From Topological Quantum Field Theories to Topological Quantum Computing

(Based on Eric Samperton's Lectures)

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A Brief History of TQFTs to TQC

From TQFT to TQC, a brief history
I. Atiyah-Witten
II. Reshetikhin-Turaev + Turaev
III. Turaev-Viro + Barrett-Westbury
IV. Kitaev-Freedman (Laid the Foundations of Topological Quantum Computing.)
V. Freedman-Kitaev-Larsen-Wang (Showed TQFTs are equivalent to the standard circuit model of BQP. Algorithms approximating Jones Polynomials of knots.)
VI. Levin-Wen (Generalization of Toric Codes to Arbitrary Unitary Fusion Categories.)

Kitaev's Motivation for Introducing Toric Codes

Freedman-Kitaev laid the foundations of TQC. Kitaev camed from the condensed matter physics side where he was thinking of ways to build fault-tolerant quantum computers using anyons, like toric code. But anyons arise in exotic QFTs.

Freedman OTOH knew the mathematical background and speculated whether NP-complete problems can be solved by QFT computers in P.

Kitaev's motivation for introducing toric codes (and generalizations to finite groups) was to address fault-tolerance USING HARDWARE. He does not use the language of TQFTs directly, but was clearly inspired by it, since anyons were understood to be "particles" that can arise in certain exotic QFTs.

I. Atiyah-Witten

1988 - Atiyah defines topological quantum field theory. This was mathematically rigorous! Uses the language of cobordisms and functors.

In mathematics, cobordism is a fundamental equivalence relation on the class of compact manifolds of the same dimension, set up using the concept of the boundary (French bord, giving cobordism) of a manifold. Two manifolds of the same dimension are cobordant if their disjoint union is the boundary of a compact manifold one dimension higher.

Atiyah's work was inspired by Witten on general and not-entirely rigorous supersymmetric quantum field theory, and Segal's axioms for conformal field theory.
In fact, $1 + 1$-dimensional CFT is equivalent to $2 + 1$ dimensional QFT but we can discuss that later.

TQFT in a nutshell

$\mathbb{K}$: A field (or other unital commutative ring.)

$\mathrm{Cob}(d)$: d-dimensional oriented cobordism category.

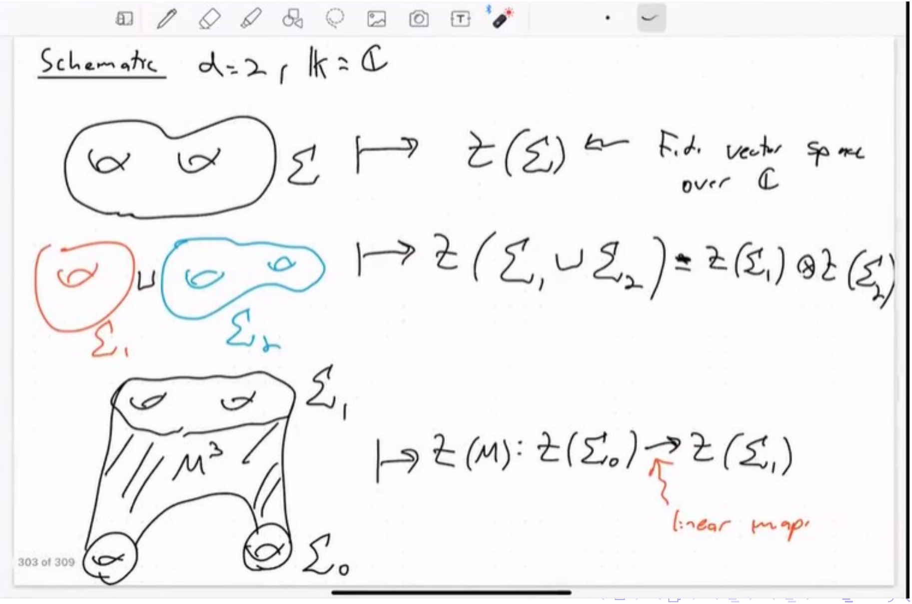
$\mathrm{Objects}(\mathrm{Cob}(d))$: Oriented, smooth, closed d-manifolds.

$\mathrm{Mor}(\mathrm{Cob}(d))$: Oriented, smooth $(d+1)$-manifolds M with $\partial M = \Sigma_0 \to \Sigma_1$. M is a morphism $M : \Sigma_0 \to \Sigma_1$.

$\otimes$ will represent a disjoint union.

A $(2+1)$-dimensional TQFT is a $\otimes$-respecting linearization of $\mathrm{Cob}(d)$, i.e., a $\otimes$-Functor $Z : \mathrm{Cob}(d) \to \mathrm{Vec}(\mathbb{K})$.

Schematic for $d = 2$ and $\mathbb{K} = \mathbb{C}$



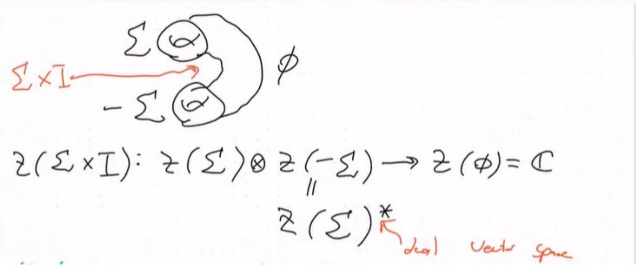
Hermitian and Unitary TQFTs

If $\mathbb{K} = \mathbb{C}$, we can ask

$$Z(-M) = Z(M)^*$$

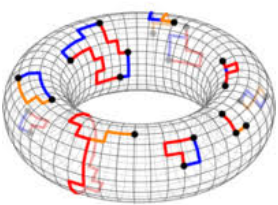
Manifold M with reversed orientation and swapped boundary pieces. For all M, if this holds, then we call the TQFT Hermitian.

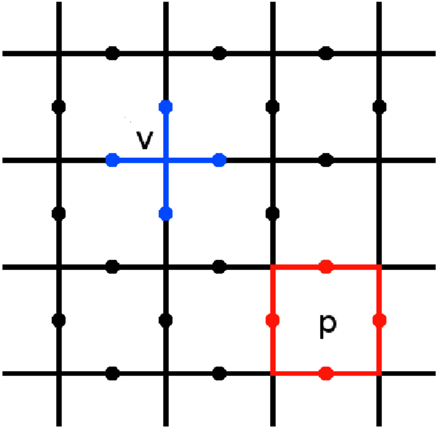
Unitarity in TQFTs



If this pairing is positive-definite and Z is Hermitian, then we say that Z is unitary. If Z is unitary then $Z(\Sigma)$ is a Hilbert space.

Coming Up Next: Kitaev's Toric Codes





The End!