Part 1: Algebras & Coalgebras

an algebra is a triple (A, μ, η) where A is a vector space and $\mu : A \otimes A \to A \quad \eta : k \to A$ are linear maps satisfying the the axioms that both of the following diagrams commute

associativity

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

unity

$$\begin{tikzcd}
k \otimes A \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

optionally, for commutative algebras

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

where $\tau_{A,A}(a \otimes a') = a' \otimes a$

Now we can define a coalgebra just by reversing all arrows. (C, Δ, ε) is a triple with C a vector space \Delta and \epsilon are linear maps satisfying the axioms

coassociativity

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & C \otimes C \otimes C
\end{tikzcd}$$

counity

$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \varepsilon"] & C \otimes k \\
& C \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

also, optionally cocommutativity

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau_{C,C}"'] & & C \otimes C \end{tikzcd}$$

$$\Delta(x) = x' \otimes x'' = x'' \otimes x'$$

Morphisms of coalgebras If we have two coalgebras: (C, Δ, ε) $(C', \Delta', \varepsilon')$ then a linear map f is a morphism of coalgebras if

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad \varepsilon = \varepsilon' \circ f$$

Which is pretty much the statement that the linear map is structure preserving.

Example: The ground coalgebra (that is, induced by the ground field). Since k is a k-vector space

$\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$

Example: Similar to the above example, k[x] can be given a coalgebra structure

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$$

$$\varepsilon(x) = 0$$
$$\varepsilon(1) = 1$$

\Delta(x) = 1 \otimes x + x \otimes 1

Fact: The dual vector space of coalgebra is an algebra. However the dual vector space of an algebra does not necessarily have a coalgebra structure unless it's finite dimensional.

Example: Let X be a set $C = k[X] = \bigoplus_{x \in X} kx$ is a vector space with basis X. Then the coalgebra structure is given by

$$\Delta(x) = x \otimes x \quad \text{and} \quad \varepsilon(x) = 1$$

The dual algebra C* is the algebra of k-valued functions on X. How does the multiplication work?

Define a linear map: $\lambda : C^* \otimes C^* \to (C \otimes C)^*$ and twist it $\overline{\lambda} : \lambda \circ \tau_{C^*, C^*}$ then

$$(ff')(x) = \mu(f \otimes f')(x) = \overline{\lambda}(f \otimes f')(\Delta(x)) = f(x) f'(x)$$

Part 2: Bialgebras

Suppose H is a vector space that simultaneously has an algebra structure (H, μ, η) and a coalgebra structure (H, Δ, ε)

Fact: The following statements are equivalent

1) The maps \mu and \eta are morphisms of coalgebras
2) the maps \Delta and \epsilon are morphisms of algebras

An H satisfying the conditions above is then called a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ and a morphism of bialgebras is just a morphism of the underlying algebra and coalgebra.

Example. $M(n) = k[x_{11}, ..., x_{nn}]$ (so there are n^2 variables). Also set

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}$$

Which define morphisms of algebras $\Delta : M(n) \to M(n) \otimes M(n)$ and $\varepsilon : M(n) \to k$

Part 3: Hopf Algebras

Important convolution: Let (A, μ, η) be an algebra and (C, Δ, ε) a coalgebra then we can define a convolution on the vector space Hom(C,A). Suppose f, g are linear maps then

$$f \star g := (\text{composition})\ C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

Definition: If H is a bialgebra, then an endomorphism S of H is an antipode if

$$S \star \mathrm{id}_H = \mathrm{id}_H \star S = \eta \circ \varepsilon$$

A bialgebra with an antipode is a Hopf algebra. And a morphism of Hopf algebras is a morphism of the bialgebra commuting with the antipodes. Antipodes, if they exist, are unique

In the finite dimensional case: Hopf algebra H with antipode S means that H* is a Hopf algebra with antipode S* (the transpose of S)

Example: Let G be a monoid and k[G] is a bialgebra. k[G] has an antipode if and only if any element x of G has an inverse-- iff G is a group.

Check: if S exists then

$$xS(x) = S(x)x = \varepsilon(x)1 = 1$$

Since this is true for all x, this means that S(x) = x^-1

SL(2) and GL(2) are Hopf algebras.

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\otimes 2}$$

Can be shown to be coassociative. For the counit we define $\varepsilon(t) = 1$

Let's demonstate an antipode for SL(2):

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If M is a group element then it can be checked:

$$M(SM) = (SM)M = \varepsilon(M)$$

Next topic: The quantum plane

- Let k be a field, and k{x,y} := the free algebra generated by x,y
- I_q two sided ideal generated by yx-qxy , q is some parameter

then

the quantum plane k_q[x,y] is the quotient algebra defined by

$$k_q[x,y] := k\{x,y\}/I_q$$

k_q[x,y] has a grading induced by the grading of the free algebra. That is for the free algebra A we have

$$A = \bigoplus_{i \in \mathbb{N}} A_i \qquad \text{and} \qquad A_i \cdot A_j \subset A_{i+j}$$

In the case of the free algebra, A_i is the subspace of words of length i. That is monomials of degree i.

Then also the ideal I_q is generated by a homogenous degree-2 element. And so the generators of the quantum plane must all be degree 1.

