Solution of the pulse width modulation problem using orthogonal polynomials and Korteweg–de Vries equations

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The mathematical underpinning of the pulse width modulation (PWM) technique lies in the attempt to represent "accurately" harmonic waveforms using only square forms of a fixed height. The accuracy can be measured using many norms, but the quality of the approximation of the analog signal (a harmonic form) by a digital one (simple pulses of a fixed high voltage level) requires of the elimination of high order harmonics in the error term. The most important practical problem is in "accurate" reproduction of sine-wave using the same number of pulses as the number of high harmonics eliminated. We describe in this paper a complete solution of the PWM problem using Padé approximations, orthogonal polynomials, and solitons. The main result of the paper is the characterization of discrete pulses answering the general PWM problem in terms of the manifold of all rational solutions to Korteweg–de Vries equations.

1. Introduction

This paper describes an analytical solution to the pulse width modulation (PWM)-inspired problem of reconstructing a set of numbers given their "odd" moments only. The problem has two formulations. The first is an algebraic problem of "sums of odd powers," where one has to determine the set $\{x_i\}$ ($i = 1 \dots n$) from the equations on sums of odd powers:

$$\sum_{i=1}^{n} x_i^{2m-1} = s_{2m-1}, m = 1 \cdots n.$$

The second is the transcendental problem of "sums of odd cosines," where angles $\{\alpha_i\}$ ($i = 1 \cdots n$) are determined from the equations:

$$\sum_{i=1}^{n} \cos(2m-1)\alpha_i = c_{2m-1}, \quad m = 1, \dots, n.$$

This transcendental version of the problem was formulated in the 1960s as a basis of the PWM method. We show here that this problem can be "analytically" solved using Padé approximation techniques, and we describe fast methods of the numerical solution needed for practical applications. The solution to this problem resulted from collaboration with our colleagues from Polytechnic University in Brooklyn, NY, D. Czarkowski and I. Selesnik (see ref. 1).

A very interesting feature of the general solution to the PWM problem lies in its connection to the classical areas of mathematics—symmetric functions, orthogonal polynomials, and the theory of completely integrable systems. The most surprising relationship is that with the Korteweg–de Vries (KdV) hierarchy of infinite dimensional Hamiltonians. We show how the complete solution of the PWM problem describes the class of all rational solutions of KdV equations.

2. Subsequences of Symmetric Functions

The problem of finding the set of n elements $\{x_i\}$ with given values of n arbitrary symmetric functions $\{s_j\}$ in x_i, $i = 1 \dots n$, is in general a very complicated one because of the nontrivial nature of the relations between symmetric functions of high degrees. Only in special cases, well described in the literature (see e.g., ref. 2), can this problem be reduced to a manageable and "exactly solvable" one. In all of these classical cases, one looks at the polynomial $P(x)$, whose roots $\{x_i\}$ are

$$P(x) = \prod_{i=1}^{n} (x - x_i).$$

This polynomial $P(x)$ is uniquely identifying the set $\{x_i\}$. The coefficients of $P(x)$ are elementary symmetric functions in x_i, which have to be determined to determine the set $\{x_i\}$. Newton's formulas for the sum of powers of x_i show that once one knows n symmetric functions

$$s_j = \sum_{i=1}^{n} x_i^{\,j}, \quad j = 1 \cdots n,$$

then one easily finds—by means of Newtons's linear recurrences—the elementary symmetric functions in x_i—the coefficients of $P(x)$—consequently finding the set $\{x_i\}$ as the set of roots of $P(x)$. The derivation of Newton's recurrences is quite simple; look at the logarithmic derivative of $P(x)$:

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}.$$

Expanding the logarithmic derivative at $x = \infty$, we get

$$\frac{P'(x)}{P(x)} = \sum_{m=0}^{\infty} \frac{s_m}{x^{m+1}}$$

for $s_m = \sum_{i=1}^{n} x_i^{\,m}$. Comparing the standard expansion of $P(x)$ at $x = \infty$ $P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$, one gets Newton's linear recursion relations between a_i and s_j:

$$k a_k = \sum_{i=1}^{k} s_i \cdot a_{k-i}.$$

An alternative derivation of relations between a_i and s_j can be done using the following symbolic relation (which we will use in later generalizations):

$$P(x) = e^{\int f(x)\,dx} = x^n e^{-\sum_{m=1}^{\infty} \frac{s_m}{mx^m}}.$$

Important combinatorial interpretations of the last identity were studied by MacMahon (3). One such identity with many interpretations is

$$a_k = \sum_{\lambda} (-1)^{\sum \lambda_i} \frac{s_1^{\lambda_1} \cdot s_2^{\lambda_2} \cdot s_3^{\lambda_3} \cdots}{1^{\lambda_1} \cdot 2^{\lambda_2} \cdot 3^{\lambda_3} \cdots \lambda_1! \cdot \lambda_2! \cdot \lambda_3! \cdots}$$

for all partitions λ where $1 \cdot \lambda_1 + 2 \cdot \lambda_2 + 3 \cdot \lambda_3 \dots = k$.

