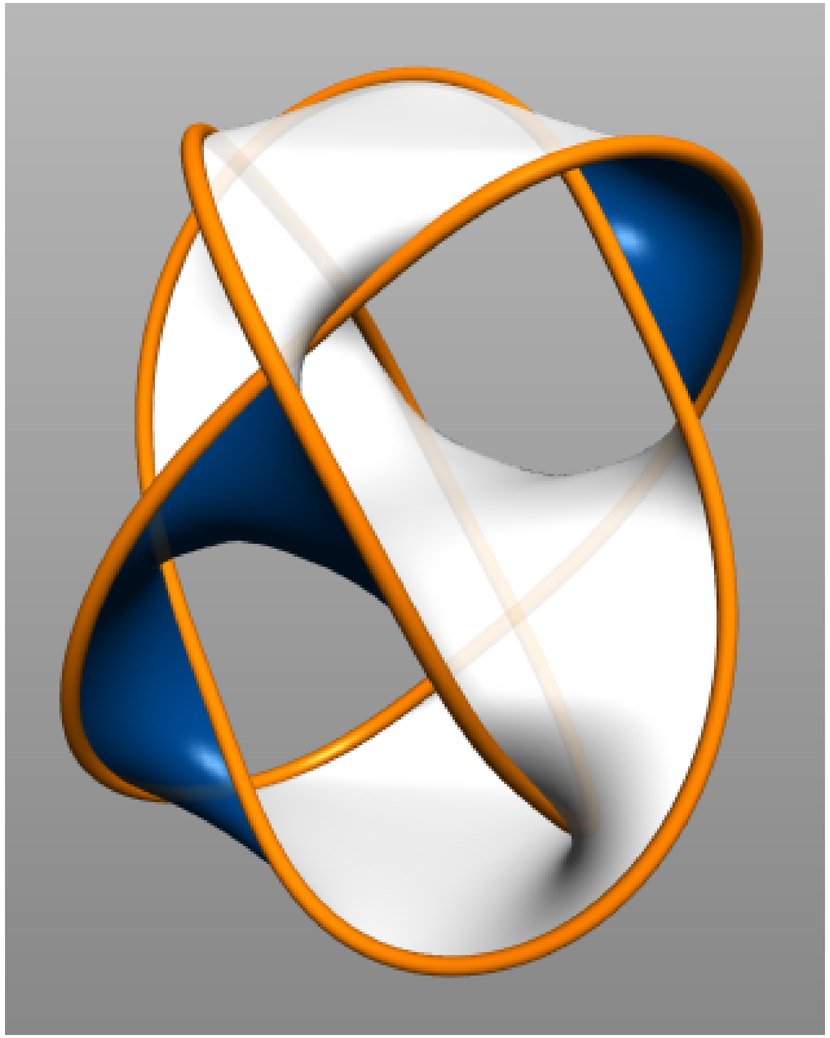


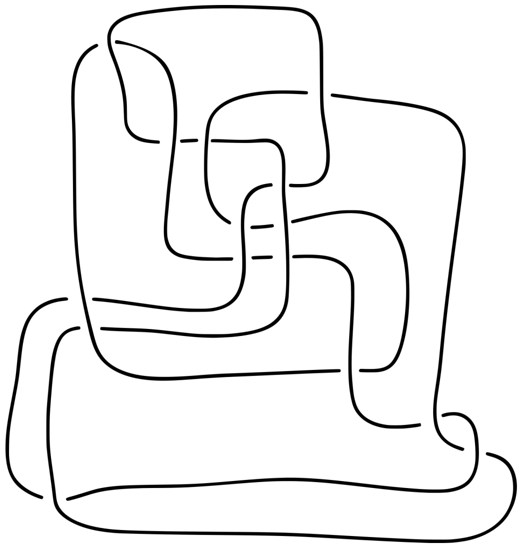
Haken's algorithm

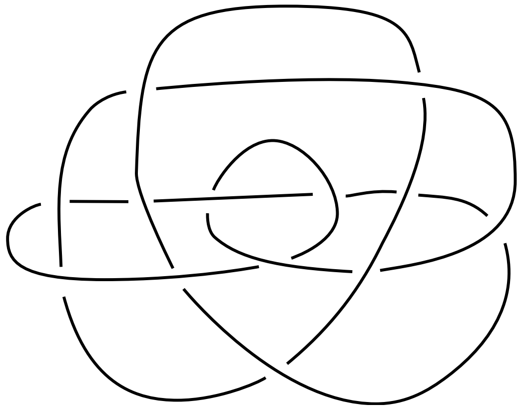
Recognizing the unknot

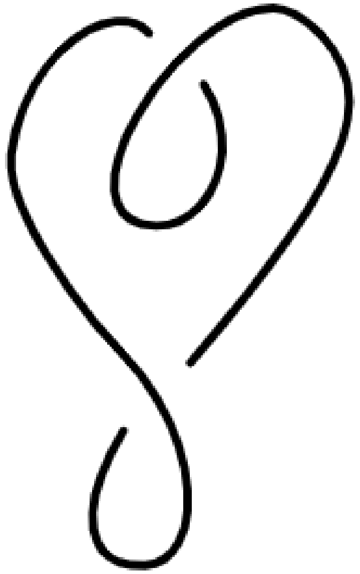
