^2}, x = x(t) = e^{2t}, y = y(t) = e^{-t}$

[Show Solution]

CHECKPOINT 4.23

Calculate dz/dt given the following functions. Express the final answer in terms of $t.$

$$z = f(x, y) = x^2 - 3xy + 2y^2, x = x(t) = 3 \sin 2t, y = y(t) = 4 \cos 2t$$

It is often useful to create a visual representation of Equation 4.29 for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (Figure 4.34). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

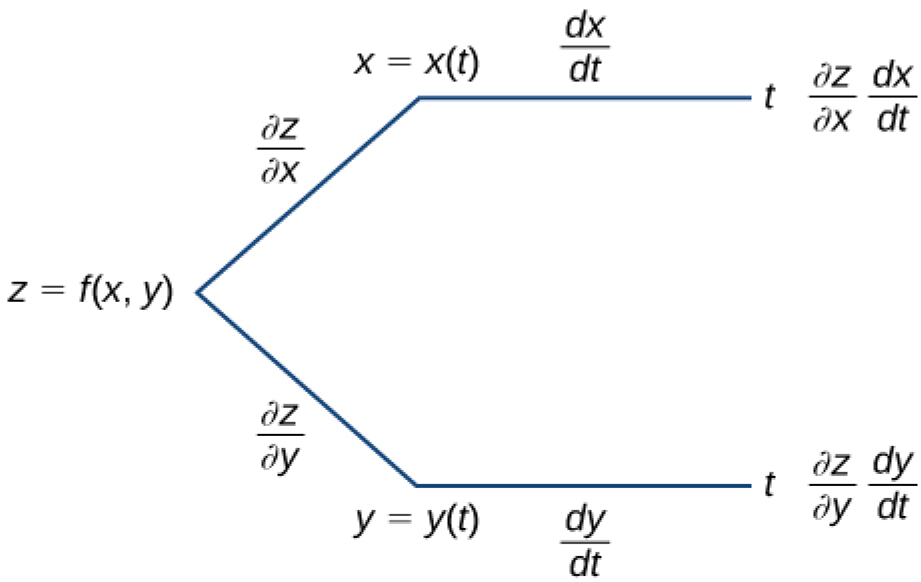


Figure 4.34 Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x, y)$. Since f has two independent variables, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $(\partial z/\partial x) \times (dx/dt)$. The bottom branch is similar: first the y branch, then the t branch. This branch is labeled $(\partial z/\partial y) \times (dy/dt)$. To get the formula for dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us Equation 4.29.

In Chain Rule for Two Independent Variables, $z = f(x, y)$ is a function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are functions of the independent variables u and v.

THEOREM 4.9

Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \tag{4.31}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{4.32}$$

We can draw a tree diagram for each of these formulas as well as follows.

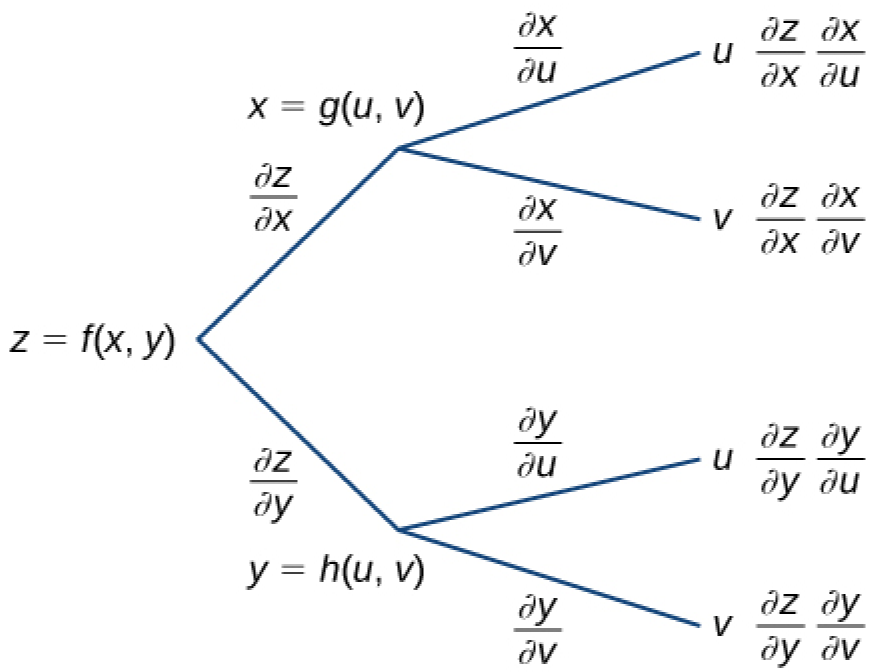


Figure 4.35 Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

To derive the formula for $\partial z/\partial u,$ start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v,$ follow only the branches that end with v and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Chain Rule for One Independent Variable, the left-hand side of the formula for the derivative is not a partial derivative, but in Chain Rule for Two Independent Variables it is. The reason is that, in Chain Rule for One Independent Variable, z is ultimately a function of t alone, whereas in Chain Rule for Two Independent Variables, z is a function of both u and v.

EXAMPLE 4.27

Using the Chain Rule for Two Variables

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x, y) = 3x^2 - 2xy + y^2, x = x(u, v) = 3u + 2v, y = y(u, v) = 4u - v.$$

[Show Solution]

CHECKPOINT 4.24

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x, y) = \frac{2x - y}{x + 3y}, x(u, v) = e^{2u}\cos 3v, y(u, v) = e^{2u}\sin 3v.$$

The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the **generalized chain rule** states.

THEOREM 4.10

Generalized Chain Rule

Let $w = f\left(x_1, x_2, \ldots, x_m\right)$ be a differentiable function of m independent variables, and for each $i \in \{1, \ldots, m\}$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j} \tag{4.33}$$

for any $j \in \{1, 2, \ldots, n\}$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

EXAMPLE 4.28

Using the Generalized Chain Rule

Calculate $\partial w / \partial u$ and $\partial w / \partial v$ using the following functions:

$$\begin{aligned} w &= f(x, y, z) = 3x^2 - 2xy + 4z^2 \\ x &= x(u, v) = e^u \sin v \\ y &= y(u, v) = e^u \cos v \\ z &= z(u, v) = e^u. \end{aligned}$$

[Show Solution]

CHECKPOINT 4.25

Calculate ∂w/∂u and ∂w/∂v given the following functions:
w=f(x,y,z)=x+2y−4z2x−y+3zx=x(u,v)=e2ucos3vy=y(u,v)=e2usin3vz=z(u,v)=e2u.