Abbreviations: PWM, pulse width modulation; KdV, Korteweg–de Vries

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What happens if, instead of classical canonical (consecutive) sets of symmetric functions (in x_i), one knows only the values of n nonconsecutive symmetric functions? The problem posed in the PWM method of power electronics, in one of its important cases, looks at the first n odd power sum symmetric functions s_{2m-1} for $m = 1 \ldots n$. The solution of that problem also leads to a sequence of linear recurrences, not among the invariants (symmetric functions) themselves but among the sequences of polynomials associated with them (these are $P_n(x) = P(x)$ as n varies). This solution is based on the Padé approximations and orthogonal polynomials.

We start with the relation between Padé approximations and the generalization of Newton relations between power and elementary symmetric functions. Let us look at the Padé approximation of the order (n, d) to the series $g(x)$ at $x = \infty$. Here the function $g(x)$ is defined via the sequence S_m:

$$g(x) = e^{-\sum_{m=1}^{\infty} \frac{S_m}{mx^m}}.$$

The definition of the Padé approximation of the order (n, d) to $g(x)$ at (the neighborhood of) $x = \infty$ is the following—it is a rational function $\frac{P_n(x)}{Q_d(x)}$ with $P_n(x)$ a polynomial of degree n and $Q_d(x)$ a polynomial of degree d—such that the expansion of $\frac{P_n(x)}{Q_d(x)}$ matches the expansion of $g(x)$ at $x = \infty$ up to the maximal order. This means that

$$\frac{P_n(x)}{Q_d(x)} - x^{n-d}g(x) = O(x^{-2d-1}) \text{ or}$$
$$P_n(x) - Q_d(x)x^{n-d}g(x) = O(x^{-d-1}).$$

After taking the logarithmic derivative of this definition, we end up with the following representation of this definition:

$$\frac{P'_n}{P_n} - \frac{Q'_d}{Q_d} = \frac{d}{dx}\log x^{n-d}g(x) + O(x^{-n-d-2}).$$

Now if we write normalized (with the leading coefficient 1) polynomials $P_n(x)$ and $Q_d(x)$ in terms of their roots:

$$P_n(x) = \prod_{i=1}^{n}(x - x_i); \quad Q_d(x) = \prod_{k=1}^{d}(x - y_k),$$

we get an identification of symmetric functions in x_i and y_k with the sequence of S_m in the definition of $g(x) = e^{-\sum_{m=1}^{\infty} \frac{S_m}{mx^m}}$. Namely, we get

$$\sum_{i=1}^{n} x_i^j - \sum_{k=1}^{d} y_k^j = S_j$$

for $j = 0 \cdots n + d$ (where $S_0 = n - d$).

In the case $d = 0$, one recovers Newton's identities. The case $d = n$—the so-called case of the "diagonal" Padé approximations—is the most interesting case. It is also the case that solves the problem of sums of odd powers. This is how it works. Consider the anti-symmetric case when $y_i = -x_i$ for $i = 1 \ldots n$ and $d = n$. In this case, $S_{2m} = 0$ for $m \geq 0$ and $S_{2m-1} = 2s_{2m-1}$. Thus we get thus Padé approximations of the order (n, n) to the following function:

$$G(x) = e^{-2\sum_{m \text{ odd}}^{\infty} \frac{s_m}{mx^m}}.$$

The Padé approximants $\frac{P_n(x)}{Q_n(x)}$ have the property

$$Q_n(x) = (-1)^n P_n(-x),$$

because $G(x)$ satisfies a functional identity: $G(-x) = 1/G(x)$. This gives us our main result.

THEOREM 1. *The solution* $\{x_i\}$ $(i = 1 \ldots n)$ *to the problem of sums of odd powers*:

$$\sum_{i=1}^{n} x_i^{2m-1} = s_{2m-1}, \quad m = 1 \cdots n,$$

is given by the roots of the numerator $P_n(x) = \prod_{i=1}^{n}(x - x_i)$ *in the Padé approximation problem of order* (n, n) *to the function*

$$G(x) = e^{-2\sum_{m \text{ odd}}^{\infty} \frac{s_m}{mx^m}}.$$

Another way to verify this approximation without specialization from the case of general sequence s_m is simply to take the expansion of the logarithmic derivative of $\frac{P_n(x)}{P_n(-x)}$ at $x = \infty$. Expanding the logarithmic derivative, and then integrating it (formally) in x, one gets a very simple identity:

$$(-1)^n \cdot \frac{P_n(x)}{P_n(-x)} = e^{-2\sum_{m \text{ odd}}^{\infty} \frac{x^{-m}}{m} \sum_{i=1}^{n} x_i^m}.$$

From this identity, *Theorem 1* follows.

