Selective Harmonic Elimination via Optimal Control Theory

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Abstract

1 Introduction

In this document, we propose a optimal control perspective of selective harmonic elimination problem (SHE) with symmetry of quarter wave. In mathematical point of view, SHE problem can be seen as search of a square wave function $f(\omega t) \mid \omega t \in (0, 2\pi)$ which have fixed a few Fourier coefficients.

In this way, the $f(\omega t)$ can be written in Fourier series as follows:

$$f(\omega t) = \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \tag{1}$$

Where a_n and b_n coefficients are:

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(\omega t) \cos(n\omega t) d(\omega t) \tag{2}$$

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} f(\omega t) \sin(n\omega t) d(\omega t) \tag{3}$$

On the other hand, the symmetry of quarter wave implies:

$$f(\omega t + \pi) = -f(\omega t) \quad t \in (0, \pi) \tag{4}$$

$$f(\omega t + \pi/2) = +f(\omega) \quad t \in (0, \pi/2) \tag{5}$$

This two conditions simplify the expressions (2) and (3), in this way:

$$a_n = 0 \mid \forall n \in \mathbb{Z} \tag{6}$$

$$b_n = \begin{cases} \frac{2}{\pi} \int_0^{\pi/2} f(\omega t) \sin(n\omega t) d(\omega t) & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases} \tag{7}$$

So, in summary $f(\omega t)$ can be written as follows:

$$f(\omega t) = \sum_{n\ odd}^{\infty} b_n \sin(n\omega t) \tag{8}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} f(\omega t)\sin(n\omega t)d(\omega t) \ \mid \ n\ odd \tag{9}$$

Now in this context, we can define a SHE problem as follows:

Problem 1.1 (SHE two levels) Given $\boldsymbol{b}_T = [b_T^1\ ,\ b_T^3\ ,\ b_T^5\ ,\ \dots\ ,\ b_T^{N/2}] \in \mathbb{R}^{N/2}$, we search a wave form $f(\omega t) \mid \omega t \in (0, \pi/2)$ such that f only can take values $\{-1, 1\}$ and its Fourier coefficients b_n satisfies $b_n = b_T^n \mid \forall n \in \{1, 3, \dots, N/2\}$.

In the typical formulation of this problem, the function $f(\omega t)$ can be represented by locations where the function $f(\omega t)$ changes its value, this locations are named switching angles. Given a some vector $\boldsymbol{b}^T$, the number of switching angles M is *a priori* unknown, so it's necessary fixed it. If we name switching angles as $\boldsymbol{\phi} = [\phi_1, \phi_2, \dots, \phi_M] \in \mathbb{R}^M$, we can simplify the expression (9) as follows:

$$b_n(\boldsymbol{\phi}) = \frac{2}{n\pi}\left[-1 + 2\sum_{i=1}^{M}(-1)^{i+1}\cos(n\phi_i)\right] \mid \forall n\ odd \tag{10}$$

With this expression, we can formulate the problem (1.1) as the next mimization problem:

$$\min_{\boldsymbol{\phi}\in\mathbb{R}^m} \sum_{n\ odd}^{N/2} (b_n(\boldsymbol{\phi}) - b_T^n)^2 \tag{11}$$

$$\text{subject to:} \begin{cases} 0 < \phi_1 \\ \phi_n < \phi_n + 2 \quad \forall n \in \{3, 5, \dots, N/2 - 2\} \\ \phi_{N/2} < \pi/2 \end{cases} \tag{12}$$

This formulation don't give a clearly procedure to choose a number of angles.

We propose consider a search of a function $f(\omega t)$ directly. In this way, instead of looking for the switching angles $\boldsymbol{\phi} \in \mathbb{R}^M$, we look for a function $f(\omega t) \in \{g(\omega t) \in L^\infty([0, \pi/2]) \ / \ |g(\omega t)| < 1\}$.

Gracias al teorema fundamental del cálculo, podemos afirmar que una función $\beta(\tau)$ definida como:

$$\beta_n(\tau) = \frac{2}{\pi} \int_0^\tau f(\omega t) \sin(n\omega t) d(\omega t) \Rightarrow \begin{cases} \frac{\partial \beta}{d\tau} & = \frac{2}{\pi} f(\tau) \sin(n\tau) \\ \beta(0) & = 0 \end{cases} \tag{13}$$

Cuando resolvemos la ecuación diferencial ordinaria (13) hasta tiempo $T = \pi/2$ obtenemos el valor del coeficiente de Fourier b_n.