EXAMPLE 4.29

Drawing a Tree Diagram

Create a tree diagram for the case when
w=f(x,y,z),x=x(t,u,v),y=y(t,u,v),z=z(t,u,v)

and write out the formulas for the three partial derivatives of w.

[Show Solution]

CHECKPOINT 4.26

Create a tree diagram for the case when

w=f(x,y),x=x(t,u,v),y=y(t,u,v)

and write out the formulas for the three partial derivatives of w.

Implicit Differentiation

Recall from Implicit Differentiation that implicit differentiation provides a method for finding dy/dx when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation x2+3y2+4y–4=0 as follows.

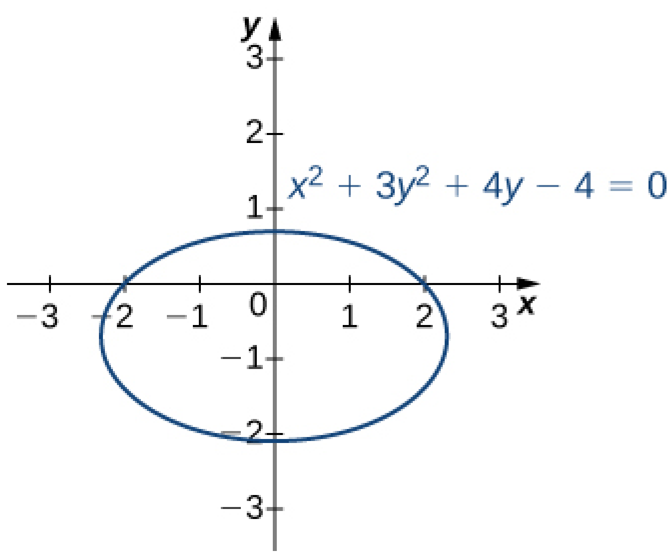


Figure 4.37 Graph of the ellipse defined by x2+3y2+4y–4=0.

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\begin{aligned}\frac{d}{dx}(x^2+3y^2+4y-4)&=\frac{d}{dx}(0)\\2x+6y\frac{dy}{dx}+4\frac{dy}{dx}&=0\\(6y+4)\frac{dy}{dx}&=-2x\\\frac{dy}{dx}&=-\frac{x}{3y+2}.\end{aligned}$$

We can also define a function $z=f(x,y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x,y)=x^2+3y^2+4y-4$. The ellipse $x^2+3y^2+4y-4=0$ can then be described by the equation $f(x,y)=0$. Using this function and the following theorem gives us an alternative approach to calculating dy/dx.

THEOREM 4.11

Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z=f(x,y)$ defines y implicitly as a function $y=g(x)$ of x via the equation $f(x,y)=0$. Then

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y} \tag{4.34}$$

provided $f_y(x,y)\neq 0$.

If the equation $f(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x}=-\frac{\partial f/\partial x}{\partial f/\partial z}\quad\text{and}\quad\frac{\partial z}{\partial y}=-\frac{\partial f/\partial y}{\partial f/\partial z} \tag{4.35}$$

as long as $f_z(x,y,z)\neq 0$.

Equation 4.34 is a direct consequence of Equation 4.31. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x,y)=0$, we can apply the chain rule to find dy/dx:

$$\begin{aligned}\frac{d}{dx}f(x,y)&=\frac{d}{dx}(0)\\\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}&=0\\\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}&=0.\end{aligned}$$

Solving this equation for dy/dx gives Equation 4.34. Equation 4.35 can be derived in a similar fashion.

Let’s now return to the problem that we started before the previous theorem. Using Implicit Differentiation of a Function of Two or More Variables and the function $f(x,y)=x^2+3y^2+4y-4$, we obtain

$$\begin{aligned}\frac{\partial f}{\partial x}&=2x\\\frac{\partial f}{\partial y}&=6y+4.\end{aligned}$$

Then Equation 4.34 gives

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{2x}{6y+4}=-\frac{x}{3y+2},$$

which is the same result obtained by the earlier use of implicit differentiation.

EXAMPLE 4.30

Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2-2xy+y^2+4x-6y-11=0$. What is the equation of the tangent line to the graph of this curve at point $(2,1)$?
b. Calculate $\partial z/\partial x$ and $\partial z/\partial y$, given $x^2e^y-yze^x=0$.

[Show Solution]

CHECKPOINT 4.27

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2+xy-y^2+7x-3y-26=0$. What is the equation of the tangent line to the graph of this curve at point $(3,-2)$?

Section 4.5 Exercises

For the following exercises, use the information provided to solve the problem.

215. Let $w(x,y,z)=xy\cos z$, where $x=t, y=t^2$, and $z=\arcsin t$. Find $\frac{dw}{dt}$.

216. Let $w(t,v)=e^{tv}$ where $t=r+s$ and $v=rs$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

217. If $w=5x^2+2y^2, x=-3s+t$, and $y=s-4t$, find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

218. If $w=xy^2, x=5\cos(2t)$, and $y=5\sin(2t)$, find $\frac{dw}{dt}$.

219. If $f(x,y)=xy, x=r\cos\theta$, and $y=r\sin\theta$, find $\frac{\partial f}{\partial r}$ and express the answer in terms of r and θ.

220. Suppose $f(x,y)=x+y, u=e^x\sin y, x=t^2$, and $y=\pi t$, where $x=r\cos\theta$ and $y=r\sin\theta$. Find $\frac{\partial f}{\partial \theta}$.

For the following exercises, find $\frac{df}{dt}$ using the chain rule and direct substitution.

221. $f(x,y)=x^2+y^2, x=t, y=t^2$

222. $f(x,y)=x^2+y^2, y=t^2, x=t$

223. $f(x,y)=xy, x=1-t, y=1+t$

224. f(x,y)=xy,x=et,y=2et

225. f(x,y)=ln(x+y), x=et,y=et

226. f(x,y)=x4, x=t,y=t

227. Let w(x,y,z)=x2+y2+z2, x=cost,y=sint, and z=et. Express w as a function of t and find dwdt directly. Then, find dwdt using the chain rule.

228. Let z=x2y, where x=t2 and y=t3. Find dzdt.

229. Let u=exsiny, where x=-ln2t and y=πt. Find dudt when x=ln2 and y=π4.

For the following exercises, find dydx using partial derivatives.