Proof of Theorem 1: First of all, the Padé approximation rational function $\frac{P_n(x)}{Q_n(x)}$ of order (n, n) is unique. Then, if $\frac{P_n(x)}{Q_n(x)}$ is a Padé approximation of order (n, n) to $G(x)$, we assume that this representation of the rational function is irreducible Then $\frac{Q_n(x)}{P_n(x)}$ is a Padé approximation of order (n, n) to $1/G(x)$, and $\frac{P_n(-x)}{Q_n(-x)}$ is a Padé approximation of order (n, n) to $G(-x)$. Because of the functional equation $G(-x) = 1/G(x)$, and the uniqueness of the Padé approximations, we get $\frac{Q_n(x)}{P_n(x)} = \frac{P_n(-x)}{Q_n(-x)}$. This equation means that $Q_n(x) = \alpha P_n(-x)$. Moreover, because the expansion of $G(x)$ at $x = \infty$ starts at 1, we have $\frac{P_n(x)}{Q_n(x)} \to 1$ as $x \to \infty$. Thus $Q_n(x) = (-1)^n P_n(-x)$. Taking into account the "main" identity

$$(-1)^n \cdot \frac{P_n(x)}{P_n(-x)} = e^{-2\sum_{m \text{ odd}}^{\infty} \frac{x^{-m}}{m} \sum_{i=1}^{n} x_i^m},$$

we can see that the right-hand side of this identity and the expansion of $G(x)$ at $x = \infty$ has to agree up to (but not including) x^{-2n-1}. This means that we have $\sum_{i=1}^{n} x_i^{2m-1} = s_{2m-1}$ for $m = 1 \cdots n$.

Because we identified the solution to the sums of odd powers problem with the numerator (or denominator) in the (diagonal) Padé approximation problem, we infer from the standard theory of continued fraction expansion that rational functions $\frac{P_n(x)}{P_n(-x)}$ are partial fractions in the continued fraction expansion of the generating function $G(x)$ at $x = \infty$. This also means (see ref. 4 for these and other facts of the theory of continued fraction expansions and orthogonal polynomials) that the sequence of polynomials $P_n(x)$ is the sequence of orthogonal polynomials and that the sequence of polynomials $P_n(x)$ satisfies three-term linear recurrence relation. Because the same recurrence is satisfied by both numerators and denominators of the partial fractions, it means that the recurrence is satisfied by two sequences—$P_n(x)$ and $(-1)^n \cdot P_n(-x)$. When the leading coefficient of $P_n(x)$ is 1, one gets a particularly simple three-term recurrence relation among $P_n(x)$:

$$P_{n+1}(x) = x \cdot P_n(x) + C_n \cdot P_{n-1}(x)$$

for $n = 0 \ldots$.

3. Complexity of PWM Computations

What is the complexity of computations of PWM polynomials and their roots—solutions of the PWM problem? One can ask the same question about all n first sums of powers. If one uses just Newton identities, the complexity is $O(n^2)$, but a much faster scheme can be found. The key to this is

$$P(x) = x^n e^{-\sum_{m=1}^{\infty} \frac{s_m}{m x^m}}.$$

Indeed, according to Brent's theorem N terms of the power series expansion of $e^{V(x)}$ can be computed in only $O(N \log N)$ steps from the power series expansion of $V(x)$ (see ref. 5, section 4.7, example 4). This algorithm requires only use of the FFT (Fast Fourier Transform) technique for computation of fast convolution.

Similar complexity considerations can be applied to the problem of fast computation of polynomials $P_n(x)$ that give the solution to the PWM problem of consecutive odd power sums. A simple $O(n^2)$ complexity algorithm provides the determination of not only the single $P_n(x)$ but also all $P_m(x)$ for $m \le n$. For large n, these algorithms become impractical. Thus one needs to use fast algorithms.

This is how a fast algorithm of computations of (all coefficients of) $P_n(x)$ of the total complexity of $O(n \log^2 n)$ works. First, one has to apply Brent's theorem to compute $O(n)$ terms of the power series expansion (at infinity) of

$$G(x) = e^{-2 \sum_{m \text{ odd}}^{\infty} \frac{s_m}{m x^m}}$$

from the first $O(n)$ terms s_m with the complexity of only $O(n \log n)$. Then one has to use fast Padé approximation algorithms. There is a variety of these algorithms, with the most popular from ref 6. Its complexity is $O(n \log^2 n)$. Thus we can compute $P_n(x)$ in at most $O(n \log^2 n)$ operations.

4. Sums of Odd Powers and Sums of Odd Cosines

The original definition of the PWM problem dealt with transcendental equations $\sum_{i=1}^{n} \cos(2m-1)\alpha_i = c_{2m-1}$ and not with algebraic equations $\sum_{i=1}^{n} x_i^{2m-1} = s_{2m-1}$. A very important contribution to the PWM problem by D. Czarkowski and I. Selesnick is in the explicit reduction of the transcendental PWM problem to the algebraic one for consecutive $m = 1, \ldots, n$. We will show now how the transcendental case is explicitly expressed using the introduced notations of $G(x)$ and $P_n(x)$. The basic transformation is $\cos\alpha_i = x_i$, or in the algebraic form: $x = (z + z^{-1})/2$, for $z = e^{\alpha}$. Now, to get from the algebraic sums of odd powers solution the transcendental one, consider the set of roots z_i, z_i^{-1}; $i = 1, \ldots, n$. Then sums of odd powers for these roots give

$$\sum_{i=1}^{n} \cos(2m-1)\alpha_i \text{ for } \cos\alpha_i = z_i.$$

Expanding each term on the right as a function of z^{-1}, we get

$$(-1)^n \cdot \frac{P_n(x)}{P_n(-x)} = e^{-2\sum_{m=1}^{\infty} T_m/m z^m},$$

where $T_m = \sum_{i=1}^{n} z_i^{\,m} + z_i^{\,-m} = 2\sum_{i=1}^{n} \cos m\alpha_i$.