The q-analogue of M(2)

Assumption for the future: q^2 =/= -1
- Two variables x,y subject to the quantum plane relation yx = qxy
- Four variables a, b, c, d commuting with x,y
- Defined new variables x', y', x'', y'' using the following relations:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Then there is an equivalence between the following relations:
* y'x' = qx'y' and y''x''=qx''y''
* ba = qab   db=qbd
ca = qac   dc = qcd
bc = cb   ad-da = (q^-1 - q)bc

Now the algebra M_q(2) is the quotient k{a,b,c,d} / I_q where I_q is generated by the 6 relations from the 2nd bullet above

Ring map f: A --> B such that f \circ \eta_A = \eta_B, f(1) = 1 preserves unit =: algebra morphism an R-point of M_q(2) is a matrix whose entires satisfy the 6 previous relations.

Fact: R-points of M_q(2) are in bijection with the set Hom_Alg(M_q(2), R) of algebra morphisms from M_q(2) to R (we send the generators a,b,c,d to A,B,C,D in R)

R-point := ((A B
C D)) \in R^4

We can also define a quantum determinant

$$\det_q := ad - q^{-1}bc = da - qbc$$

Bialgebra Structure on M_q(2)

Define algebra morphisms:

$$\Delta : M_q(2) \to M_q(2) \otimes M_q(2) \qquad \text{and} \qquad \varepsilon : M_q(2) \to k$$

uniquely determined by

$$\Delta(a) = a \otimes a + b \otimes c \qquad \Delta(b) = a \otimes b + b \otimes d$$

$$\Delta(c) = c \otimes a + d \otimes c \qquad \Delta(d) = c \otimes b + d \otimes d$$

$$\varepsilon(a) = \varepsilon(d) = 1 \qquad \varepsilon(b) = \varepsilon(c) = 0$$

M_q(2) is not commutative and not cocommutative. Also

$$\Delta(\det_q) = \det_q \otimes \det_q \qquad \varepsilon(\det_q) = 1$$

The Hopf Algebras GL_q(2), SL_q(2)

$$GL_q(2) = M_q(2)[t]/(t\det_q - 1)$$

$$SL_q(2) = M_q(2)/(\det_q - 1) = GL_q(2)/(t-1)$$

The previous counit and comultiplication are well-defined on GL_q(2) and SL_q(2) and the antipodes to make it a Hopf algebra are defined by

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

(det here is 1/determinant of the matrix times the matrix)

Certain choices make the antipode an involution

$$\begin{pmatrix} S^{2n}(a) & S^{2n}(b) \\ S^{2n}(c) & S^{2n}(d) \end{pmatrix} = \begin{pmatrix} a & q^{2n}b \\ q^{-2n}c & d \end{pmatrix} = \begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q^{-n} & 0 \\ 0 & q^n \end{pmatrix}$$

Fixing n and letting q be a root of unity of order n then these are two Hopf algebras for which the square of the antipode has order n.

Coaction on the Quantum Plane

k[x_1,...,x_n] is a polynomial algebra. If A is a commutative algebra then:

Hom(k[x_1, ..., x_n], A) ~= A^n

k[x] is called the affine line, and the set Hom(k[x],A) are called the A-points of the line. There is a similar construction for Hom(k[x_1, x_2], A) ~= A^2

Coactions & Comodules

Algebras act on modules and coalgebras coact on comodules.

Let A be an algebra, then an A-module is a pair (M,\mu_M) where M is a vector space and $\mu_M : A \otimes M \to M$ is a linear map satisfying the axioms

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu_M"'] & A \otimes M \arrow[d, "\mu_M"] \\ A \otimes M \arrow[r, "\mu_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} k \otimes M \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "\mu_M"] \\ & M \end{tikzcd}$$

Now let (C, Δ, ε) be a coalgebra. A C-comodule is a pair (N, Δ_N) where N is a vector space and $\Delta_N : N \to C \otimes N$ is a linear map called the coaction of C on N. The linear map satisfies the axioms above with renaming and arrows reversed.

Fact: k[x,y] is a comodule-algebra over the bialgebras M(2) and SL(2).
This statement has a quantum version!

Theorem: There exists a unique M_q(2)-comodule-algebra structure and a unique SL_q(2)-comod-alg structure on the quantum plane A = k_q[x,y] such that

$$\Delta_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

k_q[x,y]_n, the subspace of degree n elements of k_q[x,y] is a subcomodule and

$$k_q[x,y] = \bigoplus_n k_q[x,y]_n$$

Some final comments from Ch. 4 from Kassel on Hopf *-algebras

Say we have a complex Hopf algebra (H, ...). To make this a Hopf *-algebra we need an antilinear involution * satisfying:

1) the *-map is an algebra morphism from H into H^op, as well as a morphism of real coalgebras
2) S(S(x)*)* = x for all x \in H

A Hopf algebra H has a *-alg structure if and only if there exists an antilinear automorphism \gamma of H such that

1) \gamma is a morphism of real algebras and an antimorphism of real coalgebras
2) $\gamma^2 = (S\gamma)^2 = \mathrm{id}_H$

As you can probably guess, SL_q(2) and GL_q(2) both have Hopf *-algebra structures given by

a* = td,   b* = -qtc,   c* = -q^{-1}tb,   d* = ta,   t* = t^{-1}