230. sin(6x)+tan(8y)+5=0

231. x3+y2x−3=0

232. sin(x+y)+cos(x−y)=4

233. x2−2xy+y4=4

234. xey+yex−2x2y=0

235. x2/3+y2/3=a2/3

236. xcos(xy)+ycosx=2

237. exy+yey=1

238. x2y3+cosy=0

239. Find dzdt using the chain rule where z=3x2y3,x=t4, and y=t2.

240. Let z=3cosx−sin(xy),x=1t, and y=3t. Find dzdt.

241. Let z=e1−xy,x=t1/3, and y=t3. Find dzdt.

242. Find dzdt by the chain rule where z=cosh2(xy),x=12t, and y=et.

243. Let z=xy,x=2cosu, and y=3sinv. Find ∂z∂u and ∂z∂v.

244. Let z=ex2y, where x=uv and y=1v. Find ∂z∂u and ∂z∂v.

245. If z=xyex/y, x=rcosθ, and y=rsinθ, find ∂z∂r and ∂z∂θ when r=2 and θ=π6.

246. Find ∂w∂s if w=4x+y2+z3,x=ers2,y=ln(r+st), and z=rst2.

247. If w=sin(xyz),x=1−3t,y=e1−t, and z=4t, find ∂w∂t.

For the following exercises, use this information: A function f(x,y) is said to be homogeneous of degree n if f(tx,ty)=tnf(x,y). For all homogeneous functions of degree n, the following equation is true: x∂f∂x+y∂f∂y=nf(x,y). Show that the given function is homogeneous and verify that x∂f∂x+y∂f∂y=nf(x,y).

248. f(x,y)=3x2+y2

249. f(x,y)=x2+y2

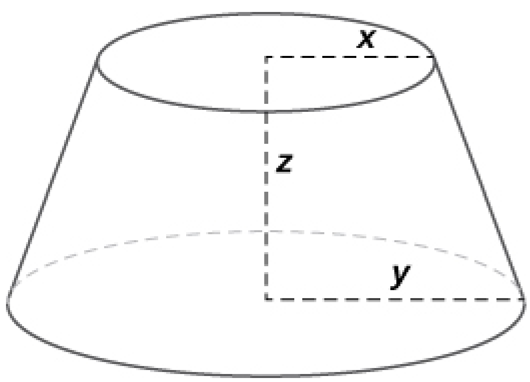
250. f(x,y)=x2y−2y3

251. The volume of a right circular cylinder is given by V(x,y)=πx2y, where x is the radius of the cylinder and *y* is the cylinder height. Suppose x and y are functions of t given by x=12t and y=13t so that xandy are both increasing with time. How fast is the volume increasing when x=2 and y=5?

252. The pressure P of a gas is related to the volume and temperature by the formula PV=kT, where temperature is expressed in kelvins. Express the pressure of the gas as a function of both V and T. Find dPdt when k=1, dVdt=2 cm^3/min, dTdt=12 K/min, V=20 cm^3, and T=20°F.

253. The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

254. The volume of a frustum of a cone is given by the formula V=13πz(x2+y2+xy), where x is the radius of the smaller circle, y is the radius of the larger circle, and z is the height of the frustum (see figure). Find the rate of change of the volume of this frustum when x=10in.,y=12in.,andz=18in.



255. A closed box is in the shape of a rectangular solid with dimensions x,y,andz. (Dimensions are in inches.) Suppose each dimension is changing at the rate of 0.5 in./min. Find the rate of change of the total surface area of the box when x=2in.,y=3in.,andz=1in.

256. The total resistance in a circuit that has three individual resistances represented by x,y, and z is given by the formula R(x,y,z)=xyzyz+xz+xy. Suppose at a given time the x resistance is 100Ω, the *y* resistance is 200Ω, and the z resistance is 300Ω. Also, suppose the x resistance is changing at a rate of 2Ω/min, the y resistance is changing at the rate of 1Ω/min, and the z resistance has no change. Find the rate of change of the total resistance in this circuit at this time.

257. The temperature T at a point (x,y) is T(x,y) and is measured using the Celsius scale. A fly crawls so that its position after t seconds is given by x=1+t and y=2+13t, where xandy are measured in centimeters. The temperature function satisfies Tx(2,3)=4 and Ty(2,3)=3. How fast is the temperature increasing on the fly's path after 3 sec?

258. The xandy components of a fluid moving in two dimensions are given by the following functions: u(x,y)=2y and v(x,y)=−2x; x≥0;y≥0. The speed of the fluid at the point (x,y) is s(x,y)=u(x,y)2+v(x,y)2. Find ∂s∂x and ∂s∂y using the chain rule.

259. Let u=u(x,y,z), where x=x(w,t),y=y(w,t),z=z(w,t),w=w(r,s),andt=t(r,s). Use a tree diagram and the chain rule to find an expression for ∂u∂r.

Learning Objectives

4.7.1. Use partial derivatives to locate critical points for a function of two variables.
4.7.2. Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.
4.7.3. Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

Critical Points

For functions of a single variable, we defined critical points as the values of the function when the derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

DEFINITION

Let $z = f(x, y)$ be a function of two variables that is defined on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a **critical point of a function of two variables** f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

EXAMPLE 4.38

Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$
b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

[Show Solution]

CHECKPOINT 4.34

Find the critical point of the function $f(x, y) = x^3 + 2xy - 2x - 4y$.

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

DEFINITION

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then *f* has a *local maximum* at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a *local maximum value*. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a *global maximum* (also called an *absolute maximum*) at (x_0, y_0).

The function f has a *local minimum* at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a *local minimum value*. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a *global minimum* (also called an *absolute minimum*) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a *local extremum* (see the following figure).

