University of Illinois at Urbana-Champaign
Dept. of Electrical and Computer Engineering

ECE 120: Introduction to Computing

Fixed- and Floating-Point Representations

In Binary, We Have A Binary Point

Let's talk about representations.

In decimal, we have a **decimal point**.

tenths

hundredths

3.1415…

In binary, we have a **binary point**.

2^{-1}'s place

2^{-2}'s place

11.001001…

2^{-3}'s place

Fixed-Point Representations Support Fractions

If we need fractions,

- we can use a **fixed-point representation**
- in which some number of bits
- come after the binary point.

For example, with 32 bits:

integer part (16 bits)	fractional part (16 bits)

Some signal processing and embedded processors use fixed-point representations.

What about Real Numbers?

A question for you:

Do we need anything else to support real numbers?

Note: Saying "yes" on the basis that there are uncountably many* real numbers is not a good answer. Integers are also infinite, and 2's complement is sufficient for practical use.

* An infinite number for each integer.

Isn't Fixed-Point Good Enough?

Let's do a calculation.

32-bit 2's complement has what range?

That's right: **[-2,147,483,648, 2,147,483,647]**.

You DID all know that, right?

I didn't. I usually write $[-2^{31}, 2^{31} - 1]$.

Let's write banking software to count pennies.

2,147,483,647 pennies is **$21,474,836.47**.

Anyone here have more? If not, we're done.

If so, use **64-bit**. You don't have that much!

Anyone Here Taking Chemistry?

But maybe you want to do your Chemistry homework?

You may need **Avogadro's number**.

Anyone remember it? **6.022×10^{23} / mol**

Sure. No problem.

10^3 is around 2^{10}, so 80 bits should work.

Who can tell me Avogadro's number to 80 bits (the first 24 decimal digits will do)?

Wikipedia May Not Help as Much as You Think!

Last I checked (July 2016!), the best known experimental value was

$$6.022140858 \times 10^{23} \text{ / mol}$$

That's only 10 digits.

So you have some serious Chemistry research to get done for your next homework!

Good luck!

Maybe we can just be close?

What about Physics?

Some have Quantum Mechanics homework?

Your computer will need **Planck's constant**.

What is it again? **6.626×10^{-27} erg-sec***

Ok. Another 90 bits after the binary point.

170 bits total.

Don't forget to find another 90 bits
(27 more decimal digits) for Avogadro.

*Use ergs, not Joules; we'll need fewer bits!

We Need More Dynamic Range, Not More Precision

Do we really need **170 bits** of precision?

Do we really need to specify the first **51 significant figures** for Avogadro's number?

Of course not!

But we do need 170 bits of range.

We need to be able to express both tiny numbers and huge numbers.

Develop a Representation Based on Scientific Notation

Let's borrow another representation from humans: scientific notation.

sign

exponent

$$+\underbrace{6.022}_{\text{mantissa/significant figures (precision)}} \times 10^{23}$$

The human representation has three parts.

Modern Computers Use Standard Floating-Point

Modern digital systems implement the IEEE 754 standard for floating-point.

A single-precision floating-point number consists of 32 bits:

sign	exponent	mantissa
(1 bit)	(8 bits)	(23 bits)

What value does a bit pattern represent?

First, let me ask you a question…

What Values Can a Leading Digit Take?

A question for you:

In the canonical form of scientific notation, what are the possible values of the leading digit?

This one

$$-\,4.123 \times 10^{45}$$

Any digit? 0-4? 1-7?

1-9 (not 0). Change exponent as needed.

What Values Can a Leading Digit Take?

Another question for you:

Same question, but now in binary.

1 (not 0). Change exponent as needed.

And one more:

How many bits do we need to store one possible answer?

The leading 1 is implicit in binary (0 bits)!

How to Calculate the Value of a Floating-Point Bit Pattern

The value represented by an IEEE single-precision floating-point bit pattern is…

sign (1 bit)	exponent (8 bits)	mantissa (23 bits)

$$(-1)^{\mathbf{sign}}\ 1.\underline{\mathbf{mantissa}} \times 2^{(\mathbf{exponent} - 127)}$$

Convert the exponent to decimal as if it were unsigned before subtracting 127.

Except that Exponents 0 and 255 have Special Meanings

That's almost correct. But **exponents 0 and 255 have special meanings**:

- 255 can mean **infinity** or **not-a-number** (**NaN**).
- 0 is a **denormalized** number: the leading implicit "1" is replaced with "0" (with power 2^{-126}), allowing the representation to capture numbers closer to 0.

Except for the fact that **the bit pattern of all 0s means 0**, these aspects are beyond the scope of our class.