Dan Tobias

Introduction

Which of these are the unknot?

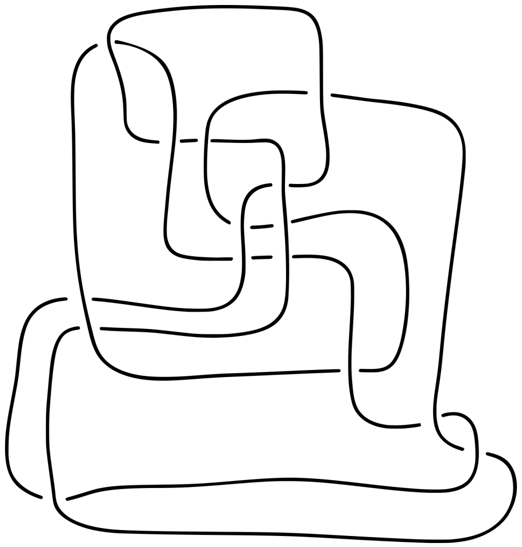


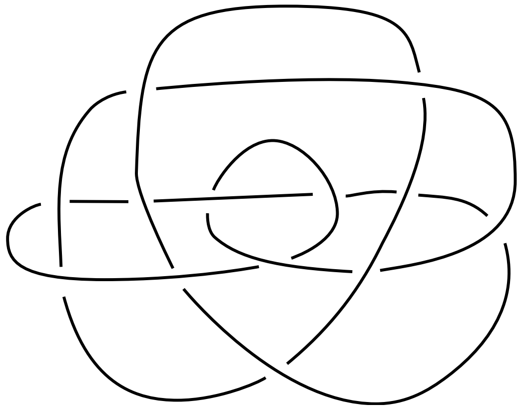


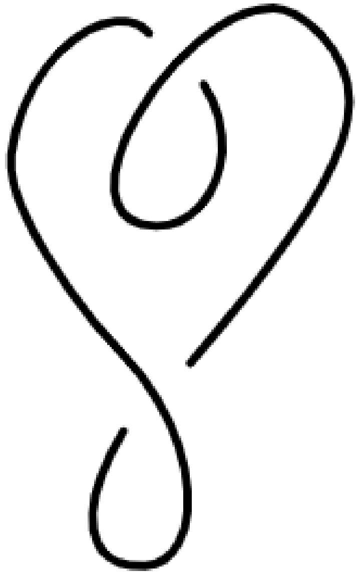


Introduction

All of them!