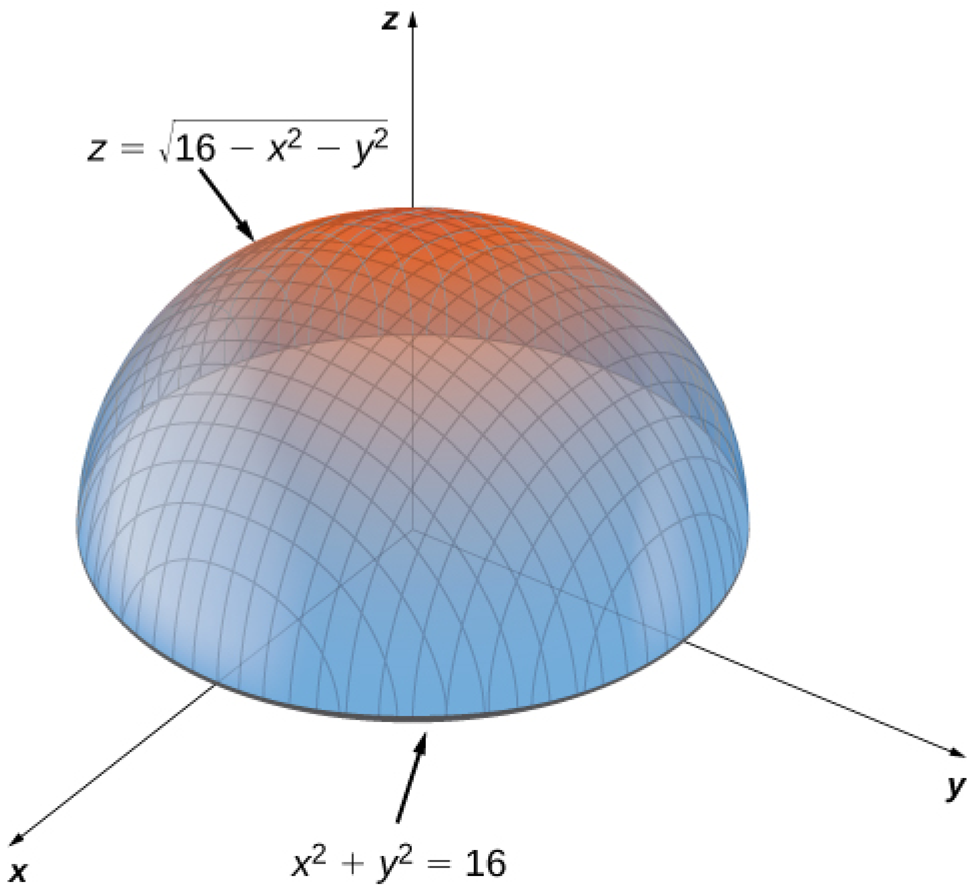


Figure 4.47 The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In Maxima and Minima, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

THEOREM 4.16

Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exists at (x_0, y_0) . If f has a local extremum at (x_0, y_0) , then (x_0, y_0) is a critical point of f.

Second Derivative Test

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point appears in the following figure.

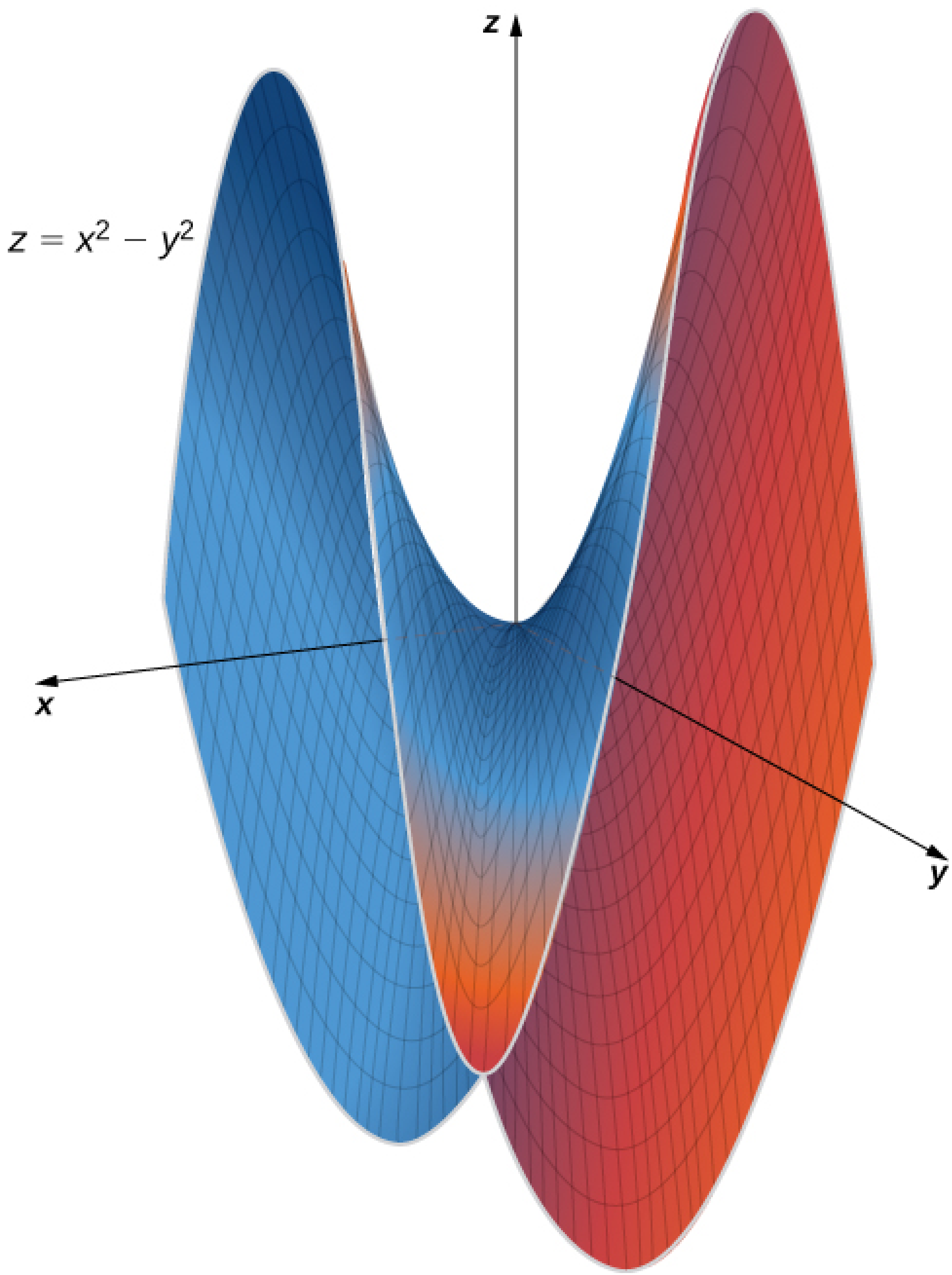


Figure 4.48 Graph of the function $z = x^2 - y^2$. This graph has a saddle point at the origin.