* *

Exponent 255 is Used for Infinity and NaN

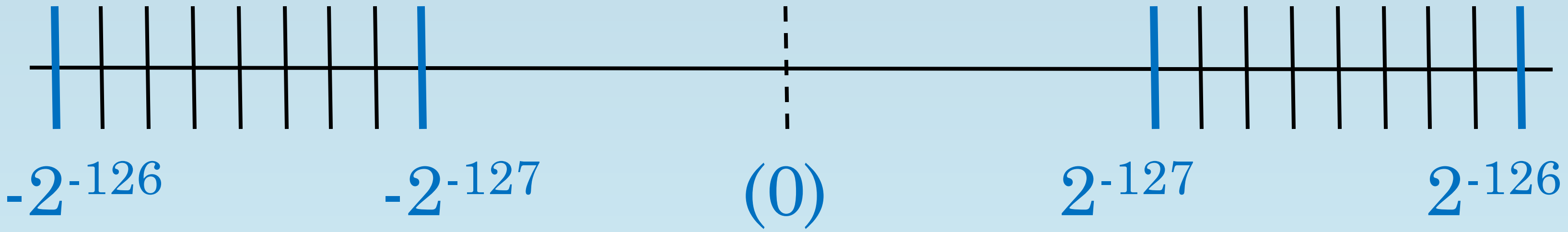
Exponent 255
- Mantissa 0
  - Sign 0: **Positive infinity**
  - Sign 1: **Negative infinity**
- Non-zero mantissa: **NaN** (Not a Number)

These special values allow the representation to have ‘correct’ answers to some problems (such as 42.0 / 0.0) and to silently track the impact of missing values and incorrect computation (such as Infinity * 0).

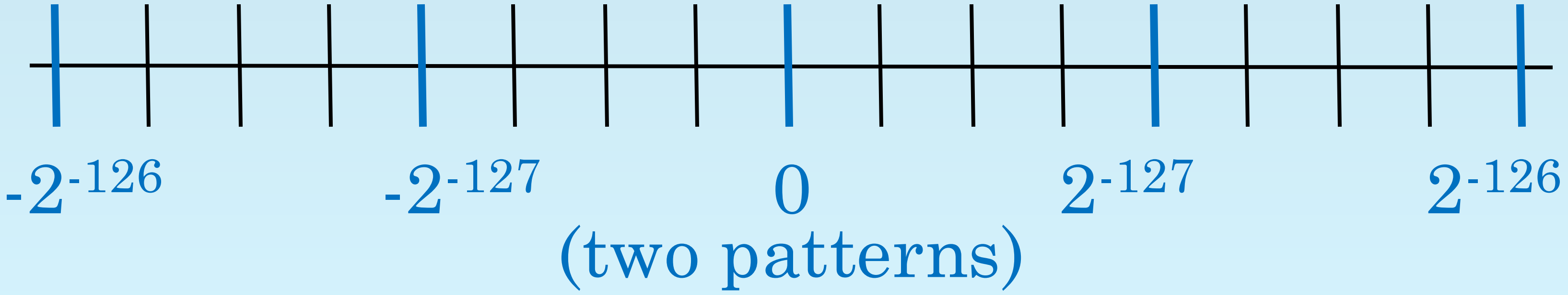
* *

Denormalization Allows Numbers Closer to 0 (and 0)

Without denormalized numbers, we have (shown with a 3-bit mantissa)…



Denormalization puts these patterns closer to 0 and gives two patterns for 0:



Converting to a Floating-Point Bit Pattern

Conversion from decimal to IEEE floating-point is not too hard:

1. Convert to binary.
2. Change to scientific notation (in binary).
3. Encode each of the three parts.

Use a Polynomial to Convert a Fraction to Binary

To convert a fraction F to binary, remember that **a fraction also corresponds to a polynomial**:

$$\mathbf{F = a_{-1}2^{-1} + a_{-2}2^{-2} + a_{-3}2^{-3} + a_{-4}2^{-4} + \ldots}$$

If we multiply both sides by 2

- the left side can only be ≥ 1
- if $a_{-1} = 1$

We can then subtract a_{-1} from both sides

and repeat to get a_{-2}, a_{-3}, a_{-4}, and so forth.

Example of Finding a Floating-Point Bit Pattern

For example, let's say that we want to find the bit pattern for **5.046875**.

We first write **5** in binary: **101**.

Now we need to convert the fraction

$$\mathbf{F = 0.046875}.$$

$0.046875 \times 2 = 0.09375 \quad (< 1, \text{ so } \mathbf{a_{-1} = 0})$

$0.09375 - 0 = 0.09375$

Example of Finding a Floating-Point Bit Pattern

Start with 0.09375.

