Exercise 49.

Consider the map $f: \mathbb{R}^2 \to \mathbb{R}^3$ by $(x,y) \mapsto (xy, x, y^2)$.
(a) Find the derivative of this map at every point.
(b) Find all regular values for this map.

Solution
(a)
Since this is stated as a map between Euclidean spaces, we can take derivatives in the "normal way". In particular, the derivative at any point (x,y) is given by the Jacobian:

$$df_{(x,y)} = \begin{pmatrix} y & x \\ 1 & 0 \\ 0 & 2y \end{pmatrix}$$

(b)
Regular values are $\mathbb{R}^3$ points p where for any $(x,y) \in f^{-1}(p)$, $df_{(x,y)}$ is a surjection. However $df_(x,y)$ is a linear map from a 2-dimensional vector space to a 3-dimensional vector space, and the rank of this operator is constrained by the dimension of its domain. Therefore we can't have a surjection to $\mathbb{R}^3$ and there are no regular values of f.

Exercise 54.
The paraboloid $z = x^2 + y^2$ intersects the plane $z = 1$ in a circle. Is this intersection transverse? Find the tangent spaces to the paraboloid and the the plane at every point in the intersection.

Solution.
It is indeed easier to do this question out of order, that is we should describe the tangent spaces first. The plane $z = 1$ is unambiguously a subset of $\mathbb{R}^3$ and it can be identified with its own tangent space. Namely, $T_p\{z=1\} = \{(0,0,1)\}^\perp$.

For the paraboloid, first notice that if $f: \mathbb{R}^3 \to \mathbb{R}$ by $(x,y,z) \mapsto x^2 + y^2 - z$ then again the differential map is always surjective, so 0 is a regular value of f and this manifold is given by $f^{-1}(0)$. Better yet, we can describe the tangent space (call the manifold $\mathcal{M}$) by $T_p\mathcal{M} = \ker df_p$.

$$df_{(x,y,z)} = \begin{pmatrix} 2x & 2y & -1 \end{pmatrix}$$

And we can describe the kernel as the plane $\{(a,b,c) \in \mathbb{R}^3 \mid 2xa + 2yb - c = 0\}$. Now this plane is only identical to $\{z = 1\}$ when $x = y = 0$, but this is not a point in the intersection of these manifolds. We can then conclude that each tangent space is a plane (not parallel to the other) and so their sum spans $\mathbb{R}^3$.

Exercise 52.

Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be the map

$$f(a,b,c,d) = (a^2 + b - \sin c + 3d, a^2 + b - \sin c + 3d)$$

Let $X \subset \mathbb{R}^2$ denote the x-axis (where $y = 0$). Prove that $f^{-1}(X)$ is a submanifold of $\mathbb{R}^4$ and give its dimension.

Solution.

The easiest way to see that this is a submanifold is to say that f is transverse to some X, and then apply a theorem which gives both that the preimage of X under f is a submanifold and the dimension of this submanifold. The truth is the preimage is mostly empty, and only the origin has a nonempty preimage. For points in the preimage

$$df_{a,b,c,d} = \begin{pmatrix} 2a & 1 & -\cos c & 3 \\ 2a & 1 & -\cos c & 3 \end{pmatrix}$$

Is a surjection onto the line $y = x$ in $\mathbb{R}^2$. The set X is a subset of $\mathbb{R}^2$ and it can be identified with its own tangent space (which is the x-axis in $\mathbb{R}^2$). Now we can see by counting dimensions that

$$\mathrm{Im}(df_x) + T_y X = T_y \mathbb{R}^2$$

Which implies that $f^{-1}(X)$ is a submanifold of dimension 1.