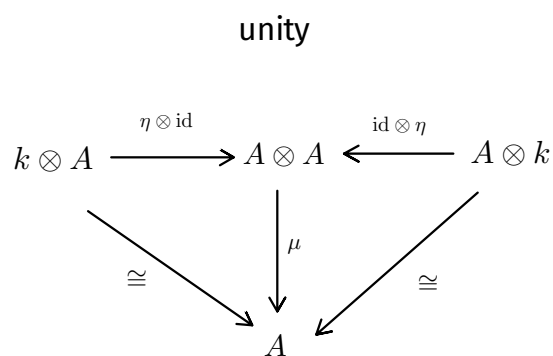
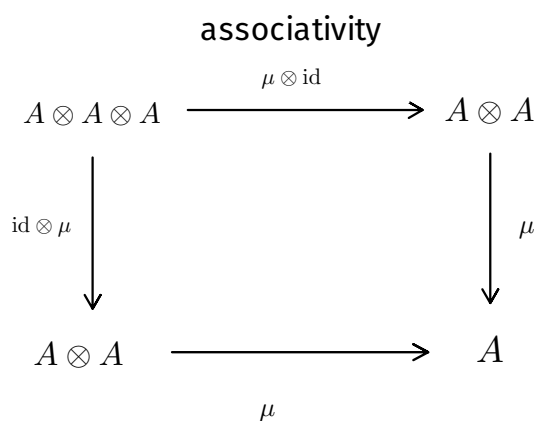
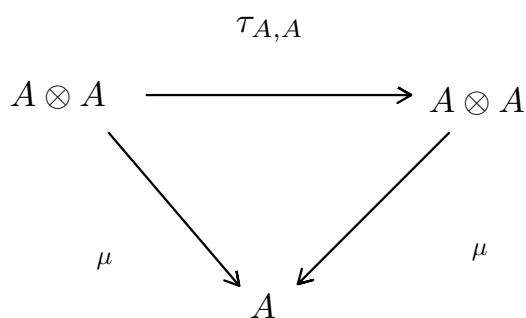


## Part 1: Algebras & Coalgebras

an algebra is a triple  $(A, \mu, \eta)$  where  $A$  is a vector space and  $\mu : A \otimes A \rightarrow A$   $\eta : k \rightarrow A$  are linear maps satisfying the the axioms that both of the following diagrams commute

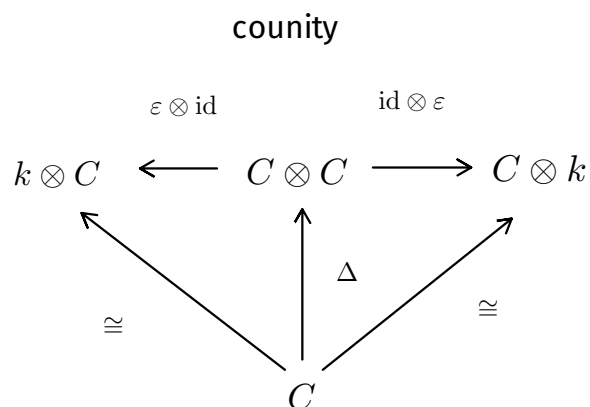
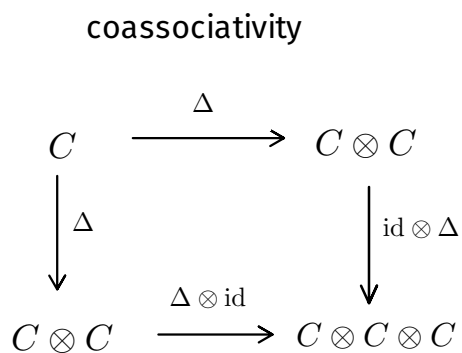


optionally, for commutative algebras



where  $\tau_{A,A}(a \otimes a') = a' \otimes a$

Now we can define a coalgebra just by reversing all arrows.  $(C, \Delta, \varepsilon)$  is a triple with  $C$  a vector space  $\Delta$  and  $\varepsilon$  are linear maps satisfying the axioms



also, optionally cocommutativity

$$\begin{array}{ccc}
 & C & \\
 \Delta \swarrow & & \searrow \Delta \\
 C \otimes C & \xrightarrow{\tau_{C,C}} & C \otimes C
 \end{array}
 \quad \Delta(x) = x' \otimes x'' = x'' \otimes x'$$