$0.09375 \times 2 = 0.1875$ (< 1, so $\mathbf{a_{-2} = 0}$)

$0.1875 - 0 = 0.1875$

$0.1875 \times 2 = 0.375$ (< 1, so $\mathbf{a_{-3} = 0}$)

$0.375 - 0 = 0.375$

$0.375 \times 2 = 0.75$ (< 1, so $\mathbf{a_{-4} = 0}$)

$0.75 - 0 = 0.75$

Example of Finding a Floating-Point Bit Pattern

Start with 0.75.

$0.75 \times 2 = 1.5$ (so $\mathbf{a_{-5} = 1}$)

$1.5 - 1 = 0.5$

$0.5 \times 2 = 1$ (so $\mathbf{a_{-6} = 1}$)

$1 - 1 = 0$ (done)

Putting the bits together, we find

$$\mathbf{F = 0.046875_{10} = 0.000011_2}$$

Example of Finding a Floating-Point Bit Pattern

Now we have converted to binary:

$$\mathbf{5.046875_{10} = 101.000011_2}$$

In binary scientific notation, we have

$$\mathbf{+\ 1.01000011 \times 2^2}$$

And, in single-precision floating point,

- the sign bit is **0**,
- the exponent is 2+127 = 129 = **10000001**,
- and the mantissa is **01000011…** (no leading 1, and 15 more 0s afterward).

Tricky Questions about Floating-Point

A question for you:

What is $2^{-30} + (1 - 1)$?

Quite tricky, I know. But yes, it's **2^{-30}**.

Another question for you:

What is $(2^{-30} + 1) - 1$?

That's right. It's **0**.

At least it is with floating-point.

Floating-Point is Not Associative

Why?

Our first sum was **$(2^{-30} + 1)$**.

To hold the integer 1, the bit pattern's exponent must be **2^0**.

But, the mantissa for single-precision floating point has only **23 bits**.

And thus represents powers down to **2^{-23}**.

The **2^{-30}** term is lost, giving **$(2^{-30} + 1) = 1$**.

So $2^{-30} + (1 - 1) \neq (2^{-30} + 1) - 1$.

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Hexadecimal, Text, and Terminology for Representations

Some Sugar-Coating for Humans

Bits are a bit of a pain. For example, try to memorize this pattern:

0001001101010101100111

But **computers always use bits**!

Humans, on the other hand,

- can use base 16,
- usually called **hexadecimal**, or **hex**,
- to make dealing with bit patterns easier.

Have you memorized the pattern? Hurry up!

Convert Hex to/from Binary in Groups of 4 Bits

Hex includes A through F to get 16 digits:

0 1 2 3 4 5 6 7 8 9 A B C D E F

$16 = 2^4$, so **each hex digit represents four bits**.

Remember:

- Use of **hex only serves to help humans write and remember bits**!
- Digital systems just use bits.

Time for a **Pop Quiz**!

Ok, what is the bit pattern?

Seriously?

Maybe you remember a few of them?

What if this is were an exam question?

Sigh.

Ok, it was **00010011010101100111**.

In hex, that's **x13567** (P&P/LC-3 hex notation—otherwise, 13567 is probably decimal!).

Can you remember that? **Please?**

Text was Historically Represented with ASCII

How do we represent text?

One early system was the American Standard Code for Information Interchange (**ASCII**).

ASCII is a 7-bit code representing
- **English letters** A-Z in both cases
- (Arabic) **digits** 0-9
- **Punctuation**
- Some **special symbols** ($, #, %, and so on)
- **Control characters** for terminals

A Few Other Text Representations

The ubiquity of the 8-bit byte gave rise to "extended" (8-bit) versions of **ASCII**.

These were not standardized.*

What about other languages?

- UIUC (NCSA) invented the browser in 1993
- and the Internet received global attention.
- **Unicode (16-bit)** includes characters for many other languages.

* There are 8-bit standard encodings for text today, but our goal is not an exhaustive list.

