Real-time Selective Harmonic Elimination/Modulation through Chebyshev polynomials

1 Problem formulation

The problem consists in eliminating or modulating certain harmonics in a square wave function $f(\tau)$, $\tau \in (0, 2\pi)$, to improve the quality of the output signal. This function $f(\tau)$ can be written in Fourier series as follows:

$$f(\tau) = \sum_{k\in\mathbb{N}} \left(a_k \sin(k\tau) + b_k \cos(k\tau)\right), \tag{1.1}$$

where the coefficients $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ are given by

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi} f(\tau)\sin(k\tau)\, d\tau \\ b_k &= \frac{1}{\pi}\int_0^{2\pi} f(\tau)\cos(k\tau)\, d\tau. \end{aligned} \tag{1.2}$$

1.1 Two levels in quarter-wave symmetry

Here we will consider the problem in two levels and in quarter-wave symmetry. This means that the function $f(\tau)$ can only assume the values $\{-1, 1\}$ and

- on the interval $(0, 2\pi)$, $f(\tau + \pi) = f(\tau)$;
- on the intervals $(0, \pi)$ and $(\pi, 2\pi)$, $f(\tau + \frac{\pi}{2}) = -f(\tau)$.

The quarter-wave symmetry yields that all the coefficients $\{a_k\}_{k\in\mathbb{N}}$ are zero. Besides, for k even the coefficients $\{b_k\}_{k\in\mathbb{N}}$ are zero as well. Hence, (1.1) becomes

$$f(\tau) = \sum_{\substack{k\in\mathbb{N} \\ k\ odd}} b_k \cos(n\tau), \tag{1.3}$$

where the Fourier coefficients $\{b_k\}$ are given by

$$b_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(\tau)\cos(k\tau)\, d\tau. \tag{1.4}$$

Moreover, in the two levels formulation, $f(\tau)$ can be represented by the locations where the function changes its value, which are usually referred to as *switching angles* and we will indicate as $\{\alpha_i\}_{i=1}^n \in (0, \frac{\pi}{2})$ with n a priori unknown. The Fourier coefficients $\{b_k\}$ can then be expressed in terms of the angles $\{\alpha_i\}_{i=1}^n$ as:

$$b_k = b_k(\alpha_1, \ldots, \alpha_n) = -\frac{4V_{dc}}{k\pi}\left[1 + 2\sum_{i=1}^n (-1)^i \cos(k\alpha_i)\right], \tag{1.5}$$

where $k \in \{1, 3, 5, 7, \ldots, 2n-1, \ldots\}$ is the harmonic order and V_{dc} is the DC-link voltage.

Our objective is to determine the switching angles α_i for which the Fourier coefficients b_k reach a specific predetermined value.

In other words, given the values of the Fourier coefficients b_k, we look for the values of α_i solving the transcendental equations (1.5). Notice that this set of equations can be easily manipulated into the form

$$\sum_{i=1}^n (-1)^{i+1}\cos(k\alpha_i) = \frac{1}{2} + \frac{k\pi b_k}{8V_{dc}},$$

leading to the following system

$$\begin{cases} \sum_{i=1}^{n}(-1)^{i+1}\cos(\alpha_i) = \frac{1}{2} + \frac{\pi b_1}{8V_{dc}} \\ \sum_{i=1}^{n}(-1)^{i+1}\cos(3\alpha_i) = \frac{1}{2} + \frac{3\pi b_3}{8V_{dc}} \\ \sum_{i=1}^{n}(-1)^{i+1}\cos(5\alpha_i) = \frac{1}{2} + \frac{5\pi b_5}{8V_{dc}} \\ \sum_{i=1}^{n}(-1)^{i+1}\cos(7\alpha_i) = \frac{1}{2} + \frac{7\pi b_7}{8V_{dc}} \\ \vdots \end{cases} \tag{1.6}$$