**Morphisms of coalgebras** If we have two coalgebras:  $(C, \Delta, \varepsilon)$   $(C', \Delta', \varepsilon')$  then a linear map  $f$  is a morphism of coalgebras if

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad \varepsilon = \varepsilon' \circ f$$

Which is pretty much the statement that the linear map is structure preserving.

**Example:** The ground coalgebra (that is, induced by the ground field). Since  $k$  is a  $k$ -vector space  $\Delta(1) = 1 \otimes 1$  and  $\varepsilon(1) = 1$

**Example:** Similar to the above example,  $k[x]$  can be given a coalgebra structure

$$\begin{aligned}
 \Delta(x^n) &= \sum_{k=0}^n \binom{n}{k} x^k \otimes x^{n-k} & \varepsilon(x) &= 0 \\
 & & \varepsilon(1) &= 1 \\
 \Delta(x) &= 1 \otimes x + x \otimes 1
 \end{aligned}$$

**Fact:** The dual vector space of coalgebra is an algebra. However the dual vector space of an algebra does not necessarily have a coalgebra structure unless it's finite dimensional.

**Example:** Let  $X$  be a set  $C = k[X] = \bigoplus_{x \in X} kx$  is a vector space with basis  $X$ . Then the coalgebra structure is given by

$$\Delta(x) = x \otimes x \quad \text{and} \quad \varepsilon(x) = 1$$

The dual algebra  $C^*$  is the algebra of  $k$ -valued functions on  $X$ . How does the multiplication work?

Define a linear map:  $\lambda : C^* \otimes C^* \rightarrow (C \otimes C)^*$  and twist it  $\bar{\lambda} : \lambda \circ \tau_{C^*, C^*}$  then

$$(ff')(x) = \mu(f \otimes f')(x) = \bar{\lambda}(f \otimes f')(\Delta(x)) = f(x)f'(x)$$

## Part 2: Bialgebras

Suppose  $H$  is a vector space that simultaneously has an algebra structure  $(H, \mu, \eta)$  and a coalgebra structure  $(H, \Delta, \varepsilon)$

**Fact:** The following statements are equivalent

- 1) The maps  $\mu$  and  $\eta$  are morphisms of coalgebras
- 2) the maps  $\Delta$  and  $\varepsilon$  are morphisms of algebras

An  $H$  satisfying the conditions above is then called a bialgebra  $(H, \mu, \eta, \Delta, \varepsilon)$  and a morphism of bialgebras is just a morphism of the underlying algebra and coalgebra.

**Example.**  $M(n) = k[x_{11}, \dots, x_{nn}]$  (so there are  $n^2$  variables). Also set

$$\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}$$

Which define morphisms of algebras  $\Delta : M(n) \rightarrow M(n) \otimes M(n)$  and  $\varepsilon : M(n) \rightarrow k$

## Part 3: Hopf Algebras

**Important convolution:** Let  $(A, \mu, \eta)$  be an algebra and  $(C, \Delta, \varepsilon)$  a coalgebra then we can define a convolution on the vector space  $\text{Hom}(C, A)$ . Suppose  $f, g$  are linear maps then

$$f \star g := (\text{composition}) \quad C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

**Definition:** If  $H$  is a bialgebra, then an endomorphism  $S$  of  $H$  is an antipode if

$$S \star \text{id}_H = \text{id}_H \star S = \eta \circ \varepsilon$$

A bialgebra with an antipode is a Hopf algebra. And a morphism of Hopf algebras is a morphism of the bialgebra commuting with the antipodes. Antipodes, if they exist, are unique

In the finite dimensional case: Hopf algebra  $H$  with antipode  $S$  means that  $H^*$  is a Hopf algebra with antipode  $S^*$  (the transpose of  $S$ )

**Example:** Let  $G$  be a monoid and  $k[G]$  is a bialgebra.  $k[G]$  has an antipode if and only if any element  $x$  of  $G$  has an inverse-- iff  $G$  is a group.

