Selective Harmonic Elimination (SHE)

The problem consists in eliminating the harmonics generated through switching of PWM converters to improve the quality of the output signal.

SHE methodology

An effective SHE methodology can be summarized in the following steps

Step 1.

To obtain the Fourier coefficients of the odd harmonics, the only existing ones due to the symmetry of the PWM waveform. By chopping the PWM waveform n times per quarter cycle, these Fourier coefficients are given by

$$b_k = -\frac{4V_{dc}}{k\pi}\left[1 - 2\sum_{i=1}^{n}(-1)^{i-1}\cos(k\alpha_i)\right] = -\frac{4V_{dc}}{k\pi}\left[1 + 2\sum_{i=1}^{n}(-1)^{i}\cos(k\alpha_i)\right] \tag{1}$$

where $k \in \{1,3,5,7,\ldots\}$ is the harmonic order, n is the total number of switching angles per quarter fundamental cycle, V_{dc} is the DC-link voltage and α_i is the optimal switching angle. Notice that (1) can be manipulated into the form

$$1 + 2\sum_{i=1}^{n}(-1)^{i}\cos(k\alpha_i) = -\frac{k\pi b_k}{4V_{dc}},$$

leading to the following set of transcendental equations

$$\begin{aligned}
&1 - 2\cos(\alpha_1) + \ldots + (-1)^n 2\cos(\alpha_n) = -\frac{\pi b_1}{4V_{dc}}\\
&1 - 2\cos(3\alpha_1) + \ldots + (-1)^n 2\cos(3\alpha_n) = -\frac{3\pi b_3}{4V_{dc}}\\
&\vdots\\
&1 - 2\cos(k\alpha_1) + \ldots + (-1)^n 2\cos(k\alpha_n) = -\frac{k\pi b_k}{4V_{dc}}
\end{aligned} \tag{2}$$

Step 2.

To convert the transcendental equations (2) into algebraic ones. This is done in the following way:

2.1 We apply the change of variables $x_k = -(-1)^k\cos(\alpha_k)$ to the first equation in (2), obtaining

$$1 - 2x_1 - \ldots - 2x_n = -\frac{\pi b_1}{4V_{dc}} \quad\longrightarrow\quad x_1 + \ldots + x_n = \frac{1}{2} + \frac{\pi b_1}{8V_{dc}} =: s_1.$$

2.2 By considering the Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x)$$

and their trigonometric definition

$$T_j(x) = \cos(j\arccos(x)), \quad |x| \le 1,$$

we have

$$\begin{aligned}
&T_0(\cos(\alpha_i)) = 1\\
&T_1(\cos(\alpha_i)) = \cos(\alpha_i)\\
&T_2(\cos(\alpha_i)) = \cos(2\alpha_i) = 2\cos(\alpha_i)T_1(\cos(\alpha_i)) - T_0(\cos(\alpha_i)) = 2\cos^2(\alpha_i) - 1\\
&T_3(\cos(\alpha_i)) = \cos(3\alpha_i) = 2\cos(\alpha_i)T_2(\cos(\alpha_i)) - T_1(\cos(\alpha_i)) = 4\cos^3(\alpha_i) - 3\cos(\alpha_i)
\end{aligned}$$

Hence

$$\begin{aligned}
&1-2\cos(3\alpha_1)+\ldots+(-1)^n 2\cos(3\alpha_n)\\
&\quad=1-2\Big(4\cos^3(\alpha_1)-3\cos(\alpha_1)\Big)+\ldots+(-1)^n 2\Big(4\cos^3(\alpha_n)-3\cos(\alpha_n)\Big)\\
&\quad=1-8\Big(\cos^3(\alpha_1)+\ldots+(-1)^n\cos^3(\alpha_n)\Big)+6\Big(\cos(\alpha_1)+\ldots+(-1)^n\cos^3(\alpha_n)\Big).
\end{aligned}$$

Repeating the same change of variables as before, we get

$$\begin{aligned}
&1-2\cos(3\alpha_1)+\ldots+(-1)^n 2\cos(3\alpha_n)\\
&\quad=1-8(x_1^3+\ldots+x_n^3)+6(x_1+\ldots+x_n)=1-8(x_1^3+\ldots+x_n^3)+6s_1.
\end{aligned}$$

This yields

$$x_1^3+\ldots+x_n^3=\frac{1}{8}+\frac{3}{4}s_1+\frac{3\pi b_3}{32V_{dc}}=\frac{1}{2}+\frac{3\pi b_1}{32V_{dc}}+\frac{3\pi b_3}{32V_{dc}}=:s_3 \tag{3}$$

2.3 Iterating this procedure, we get

$$x_1+\ldots+x_n=s_1 \tag{4}$$
$$x_1^3+\ldots+x_n^3=s_3 \tag{5}$$
$$\vdots \tag{6}$$
$$x_1^{2n-1}+\ldots+x_n^{2n-1}=s_{2n-1} \tag{7}$$

where the coefficients s_ℓ are obtained through the recursive formula

$$T_{2n-1}(x)|_{x^{2n-1}=s_{2n-1}}=\frac{1}{2}\left(1+\frac{\pi b_{2n-1}}{4V_{dc}}\right).$$

Step 2.

