MSE

Vi har:

$$\begin{aligned}
MSE(\hat{\theta}) &= E\left((\hat{\theta}-\theta)^2\right) \\
&= E\left((\hat{\theta}-E(\hat{\theta})+E(\hat{\theta})-\theta)^2\right) \\
&= E\left((\hat{\theta}-E(\hat{\theta}))^2 + 2(\hat{\theta}-E(\hat{\theta}))(E(\hat{\theta})-\theta) + (E(\hat{\theta})-\theta)^2\right) \\
&= \underbrace{E\left((\hat{\theta}-E(\hat{\theta}))^2\right)}_{Var(\hat{\theta})} + 2E(\hat{\theta}-E(\hat{\theta}))\cdot(E(\hat{\theta})-\theta) + E\left((E(\hat{\theta})-\theta)^2\right) \\
&= Var(\hat{\theta}) + 2\underbrace{(E(\hat{\theta})-E(\hat{\theta}))}_{=0}\cdot(E(\hat{\theta})-\theta) + (\underbrace{E(\hat{\theta})-\theta}_{Bias(\hat{\theta})})^2 \\
&= Var(\hat{\theta}) + (Bias(\hat{\theta}))^2 = \text{varians} + \text{skjevhet}^2
\end{aligned}$$

Eksempel: $Y \sim Bin(n,p)$, $\hat{p} = \frac{Y}{n}$

Vi har:

$$\underline{E(\hat{p})} = E\left(\frac{Y}{n}\right) = \frac{1}{n}E(Y) = \frac{1}{\cancel{n}}\cancel{n}p = \underline{p}$$

Altså er $\hat{p}$ forventningsrett for p.

<u>Uavhengige identisk fordelte variable</u>

$X_1, \dots, X_n$ er uif med forventning μ og varians σ^2.

Vi har:

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\underbrace{E(X_i)}_{\mu} = \frac{1}{\cancel{n}}\cdot\cancel{n}\mu = \underline{\underline{\mu}}$$

$$E(S^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right) = \frac{1}{n-1}E\left(\sum_{i=1}^{n}X_i^2 - 2\bar{X}\overbrace{\sum_{i=1}^{n}X_i}^{n\cdot\bar{X}} + n\bar{X}^2\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}E(X_i^2) - 2nE(\bar{X}^2) + nE(\bar{X}^2)\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}E(X_i^2) - nE(\overbrace{\bar{X}^2}^{(\frac{1}{n}\sum X_i)^2 = \frac{1}{n^2}\cdot(\sum X_i)^2})\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}E(X_i^2) - \cancel{n}\cdot\frac{1}{n^{\cancel{2}}}E\left(\left(\sum_{i=1}^{n}X_i\right)^2\right)\right)$$

Husk at $Var(Y) = E(Y^2) - (E(Y))^2 \longrightarrow E(Y^2) = \underline{Var(Y) + (E(Y))^2}$

Vi får:

$$E(S^2) = \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(\underbrace{Var(X_i)}_{\sigma^2} + (\underbrace{E(X_i)}_{\mu})^2\right) - \frac{1}{n}\left(Var\left(\sum_{i=1}^{n}X_i\right) + \left(E\left(\sum_{i=1}^{n}X_i\right)\right)^2\right)\right)$$

Da $X_1, \dots, X_n$ er uif, er $Var\left(\sum_{i=1}^{n}X_i\right) = \sum_{i=1}^{n}\underbrace{Var(X_i)}_{\sigma^2} = n\sigma^2$

og $E\left(\sum_{i=1}^{n}X_i\right) = \sum_{i=1}^{n}\underbrace{E(X_i)}_{\mu} = n\mu$

Vi får:

$$E(S^2) = \frac{1}{n-1}\left(n\sigma^2 + n\mu^2 - \frac{1}{n}\left(n\sigma^2 + n^2\mu^2\right)\right)$$

$$= \frac{1}{n-1}\left(n\sigma^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2}\right) = \frac{1}{\cancel{n-1}}\cancel{(n-1)}\sigma^2 = \underline{\underline{\sigma^2}}$$

Altså er $\bar{X}$ forventningsrett for μ og S^2 forventningsrett for σ^2.

Eksempel: $Y \sim \text{Bin}(n, p)$ og $\hat{p} = \frac{Y}{n}$.

Vi har:

$$\sigma_{\hat{\theta}}^2 = V(\hat{p}) = V\left(\frac{Y}{n}\right) = \frac{1}{n^2} \cdot V(Y) = \frac{1}{n^{\cancel{2}}} \cancel{n} \cdot p \cdot (1-p) = \frac{p(1-p)}{n},$$

slik at

$$\sigma_{\hat{\theta}} = \sqrt{V(\hat{p})} = \sqrt{\frac{p(1-p)}{n}}.$$

For meningsmålingen i Eks. 1 i innledningen får vi:

$$s_{\hat{p}} = \sqrt{\frac{0{,}162(1-0{,}162)}{8208}} \simeq 0{,}00407$$