Check: if  $S$  exists then

$$xS(x) = S(x)x = \varepsilon(x)1 = 1$$

Since this is true for all  $x$ , this means that  $S(x) = x^{-1}$

SL(2) and GL(2) are Hopf algebras.

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\otimes 2}$$

Can be shown to be coassociative. For the counit we define  $\varepsilon(t) = 1$

Let's demonstrate an antipode for SL(2):

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If M is a group element then it can be checked:

$$M(SM) = (SM)M = \varepsilon(M)$$

## Next topic: The quantum plane

- Let  $k$  be a field, and  $k\{x,y\} :=$  the free algebra generated by  $x,y$
- $I_q$  two sided ideal generated by  $yx-qxy$ ,  $q$  is some parameter

then

the quantum plane  $k_q[x,y]$  is the quotient algebra defined by

$$k_q[x,y] := k\{x,y\}/I_q$$

$k_q[x,y]$  has a grading induced by the grading of the free algebra. That is for the free algebra  $A$  we have

$$A = \bigoplus_{i \in \mathbb{N}} A_i \quad \text{and} \quad A_i \cdot A_j \subset A_{i+j}$$

In the case of the free algebra,  $A_i$  is the subspace of words of length  $i$ . That is monomials of degree  $i$ .

Then also the ideal  $I_q$  is generated by a homogenous degree-2 element. And so the generators of the quantum plane must all be degree 1.

## The $q$ -analogue of $M(2)$

Assumption for the future:  $q^2 \neq -1$

- Two variables  $x,y$  subject to the quantum plane relation  $yx = qxy$
- Four variables  $a, b, c, d$  commuting with  $x,y$
- Defined new variables  $x', y', x'', y''$  using the following relations:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Then there is an equivalence between the following relations:

$$* y'x' = qx'y' \text{ and } y''x'' = qx''y''$$

$$* ba = qab \quad db = qbd$$

$$ca = qac \quad dc = qcd$$

$$bc = cb \quad ad - da = (q^{-1} - q)bc$$

Now the algebra  $M_q(2)$  is the quotient  $k\{a,b,c,d\} / I_q$  where  $I_q$  is generated by the 6 relations from the 2nd bullet above

Ring map  $f: A \rightarrow B$  such that  $f \circ \eta_A = \eta_B, f(1) = 1$  preserves unit  $\Rightarrow$  algebra morphism

an R-point of  $M_q(2)$  is a matrix whose entries satisfy the 6 previous relations.

Fact: R-points of  $M_q(2)$  are in bijection with the set  $\text{Hom\_Alg}(M_q(2), R)$  of algebra morphisms from  $M_q(2)$  to  $R$  (we send the generators  $a, b, c, d$  to  $A, B, C, D$  in  $R$ )

$$\text{R-point} := \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \in R^4$$

We can also define a quantum determinant

$$\det_q := ad - q^{-1}bc = da - qbc$$

## Bialgebra Structure on $M_q(2)$

Define algebra morphisms:

$$\Delta : M_q(2) \rightarrow M_q(2) \otimes M_q(2) \quad \text{and} \quad \varepsilon : M_q(2) \rightarrow k$$

uniquely determined by

$$\Delta(a) = a \otimes a + b \otimes c \quad \Delta(b) = a \otimes b + b \otimes d$$

$$\Delta(c) = c \otimes a + d \otimes c \quad \Delta(d) = c \otimes b + d \otimes d$$

$$\varepsilon(a) = \varepsilon(d) = 1 \quad \varepsilon(b) = \varepsilon(c) = 0$$

$M_q(2)$  is not commutative and not cocommutative. Also

$$\Delta(\det_q) = \det_q \otimes \det_q \quad \varepsilon(\det_q) = 1$$

## The Hopf Algebras $GL_q(2)$ , $SL_q(2)$

$$GL_q(2) = M_q(2)[t]/(t \det_q - 1)$$

$$SL_q(2) = M_q(2)/(\det_q - 1) = GL_q(2)/(t - 1)$$

The previous counit and comultiplication are well-defined on  $GL_q(2)$  and  $SL_q(2)$  and the antipodes to make it a Hopf algebra are defined by

