• Eksempel: $X_1, \dots, X_n \overset{uif}{\sim} \text{Gamma}(\alpha, \beta)$

Vi har: $f(x;\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$, $x > 0$

$\longrightarrow \log f(x;\alpha,\beta) = -\alpha\log(\beta) - \log(\Gamma(\alpha)) + (\alpha-1)\log(x) - \frac{x}{\beta}$

Vi får:

$$\frac{\partial \log f(x;\alpha,\beta)}{\partial\alpha} = -\log(\beta) - \frac{\partial}{\partial\alpha}\log(\Gamma(\alpha)) + \log(x)$$

$$\frac{\partial \log f(x;\alpha,\beta)}{\partial\beta} = -\frac{\alpha}{\beta} + \frac{x}{\beta^2}$$

$$\frac{\partial^2 \log f(x;\alpha,\beta)}{\partial\alpha^2} = -\frac{\partial^2}{\partial\alpha^2}\log(\Gamma(\alpha)) = -\psi_1(\alpha) \quad \text{(trigammafunksjonen)}$$

$$\frac{\partial^2 \log f(x;\alpha,\beta)}{\partial\beta^2} = \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3}$$

$$\frac{\partial^2 \log f(x;\alpha,\beta)}{\partial\alpha\,\partial\beta} = -\frac{1}{\beta}$$

Vi får dermed:

$$I_{11}(\alpha,\beta) = -E\left(\frac{\partial^2 \log f(X;\alpha,\beta)}{\partial\alpha^2}\right) = E(\psi_1(\alpha)) = \psi_1(\alpha)$$

$$I_{22}(\alpha,\beta) = -E\left(\frac{\partial^2 \log f(X;\alpha,\beta)}{\partial\beta^2}\right) = E\left(-\frac{\alpha}{\beta^2} + \frac{2X}{\beta^3}\right)$$

$$= -\frac{\alpha}{\beta^2} + \frac{2}{\beta^3}\underbrace{E(X)}_{\alpha\beta}$$

$$= -\frac{\alpha}{\beta^2} + \frac{2}{\beta^{\cancel{3}2}}\alpha\cancel{\beta} = \frac{\alpha}{\beta^2}$$

$$I_{12}(\alpha,\beta) = -E\left(\frac{\partial^2 \log f(X;\alpha,\beta)}{\partial\alpha\,\partial\beta}\right) = E\left(\frac{1}{\beta}\right) = \frac{1}{\beta}$$

$$\longrightarrow I(\alpha,\beta) = \begin{pmatrix} \psi_1(\alpha) & 1/\beta \\ 1/\beta & \alpha/\beta^2 \end{pmatrix}$$

$$\longrightarrow I(\alpha,\beta)^{-1} = \frac{1}{\psi_1(\alpha)\cdot\frac{\alpha}{\beta^2} - \frac{1}{\beta}\cdot\frac{1}{\beta}} \begin{pmatrix} \alpha/\beta^2 & -\frac{1}{\beta} \\ -\frac{1}{\beta} & \psi_1(\alpha) \end{pmatrix}$$

$$= \frac{\beta^2}{\alpha\cdot\psi_1(\alpha) - 1} \begin{pmatrix} \frac{\alpha}{\beta^2} & -\frac{1}{\beta} \\ -\frac{1}{\beta} & \psi_1(\alpha) \end{pmatrix}$$

$$= \frac{1}{\alpha\psi_1(\alpha) - 1} \begin{pmatrix} \alpha & -\beta \\ -\beta & \beta^2\psi_1(\alpha) \end{pmatrix}$$

Det betyr at

$$\hat\alpha \overset{\text{tiln.}}{\sim} N\left(\alpha, \frac{1}{n}\frac{\alpha}{\alpha\psi_1(\alpha) - 1}\right) \text{ og } \hat\beta \overset{\text{tiln.}}{\sim} N\left(\beta, \frac{1}{n}\frac{\beta^2\psi_1(\alpha)}{\alpha\psi_1(\alpha) - 1}\right)$$

Konfidensintervaller

Vi har $X_1, \ldots, X_n \overset{uif}{\sim} N(\mu, \sigma^2)$

Da er $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \sim N(\mu, \frac{\sigma^2}{n})$

og $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

Vi får:

$$P\left(-1{,}96 \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq 1{,}96\right) = 0{,}95$$

$$\rightarrow P\left(-1{,}96 \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{X} - \mu \leq 1{,}96 \cdot \frac{\sigma}{\sqrt{n}}\right) = 0{,}95$$

$$\rightarrow P\left(-\bar{X} - 1{,}96 \frac{\sigma}{\sqrt{n}} \leq -\mu \leq -\bar{X} + 1{,}96 \frac{\sigma}{\sqrt{n}}\right) = 0{,}95$$

$$\rightarrow P\left(\bar{X} + 1{,}96 \frac{\sigma}{\sqrt{n}} \geq \mu \geq \bar{X} - 1{,}96 \frac{\sigma}{\sqrt{n}}\right) = 0{,}95$$

• Eksempel: $X_1, \dots, X_n \overset{uif}{\sim} N(\mu, \sigma^2)$ med μ kjent og σ^2 ukjent.