This means that the function $G(x)$ (or its Padé approximation) that determines the solution of the algebraic sums of odd powers problem (in x) can be reduced to the transcendental sums of odd cosines problem (in z). Specifically, $G(x)$, as a function of z, has the following form:

$$G(x) = e^{-4\sum_{m \text{ odd}}^{\infty} \frac{c_m}{m z^m}},$$

where the sequence $\{c_m\}$ arises from the following general transcendental sums of odd cosines problem:

$$\sum_{i=1}^{n} \cos(2m-1)\alpha_i = c_{2m-1}, \quad m = 1, \ldots, n,$$

corresponding to the algebraic sums of odd powers problem:

$$\sum_{i=1}^{n} x_i^{\,2m-1} = s_{2m-1}, \quad m = 1, \ldots, n,$$

with $x_i = \cos\alpha_i$ for $i = 1, \ldots, n$.

Notice that the "explicit expression" for x_i (α_i) or $P_n(x)$ simply means that the continued fraction expansion of $G(x)$ (in x or z at infinity) is known "explicitly," or equivalently that the coefficients C_n in the main three-term recurrence describing $P_n(x)$ are "explicit" (i.e., classical elementary or transcendental) functions of n. From this point of view, the main PWM transcendental problem

$$\sum_{i=1}^{n} \cos\alpha_i = a, \quad \sum_{i=1}^{n} \cos(2m-1)\alpha_i = 0, m = 2, \ldots, n$$

is not "explicitly solvable" because the corresponding function $G(x)$ (see the expression above in terms of z and the sequence $\{c_m\}$),

$$G_a(x) = e^{-4a(x-\sqrt{x^2-1})},$$

does not have an "explicit" continued fraction expansion at $x = \infty$. On the other hand, a very similar algebraic problem,

$$\sum_{i=1}^{n} x_i = a; \ \sum_{i=1}^{n} x_i^{\,2m-1} = 0; \ m = 2, \ldots, n,$$

does have an "explicit" solution, because the corresponding function $e^{-2a/x}$ has a classical continued fraction expansion derived by Euler. The polynomials $P_n(x)$ arising in this special algebraic problem are well-known as Bessel polynomials.

5. Simple Algorithm of Polynomial Computations

A simple algorithm of computation of all $P_m(x)$ for all $m = 0, \ldots, n$, having the complexity of $O(n^2)$ is easy to describe. Let us look at the expansion of $G(x)$ at $x = \infty$.

$$G(x) = \sum_{k=0}^{\infty} (-1)^k c_k \cdot x^{-k}.$$

By *Theorem 1*, polynomials $P_n(x)$ are defined from the Padé approximation problem to $G(x)$, i.e., the remainder function

$$R_n(x) = P_n(-x) \cdot G(x) - (-1)^n \cdot P_n(x)$$

has the following expansion at $x = \infty$: $R_n(x) = O(\frac{1}{x^{n+1}})$. Because $P_n(x)$ are orthogonal polynomials, they satisfy

$$P_{n+1}(x) = x \cdot P_n(x) + C_n \cdot P_{n-1}(x)$$

for $n = 0 \ldots$. The initial conditions can be chosen here as: $P_{-1} = 1$, $P_0 = 1$. Let us write $P_n(x)$ in terms of its coefficients: $P_n(x) = \sum_{i=0}^{n} P_{n,i} \cdot x^i$.

If $P_{n-1}(x)$ and $P_n(x)$ are known, then to determine the single unknown C_n, and thus $P_{n+1}(x)$, we have to look at the coefficient at x^{-n} of $R_n(x)$. Assume that the leading coefficient of $P_n(x)$ is 1. Looking at the coefficient at x^{-n} in the expansion of $R_n(x)$, we get the following expression for C_n:

$$C_n = -\frac{\sum_{j=0}^{n} P_{n,j} \cdot c_{n+j+1}}{\sum_{i=0}^{n-1} P_{n-1,i} \cdot c_{n+i}}.$$

Once C_n is determined, the coefficients $P_{n+1,i}$ are easily determined recursively.

For this algorithm to work, one needs coefficients c_i in the expansion of $G(x)$. As we know, the complexity of Brent's algorithm of computations of c_i up to $O(n)$ that uses FFT is $O(n \log n)$.