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

(det here is  $1/\text{determinant of the matrix times the matrix}$ )

Certain choices make the antipode an involution

$$\begin{pmatrix} S^{2n}(a) & S^{2n}(b) \\ S^{2n}(c) & S^{2n}(d) \end{pmatrix} = \begin{pmatrix} a & q^{2n}b \\ q^{-2n}c & d \end{pmatrix} = \begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q^{-n} & 0 \\ 0 & q^n \end{pmatrix}$$

Fixing  $n$  and letting  $q$  be a root of unity of order  $n$  then these are two Hopf algebras for which the square of the antipode has order  $n$ .

## Coaction on the Quantum Plane

$k[x_1, \dots, x_n]$  is a polynomial algebra. If  $A$  is a commutative algebra then:

$$\text{Hom}(k[x_1, \dots, x_n], A) \cong A^n$$

$k[x]$  is called the affine line, and the set  $\text{Hom}(k[x], A)$  are called the  $A$ -points of the line. There is a similar construction for  $\text{Hom}(k[x_1, x_2], A) \cong A^2$

## Coactions & Comodules

Algebras act on modules and coalgebras coact on comodules.

Let  $A$  be an algebra, then an  $A$ -module is a pair  $(M, \mu_M)$  where  $M$  is a vector space and  $\mu_M : A \otimes M \rightarrow M$  is a linear map satisfying the axioms

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mu \otimes \text{id}} & A \otimes M \\ \text{id} \otimes \mu_M \downarrow & & \downarrow \mu_M \\ A \otimes M & \xrightarrow{\mu_M} & M \end{array} \qquad \begin{array}{ccc} k \otimes M & \xrightarrow{\eta \otimes \text{id}} & A \otimes M \\ & \searrow \cong & \downarrow \mu_M \\ & & M \end{array}$$

Now let  $(C, \Delta, \varepsilon)$  be a coalgebra. A  $C$ -comodule is a pair  $(N, \Delta_N)$  where  $N$  is a vector space and  $\Delta_N : N \rightarrow C \otimes N$  is a linear map called the coaction of  $C$  on  $N$ . The linear map satisfies the axioms above with renaming and arrows reversed.

Fact:  $k[x, y]$  is a comodule-algebra over the bialgebras  $M(2)$  and  $SL(2)$ .

This statement has a quantum version!

Theorem: There exists a unique  $M_q(2)$ -comodule-algebra structure and a unique  $SL_q(2)$ -comod-alg structure on the quantum plane  $A = k_q[x, y]$  such that

$$\Delta_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

$k_q[x,y]_n$ , the subspace of degree  $n$  elements of  $k_q[x,y]$  is a subcomodule and

$$k_q[x,y] = \bigoplus_n k_q[x,y]_n$$

### Some final comments from Ch. 4 from Kassel on Hopf $\ast$ -algebras

Say we have a complex Hopf algebra  $(H, \dots)$ . To make this a Hopf  $\ast$ -algebra we need an antilinear involution  $\ast$  satisfying:

- 1) the  $\ast$ -map is an algebra morphism from  $H$  into  $H^{\text{op}}$ , as well as a morphism of real coalgebras
- 2)  $S(S(x)\ast)\ast = x$  for all  $x \in H$

A Hopf algebra  $H$  has a  $\ast$ -alg structure if and only if there exists an antilinear automorphism  $\gamma$  of  $H$  such that

- 1)  $\gamma$  is a morphism of real algebras and an antimorphism of real coalgebras
- 2)  $\gamma^2 = (S\gamma)^2 = \text{id}_H$

As you can probably guess,  $SL_q(2)$  and  $GL_q(2)$  both have Hopf  $\ast$ -algebra structures given by

$$a^\ast = td, \quad b^\ast = -qtc, \quad c^\ast = -q^{-1}tb, \quad d^\ast = ta, \quad t^\ast = t^{-1}$$