Terminology: Representations vs. Data Types

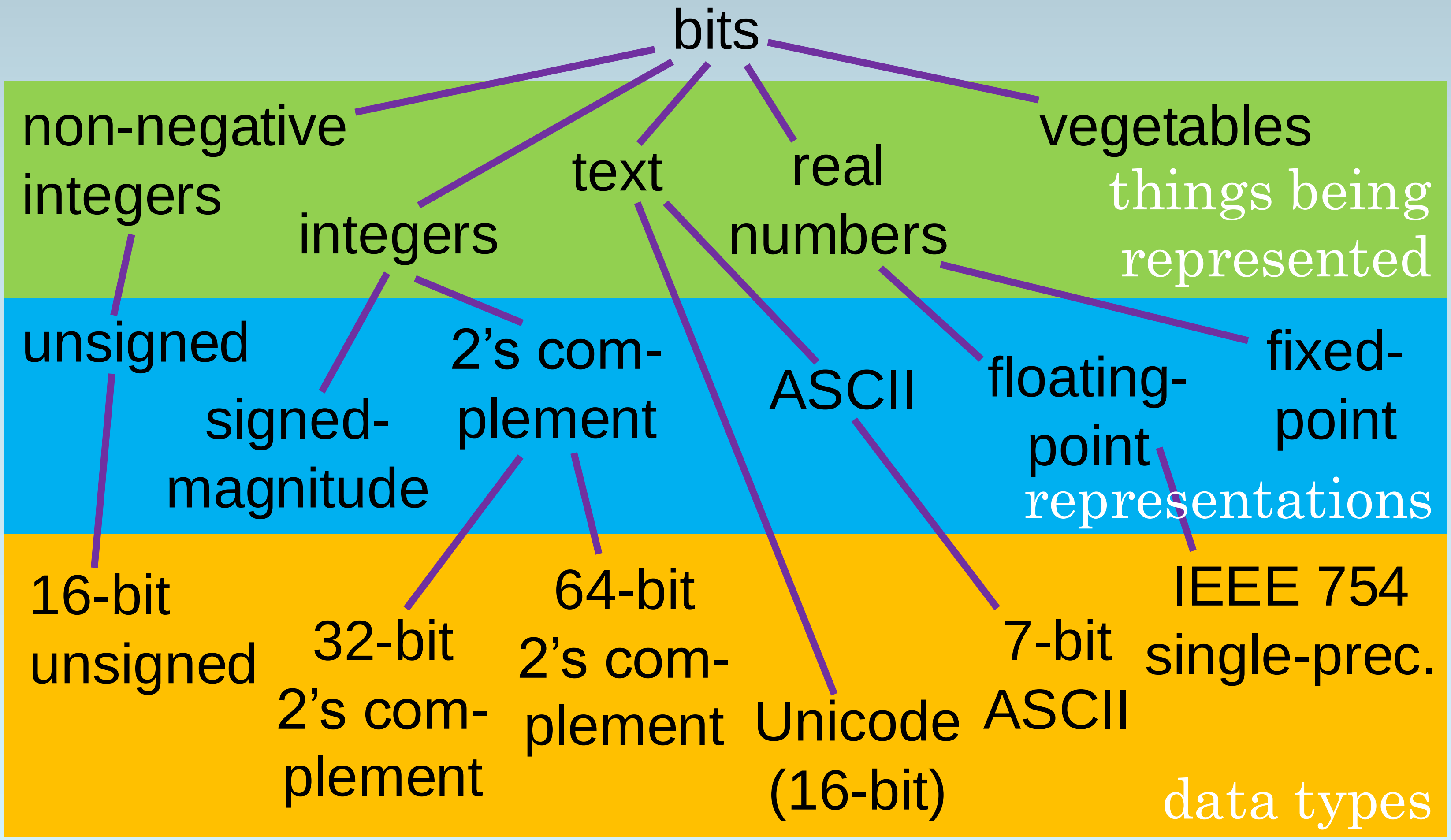
We will try to differentiate between

- **representation**: ways of encoding specific types of information into bit patterns
- **data type**: a specific number of bits encoded with a specific representation

Examples of data types include: **8-bit unsigned**, **16-bit 2's complement**, **IEEE 754 single-precision floating point**

High-level languages such as C associate values with data types.

Illustration of a Representation Taxonomy



Remember: Computers Do Not "Understand" Bits

Human text usually in **ASCII** or **Unicode**
- human-readable files
- your typing
- text printed for you to read

Computer do not "understand" what the bits mean.

Computers Always Do What They're Told

For example, what does a computer do if someone tells it …

- to add the **ASCII** character "3" (`0110011`)
- to the **ASCII** character "2" (`0110010`)?

The computer adds them!

Using an adder…

Natural log just got simpler!

```
   11   1
   0110011  ("3")
 + 0110010  ("2")
 ----------
   1100101  ("e")
```

Computers Require Explicit Instructions

To get the "right" answer, someone (a human) must tell the computer

- to convert the **ASCII** to **unsigned** or **2's complement**
- to add the converted values, and
- to convert the sum back to **ASCII**!

Second-Chance **Pop Quiz**!

Ok, what is the number in hex?

x13567

Memorizing numbers is not a learning objective in ECE120.

But you probably get the point of the exercise.

Hex makes it easier to deal with bits.

(You may find hex harder to use for arithmetic and logic calculations, though.)