Often the complexity is bounded only by $O(n)$. For example, in the most interesting case of the PWM problem,

$$G(x) = G_a(x) = e^{-4a(x-\sqrt{x^2-1})},$$

the algorithm of computing c_i is a very simple one of complexity $O(n)$ only. This algorithm follows our general power series algorithms: notice that $G_a(x)$ satisfies the second order linear differential equation (with singularities at $x = -1$, 1, ∞ and an apparent singularity at $x = 0$)

$$y''x(x^2-1) + y'(8ax(x^2-1)) + y(4a(1-4ax)) = 0.$$

If we look at the expansion of $G_a(x)$ at $x = \infty$: $G_a(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{x^n}$, then we get the four-term recurrence on c_n: $c_{n+2} = \frac{1}{8a(n+2)} \cdot (c_{n+1}((n+1)^2 + (n+1) - 16a^2) + c_n(8an + 4a) + c_{n-1}(-(n-1)^2 - 2(n-1)))$. The initial conditions are $c_n = 0$ for $n < 0$ and $c_0 = 1$, $c_1 = 2a$, $c_2 = 2a^2, \ldots$. From these values and the equation for c_n, one derives the C_n factors in the three-term recurrence for orthogonal polynomials $P_n(x)$. Here are a few initial C_n:

$$C_0 = -a, \quad C_1 = \frac{4a^2-3}{12}, \quad C_2 = \frac{45 - 60a^2 + 16a^4}{60(4a^2-3)}.$$

Here all C_n are rational functions in a of a rather special structure. Because the case of the continued fraction expansion of $G_a(x)$ is not explicitly solvable, C_n is not a "known" function of a and n. An important consequence of the "unsolvability" of C_n is the growth of coefficients of C_n as rational functions in a with integer coefficients. According to standard conjectures about explicit and nonexplicit continued fraction expansions (see ref. 7), the coefficients of C_n as the rational function in a over $\mathbf{Z}$ are growing as $e^{O(n^2)}$ for large n. In fact, for $n \geq 12$, the coefficients of C_n in a are large integers. This makes it impractical to precompute with full accuracy C_n for large n. It is also unnecessary to analytically determine $C_n = C_n(a)$ explicitly, because we need to know $C_n(a)$ only in the range of a that is significant for applications; this is the range where the weight of orthogonal polynomials $P_n(x)$ is positive.

6. Fast Algorithms of the Solution to the PWM Problems

We already know that while the slow algorithms of computation of $P_n(x)$ can be completed in $O(n^2)$ operations, the fast FFT-algorithms can be completed in $O(n \log^2 n)$ operations. After $P_n(x)$ is determined, we need, in addition, to determine the set $\{x_i\}$ of all roots of $P_n(x)$. One can use general methods of computation of roots of univariant polynomials, which would bring the overall minimal complexity higher. We do not need to do it in our case because the orthogonality properties of $P_n(x)$ allow us to have fast and numerically stable methods of computing $\{x_i\}$. We present one such algorithm, which is suitable for both the moderate and large ranges of n.

For these fast algorithms, we use fast polynomial evaluation: for any given set of N points $\{X_i\}$ and a polynomial $P(X)$ of degree N in X, one needs at most $O(N \log^2 N)$ operations to evaluate $P(X_i)$ for all $i = 1, \ldots, N$. For orthogonal polynomials $P_n(x)$, arising from the algebraic version of the PWM problem, we can use fast algorithms of evaluation at n points of $P_n(x)$ and $P_n'(x)$ to get n Newton–Raphson approximations running at the same time:

$$x_i = x_i - \frac{P_n(x_i)}{P_n'(x_i)}; \quad i = 1, \ldots, n.$$

To get rapid (geometric) rate of convergence of this algorithm, one needs initial conditions of x_i, corresponding to centers of intervals separating the roots of $P_n(x)$. Such an approximation can be rigorously derived using the classical properties of orthogonal polynomials (see ref. 4). Thus one can chose $O(n/\epsilon)$ total starting points x in the Newton–Raphson iterations with the property that any real root x_i is within the distance ϵ/n from at least one starting point x of the iteration. In the case of a fixed machine precision, the number of iteration is constant, providing us with the fast algorithm of computing the set $\{x_i\}$ with at most $O(n \log^2 n)$ operations.

7. Solvable Extensions of the PWM Problem

A variety of extensions of the sums of odd powers problem can be solved using Padé approximation techniques. For this, one uses methods of generalized graded Padé approximations that we developed. Some of these problems arise in practical applications of signal processing. A particular example of the problem, generalizing the sums of odd powers, is the problem where for a given n and $N \geq 1$ one knows consecutive sums of powers of $\{x_i\}$ $(i = 1, \ldots, n)$ except every Nth one (i.e., $1, \ldots, N-1, N+1, \ldots$). This problem is analytically solved using simultaneous Padé approximations to $N-1$ functions in a way identical to the one presented above for $N = 2$.

8. Completely Integrable Equations

In connection with Padé approximation solution to the sums of odd powers problem, we can ask what "nonclassical" objects this solution is built from. We succeeded in identifying these objects with well studied in recent "soliton equations" and their solutions—isospectral deformation equations of the full KdV hierarchy and their Bäcklund transformations.

