Momentmetoden

- Eksempel: $X_1, \dots, X_n \overset{uif}{\sim} \text{Bernoulli}(p)$. Da er $Y = \sum_{i=1}^{n} X_i \sim \text{Bin}(n,p)$.

Vi har:

$$E(X) = p$$

Momentestimatoren for p er da gitt ved:

$$\frac{1}{n} \sum_{i=1}^{n} X_i = \frac{Y}{n} = \hat{p}.$$

- Eksempel: $X_1, \dots, X_n \overset{uif}{\sim} \text{Gamma}(\alpha, \beta)$.

Vi har: $E(X) = \alpha\beta$ og $E(X^2) = \text{Var}(X) + (E(X))^2 = \alpha\beta^2 + (\alpha\beta)^2 = \beta^2(\alpha+1)\alpha$

Vi får:

$$\hat{\alpha}\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \quad \text{og} \quad \hat{\beta}^2(\hat{\alpha}+1)\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\longrightarrow \hat{\alpha} = \frac{\bar{X}}{\hat{\beta}}$$

$$\longrightarrow \hat{\beta}^2\left(\frac{\bar{X}}{\hat{\beta}} + 1\right)\frac{\bar{X}}{\hat{\beta}} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\longrightarrow \underbrace{\bar{X}^2 + \hat{\beta}\bar{X}}_{\bar{X}(\bar{X}+\hat{\beta})} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\longrightarrow \bar{X} + \hat{\beta} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\bar{X}}$$

$$\longrightarrow \hat{\beta} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\bar{X}} - \bar{X} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2}{\bar{X}}$$

$$\longrightarrow \hat{\alpha} = \frac{\bar{X}}{\hat{\beta}} = \frac{\bar{X}}{\frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2}{\bar{X}}} = \frac{\bar{X}^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2}$$

Maksimum likelihood-estimatoren

- Eksempel: $X_1, \dots, X_n \overset{uif}{\sim}$ Bernoulli(p).

Vi har observerte data $x_1, \dots, x_n$.

Sannsynligheten for å ha observert $x_1, \dots, x_n$ er da

$$f(x_1, \dots, x_n; p) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$
$$\overset{uif}{=} \prod_{i=1}^n P(X_i = x_i)$$
$$= \prod_{i=1}^n p^{x_i}(1-p)^{1-x_i} = p^{\overbrace{\sum_{i=1}^n x_i}^{y}} \cdot (1-p)^{n - \overbrace{\sum_{i=1}^n x_i}^{y}} = p^y (1-p)^{n-y},$$

der $y = \sum_{i=1}^n x_i$.

Vi vil finne den verdien av p som maksimerer denne sannsynligheten. Vi skal altså maksimere

$$f(x_1, \dots, x_n; p) = p^y (1-p)^{n-y} \quad \text{mhp } p.$$

Det er det samme som å maksimere

$$\log(f(x_1, \dots, x_n; p)) = \log(p^y (1-p)^{n-y}) = \log(p^y) + \log((1-p)^{n-y})$$
$$= y \log(p) + (n-y)\log(1-p)$$

Makspunktet finner vi ved å derivere mhp p og sette lik 0:

$$\frac{d}{dp}\log(f(x_1, \dots, x_n; p)) = \frac{d}{dp}\left(y\log(p) + (n-y)\log(1-p)\right)$$
$$= \frac{y}{p} + n-y\left(-\frac{1}{1-p}\right) = \frac{y}{p} - \frac{n-y}{1-p} = 0$$

$$\rightarrow \frac{y}{p} = \frac{n-y}{1-p}$$
$$\rightarrow y - \cancel{yp} = np - \cancel{yp}$$
$$\rightarrow p = \frac{y}{n}.$$

Maksimum likelihood-estimatet for p er $\hat{p} = \frac{y}{n}$, og den tilsvarende estimatoren er $\hat{p} = \frac{Y}{n}$.