We have reduced our transcendental equations (2) to a series of polynomial equations in the form

$$\sum_{i=1}^{n}x_i^{2\ell-1}=s_{2\ell-1},\quad \ell=1,\ldots,n. \tag{8}$$

To solve (8), we look at the polynomial $P(x)$ whose roots are $\{x_i\}_{i=1}^n$, that is,

$$P(x)=\prod_{i=1}^{n}(x-x_i)=p_0x^n+p_1x^{n-1}+\ldots+p_n=\sum_{m=0}^{n}p_mx^{n-m}.$$

Expanding the logarithmic derivative at $x=\infty$, we get

$$\frac{d}{dx}\ln(P(x))=\frac{P'(x)}{P(x)}=\sum_{\ell\geq 0}\frac{s_\ell}{x^{\ell+1}}=\frac{s_0}{x}+\sum_{\ell\geq 1}\frac{s_\ell}{x^{\ell+1}}$$

where, according to (8), $s_\ell=\sum_{i=1}^n x_i^\ell$. Integrating in the variable x, and observing that $s_0=n$, we obtain

$$\begin{aligned}
\ln(P(x))&=\int\left(\frac{s_0}{x}+\sum_{\ell\geq 1}\frac{s_\ell}{x^{\ell+1}}\right)dx\\
&=s_0\ln(x)+\sum_{\ell\geq 1}s_\ell\int x^{-\ell-1}\,dx=n\ln(x)-\sum_{\ell\geq 1}\frac{s_\ell}{\ell x^\ell}=\ln(x^n)-\sum_{\ell\geq 1}\frac{s_\ell}{\ell x^\ell}.
\end{aligned}$$

Hence

$$P(x) = \exp\left(\ln(x^n) - \sum_{\ell\geq 1} \frac{s_\ell}{\ell x^\ell}\right) = x^n \exp\left(-\sum_{\ell\geq 1} \frac{s_\ell}{\ell x^\ell}\right)$$

Moreover, we can get rid of the multiplicative factor x^n by dividing the above expression by $P(-x)$. This yields

$$P(x) = (-1)^n P(-x) G\left(\frac{1}{x}\right), \tag{9}$$

where

$$G\left(\frac{1}{x}\right) = \exp\left(V\left(\frac{1}{x}\right)\right) = \exp\left(-2\sum_{\substack{\ell\geq 1\\ \ell\ odd}} \frac{s_\ell}{\ell x^\ell}\right). \tag{10}$$

Besides, it is possible to determine an explicit expression of $G(1/x)$ as follows:

1. First of all, let us introduce the series expansion of $G(x)$ and $V(x)$:

$$\begin{aligned} G(x) &= g_0 + g_1 x + g_2 x^2 + \ldots = \sum_{m\geq 0} g_m x^m \\ V(x) &= v_0 + v_1 x + v_2 x^2 + \ldots = \sum_{m\geq 0} v_m x^m. \end{aligned}$$

2. From (10), we get

$$V\left(\frac{1}{x}\right) = -2\sum_{\substack{\ell\geq 1\\ \ell\ odd}} \frac{s_\ell}{\ell x^\ell} \quad\longrightarrow\quad V(x) = -2\sum_{\substack{\ell\geq 1\\ \ell\ odd}} \frac{s_\ell}{\ell} x^\ell.$$

   Thus, the coefficients v_m are given by $v_m = -2\frac{s_m}{m}$, for m odd, and $v_m = 0$, for m even.

3. From the expression $G(x) = \exp(V(x))$, we obtain

$$\frac{dG(x)}{dx} = \frac{dV(x)}{dx} G(x).$$

   Expanding both sides of the identity above, we get

$$\begin{aligned} g_1 + 2g_2 x + 3g_3 x^2 + \ldots &= (v_1 + 2v_2 x + 3v_3 x^2 + \ldots)(g_0 + g_1 x + g_2 x^2 + \ldots) \\ &= (v_1 g_0) + (2v_2 g_0 + v_1 g_1)x + (3v_3 g_0 + 2v_2 g_1 + v_1 g_2)x^2 + \ldots \end{aligned}$$

   By equating the coefficients of the left and right-hand side, we thus find that

$$g_0 = 1, \quad g_m = \sum_{k=1}^{m} \frac{k}{m} v_k g_{m-k} \quad \text{for } m \geq 1.$$

   Since we already know the coefficients v_m, the coefficients g_m are now fully determined.

From (9) we now have

$$P(x) = (-1)^n P(-x) G\left(\frac{1}{x}\right) = (-1)^n P(-x)\left(g_0 + \frac{g_1}{x} + \frac{g_2}{x^2} + \ldots\right),$$

which gives

$$\sum_{m=0}^{n} p_m x^{n-m} = (-1)^n \left(\sum_{m=0}^{n} p_m (-x)^{n-m}\right)\left(\sum_{m\geq 0} g_m x^{-m}\right)$$
$$= \left(\sum_{m=0}^{n} (-1)^m p_m x^{n-m}\right)\left(\sum_{m\geq 0} g_m x^{-m}\right).$$

By equating once again the coefficients of the left and right-hand side, we thus find the expressions for $\{p_m\}_{m=0}^{n}$. This gives us the analytic expression of the polynomial $P(x)$. The solutions to (4) will be the roots of this polynomial, and the switching angles $\{\alpha_k\}_{k=1}^{n}$ are then determined by inverting the identity

$$x_k = -(-1)^k \cos(\alpha_k).$$