In this graph, the origin is a saddle point. This is because the first partial derivatives of $f(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore the vertical trace corresponding to $y = 0$ is $z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

DEFINITION

Given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_0(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

THEOREM 4.17

Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2. \tag{4.43}$$

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

See Figure 4.49.

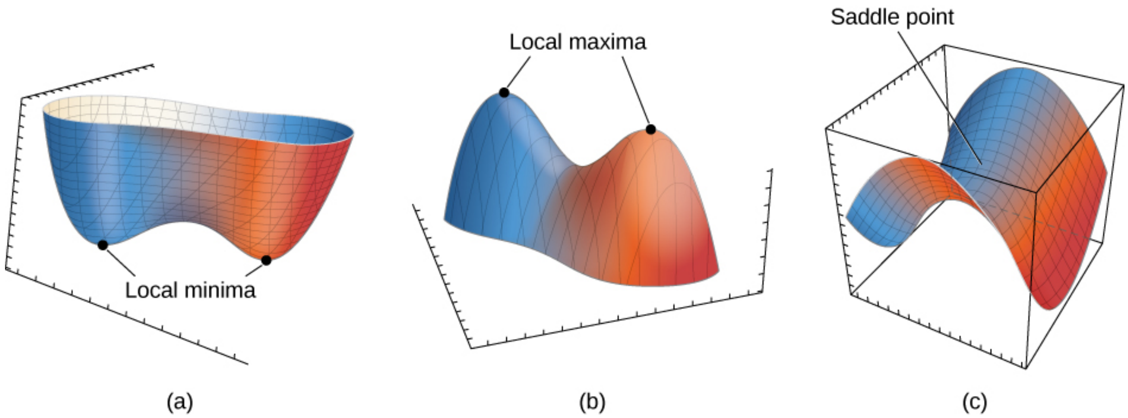


Figure 4.49 The second derivative test can often determine whether a function of two variables has a local minima (a), a local maxima (b), or a saddle point (c).

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

PROBLEM-SOLVING STRATEGY: USING THE SECOND DERIVATIVE TEST FOR FUNCTIONS OF TWO VARIABLES

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$ for each critical point of f.
3. Apply Second Derivative Test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

EXAMPLE 4.39

Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$
b. $g(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

[Show Solution]

CHECKPOINT 4.35

Use the second derivative to find the local extrema of the function

$$f(x, y) = x^3 + 2xy - 6x - 4y^2.$$

Absolute Maxima and Minima

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is *bounded* if all the points in that set can be contained within a ball (or disk) of finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

THEOREM 4.18

Extreme Value Theorem

A continuous function $f(x, y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Now that we know any continuous function f defined on a closed, bounded set attains its extreme values, we need to know how to find them.

THEOREM 4.19

Finding Extreme Values of a Function of Two Variables

Assume $z = f(x, y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

i. The values of f at the critical points of f in D.
ii. The values of f on the boundary of D.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of D, then it is located at an interior point of D. But an interior point (x_0, y_0) of D that's an absolute extremum is also a local extremum; hence, (x_0, y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on D are the extreme values of f on the interior or boundary of D.

PROBLEM-SOLVING STRATEGY: FINDING ABSOLUTE MAXIMUM AND MINIMUM VALUES

Let $z = f(x, y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps $2 \text{ and } 3$.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in Example 4.40. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the set D is a more complicated curve defined by a function $g(x, y) = c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary. The method of Lagrange multipliers is introduced in Lagrange Multipliers.

EXAMPLE 4.40

Finding Absolute Extrema

Use the problem-solving strategy for finding absolute extrema of a function to determine the absolute extrema of each of the following functions:

a. $f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \le x \le 4$ and $0 \le y \le 2$
b. $g(x, y) = x^2 + y^2 + 4x - 6y$ on the domain defined by $x^2 + y^2 \le 16$

[Show Solution]

CHECKPOINT 4.36

Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \le x \le 2$ and $-1 \le y \le 3$.

EXAMPLE 4.41

Chapter Opener: Profitable Golf Balls

Figure 4.56 (credit: modification of work by oatsy40, Flickr)

Pro-T company has developed a profit model that depends on the number *x* of golf balls sold per month (measured in thousands), and the number of hours per month of advertising *y*, according to the function

$$z = f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50{,}000$, and the maximum number of hours of advertising that can be purchased is 25. Find the values of x and y that maximize profit, and find the maximum profit.

[Show Solution]

Section 4.7 Exercises

For the following exercises, find all critical points.

310. $f(x, y) = 1 + x^2 + y^2$

311. $f(x, y) = (3x - 2)^2 + (y - 4)^2$

312. $f(x, y) = x^4 + y^4 - 16xy$

313. $f(x, y) = 15x^3 - 3xy + 15y^3$

For the following exercises, find the critical points of the function by using algebraic techniques (completing the square) or by examining the form of the equation. Verify your results using the partial derivatives test.

314. $f(x, y) = \sqrt{x^2 + y^2 + 1}$

315. $f(x, y) = -x^2 - 5y^2 + 8x - 10y - 13$

316. $f(x, y) = x^2 + y^2 + 2x - 6y + 6$

317. $f(x, y) = \sqrt{x^2 + y^2} + 1$

For the following exercises, use the second derivative test to identify any critical points and determine whether each critical point is a maximum, minimum, saddle point, or none of these.

318. $f(x, y) = -x^3 + 4xy - 2y^2 + 1$

319. $f(x, y) = x^2y^2$

320. $f(x, y) = x^2 - 6x + y^2 + 4y - 8$

321. $f(x, y) = 2xy + 3x + 4y$

322. $f(x, y) = 8xy(x + y) + 7$

323. $f(x, y) = x^2 + 4xy + y^2$

324. $f(x, y) = x^3 + y^3 - 300x - 75y - 3$

325. $f(x, y) = 9 - x^4y^4$

326. $f(x, y) = 7x^2y + 9xy^2$

327. $f(x, y) = 3x^2 - 2xy + y^2 - 8y$

328. $f(x, y) = 3x^2 + 2xy + y^2$

329. $f(x, y) = y^2 + xy + 3y + 2x + 3$

330. $f(x, y) = x^2 + xy + y^2 - 3x$

331. $f(x, y) = x^2 + 2y^2 - x^2y$

332. $f(x, y) = x^2 + y - e^y$

333. $f(x, y) = e^{-(x^2+y^2+2x)}$

334. $f(x, y) = x^2 + xy + y^2 - x - y + 1$

335. $f(x, y) = x^2 + 10xy + y^2$

336. $f(x, y) = -x^2 - 5y^2 + 10x - 30y - 62$

337. $f(x, y) = 120x + 120y - xy - x^2 - y^2$

338. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

339. $f(x, y) = x^2 + x - 3xy + y^3 - 5$

340. $f(x, y) = 2xye^{-x^2-y^2}$

For the following exercises, determine the extreme values and the saddle points. Use a CAS to graph the function.