2 Resolution of the transcendental equations

Following the approach of [2, 3], to solve (1.6) we are going to transform this set of transcendental equations in algebraic ones. To this end, from now on we will make the convention that the number n of switching angles coincides with the number of harmonics we want to eliminate or modulate. With this convention, the system (1.6) becomes

$$\begin{cases} \sum_{i=1}^{n}(-1)^{i+1}\cos(\alpha_i) = \frac{1}{2} + \frac{\pi b_1}{8V_{dc}} \\ \sum_{i=1}^{n}(-1)^{i+1}\cos(3\alpha_i) = \frac{1}{2} + \frac{3\pi b_3}{8V_{dc}} \\ \sum_{i=1}^{n}(-1)^{i+1}\cos(5\alpha_i) = \frac{1}{2} + \frac{5\pi b_5}{8V_{dc}} \\ \vdots \\ \sum_{i=1}^{n}(-1)^{i+1}\cos((2n-1)\alpha_i) = \frac{1}{2} + \frac{(2n-1)\pi b_{2n-1}}{8V_{dc}} \end{cases} \tag{2.1}$$

The strategy for solving (2.1) consists of two main steps:

Step 1. We first transform the transcendental equation into algebraic ones by applying the change of variables

$$x_i = (-1)^{i+1}\cos(\alpha_i), \quad i = 1, 2, \ldots, n. \tag{2.2}$$

Notice that, since $\alpha_i \in (0, \frac{\pi}{2})$ for any $i = 1, \ldots, 2$, the above transformation is one-to-one. By means of (2.2), we obtain from (2.1) a system in the form

$$\begin{cases} x_1 + x_2 + \ldots + x_n = s_1 \\ x_1^3 + x_2^3 + \ldots + x_n^3 = s_3 \\ x_1^5 + x_2^5 + \ldots + x_n^5 = s_5 \\ \vdots \\ x_1^{2n-1} + x_2^{2n-1} + \ldots + x_n^{2n-1} = s_{2n-1} \end{cases} \tag{2.3}$$

where the coefficient $\{s_{2\ell-1}\}_{\ell=1}^{n}$ depend on the Fourier coefficients $\{b_{2\ell-1}\}_{\ell=1}^{n}$ as follows:

$$\begin{aligned} s_1 &= s_1(b_1) \\ s_3 &= s_3(b_1, b_3) \\ s_5 &= s_5(b_1, b_3, b_5) \\ &\vdots \\ s_{2\ell-1} &= s_{2\ell-1}(b_1, b_3, b_5, \dots, b_{2\ell-1}) \end{aligned} \tag{2.4}$$

A more detailed presentation of the above procedure will be given in the Appendix A.

Remark 1. It is important to remark that, according to (2.4), $s_{2\ell-1}$ depends on all the Fourier coefficients $b_1\, b_3, \dots, b_{2\ell-1}$. As soon as one of these coefficients is unknown, the corresponding value $s_{2\ell-1}$ is unknown too, as well as all the successive ones $\{s_k\}_{k>2\ell-1}$.

Step 2. After Step 1, we reduced our original system (2.1) to a set of sums of odd powers

$$\sum_{i=1}^{n} x_i^{2\ell-1} = s_{\ell-1}, \quad \ell = 1, \dots, n. \tag{2.5}$$

Moreover, we know for instance from [1, Theorem 1] that the solution $\{x_i\}_{i=1}^{n}$ is determined as the roots of a polynomial of degree n

$$p(x) = \prod_{i=1}^{n}(x - x_i) = \sum_{m=0}^{n} p_m x^{n-m},$$

whose coefficients $\{p_m\}_{m=0}^{n}$ which can be computed in terms of $\{s_{2\ell-1}\}_{\ell=1}^{n}$ (see Appendix B for more detail). In other words, our original problem (2.1) has now been reduced to the computation of the set of coefficients $\{p_m\}_{m=0}^{n}$ which identify univocally the polynomial p and in computing its roots.