- Thistlethwaite unknot
- Ochiai's unknot
- Simple unknot

Introduction

If K is a knot in S^3, how can we decide if K is the unknot?

- We want to focus on determining if our knot bounds an embedded disk since the only knot which does so is the unknot
- Analyze the knot complement using normal surface theory to find this disk
- Recast this problem as an integer programming problem, and examine the solutions to see if we bound a disk

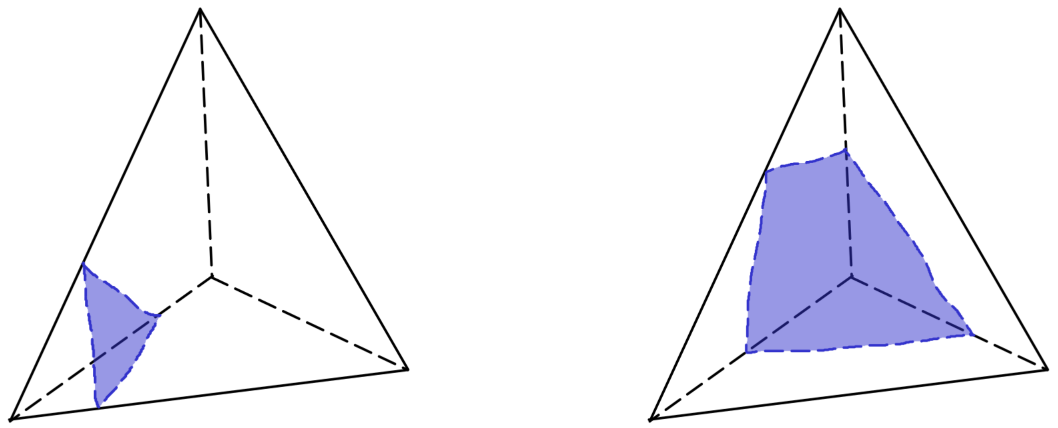
Introduction

1. Triangulate the knot compliment $S^3 \setminus K$
2. Construct finitely many fundamental solutions to the normal surface equations
3. Filter out the solutions which don't mean the quadrilateral condition
4. Check if any of the solutions are disks using Euler characteristic
5. Check if the boundary of this disk is essential on the boundary of knot compliment

Normal surfaces

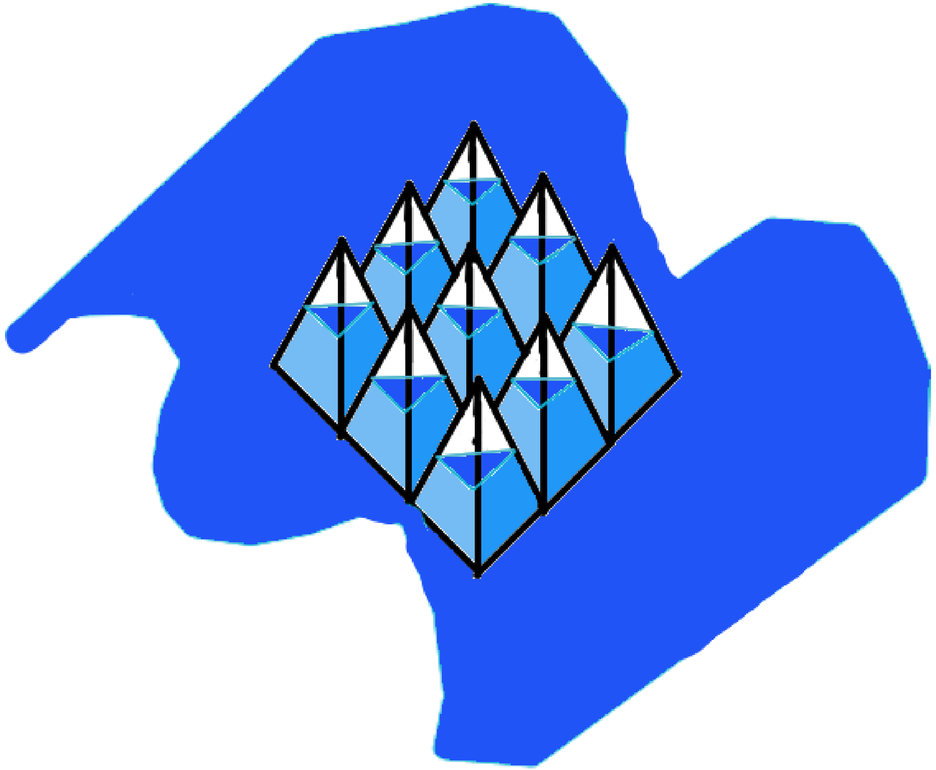
Start by defining two types of normal disks:

- Normal triangles are disks in a 3-simplex which meets three edges and three faces of the simplex
- Normal quadrilaterials are disks in a 3-simplex which meet four faces and four edges



Normal surfaces

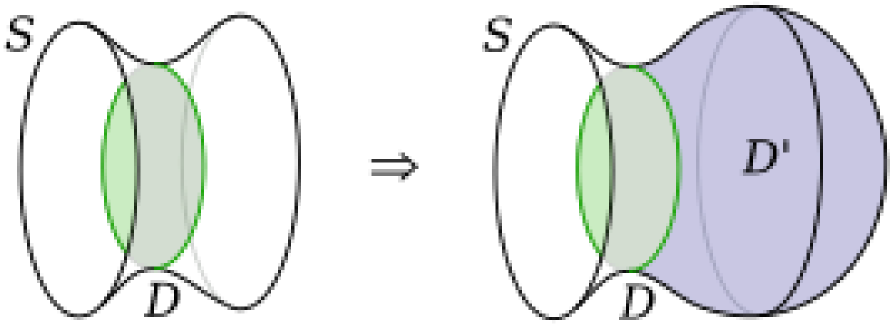
A normal surface in a triangulated 3-manifold then is one which intersects each 3-simplex in a disjoint union of normal triangles and quadrilaterals.

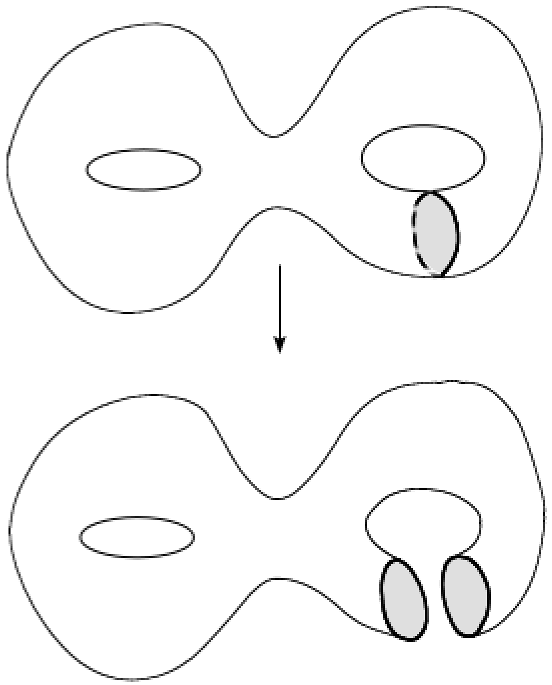


Normal surfaces

- A compressing disk for a surface F is an embedded disk where the boundary of the disk lies in ∂F
- If the boundary of the disk doesn't bound a disk on F, then it is a non-trivial compressing disk
- A surface with only trivial compressing disks is called incompressible