341. **[T]** $f(x, y) = ye^x - e^y$

342. **[T]** $f(x, y) = x\sin(y)$

343. **[T]** $f(x, y) = \sin(x)\sin(y), x \in (0, 2\pi), y \in (0, 2\pi)$

Find the absolute extrema of the given function on the indicated closed and bounded set R.

344. $f(x, y) = xy - x - 3y$; R is the triangular region with vertices $(0, 0), (0, 4),$ and $(5, 0)$.

345. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2y + 1$ on the region $R = \left\{(x, y) \middle| x^2 + y^2 \le 4\right\}$.

346. $f(x, y) = x^3 - 3xy - y^3$ on $R = \{(x, y): -2 \le x \le 2, -2 \le y \le 2\}$

347. $f(x, y) = \frac{-2y}{x^2+y^2+1}$ on $R = \left\{(x, y): x^2 + y^2 \le 4\right\}$

348. Find three positive numbers the sum of which is 27, such that the sum of their squares is as small as possible.

349. Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

350. Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the plane $x + y + z = 1$.

351. The sum of the length and the girth (perimeter of a cross-section) of a package carried by a delivery service cannot exceed 108 in. Find the dimensions of the rectangular package of largest volume that can be sent.

352. A cardboard box without a lid is to be made with a volume of 4 ft^3. Find the dimensions of the box that requires the least amount of cardboard.

353. Find the point on the surface $f(x, y) = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$. Identify the point on the plane.

354. Find the point in the plane $2x - y + 2z = 16$ that is closest to the origin.

355. A company manufactures two types of athletic shoes: jogging shoes and cross-trainers. The total revenue from x units of jogging shoes and y units of cross-trainers is given by $R(x, y) = -5x^2 - 8y^2 - 2xy + 42x + 102y$, where x and y are in thousands of units. Find the values of *x* and *y* to maximize the total revenue.

356. A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

357. Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 cm.

Learning Objectives

4.8.1. Use the method of Lagrange multipliers to solve optimization problems with one constraint.
4.8.2. Use the method of Lagrange multipliers to solve optimization problems with two constraints.

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

Lagrange Multipliers

Example 4.41 was an applied situation involving maximizing a profit function, subject to certain **constraints.** In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month $(x),$ and a maximum number of advertising hours that could be purchased per month $(y).$ Suppose these were combined into a budgetary constraint, such as $20x + 4y \leq 216,$ that took into account the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is, still, to maximize profit, but now there is a different type of constraint on the values of x and $y.$ This constraint, when combined with the profit function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2,$ is an example of an **optimization problem**, and the function $f(x, y)$ is called the **objective function**. A graph of various level curves of the function $f(x, y)$ follows.

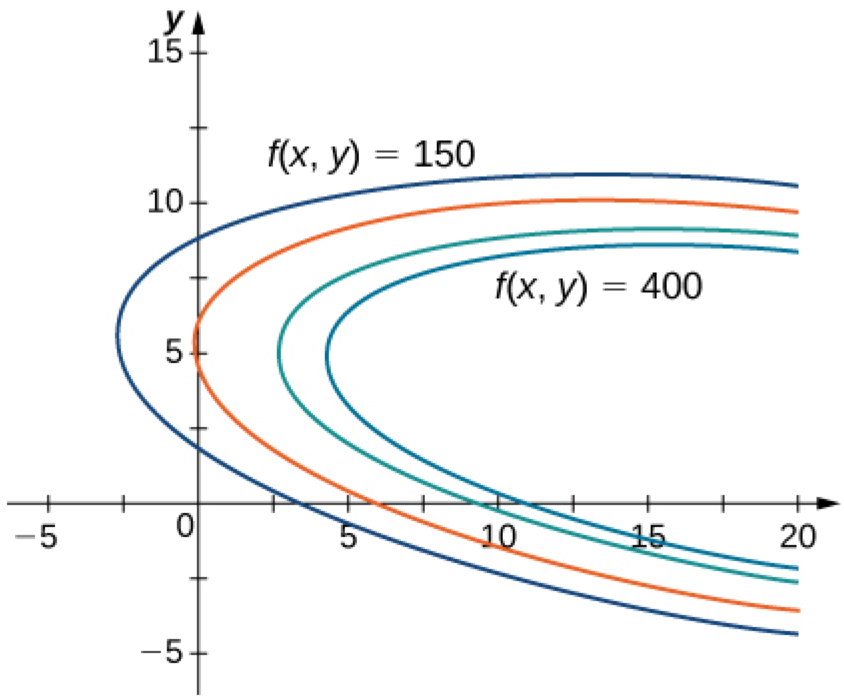


Figure 4.59 Graph of level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and $400.$

In Figure 4.59, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there was no restriction on the number of golf balls the company could produce, or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would be not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x + 4y \leq 216.$ To see how this constraint interacts with the profit function, Figure 4.60 shows the graph of the line $20x + 4y = 216$ superimposed on the previous graph.

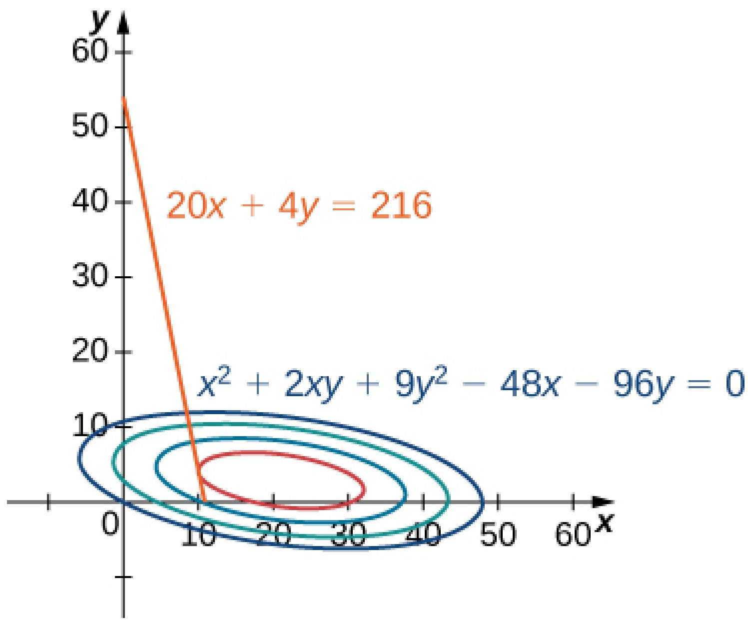


Figure 4.60 Graph of level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and $395.$ The red graph is the constraint function.