3 Simulation experiments

Let us present some simulation experiments for the procedure we just described. In what follows, we will consider two specific cases:

1. **Selective Harmonic Elimination**: we set the first Fourier coefficient $b_1 = 0.5m_a$ for different values of the modulation index $m_a \in (0.01, 1.05)$ and we eliminate the third, fifth and seventh Fourier coefficients (that is, $b_3 = b_5 = b_7 = 0$).
2. **Selective Harmonic Modulation**: we set the first Fourier coefficient $b_1 = 0.5m_a$ for different values of the modulation index $m_a \in (0.01, 1.05)$, $b_3 = 0.05$ and we eliminate the fifth ans seventh Fourier coefficients (that is, $b_5 = b_7 = 0$).

In both cases, (2.3) converts in the following system of four non-linear equations in four variables

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = s_1 \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 = s_3 \\ x_1^5 + x_2^5 + x_3^5 + x_4^5 = s_5 \\ x_1^7 + x_2^7 + x_3^7 + x_4^7 = s_7. \end{cases} \tag{3.1}$$

3.1 Selective Harmonic Elimination

We start with the Selective Harmonic Elimination problem, in which we want the Fourier coefficients (b_1, b_3, b_5, b_7) to match the target

$$(b_1^T, b_3^T, b_5^T, b_7^T) = (0.5m_a, 0, 0, 0), \quad m_a \in (0.01, 1.05).$$

In our simulations, we considered a 105-points discretization of the interval $(0.01, 1.05)$,

$$0.01 = m_{a,1} < m_{a,2} < \ldots < m_{a,i} < m_{a,i+1} < \ldots < m_{a,105} = 1.05$$

with $m_{a,i} = 0.01 + (i-1)\Delta m_a$, $i = 1, \ldots, 105$, $\Delta m_a = 10^{-2}$. For each value of $m_{a,i}$, we computed the corresponding vector of target Fourier coefficients

$$\begin{aligned}
\mathbf{b}_1^T &= (b_{1,1}^T, b_{1,2}^T, b_{1,3}^T, \ldots, b_{1,105}^T) = 0.5(m_{a,1}, m_{a,2}, m_{a,3}, \ldots, m_{a,105}) \in \mathbb{R}^{105},\\
\mathbf{b}_3^T &= (b_{3,1}^T, b_{3,2}^T, b_{3,3}^T, \ldots, b_{3,105}^T) = (0, 0, 0, \ldots, 0) \in \mathbb{R}^{105},\\
\mathbf{b}_5^T &= (b_{5,1}^T, b_{5,2}^T, b_{5,3}^T, \ldots, b_{5,105}^T) = (0, 0, 0, \ldots, 0) \in \mathbb{R}^{105},\\
\mathbf{b}_7^T &= (b_{7,1}^T, b_{7,2}^T, b_{7,3}^T, \ldots, b_{7,105}^T) = (0, 0, 0, \ldots, 0) \in \mathbb{R}^{105},
\end{aligned}$$

and used these values to obtain the four coefficients

$$s_1 = s_1(b_{1,i}^T), \quad s_3 = s_3(b_{1,i}^T, b_{3,i}^T), \quad s_5 = s_5(b_{1,i}^T, b_{3,i}^T, b_{5,i}^T), \quad s_7 = s_7(b_{1,i}^T, b_{3,i}^T, b_{5,i}^T, b_{7,i}^T), \quad i = 1, \ldots, 105,$$

following the procedure presented in Appendix A.

We then solved the corresponding algebraic system (3.1) through the procedure presented in Appendix B, obtaining the switching angles

$$\begin{aligned}
\boldsymbol{\alpha}_1 &= (\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \ldots, \alpha_{1,105}) \in \mathbb{R}^{105},\\
\boldsymbol{\alpha}_2 &= (\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \ldots, \alpha_{2,105}) \in \mathbb{R}^{105},\\
\boldsymbol{\alpha}_3 &= (\alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3}, \ldots, \alpha_{3,105}) \in \mathbb{R}^{105},\\
\boldsymbol{\alpha}_4 &= (\alpha_{4,1}, \alpha_{4,2}, \alpha_{4,3}, \ldots, \alpha_{4,105}) \in \mathbb{R}^{105},
\end{aligned}$$