The only parameter left in our solution is the coefficient factor C_n of the three-term recurrence for polynomials P_n:

$$P_{n+1}(x) = x \cdot P_n(x) + C_n \cdot P_{n-1}(x). \quad [1]$$

This parameter C_n is a function of n and of the whole generating sequence $\{s_m : m - \text{odd}\}$ of values of odd symmetric functions. It is C_n that is a solution of the KdV type hierarchy of completely integrable p.d.e.s (partial differential equations) in variables s_m. In addition to p.d.e.s in s_m, the parameter C_n satisfies difference-differential equations in n and each of s_m. The formal derivation of the full hierarchy of such equations is based on ref. 8. To see how one can derive them, start with Eq. **1**. Think of x as a spectral parameter: $x = \lambda$. Then Eq. **1** is an eigenvalue problem for the difference operator in n. The next step is the realization that polynomials P_n satisfy linear differential equations in each of the variables s_m. Once such a differential equation in s_m is derived, one looks at the consistency condition of this equation and Eq. **1**. Such a consistency condition is a classical isospectral deformation condition. This consistency condition implies a nonlinear difference-differential equation on C_n in n and s_m (for any odd m). The resulting nonlinear equation belongs to a completely integrable class. Eliminating n for s_m and $s_{m'}$, one gets a KdV type p.d.e. on C_n in s_m and $s_{m'}$.

To see how these are derived, let us look at the case of $m = 1$ and the variable s_1. The corresponding partial differential equation on P_n is

$$x \cdot P_{n,s_1} = P_n + E_n \cdot P_{n-1}, \quad [2]$$

where parameter E_n does not depend on x. The consistency condition between [1] and [2] leads to the following two difference-differential equations

$$C_n = E_n \cdot E_{n+1}; \quad C_{n,s_1} = E_{n+1} - E_n. \quad [3]$$

We also summarize here all relationships with Toeplitz determinants in the expansion

$$G(x) = \sum_{m=0}^{\infty} \frac{c_m}{x^m}$$

of the approximated generating function of sums of odd powers $\{s_{2m-1}\}$:

$$G = e^{-2\sum_{m\text{odd}}^{\infty} \frac{s_m}{m x^m}}.$$

Specifically, we use the standard representation of the main and the auxiliary Toeplitz determinants:

$$D_n = \det(c_{i+j})_{i,j=0}^{n-1}; \quad \Delta_n = \det(c_{i+j+1})_{i,j=0}^{n-1}.$$

The relationships between C_n and the determinants are the following ones:

$$C_n = -\frac{\Delta_{n+1}\cdot\Delta_{n-1}}{\Delta_n^2}; \quad \Delta_n^2 = (-1)^n 2D_n \cdot D_{n+1}.$$

The expression of E_n in the difference-differential equations above is:

$$E_n^2 = -\frac{D_{n+1}\cdot D_{n-1}}{D_n^2}.$$

The main object D_n is expressed in terms of the τ-function (typical notation for KdV equations):

$$D_n = \tau_n^2,$$

where τ_n is a polynomial in s_{2m-1}. We need, however, to normalize the polynomial τ_n, so that the leading power of s_1 (and it is $s_1^{n(n-1)/2}$) would have a coefficient of 1. In this case we can write more accurately

$$D_n = r_n \cdot \tau_n^2,$$

The rational numbers r_n are easily determined using the following expressions:

$$E_n = -\frac{1}{2n-1}\cdot\frac{\tau_{n+1}\cdot\tau_{n-1}}{\tau_n^2}; \qquad C_n = \frac{1}{4n^2-1}\cdot\frac{\tau_{n+2}\cdot\tau_{n-1}}{\tau_n\cdot\tau_{n+1}}.$$

This implies that $r_{n+1} = -r_n^2/((2n-1)^2 \cdot r_{n-1})$. Substituting these expressions to the equations in **3**, we get the difference differential equation on τ_n in n and s_1.

9. KdV Hierarchy

Explicit recursion relations defining all commuting (higher) KdV flows are well known. These relations connect the infinite sequence of conserved quantities H_m of the original KdV equation; these are the (infinite dimensional) Hamiltonians of the higher KdV equations, with the vector flows X_m of the (mth) KdV equation. The mth KdV is

$$u_{t_m} = X_m(u); \quad \text{for } X_m(u) = \partial_x \frac{\delta H_m}{\delta u}.$$

$\delta H_m/\delta u$ is the gradient of the functional H_m of u, e.g., for the first KdV Hamiltonian (the **actual** KdV equation), we gave:

$$\frac{\delta H_2}{\delta u} = \frac{3}{2}u^2 - \frac{1}{2}u'', \quad \text{for } H_2 = \int\left(\frac{1}{2}u^3 + \frac{1}{4}(u')^2\right)dx.$$

The flows $X_m(u)$ are commuting (as Poisson structures induced by H_m). Recursion connecting successive Hamiltonians are relatively simple:

$$X_m(u) = \partial_x\frac{\delta H_m}{\delta u} = \left(-\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x\right)\frac{\delta H_{m-1}}{\delta u}.$$

One can also write all higher KdV flows X_m in the explicit form using this recursion as follows:

$$X_m = N_u^m X_1; \quad X_1(u) = u_x;$$
$$N_u = -\frac{1}{2}\partial_x^2 + 2u + u_x\partial_x^{-1}.$$

Do not confuse x variable in KdV equations with x used in the sums of odd powers PWM problem. The variable x in the PWM problem is the **spectral** variable, usually denoted as λ, so in sections devoted to KdV, we will use this notation. Thus the main generating function will be denoted in these notations as: $G = e^{-2\sum_{m\text{odd}}^{\infty}\frac{s_m}{m\lambda^m}}$, and the polynomials whose roots solve the PWM problem will be denoted as $P_n(\lambda)$.