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the red line in Figure 4.60. Inspection of this graph reveals that this point exists where the line is tangent to the level curve of $f.$ Trial and error reveals that this profit level seems to be around $395,$ when x and y are both just less than $5.$ We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers**.

THEOREM 4.20

Method of Lagrange Multipliers: One Constraint

Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x, y) = 0.$ Suppose that $f,$ when restricted to points on the curve $g(x, y) = 0,$ has a local extremum at the point (x_0, y_0) and that $\nabla g(x_0, y_0) \neq 0$. Then there is a number λ called a **Lagrange multiplier**, for which

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Proof

Assume that a constrained extremum occurs at the point (x_0, y_0). Furthermore, we assume that the equation $g(x, y) = 0$ can be smoothly parameterized as

$$x = x(s) \text{ and } y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Therefore, the quantity $z = f(x(s), y(s))$ has a relative maximum or relative minimum at $s = 0$, and this implies that $\frac{dz}{ds} = 0$ at that point. From the chain rule,

$$\frac{dz}{ds} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}\right) \cdot \left(\frac{\partial x}{\partial s}\hat{\mathbf{i}} \cdot \frac{\partial y}{\partial s}\hat{\mathbf{j}}\right) = 0,$$

where the derivatives are all evaluated at $s = 0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector $\mathrm{T}(0)$ to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0$, it follows from this equation that

$$\nabla f(x_0, y_0) \cdot \mathrm{T}(0) = 0,$$

which implies that the gradient is either $\mathbf{0}$ or is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x, y) = 0$ is a level curve for the function $g(x, y)$ so that if $\nabla g(x_0, y_0) \neq 0$ then $\nabla g(x_0, y_0)$ is normal to this curve at (x_0, y_0) It follows, then, that there is some scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

□

To apply Method of Lagrange Multipliers: One Constraint to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

PROBLEM-SOLVING STRATEGY: STEPS FOR USING LAGRANGE MULTIPLIERS

1. Determine the objective function $f(x, y)$ and the constraint function $g(x, y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:

$$\begin{aligned} \nabla f\left(x_0, y_0\right) &= \lambda \nabla g\left(x_0, y_0\right) \\ g\left(x_0, y_0\right) &= 0. \end{aligned}$$

3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

EXAMPLE 4.42

Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x, y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$.

[Show Solution]

CHECKPOINT 4.37

Use the method of Lagrange multipliers to find the maximum value of $f(x, y) = 9x^2 + 36xy - 4y^2 - 18x - 8y$ subject to the constraint $3x + 4y = 32$.

Let's now return to the problem posed at the beginning of the section.

EXAMPLE 4.43

Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x + 4y = 216.$ Find the values of x and y that maximize profit, and find the maximum profit.

[Show Solution]

CHECKPOINT 4.38

A company has determined that its production level is given by the Cobb-Douglas function $f(x, y) = 2.5x^{0.45}y^{0.55}$ where x represents the total number of labor hours in 1 year and y represents the total capital input for the company. Suppose 1 unit of labor costs $\$40$ and 1 unit of capital costs $\$50.$ Use the method of Lagrange multipliers to find the maximum value of $f(x, y) = 2.5x^{0.45}y^{0.55}$ subject to a budgetary constraint of $\$500,000$ per year.

In the case of an optimization function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. An example of an optimization function with three variables could be the Cobb-Douglas function in the previous example: $f(x, y, z) = x^{0.2}y^{0.4}z^{0.4},$ where x represents the cost of labor, y represents capital input, and z represents the cost of advertising. The method is the same as for the method with a function of two variables; the equations to be solved are

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= 0. \end{aligned}$$

EXAMPLE 4.44

Lagrange Multipliers with a Three-Variable Optimization Function

Find the minimum of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1$.

[Show Solution]

CHECKPOINT 4.39

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x + y + z$$

subject to the constraint $x^2 + y^2 + z^2 = 1$.

Problems with Two Constraints

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the optimization function, w is a function of three variables:

$$w = f(x, y, z)$$

and it is subject to two constraints:

$$g(x, y, z) = 0 \text{ and } h(x, y, z) = 0.$$

There are two Lagrange multipliers, λ_1 and λ_2, and the system of equations becomes

$$\begin{aligned}
\nabla f\left(x_0, y_0, z_0\right) &= \lambda_1 \nabla g\left(x_0, y_0, z_0\right) + \lambda_2 \nabla h\left(x_0, y_0, z_0\right) \\
g\left(x_0, y_0, z_0\right) &= 0 \\
h\left(x_0, y_0, z_0\right) &= 0.
\end{aligned}$$

EXAMPLE 4.45

Lagrange Multipliers with Two Constraints

Find the maximum and minimum values of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $z^2 = x^2 + y^2$ and $x + y - z + 1 = 0$.

[Show Solution]

CHECKPOINT 4.40

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $2x + y + 2z = 9$ and $5x + 5y + 7z = 29$.

Section 4.8 Exercises

For the following exercises, use the method of Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraints.