Finally, with these angles we built the function $f(\tau)$ and we employed the formula (1.4) to obtain the Fourier coefficients

$$\begin{aligned}
\boldsymbol{b}_1 &= (b_{1,1}, b_{1,2}, b_{1,3}, \ldots, b_{1,105}) \in \mathbb{R}^{105},\\
\boldsymbol{b}_3 &= (b_{3,1}, b_{3,2}, b_{3,3}, \ldots, b_{3,105}) \in \mathbb{R}^{105},\\
\boldsymbol{b}_5 &= (b_{5,1}, b_{5,2}, b_{5,3}, \ldots, b_{5,105}) \in \mathbb{R}^{105},\\
\boldsymbol{b}_7 &= (b_{7,1}, b_{7,2}, b_{7,3}, \ldots, b_{7,105}) \in \mathbb{R}^{105},
\end{aligned}$$

and we compared them with the target vectors $\mathbf{b}_{2\ell-1}^T$, $\ell = 1, \ldots, 4$, by computing the quadratic error

$$e_{2\ell-1} = \|\mathbf{b}_{2\ell-1}^T - \mathbf{b}_{2\ell-1}\|^2.$$

These errors are displayed in Figure 1.

Finally, figure 2 shows the behavior of the function $f(\tau)$ with respect to the modulation index and the switching angles.

3.2 Selective Harmonic Modulation

We now consider the Selective Harmonic Elimination problem, in which we want the Fourier coefficients (b_1, b_3, b_5, b_7) to match the target

$$(b_1^T, b_3^T, b_5^T, b_7^T) = (0.5m_a, 0.05, 0, 0), \quad m_a \in (0.01, 1.12).$$

As before, we considered a 112-points discretization of the interval $(0.01, 1.12)$,

$$0.01 = m_{a,1} < m_{a,2} < \ldots < m_{a,i} < m_{a,i+1} < \ldots < m_{a,112} = 1.12$$

with $m_{a,i} = 0.01 + (i-1)\Delta m_a$, $i = 1, \ldots, 112$, $\Delta m_a = 10^{-2}$. For each value of $m_{a,i}$, we computed the corresponding vector of target Fourier coefficients

$$\begin{aligned}
\mathbf{b}_1^T &= (b_{1,1}^T, b_{1,2}^T, b_{1,3}^T, \ldots, b_{1,112}^T) = 0.5(m_{a,1}, m_{a,2}, m_{a,3}, \ldots, m_{a,112}) \in \mathbb{R}^{112},\\
\mathbf{b}_3^T &= (b_{3,1}^T, b_{3,2}^T, b_{3,3}^T, \ldots, b_{3,112}^T) = (0.05, 0.05, 0.05, \ldots, 0.05) \in \mathbb{R}^{112},\\
\mathbf{b}_5^T &= (b_{5,1}^T, b_{5,2}^T, b_{5,3}^T, \ldots, b_{5,112}^T) = (0, 0, 0, \ldots, 0) \in \mathbb{R}^{112},\\
\mathbf{b}_7^T &= (b_{7,1}^T, b_{7,2}^T, b_{7,3}^T, \ldots, b_{7,112}^T) = (0, 0, 0, \ldots, 0) \in \mathbb{R}^{112},
\end{aligned}$$