Further, the identification of odd moments s_{2m-1} with the canonical variables of p.d.e.s in the KdV hierarchy, is the following one: $x = s_1$; and for the higher flows' "time variables" t_m, we have $t_m = \frac{s_{2m-1}}{(2m-1)2^{m-1}}$ (so the standard KdV time is $t = s_3/6$).

10. KdV Rational Solutions describe PWM

The relationship between the KdV hierarchy of equations and the general sums of odd powers problem is a very interesting one. Roughly speaking, the parameter C_n in the recurrence relation defining the orthogonal polynomials in the sums of odd powers problem, as functions of odd moments s_{2m-1}, satisfies all p.d.e.s in the KdV hierarchy, with $x = s_1$ and the higher flows' time variables t_m being $t_m = \frac{s_{2m-1}}{(2m-1)2^{m-1}}$.

Moreover (and this is what distinguishes the sums of odd powers case and completely characterizes it in terms of KdV equations), rational solutions to the KdV equation and to the full KdV hierarchy are completely described by the solution to the sums of odd powers problem.

The class of all rational solutions to KdV, which have been well studied since 1977 and are still being investigated today, has many interesting and important properties—this class is a limit case of famous N-soliton solutions corresponding to special rational curves; it has a famous many-particle interpretation in terms of dynamics of poles of these solutions in x- (and t_m-) planes, etc.

The specific KdV relationship is the following. As above we look at the orthogonal polynomials $P_n(\lambda)$ representing the Padé approximants to the generating function G of the sequence of odd moments s_{2m-1}:

$$G = e^{-2\sum_{m\text{odd}}^{\infty}\frac{s_m}{m\lambda^m}},$$

and satisfying the three-term recurrence

$$P_{n+1}(\lambda) = \lambda\cdot P_n(\lambda) + C_n\cdot P_{n-1}(\lambda).$$

"Explicit" expressions for $P_n(\lambda)$ and C_n involve (as above) Toeplitz determinants in the coefficients c_m of the expansion of G at $\lambda = \infty$:

$$G_a(\lambda) = \sum_{m=0}^{\infty}\frac{c_m}{\lambda^m}.$$

We use the same notations as above:

$$D_n = \det(c_{i+j})_{i,j=0}^{n-1}; \quad \Delta_n = \det(c_{i+j+1})_{i,j=0}^{n-1};$$
$$C_n = -\frac{\Delta_{n+1}\cdot\Delta_{n-1}}{\Delta_n^2}; \quad \Delta_n^2 = (-1)^n 2D_n\cdot D_{n+1}.$$

The main object D_n is expressed in terms of the τ-function of KdV-type equations:

$$D_n = \tau_n^2,$$

where τ_n is a polynomial in s_{2m-1}. The KdV solutions are expressed in terms of the u potential, which is very simply related to τ as follows:

$$u_n = -2\partial_x^2(\log\tau_n) = -\partial_x^2(\log D_n).$$

The relationship between the KdV hierarchy and the sums of odd powers recurrences is the following one.

THEOREM 2. *If the generating sequence of odd moments* $\{s_{2m-1}\}$ *is considered as a sequence of independent variables, then with the identification* $x = s_1$, *and* $t_m = \frac{s_{2m-1}}{(2m-1)2^{m-1}}$, *the* τ *functions* $\tau_n = \sqrt{D_n}$ *are* **all** *rational solutions of the KdV equation (and all commuting higher KdV equations). The polynomials* τ_n *are characterized by their degree in* x: *it is* $\frac{n(n-1)}{2}$.

An important consequence of this theorem is the characterization of the full manifold of rational solutions as explicit functions of actual higher KdV natural parameters t_m (see above the identification $t_m = \frac{s_{2m-1}}{(2m-1)2^{m-1}}$ with odd moments variables). This completes the study of rational solutions of KdV (9).

We write explicitly the first few τ_n, normalizing them (for uniqueness) with the coefficient at $x^{n(n-1)/2}$ being 1 (remember that $x = s_1$):

$\tau_2 = s_1$; $\tau_3 = s_1^3 - s_3$; $\tau_4 = s_1^6 - 5s_1^3 s_3 + 5s_3^2 + 9s_1 s_5$; $\tau_5 = s_1^{10} - 15s_1^7 s_3 - 175 s_1 s_3^3 + 63(5s_1^2 s_3 + s_1^5 - 3s_5)s_5 + 225(s_3 - s_1^3)$.

The recurrence that these normalized polynomials τ_n satisfy is a known one (it also follows from the difference-differential equation on C_n in n and s_1), which contains a crucial ambiguity, hiding in constants of integration the explicit dependency on s_i:

$$\tau_{n+1,\,x} \cdot \tau_{n-1} - \tau_{n+1} \cdot \tau_{n-1,\,x} = (2n-1) \cdot \tau_n^2.$$

Theorem 2 and the direct relation to the Padé approximation problem to G provides a theory of rational solutions to the KdV hierarchy that is much more simple than all other descriptions (we refer to refs. 10 and 9 for an original exposition, and to ref. 11 for the modern presentation and review).