Este se puede ver como una ecuación diferencial ordinaria controlada donde los estados del sistema son β_n y el control es $f(\tau) \mid \tau \in [0, \pi/2]$. Entonces se puede plantear el problema de control cuyo objetivo es llevar el sistema $\beta_n(\tau)$ desde el estado nulo hasta b_T^n para cada $n \in \{1, 3, 5, \dots, N/2\}$ en un tiempo final $\tau_f = \pi/2$

2 Optimal control formulation

Problem 2.1 Given $\boldsymbol{b}_T \in \mathbb{R}^{n_b}$, we define a cost functional in this way:

$$J[f(\tau)] = \left[||\boldsymbol{b}_T - \boldsymbol{\beta}(T)||^2 - \epsilon \int_0^{\pi/2} ||f(\tau)||^2 d\tau \right] \tag{14}$$

where $\boldsymbol{\beta}(\tau) = [\beta_1(\tau) \ \beta_3(\tau) \ \dots \ \beta_{N/2}(\tau)]^T$, $||.||$ is a euclidean norm and ϵ is a penalization parameter to maximized the norm of control $f(\tau)$. This maximization and constraint $|f(\tau)| < 1$, produce a *bang-bang* control.

So, the optimal control problem can be write:

$$\min_{|f(\tau)|<1} J[f(\tau)] \tag{15}$$

$$\text{suject to: } \begin{cases} \frac{d\beta_n}{d\tau} = (2/\pi) \sin(n\tau) f(\tau) \quad \tau \in [0, \pi/2] \\ \beta_n(0) = 0 \end{cases} \tag{16}$$