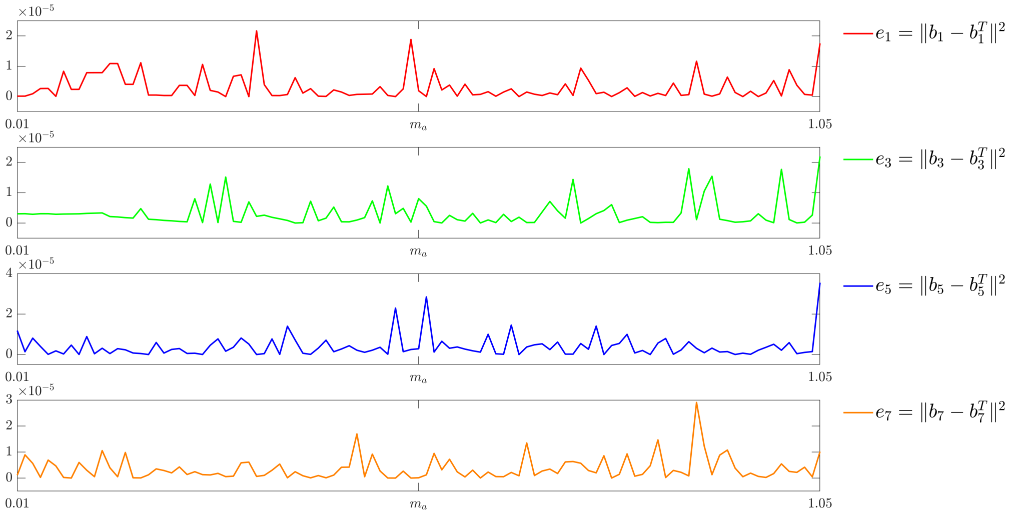

Figure 1: Error $e_{2\ell-1} = \|\mathbf{b}_{2\ell-1}^T - \mathbf{b}_{2\ell-1}\|^2$, $\ell = 1, \ldots, 4$.

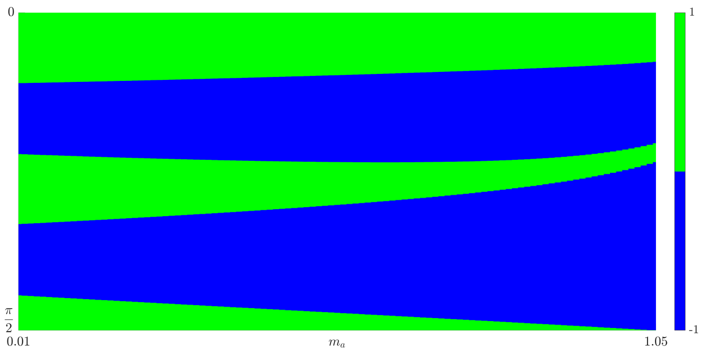

Figure 2: Behavior of the function $f(\tau)$ for the Selective harmonic Elimination problem with respect to the modulation index (horizontal axis) and the switching angles (vertical axis). The blue region indicates when f takes the value -1, while in the green region f takes the value 1.

and used these values to obtain the four coefficients

$$s_1 = s_1(b_{1,i}^T), \quad s_3 = s_3(b_{1,i}^T, b_{3,i}^T), \quad s_5 = s_5(b_{1,i}^T, b_{3,i}^T, b_{5,i}^T), \quad s_7 = s_7(b_{1,i}^T, b_{3,i}^T, b_{5,i}^T, b_{7,i}^T), \quad i = 1, \ldots, 112,$$

following the procedure presented in Appendix A.

We then solved again the corresponding algebraic system (3.1) through the procedure presented in Appendix B, obtaining the switching angles

$$\begin{aligned}
\boldsymbol{\alpha}_1 &= (\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \ldots, \alpha_{1,112}) \in \mathbb{R}^{112}, \\
\boldsymbol{\alpha}_2 &= (\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \ldots, \alpha_{2,112}) \in \mathbb{R}^{112}, \\
\boldsymbol{\alpha}_3 &= (\alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3}, \ldots, \alpha_{3,112}) \in \mathbb{R}^{112}, \\
\boldsymbol{\alpha}_4 &= (\alpha_{4,1}, \alpha_{4,2}, \alpha_{4,3}, \ldots, \alpha_{4,112}) \in \mathbb{R}^{112},
\end{aligned}$$