Normal surfaces





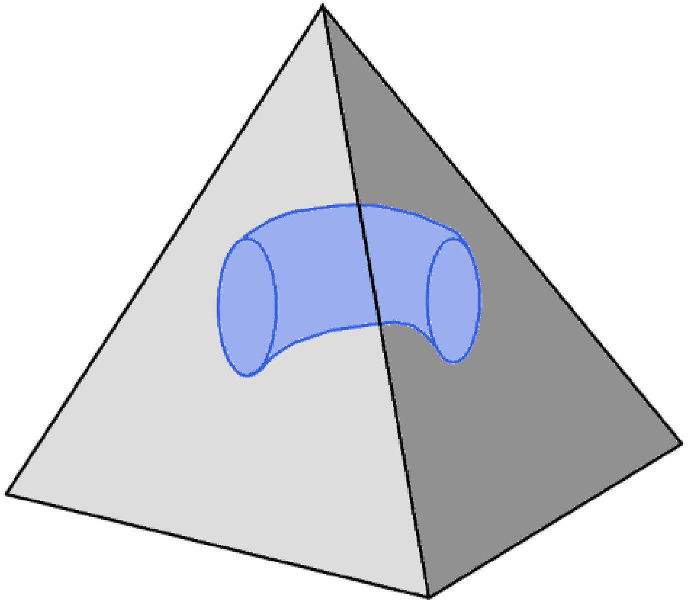
an incompressible surface and a compression of a torus

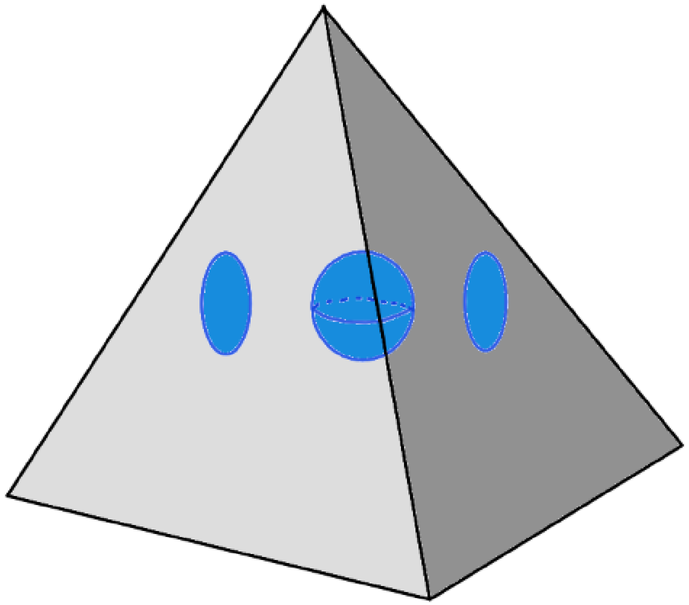
Normal surfaces

- If F is an embedded surface in M^3 then F intersects tetrahedrons T in the triangulation in a collection of subsurfaces. Each intersection of the surface with ∂T will be a simple closed curve, disjoint from the others
- Each closed curve cuts ∂T into two disks. The Loop theorem then guarantees that we have a compressing disk in T which allow us to create a new surface whose components meet T in disks and 2-spheres
- We discard the 2-sphere components since T is a ball so irreducible and $T \# S^2 \cong T$
- Curves contained in a face of a tetrahedron can be isotoped across that face and eliminated