Det kan vises at (vises senere i Kap 6.4):

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{1}{\sigma^2}\cdot n \cdot \underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2}_{\hat{\sigma}^2} = \frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_n,$$

der $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$ er forventningsrett for σ^2 når μ er kjent. Vi får:

$$P\Big(\underbrace{\chi^2_{1-\alpha/2,n}}_{a} \le \underbrace{\frac{n\hat{\sigma}^2}{\sigma^2}}_{h(X_1,\dots,X_n;\sigma^2)} \le \underbrace{\chi^2_{\alpha/2,n}}_{b}\Big) = 1-\alpha$$

Vi omformer ulikheten:

$$\chi^2_{1-\alpha/2,n} \le \frac{n\hat{\sigma}^2}{\sigma^2} \le \chi^2_{\alpha/2,n}$$

$$\longrightarrow \frac{1}{n\hat{\sigma}^2}\chi^2_{1-\alpha/2,n} \le \frac{1}{\sigma^2} \le \frac{1}{n\hat{\sigma}^2}\chi^2_{\alpha/2,n}$$

$$\longrightarrow \frac{n\hat{\sigma}^2}{\chi^2_{1-\alpha/2,n}} \ge \sigma^2 \ge \frac{n\hat{\sigma}^2}{\chi^2_{\alpha/2,n}}$$

Da blir $100\cdot(1-\alpha)\%$ konfidensintervall for σ^2 gitt ved

$$\left(\frac{n\hat{\sigma}^2}{\chi^2_{\alpha/2,n}}, \frac{n\hat{\sigma}^2}{\chi^2_{1-\alpha/2,n}}\right)$$

• Eksempel: $X_1, \dots, X_n \overset{uif}{\sim}$ Bernoulli(p)

Vi har estimatoren $\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n}$. Vi vet at

$E(\hat{p}) = p$ og $V(\hat{p}) = \frac{p(1-p)}{n}$, samt at

$$\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \overset{tiln.}{\sim} N(0,1)$$

for $np \geq 10$ og $n(1-p) \geq 10$.

Vi har:

$$P\left(-z_{\alpha/2} \leq \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \leq z_{\alpha/2}\right) \simeq 1-\alpha$$

Vi får:

$$-z_{\alpha/2} \leq \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \leq z_{\alpha/2}$$

$$\longrightarrow -z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} \leq \hat{p}-p \leq z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

$$\longrightarrow (\hat{p}-p)^2 \leq z_{\alpha/2}^2 \frac{p(1-p)}{n}$$

$$\longrightarrow \hat{p}^2 - 2\hat{p}\cdot p + p^2 \leq \frac{z_{\alpha/2}^2}{n}\cdot p - \frac{z_{\alpha/2}^2}{n}\cdot p^2$$

$$\longrightarrow \left(1 + \frac{z_{\alpha/2}^2}{n}\right)p^2 - \left(2\hat{p} + \frac{z_{\alpha/2}^2}{n}\right)p + \hat{p}^2 \leq 0$$

Vi bytter ut $\leq$ med $=$ og løser for p:

$$p = \frac{2\hat{p} + z_{\alpha/2}^2/n \pm \sqrt{(2\hat{p} + z_{\alpha/2}^2/n)^2 - 4\cdot\hat{p}^2\left(1+\frac{z_{\alpha/2}^2}{n}\right)}}{2\cdot\left(1+\frac{z_{\alpha/2}^2}{n}\right)}$$

$$= \frac{\hat{p} + z_{\alpha/2}^2/2n}{1+\frac{z_{\alpha/2}^2}{n}} \pm \frac{1}{1+\frac{z_{\alpha/2}^2}{n}}\sqrt{\cancel{\hat{p}^2} + \hat{p}\frac{z_{\alpha/2}^2}{n} + \frac{z_{\alpha/2}^4}{4n^2} - \cancel{\hat{p}^2} - \hat{p}^2\cdot\frac{z_{\alpha/2}^2}{n}}$$

$$= \frac{\hat{p} + z_{\alpha/2}^2/2n}{1+\frac{z_{\alpha/2}^2}{n}} \pm \frac{1}{1+\frac{z_{\alpha/2}^2}{n}}\sqrt{\frac{z_{\alpha/2}^2}{n}\hat{p}(1-\hat{p}) + \frac{z_{\alpha/2}^4}{4n^2}}$$

Da blir $100\cdot(1-\alpha)\%$ konfidensintervall for p:

$$\frac{\hat{p} + z_{\alpha/2}^2/2n}{1+z_{\alpha/2}^2/n} \pm \frac{1}{1+\frac{z_{\alpha/2}^2}{n}}\sqrt{\frac{z_{\alpha/2}^2}{n}\hat{p}(1-\hat{p}) + \frac{z_{\alpha/2}^4}{4n^2}}$$