Selective Harmonic Elimination (SHE)

The problem consists in eliminating the harmonics generated through switching of PWM converters to improve the quality of the output signal.

The starting point for the SHE methodology consists in obtaining the Fourier coefficients of the odd harmonics, the only existing ones due to the symmetry of the PWM waveform. By chopping the PWM waveform n times per half cycle, these Fourier coefficients are given by

$$a_k = -\frac{4V_{dc}}{k\pi}\sum_{i=1}^{2n}(-1)^{i+1}\sin(k\alpha_i), \qquad b_k = -\frac{4V_{dc}}{k\pi}\left[1-\sum_{i=1}^{2n}(-1)^{i}\cos(k\alpha_i)\right], \tag{1}$$

where $k \in \{1, 3, 5, \ldots 2n-1\}$ is the harmonic order, $2n$ is the total number of switching angles per half fundamental cycle, V_{dc} is the DC-link voltage and α_i is the optimal switching angle. Notice that (1) can be easily manipulated into the form

$$\begin{aligned} &-\frac{k\pi a_k}{4V_{dc}} = \sum_{i=1}^{2n}(-1)^{i+1}\sin(k\alpha_i), \quad -\frac{k\pi b_k}{4V_{dc}} = 1-\sum_{i=1}^{2n}(-1)^{i}\cos(k\alpha_i), \\ &p = 1, 3, 5, \ldots, 2n-1, \end{aligned}$$

leading to the following sets of $2n$ transcendental equations

$$\begin{cases} \sin(\alpha_1) - \sin(\alpha_2) + \ldots + (-1)^{2n+1}\sin(\alpha_{2n}) = -\dfrac{\pi a_1}{4V_{dc}} \\ \sin(3\alpha_1) - \sin(3\alpha_2) + \ldots + (-1)^{2n+1}\sin(3\alpha_{2n}) = -\dfrac{3\pi a_3}{4V_{dc}} \\ \vdots \\ \sin((2n-1)\alpha_1) - \sin((2n-1)\alpha_2) + \ldots + (-1)^{2n+1}\sin((2n-1)\alpha_{2n}) = -\dfrac{n\pi a_{2n-1}}{4V_{dc}} \end{cases} \tag{2}$$

$$\begin{cases} 1-\cos(\alpha_1) + \cos(\alpha_2) + \ldots + (-1)^{2n}\cos(\alpha_{2n}) = -\dfrac{\pi b_1}{4V_{dc}} \\ 1-\cos(3\alpha_1) + \cos(3\alpha_2) + \ldots + (-1)^{2n}\cos(3\alpha_{2n}) = -\dfrac{3\pi b_3}{4V_{dc}} \\ \vdots \\ 1-\cos((2n-1)\alpha_1) + \cos((2n-1)\alpha_2) + \ldots + (-1)^{2n}\cos((2n-1)\alpha_{2n}) = -\dfrac{(2n-1)\pi b_{2n-1}}{4V_{dc}} \end{cases} \tag{3}$$

The next step is to look for the switching angles $\{\alpha_i\}_{i=1}^{2n}$ such that the transcendental equations (2) and (3) are satisfied at the same time.

To this end, we can first observe that, if we apply the change of variables $\beta_i = \alpha_i - \frac{\pi}{2}$, $i = 1, \ldots, 2n$, for k odd, we have

$$\sin(k\alpha_i) = \sin\left(k\left(\beta_i + \frac{\pi}{2}\right)\right) = (-1)^{k+1}\cos(k\beta_i) = \cos(k\beta_i).$$

Hence, system (2) becomes

$$\begin{cases} \cos(\beta_1) - \cos(\beta_2) + \ldots + (-1)^{2n+1}\cos(\beta_{2n}) = -\dfrac{\pi a_1}{4V_{dc}} \\ \cos(3\beta_1) - \cos(3\beta_2) + \ldots + (-1)^{2n+1}\cos(3\beta_{2n}) = -\dfrac{3\pi a_3}{4V_{dc}} \\ \vdots \\ \cos((2n-1)\beta_1) - \cos((2n-1)\beta_2) + \ldots + (-1)^{2n+1}\cos((2n-1)\beta_{2n}) = -\dfrac{(2n-1)\pi a_{2n-1}}{4V_{dc}} \end{cases} \tag{4}$$