$$\forall n \in \{1, 3, 5, \dots, N/2\}$$

3 Numerical Results

Sea $\boldsymbol{b}_T = [b_T^1, b_T^5, b_T^7, b_T^{11}] = [MI, 0, 0, 0, 0]$

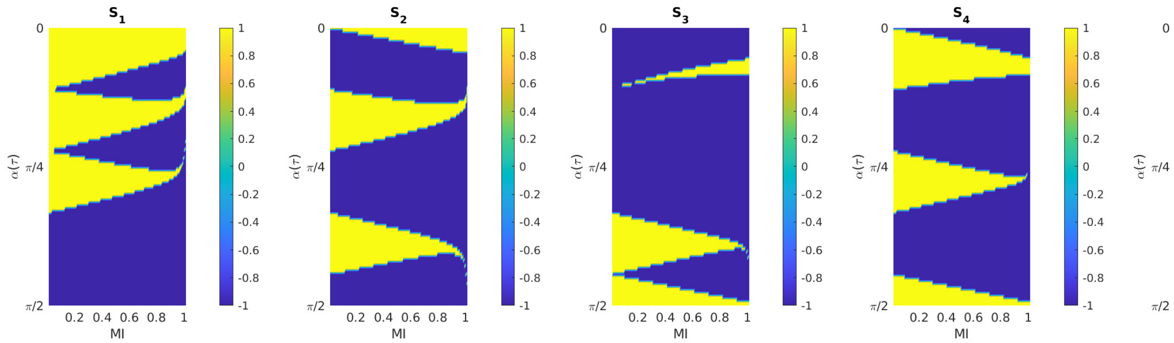

Figure 1: Solutions

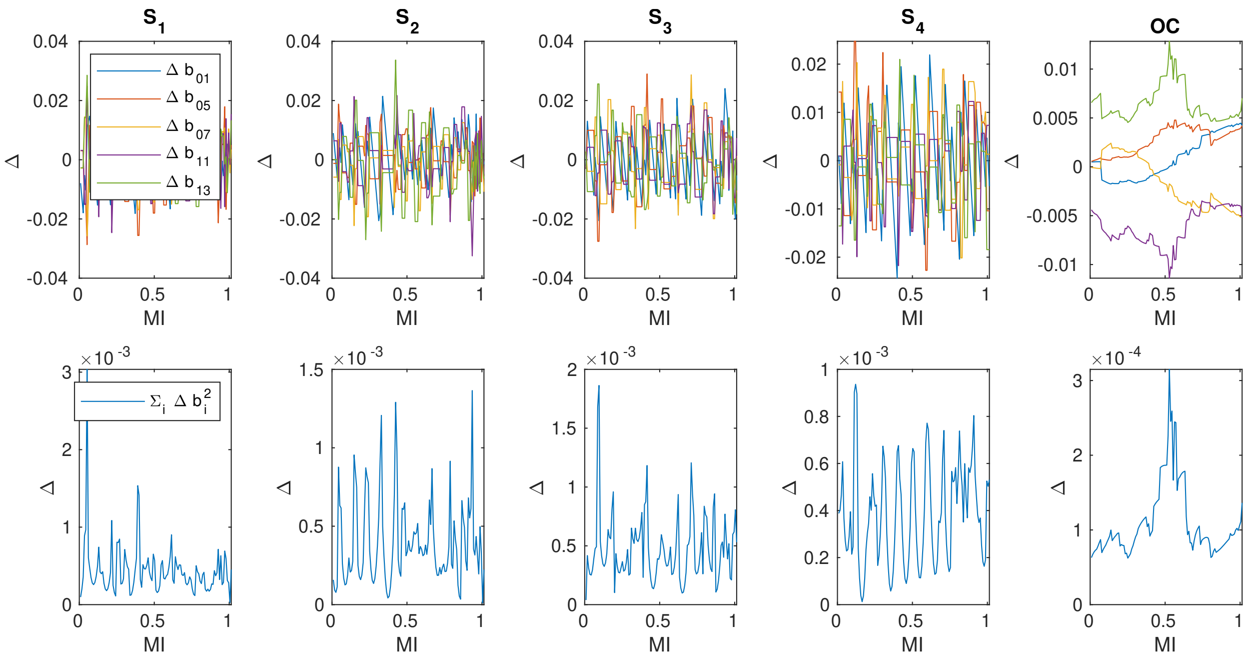

Figure 2: Errors

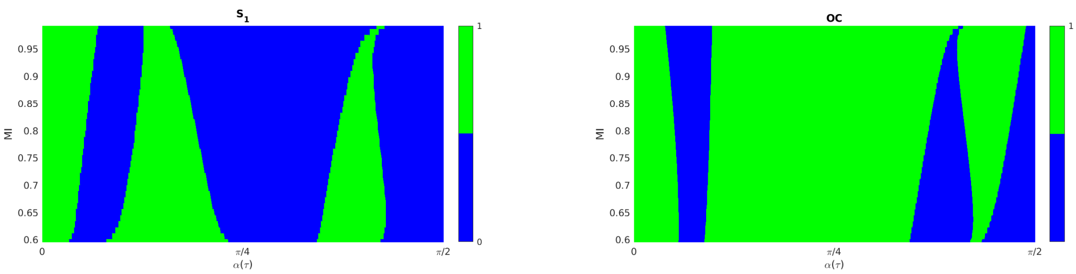

Figure 3: Solutions

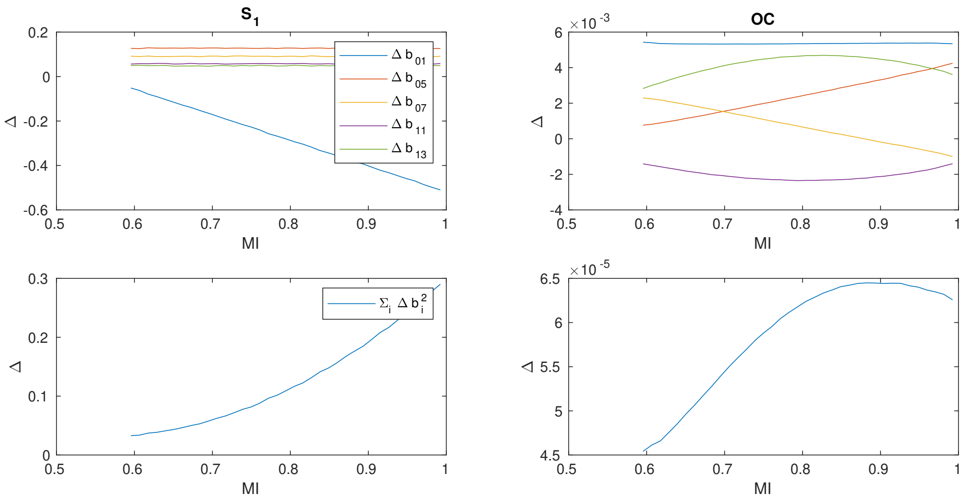

Figure 4: Errors

A Other

$$\int_0^{\pi/2} ||f(\tau)||^2 d\tau = \pi/2 \tag{17}$$

$$\int_0^{\pi/2} ||f(\tau)||^2 d\tau = \pi/2 \int_0^{\pi/2} || \sum_{n\ odd}^{\infty} b_n \sin(n\tau)||^2 d\tau = \pi/2 \tag{18}$$

$$\int_0^{\pi/2} \sum_{n,n'\ odd}^{\infty} b_n b'_n \sin(n\tau)\sin(n'\tau) d\tau = \pi/2 \tag{19}$$

$$\sum_{n,n'\ odd}^{\infty} b_n b'_n \int_0^{\pi/2} \sin(n\tau)\sin(n'\tau) d\tau = \pi/2 \tag{20}$$

$$?? \tag{21}$$

$$\frac{\pi}{4} \sum_{n\ odd}^{\infty} b_n^2 = \pi/2 \tag{22}$$

$$\sum_{n\ odd}^{\infty} b_n^2 = 2 \tag{23}$$

References