Normal surfaces

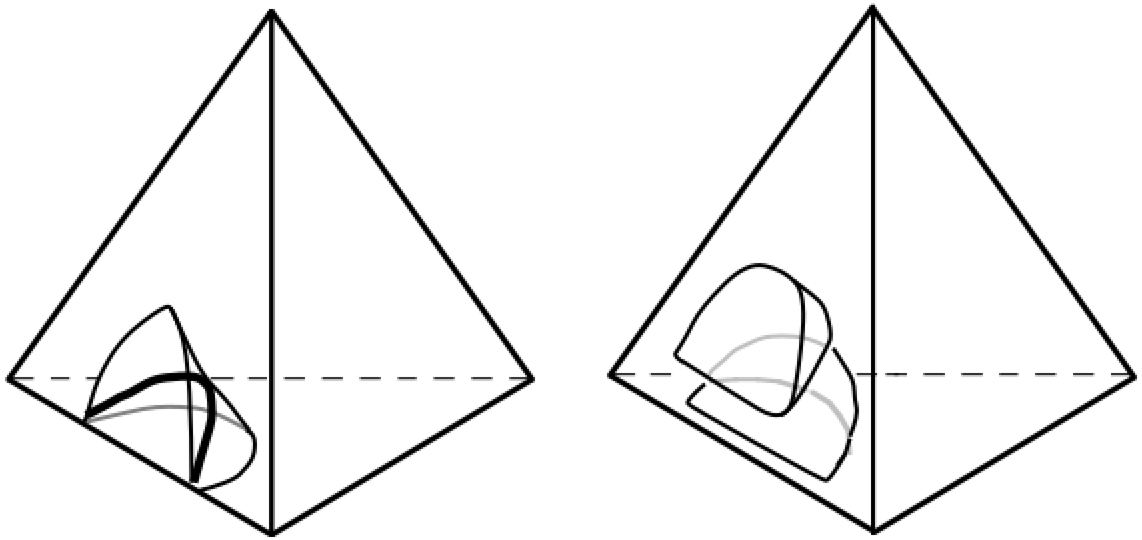
Compression of a piece of a torus in a tetrahedron





Normal surfaces

Consider adding context for this isotope



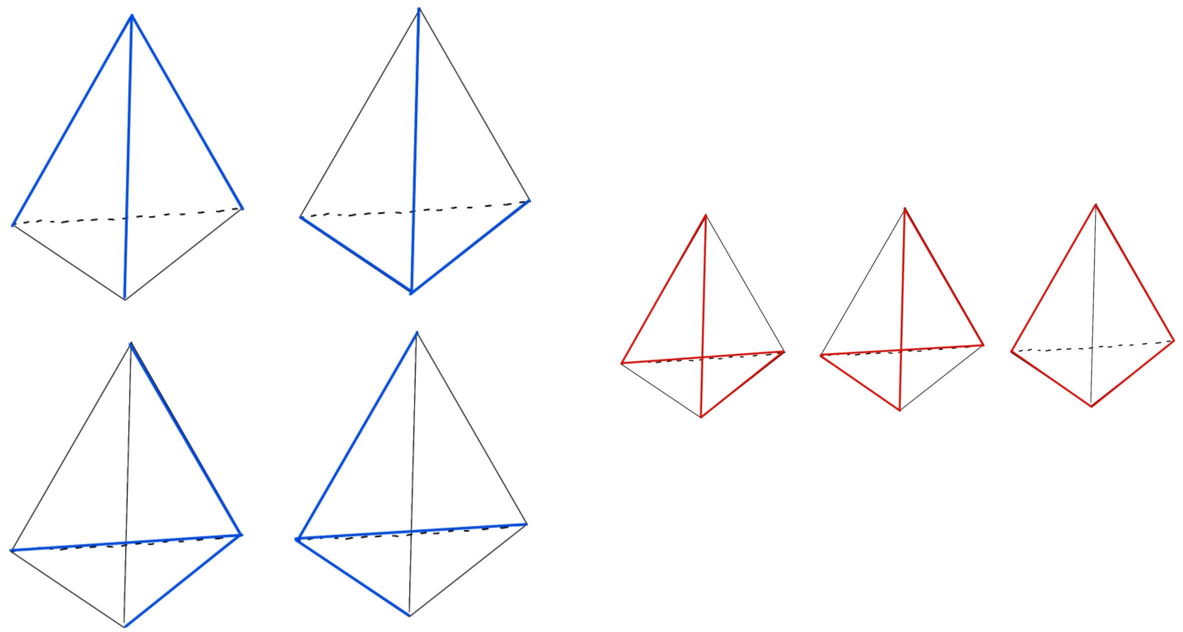
Normal surfaces

- Now we can repeat these processes for all tetrahedron to be left with surfaces which intersect each edge of ∂T at most once
- Disks which meet each edge at most once can only be normal triangles or quadrilaterals

And so we can decompose an embedded surface into a disjoint union of normal surfaces in M^3!