In order to solve systems (3) and (4), we are going to transform them into algebraic ones. Let us start with (4). The procedure to follows summarizes in three steps:

Step 1: we apply the change of variables $x_i = (-1)^{i+1}\cos(\beta_i)$, $i = 1, \dots, 2n$ to the first equation in (2), obtaining

$$x_1 + x_2 + \ldots + x_{2n} = -\frac{\pi a_1}{4V_{dc}} =: r_1.$$

Step 2: by considering the Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x)$$

and their trigonometric definition

$$T_j(x) = \cos(j\arccos(x)), \quad |x| \leq 1,$$

we have for all $i = 1, \dots, 2n$

$$\begin{aligned}
&T_0(\cos(\beta_i)) = 1\\
&T_1(\cos(\beta_i)) = \cos(\alpha_i)\\
&T_2(\cos(\beta_i)) = \cos(2\beta_i) = 2\cos(\beta_i)T_1(\cos(\beta_i)) - T_0(\cos(\beta_i)) = 2\cos^2(\beta_i) - 1\\
&T_3(\cos(\beta_i)) = \cos(3\beta_i) = 2\cos(\beta_i)T_2(\cos(\beta_i)) - T_1(\cos(\beta_i)) = 4\cos^3(\beta_i) - 3\cos(\beta_i)
\end{aligned}$$

Hence

$$\begin{aligned}
\cos(3\beta_1) - \cos(3\beta_2) + &\ldots + (-1)^{2n+1}\cos(3\beta_{2n})\\
&= \Big(4\cos^3(\beta_1) - 3\cos(\beta_1)\Big) - \Big(4\cos^3(\beta_2) - 3\cos(\beta_2)\Big)\\
&\quad + \ldots + (-1)^{2n+1}\Big(4\cos^3(\beta_{2n}) - 3\cos(\beta_{2n})\Big)\\
&= 4\Big(\cos^3(\beta_1) - \cos^3(\beta_2) + \ldots + (-1)^{2n+1}\cos^3(\beta_{2n})\Big)\\
&\quad + 3\Big(\cos(\beta_1) - \cos(\beta_2) + \ldots + (-1)^{2n+1}\cos(\beta_{2n})\Big).
\end{aligned}$$

Using again the change of variables $x_i = (-1)^{i+1}\cos(\beta_i)$, $i = 1, \dots, 2n$, and noticing that

$$x_i^3 = (-1)^{3i+3}\cos^3(\beta_i) = (-1)^{i+1}\cos^3(\beta_i), \quad i = 1, \dots, 2n$$

we get

$$\begin{aligned}
-\frac{3\pi a_3}{4V_{dc}} &= \cos(3\beta_1) - \cos(3\beta_2) + \ldots + (-1)^{2n+1}\cos(3\beta_{2n})\\
&= 4(x_1^3 + x_2^3 + \ldots + x_{2n}^3) + 3(x_1 + x_2 + \ldots + x_{2n}) = 4(x_1^3 + x_2^3 + \ldots + x_{2n}^3) + 3r_1.
\end{aligned}$$

This yields

$$x_1^3 + x_2^3 + \ldots + x_{2n}^3 = -\frac{1}{4}\left(\frac{3\pi a_3}{4V_{dc}} + 3r_1\right) =: r_3 \tag{5}$$

Step 3: iterating this procedure, we get

$$\begin{aligned}
&x_1 + x_2 + \ldots + x_{2n} = r_1\\
&x_1^3 + x_2^3 + \ldots + x_{2n}^3 = r_3\\
&\vdots\\
&x_1^{2n-1} + x_2^{2n-1} + \ldots + x_{2n}^{2n-1} = r_{2n-1}
\end{aligned} \tag{6}$$

The transformation of (3) in a set of algebraic equations can be obtained through an analogous procedure as follows:

Step 1: we apply the change of variables $y_i = (-1)^i \cos(\alpha_i)$, $i = 1, \dots, 2n$ to the first equation in (3), obtaining

$$y_1 + y_2 + \ldots + y_{2n} = s_1.$$

Step 2: by considering again the Chebyshev polynomials and proceeding as before, we get

$$y_1^3 + y_2^3 + \ldots + y_{2n}^3 = s_3 \tag{7}$$

Step 3: iterating this procedure, we get

$$\begin{aligned} & y_1 + y_2 + \ldots + y_{2n} = s_1 \\ & y_1^3 + y_2^3 + \ldots + y_{2n}^3 = s_3 \\ & \vdots \\ & y_1^{2n-1} + y_2^{2n-1} + \ldots + y_{2n}^{2n-1} = s_{2n-1} \end{aligned} \tag{8}$$

Summarizing, we have shown that solving (3) and (4) amounts to find the solution $\{x_i\}_{i=1}^{2n}$ and $\{y_i\}_{i=1}^{2n}$ of (6) and (8).

At this regards, let us notice that from the identities $x_i = (-1)^{i+1}\cos(\beta_i)$ and $y_i = (-1)^i \cos(\alpha_i)$ we can obtain

$$\begin{aligned} & x_i = (-1)^{i+1}\cos(\beta_i) = (-1)^{i+1}\cos\left(\alpha_i - \frac{\pi}{2}\right) = (-1)^{i+1}\sin(\alpha_i) \quad \longrightarrow \quad \alpha_i = \arcsin\left((-1)^{i+1}x_i\right) \\ & y_i = (-1)^i\cos(\alpha_i) = (-1)^i\cos\left(\arcsin\left((-1)^{i+1}x_i\right)\right) = (-1)^i\sqrt{1 - x_i^2} \end{aligned}$$

Hence, it will be sufficient to solve only (6) to determine $\{x_i\}_{i=1}^{2n}$ and $\{y_i\}_{i=1}^{2n}$ will then be given by the above relation.

Let us then discuss the resolution of (6). Firstly, notice that the system can be written in a compact form as

$$\sum_{i=1}^{2n} x_i^k = r_k. \tag{9}$$

To solve (9), we look at the polynomial $P(x)$ whose roots are $\{x_i\}_{i=1}^{2n}$, that is,

$$P(x) = \prod_{i=1}^{2n}(x - x_i) = \sum_{m=0}^{2n} p_m x^{2n-m}.$$

Notice that a simple computation gives

$$\frac{P'(x)}{P(x)} = \frac{d}{dx}\ln(P(x)) = \sum_{i=1}^{2n}\frac{1}{x - x_i}.$$

Expanding this logarithmic derivative at $x = \infty$, we get

$$\frac{d}{dx}\ln(P(x)) = \sum_{\ell \geq 0}\frac{r_\ell}{x^{\ell+1}} = \frac{r_0}{x} + \sum_{\ell \geq 1}\frac{r_\ell}{x^{\ell+1}}$$

where, according to (9), $r_\ell = \sum_{i=1}^{2n} x_i^\ell$. Integrating in the variable x, and observing that $r_0 = 2n$, we obtain

$$\begin{aligned} \ln(P(x)) &= \int\left(\frac{r_0}{x} + \sum_{\ell \geq 1}\frac{r_\ell}{x^{\ell+1}}\right) dx \\ &= r_0\ln(x) + \sum_{\ell \geq 1} r_\ell \int x^{-\ell-1}\, dx = 2n\ln(x) - \sum_{\ell \geq 1}\frac{r_\ell}{\ell x^\ell} = \ln(x^{2n}) - \sum_{\ell \geq 1}\frac{r_\ell}{\ell x^\ell}. \end{aligned}$$

Hence

$$P(x) = \exp\left(\ln(x^{2n}) - \sum_{\ell\geq 1} \frac{r_\ell}{\ell x^\ell}\right) = x^{2n}\exp\left(-\sum_{\ell\geq 1}\frac{r_\ell}{\ell x^\ell}\right).$$

Moreover, we can get rid of the multiplicative factor x^{2n} by dividing the above expression by $P(-x)$. This yields

$$P(x) = P(-x)G\left(\frac{1}{x}\right), \tag{10}$$

where

$$G\left(\frac{1}{x}\right) = \exp\left(V\left(\frac{1}{x}\right)\right) = \exp\left(-2\sum_{\substack{\ell\geq 1\\ \ell\ odd}}\frac{r_\ell}{\ell x^\ell}\right). \tag{11}$$