358. $f(x, y) = x^2 y; x^2 + 2y^2 = 6$

359. $f(x, y, z) = xyz, x^2 + 2y^2 + 3z^2 = 6$

360. $f(x, y) = xy; 4x^2 + 8y^2 = 16$

361. $f(x, y) = 4x^3 + y^2; 2x^2 + y^2 = 1$

362. $f(x, y, z) = x^2 + y^2 + z^2, x^4 + y^4 + z^4 = 1$

363. $f(x, y, z) = yz + xy, xy = 1, y^2 + z^2 = 1$

364. $f(x, y) = x^2 + y^2, (x - 1)^2 + 4y^2 = 4$

365. $f(x, y) = 4xy, \frac{x^2}{9} + \frac{y^2}{16} = 1$

366. $f(x, y, z) = x + y + z, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

367. $f(x, y, z) = x + 3y - z, x^2 + y^2 + z^2 = 4$

368. $f(x, y, z) = x^2 + y^2 + z^2, xyz = 4$

369. Minimize $f(x, y) = x^2 + y^2$ on the hyperbola $xy = 1$.

370. Minimize $f(x, y) = xy$ on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

371. Maximize $f(x, y, z) = 2x + 3y + 5z$ on the sphere $x^2 + y^2 + z^2 = 19$.

372. Maximize $$\begin{aligned} f(x, y) &= x^2 - y^2; x > 0, y > 0; \\ g(x, y) &= y - x^2 = 0 \end{aligned}$$

373. The curve $x^3 - y^3 = 1$ is asymptotic to the line $y = x$. Find the point(s) on the curve $x^3 - y^3 = 1$ farthest from the line $y = x$.

374. Maximize $U(x, y) = 8x^{4/5}y^{1/5}; 4x + 2y = 12$

375. Minimize $f(x, y) = x^2 + y^2, x + 2y - 5 = 0$.

376. Maximize $f(x, y) = \sqrt{6 - x^2 - y^2}, x + y - 2 = 0$.

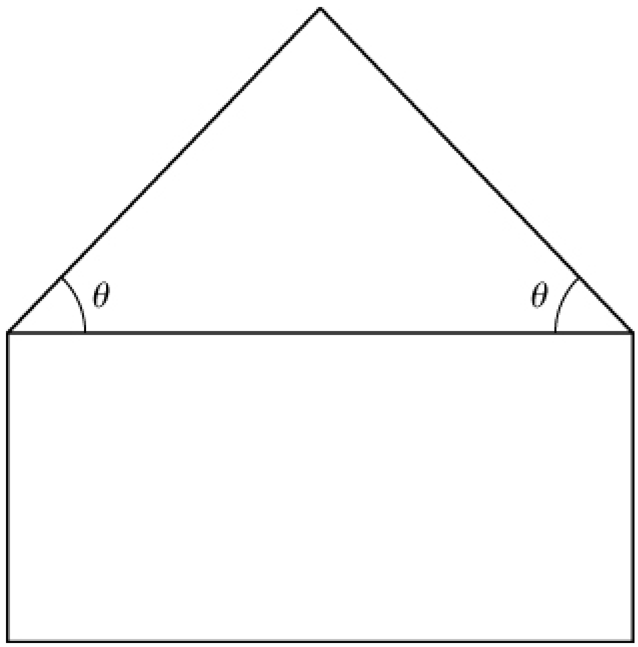
377. Minimize $f(x, y, z) = x^2 + y^2 + z^2, x + y + z = 1$.

378. Minimize $f(x, y) = x^2 - y^2$ subject to the constraint $x - 2y + 6 = 0$.

379. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ when $x + y + z = 9$ and $x + 2y + 3z = 20$.

For the next group of exercises, use the method of Lagrange multipliers to solve the following applied problems.

380. A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the diagram. If the perimeter of the pentagon is 10 in., find the lengths of the sides of the pentagon that will maximize the area of the pentagon.

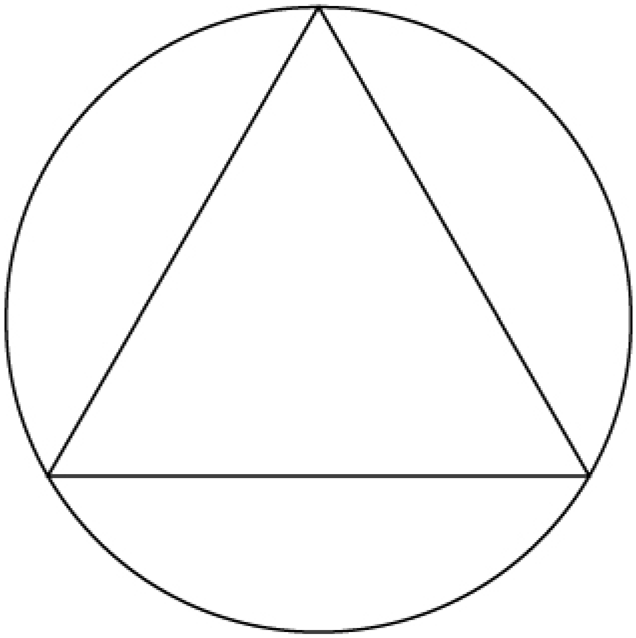


381. A rectangular box without a top (a topless box) is to be made from 12 ft^2 of cardboard. Find the maximum volume of such a box.

382. Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

383. Find the point on the surface $x^2 - 2xy + y^2 - x + y = 0$ closest to the point $(1, 2, -3)$.

384. Show that, of all the triangles inscribed in a circle of radius R (see diagram), the equilateral triangle has the largest perimeter.



385. Find the minimum distance from point $(0, 1)$ to the parabola $x^2 = 4y.$

386. Find the minimum distance from the parabola $y = x^2$ to point $(0, 3).$

387. Find the minimum distance from the plane $x + y + z = 1$ to point $(2, 1, 1).$

388. A large container in the shape of a rectangular solid must have a volume of 480 m^3. The bottom of the container costs \$5/m^2 to construct whereas the top and sides cost \$3/m^2 to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has the minimum cost.

389. Find the point on the line $y = 2x + 3$ that is closest to point $(4, 2).$

390. Find the point on the plane $4x + 3y + z = 2$ that is closest to the point $(1, -1, 1).$

391. Find the maximum value of $f(x, y) = \sin x \sin y,$ where x and y denote the acute angles of a right triangle. Draw the contours of the function using a CAS.

392. A rectangular solid is contained within a tetrahedron with vertices at

$(1, 0, 0), (0, 1, 0), (0, 0, 1),$ and the origin. The base of the box has dimensions $x, y,$ and the height of the box is $z.$ If the sum of $x, y,$ and z is 1.0, find the dimensions that maximizes the volume of the rectangular solid.

393. **[T]** By investing *x* units of labor and *y* units of capital, a watch manufacturer can produce $P(x, y) = 50x^{0.4}y^{0.6}$ watches. Find the maximum number of watches that can be produced on a budget of $\$20,000$ if labor costs \$100/unit and capital costs \$200/unit. Use a CAS to sketch a contour plot of the function.