t-fordelingen

Hvis $Z \sim N(0,1)$ og $U \sim \chi^2_\nu$ er uavhengige, så er

$T = \frac{Z}{\sqrt{U/\nu}} \sim t_\nu$. For å finne tettheten ser vi først

på den kumulative fordelingen:

$$F(t) = P(T \le t) = P\left(\frac{Z}{\sqrt{U/\nu}} \le t\right) = P\left(Z \le t\sqrt{\frac{U}{\nu}}\right)$$

$$= \int_0^\infty \int_{-\infty}^{t\sqrt{u/\nu}} f(u,z)\, dz\, du$$

$$\overset{\text{uavh.}}{=} \int_0^\infty \int_{-\infty}^{t\sqrt{u/\nu}} f_1(u) \cdot f_2(z)\, dz\, du$$

Vi får:

$$f(t) = \frac{d}{dt} F(t) = \frac{d}{dt} \int_0^\infty \int_{-\infty}^{t\sqrt{u/\nu}} f_1(u)\, f_2(z)\, dz\, du$$

$$= \int_0^\infty \frac{d}{dt} \int_{-\infty}^{t\sqrt{u/\nu}} f_1(u) \cdot f_2(z)\, dz\, du$$

$$\frac{d}{dt} \int_0^t f(y)\, dy = f(t)$$

$$= \int_0^\infty \sqrt{\frac{u}{\nu}} \cdot f_U(u) f_2\left(t\sqrt{\frac{u}{\nu}}\right) du$$

$$= \int_0^\infty \sqrt{\frac{u}{\nu}} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} u^{\nu/2-1} e^{-\frac{u}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(t\sqrt{\frac{u}{\nu}}\right)^2} du$$

$$= \frac{1}{\sqrt{\nu}\, 2^{\nu/2}\, \Gamma(\nu/2) \sqrt{2\pi}} \int_0^\infty u^{\nu/2 - \frac{1}{2}} \cdot e^{-\frac{u}{2} - \frac{t^2 u}{2\nu}} du$$

$$= \frac{\left(\frac{2}{1+t^2/\nu}\right)^{\frac{\nu+1}{2}} \cdot \Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu}\, 2^{\nu/2}\, \Gamma\left(\frac{\nu}{2}\right) \sqrt{2\pi}} \underbrace{\int_0^\infty \frac{1}{\left(\frac{2}{1+t^2/\nu}\right)^{\frac{\nu+1}{2}} \cdot \Gamma\left(\frac{\nu+1}{2}\right)} u^{\frac{\nu+1}{2}-1} \cdot e^{-\frac{u}{2/(1+t^2/\nu)}} du}_{\text{Gamma}\left(\frac{\nu+1}{2},\, \frac{2}{1+t^2/\nu}\right) \;=\; 1}$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < t < \infty$$

Hvis $X_1, \dots, X_n \overset{\text{uif}}{\sim} N(\mu, \sigma^2)$, så er $T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$.

Bevis:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\overbrace{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}^{\sim N(0,1)}}{\sqrt{S^2/\cancel{n} \,/\, \sigma^2/\cancel{n}}} = \frac{\overbrace{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}^{\sim N(0,1)}}{\sqrt{\underbrace{\frac{(n-1)S^2}{\sigma^2}}_{\sim \chi^2_{n-1}} / (n-1)}} = \frac{Z}{\sqrt{U/(n-1)}},$$

der $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ og $U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, og

Z og U er uavhengige da $\bar{X}$ og S^2 er det.

Altså er $T \sim t_{n-1}$.

Vi skal vise at $E(T) = 0$, $\nu > 1$ og $V(T) = \frac{\nu}{\nu-2}$, $\nu > 2$

Vi trenger k-te moment i χ^2_ν-fordelingen. La $U \sim \chi^2_\nu$. Da er:

$$E(U^k) = \int_{-\infty}^{\infty} u^k f(u)\,du = \int_0^{\infty} u^k \cdot \frac{1}{2^{\nu/2}\Gamma(\nu/2)} u^{\nu/2-1} e^{-\frac{u}{2}}\,du$$

$$= \frac{2^{\nu/2+k}\Gamma(\nu/2+k)}{2^{\nu/2}\Gamma(\nu/2)} \underbrace{\int_0^{\infty} \underbrace{\frac{1}{2^{\frac{\nu}{2}+k}\Gamma(\frac{\nu}{2}+k)} u^{\nu/2+k-1} e^{-\frac{u}{2}}}_{\text{Gamma}(\frac{\nu}{2}+k,\,2)}\,du}_{=1,\ k > -\frac{\nu}{2}}$$

$$= \frac{2^k \Gamma(\frac{\nu}{2}+k)}{\Gamma(\frac{\nu}{2})}$$

Vi har:

$$E(T) = E\left(\frac{Z}{\sqrt{U/\nu}}\right) = E\left(Z \cdot \left(\frac{U}{\nu}\right)^{-\frac{1}{2}}\right)$$

$$\overset{\text{uavh.}}{=} \underbrace{E(Z)}_{=0} \cdot \underbrace{E\left(\left(\frac{U}{\nu}\right)^{-\frac{1}{2}}\right)}_{= \frac{2^{-\frac{1}{2}}\Gamma(\frac{\nu}{2}-\frac{1}{2})}{\nu^{-\frac{1}{2}}\Gamma(\frac{\nu}{2})},\ \text{hvis } -\frac{1}{2} > -\frac{\nu}{2} \text{ altså } \nu > 1}$$

Altså er $E(T) = 0$ når $E\left(\left(\frac{U}{\nu}\right)^{-\frac{1}{2}}\right)$ eksisterer, altså når $\nu > 1$.

Vi har:

$$V(T) = E(T^2) - \underbrace{(E(T))^2}_{=0,\ \nu>1} = E(T^2), \quad \nu > 1$$

$$= E\left(\left(\frac{Z}{\sqrt{U/\nu}}\right)^2\right)$$

$$= E\left(Z^2 \cdot \left(\frac{U}{\nu}\right)^{-1}\right)$$

$$\overset{\text{uavh.}}{=} E(Z^2) \cdot E\left(\left(\frac{U}{\nu}\right)^{-1}\right)$$

Videre er:

$$E(Z^2) = \underbrace{V(Z)}_{=1} + (\underbrace{E(Z)}_{=0})^2 = 1$$

$$E\left(\left(\frac{U}{\nu}\right)^{-1}\right) = \nu \cdot E(U^{-1}) = \nu \cdot \frac{2^{-1}\cdot\Gamma(\frac{\nu}{2}-1)}{\Gamma(\frac{\nu}{2})}, \quad \text{så sant } -1 > -\frac{\nu}{2} \text{ altså } \nu > 2$$

$$= \frac{\nu}{2} \cdot \frac{\cancel{\Gamma(\frac{\nu}{2}-1)}}{\underbrace{(\frac{\nu}{2}-1)\cdot\cancel{\Gamma(\frac{\nu}{2}-1)}}_{\Gamma(\frac{\nu}{2})}}$$

$$= \frac{\nu}{\cancel{2}} \cdot \frac{1}{\frac{\nu-2}{\cancel{2}}} = \frac{\nu}{\nu-2}, \quad \nu > 2$$

Vi får:

$$V(T) = \underbrace{E(Z^2)}_{=1} \cdot \underbrace{E\left(\left(\frac{U}{\nu}\right)^{-1}\right)}_{\frac{\nu}{\nu-2}} = \frac{\nu}{\nu-2}, \quad \text{så sant } \nu > 2.$$

F-fordelingen

Hvis $X_1, \dots, X_m \overset{uif}{\sim} N(\mu_1, \sigma_1^2)$ og $Y_1, \dots, Y_n \overset{uif}{\sim} N(\mu_2, \sigma_2^2)$

og $X_1, \dots, X_m$ og $Y_1, \dots, Y_n$ er uavhengige, er

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{m-1,n-1}.$$

Bevis:

Vi har:

$$F = \frac{\overbrace{\frac{(m-1)S_1^2}{\sigma_1^2}}^{\sim \chi^2_{m-1}} \Big/ (m-1)}{\underbrace{\frac{(n-1)S_2^2}{\sigma_2^2}}_{\sim \chi^2_{n-1}} \Big/ (n-1)} = \frac{U_1/(m-1)}{U_2/(n-1)},$$

der $U_1 \sim \chi^2_{m-1}$ og $U_2 \sim \chi^2_{n-1}$ er uavhengige.
Altså er $F \sim F_{m-1,n-1}$.

Prediksjonsintervall:

$X_1, \dots, X_n \overset{uif}{\sim} N(\mu, \sigma^2)$ og $X_{n+1} \sim N(\mu, \sigma^2)$ er uavhengige. Da er $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, slik at $\bar{X} - X_{n+1}$ også må være normalfordelt. Videre er $\bar{X}$ og X_{n+1} uavhengige, slik at:

$$E(\bar{X} - X_{n+1}) = \underbrace{E(\bar{X})}_{\mu} - \underbrace{E(X_{n+1})}_{\mu} = \mu - \mu = 0$$

$$V(\bar{X} - X_{n+1}) \overset{uavh.}{=} \underbrace{V(\bar{X})}_{\frac{\sigma^2}{n}} + (-1)^2 \cdot \underbrace{V(X_{n+1})}_{\sigma^2} = \frac{\sigma^2}{n} + \sigma^2 = \sigma^2\left(1 + \frac{1}{n}\right).$$