11. Equations in Nonintegrable PWM Cases

Whenever the function G that is expanded into its continued fraction does not satisfy a Riccatti equation over $\mathbf{C}(x)$, there is no simple Painlevé equation/recurrence on partial quotients C_n. Most G fall into this category, and examples of G from PWM problems are not integrable either. The most interesting example of G depends on the parameter a (voltage level):

$$G = e^{-4a(x - \sqrt{x^2-1})}$$

expanded at $x = \infty$. The corresponding partial quotient $C_n = C_n(a)$ has to be determined as a function of a in order to compute solutions $P_n(x)$ of the main PWM problem. We show that $C_n(a)$, though not a Painlevé function, satisfies an algebraic difference-differential equation in n and a.

We start with the definition of the Padé approximation to G:

$$q_n \cdot G - p_n = O(x^{-n-1}),$$

where $q_n = q_n(x)$, $p_n = p_n(x)$ are polynomials of degree n in x, and $p_n(x) = (-1)^n q_n(-x)$. From the expansion of G at $x = \infty$, one gets leading coefficients of q_n: $q_n = x^n + ax^{n-1} + q_{2,\,n}x^{n-2} + \ldots$, where $q_{2,\,n+1} = q_{2,\,n} + C_i$, or $q_{2,\,n} = \sum_{i=1}^{n-1} C_i$. G satisfies a linear p.d.e. over $\mathbf{Q}(x, a)$:

$$L \cdot G_x = 4aG \text{ for } L = (x^2 - 1) \cdot d_x - a \cdot x \cdot d_a.$$

We can differentiate the definition of the Padé approximation and get

$$Q_n^{(1)} \cdot G - P_n^1 = O(x^{-n});$$
$$Q_n^{(1)} = (x^2 - 1) q_{n,\,x} - a x q_{n,\,a} + 4a q_n;$$
$$P_n^{(1)} = (x^2 - 1) p_{n,\,x} - a x p_{n,\,a}.$$

Because of orthogonality properties of $q_n(x)$, we can express $Q_n^{(1)}$ as a linear combination of only a few of q_m values:

$$Q_n^{(1)} = \alpha_n \cdot q_{n-1} + 2a \cdot q_n + n \cdot x \cdot q_n;$$
$$\alpha_n = -2 \cdot \sum_{i=1}^{n-1} C_i - a \cdot \sum_{i=1}^{n-1} C_{i,\,a} + 2a^2 - n.$$

As a result, we have a linear partial difference-differential equation on q_n in n, a, and x. This equation is compatible with original three-term linear recurrence on q_n. The consistency condition becomes our new equation on $C_n(a)$.

This leads to a new equation on α_n and C_n:

$$\alpha_{n+1} = \alpha_{n-1} \cdot C_n / C_{n-1} - 1.$$

We get a differential equation in a on C_n:

$$C_{n,\,a} = -\xi_{n-1} \cdot C_n + \frac{\alpha_n}{a}; \quad \xi_{n-1} = \frac{1}{a} \cdot \left(\frac{\alpha_{n-1}}{C_{n-1}} + 2 \right).$$

Its solution can be written in quadratures as follows:

$$C_n = \left(e^{-\int \xi_{n-1}} \right) \cdot \left(\int \frac{\alpha_n}{a} \cdot e^{\int \xi_{n-1}} \right).$$

The initial condition on C_n as a function of a is $C_n\,|_{a=0} = -1/4$. One can also present a detailed algebraic analysis of C_n as a rational function of a in terms of Toeplitz determinants in coefficients c_k of G at $x = \infty$.

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1. Chudnovsky, D. V., Chudnovsky, G. V., Czarkowski, D. & Selesnik, I. (2000) *IEEE Comm.*, in press.
2. Littlewood, D. E. (1958) *The Theory of Group Characters* (Oxford Univ. Press, London).
3. MacMahon, P. A. (1915) *Combinatory Analysis, v.I, II* (Cambridge Univ. Press, Cambridge, U.K.).
4. Szego, G. (1978) *Orthogonal Polynomials* (Am. Math. Soc., Providence, RI).
5. Knuth, D. (1981) *The Art of Computer Programming* (Addison–Wesly, Reading, MA), Vol. 2.
6. Brent, R., Gustavson, F. & Yun, D. (1980) *J. Algorithms* **1** 259–295.
7. Chudnovsky, D. V. & Chudnovsky, G. V. (1989) *Proc. Symp. Pure Math.* (Am. Math. Soc. Providence, RI) Vol. 49, Part 2, pp. 167–232.
8. Chudnovsky, D. V. & Chudnovsky, G. V. (1983) *Proc. Natl. Acad. Sci. USA* **80**, 1774–1777.
9. Chudnovsky, D. V. & Chudnovsky, G. V. (1977) *Nuovo Cimento* **40B**, 339–353.
10. Airault, H., McKean, H. P. & Moser, J. (1977) *Comm. Pure Appl. Math* **30**, 95–148.
11. Zubelli, J. P., Falqui, G., Magri, F. & Padroni, M. (1999) *Theor. Math. Physics*, in press.