Besides, it is possible to determine an explicit expression of $G(1/x)$ as follows:

Step 1: first of all, let us introduce the series expansion of $V(x)$:

$$V(x) = v_0 + v_1 x + v_2 x^2 + \ldots = \sum_{\ell\geq 0} v_\ell x^\ell.$$

Moreover, from (11) we get

$$V\left(\frac{1}{x}\right) = -2\sum_{\substack{\ell\geq 1\\ \ell\ odd}}\frac{r_\ell}{\ell x^\ell} \quad\longrightarrow\quad V(x) = -2\sum_{\substack{\ell\geq 1\\ \ell\ odd}}\frac{r_\ell}{\ell}x^\ell.$$

Thus,

$$V(x) = -2\sum_{\substack{\ell\geq 1\\ \ell\ odd}}\frac{r_\ell}{\ell}x^\ell = \sum_{\ell\geq 0} v_\ell x^\ell$$

and, equating the coefficients of the same order, we have

$$\begin{cases} v_\ell = -2\frac{r_\ell}{\ell}, & \text{for } \ell \text{ odd} \\ v_\ell = 0, & \text{for } \ell \text{ even.} \end{cases}$$

Step 2: let us introduce the series expansion of $G(x)$

$$G(x) = g_0 + g_1 x + g_2 x^2 + \ldots = \sum_{\ell\geq 0} g_\ell x^\ell$$

and notice that, from the expression $G(x) = \exp(V(x))$, we obtain

$$\frac{dG(x)}{dx} = \frac{dV(x)}{dx}G(x).$$

Expanding both sides of the identity above, we get

$$\begin{aligned} g_1 + 2g_2 x + 3g_3 x^2 + \ldots &= (v_1 + 2v_2 x + 3v_3 x^2 + \ldots)(g_0 + g_1 x + g_2 x^2 + \ldots) \\ &= (v_1 g_0) + (2v_2 g_0 + v_1 g_1)x + (3v_3 g_0 + 2v_2 g_1 + v_1 g_2)x^2 + \ldots \end{aligned}$$

By setting $g_0 = 1$ (which yields $v_0 = \ln(g_0) = 0$) and equating the coefficients of the left and right-hand side, we thus find that

$$g_0 = 1, \quad g_\ell = \sum_{k=1}^{\ell}\frac{k}{\ell}v_k g_{\ell-k} = \sum_{\substack{k=1\\ k\ odd}}^{\ell}\frac{k}{\ell}v_k g_{\ell-k} \quad \text{for } \ell\geq 1.$$

Since we already know the coefficients v_ℓ, the coefficients g_ℓ are now fully determined.

From (10) we now have

$$P(x) = P(-x)G\left(\frac{1}{x}\right) = P(-x)\left(g_0 + \frac{g_1}{x} + \frac{g_2}{x^2} + \dots\right),$$

which gives

$$\begin{aligned}
&\sum_{m=0}^{2n} p_m x^{2n-m} \\
&\quad = \left(\sum_{m=0}^{2n} p_m (-x)^{2n-m}\right)\left(\sum_{\ell \geq 0} g_\ell x^{-\ell}\right) = \left(\sum_{m=0}^{2n} (-1)^{2n-m} p_m x^{2n-m}\right)\left(\sum_{m \geq 0} g_m x^{-m}\right) \\
&\quad = \left(\sum_{m=0}^{2n} (-1)^m p_m x^{2n-m}\right)\left(\sum_{m \geq 0} g_m x^{-m}\right).
\end{aligned}$$

By equating once again the coefficients of the left and right-hand side, we thus find the expressions for $\{p_m\}_{m=0}^{2n}$. This gives us the analytic expression of the polynomial $P(x)$. The solutions to (6) will be the roots of this polynomial, which can be approximated through the Newton method. The switching angles $\{\alpha_i\}_{i=1}^{2n}$ are then determined as

$$\alpha_i = \arcsin\left((-1)^{i+1} x_i\right), \quad i = 1, \dots, 2n.$$