Normal surfaces

We can breakdown the behavior in each tetrahedron by noticing that there are only 7 types of disks which can completely characterize the normal surface



Normal surfaces

Now assign a vector $(x_1, x_2, \ldots, x_7)$ that corresponds to the normal surface where x_i is the number of disks of type i that occur in across all tetrahedrons.

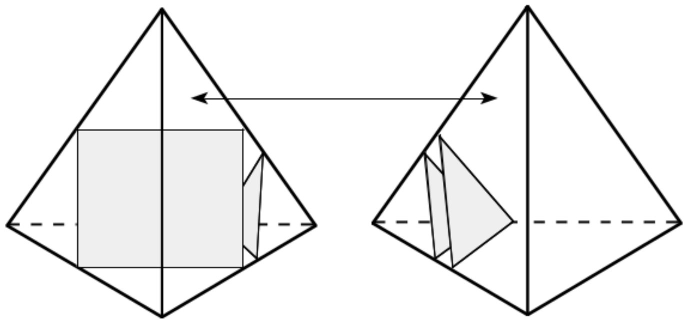
- In the reverse direction, say we have some vector X as described, this won't exactly result in a properly embedded surface, so we need to add some conditions
- One condition is that the disks in adjacent tetrahedrons are appropriately stitched together—this yields the normal surface equations
- A tetrahedron can only contain at most one type of quadrilateral—this yields the quadrilateral condition

Regular exchanges

To be filled in

Integer programming

- There are matching equations of the form $x_{i_1} + x_{i_2} = x_{i_3} + x_{i_4}$ since each face of a tetrahedron has arcs which are created by only one type of triangle and one type of quadrilateral. We can add the condition $x_i \geq 0$ to get the *normal surface equations*
- Having at most one type of quadrilaterial just means that if some $x_i \neq 0$ it forces some other x_j to be 0



Integer Programming

Fundamental solutions

Suppose A, B, and C are solutions to the normal surface equations and C is an embedded normal surface with $C = A + B$

- Then A and B also give rise to embedded normal surfaces
- Previous bullet implies $\chi(C) = \chi(A) + \chi(B)$
- If there does not exist A and B where $C = A + B$ then C is called fundamental

Integer Programming

Now that we know that all solutions can be broken into fundamental solutions we have the following important fact:

- There are only finitely many fundamental solutions

Finitely many fundamental solutions

- The solutions to the normal surface equations form a cone, and we can extend the solutions over the reals to get a cone in $\mathbb{R}^n_{\geq 0}$
- This cone intersects the convex simplex $\sum x_i = 1$ with $x_i \geq 0$ in a simplex with finitely many vertices v_j
- The integer solutions to the normal surface equations can be expressed as a rational linear combination of these v_j, similarly the integer multiples of v_j, $V_j = \sum \lambda_j v_j$ form a rational basis for all normal surfaces. Which means for a normal surface $X = \sum t_j V_j$

Finitely many fundamental solutions

- Take the compact set S of real solutions to the normal surface equations in the form $s = \sum t_j V_j, 0 \leq t_j \leq 1, 1 \leq j \leq n$
- Compactness guarantees only a finite number of integral points
- If X is an integral solution not in S, then for $t_k > 1$ $X = V_k + (\sum t_j V_j - V_k)$
- This implies all fundamental solutions are in S
- Then we just filter the solutions which meet the quadrilateral condition to get an embedded surface

References

- Joels paper (for graphics and understanding)