Finally, with these angles we built the function $f(\tau)$ and we employed the formula (1.4) to obtain the Fourier coefficients

$$\begin{aligned}
\boldsymbol{b}_1 &= (b_{1,1}, b_{1,2}, b_{1,3}, \ldots, b_{1,112}) \in \mathbb{R}^{112}, \\
\boldsymbol{b}_3 &= (b_{3,1}, b_{3,2}, b_{3,3}, \ldots, b_{3,112}) \in \mathbb{R}^{112}, \\
\boldsymbol{b}_5 &= (b_{5,1}, b_{5,2}, b_{5,3}, \ldots, b_{5,112}) \in \mathbb{R}^{112}, \\
\boldsymbol{b}_7 &= (b_{7,1}, b_{7,2}, b_{7,3}, \ldots, b_{7,112}) \in \mathbb{R}^{112},
\end{aligned}$$

and we compared them with the target vectors $\mathbf{b}_{2\ell-1}^T$, $\ell = 1, \dots, 4$, by computing the quadratic error

$$e_{2\ell-1} = \|\mathbf{b}_{2\ell-1}^T - \mathbf{b}_{2\ell-1}\|^2.$$

These errors are displayed in Figure 3.

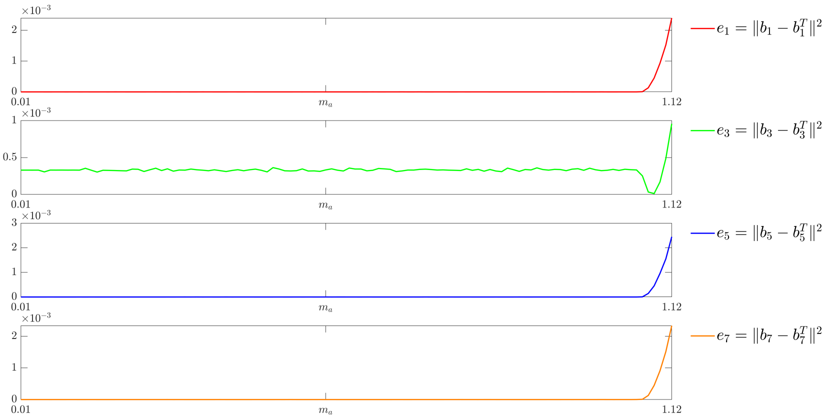

Figure 3: Error $e_{2\ell-1} = \|\mathbf{b}_{2\ell-1}^T - \mathbf{b}_{2\ell-1}\|^2$, $\ell = 1, \dots, 4$.

Finally, figure 4 shows the behavior of the function $f(\tau)$ with respect to the modulation index and the switching angles.

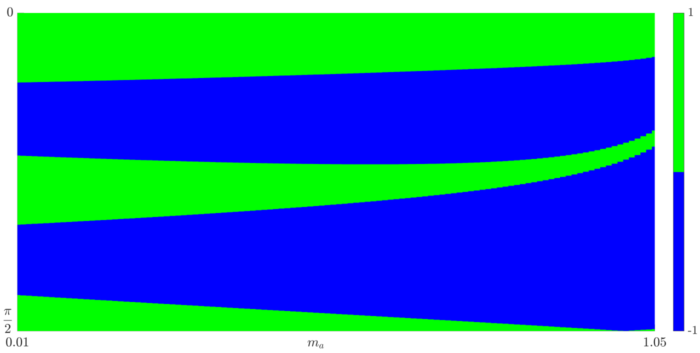

Figure 4: Behavior of the function $f(\tau)$ for the Selective Harmonic Modulation problem with respect to the modulation index (horizontal axis) and the switching angles (vertical axis). The blue region indicates when f takes the value -1, while in the green region f takes the value 1.

A Transformation of (2.1) in a set of algebraic equations

As we said, in order to solve system (2.1), we shall transform the transcendental equations in algebraic ones. The procedure to apply summarizes as follows:

Step 1. We apply the changes of variables (2.2) to the first equation in (2.1), obtaining

$$x_1 + x_2 + \ldots + x_n = \frac{1}{2} + \frac{\pi b_1}{8V_{dc}} =: s_1.$$

Step 2. By considering the Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x)$$

and their trigonometric definition

$$T_j(x) = \cos(j\arccos(x)), \quad |x| \le 1,$$

we have for all $i = 1, 2 \ldots, n$

$$
\begin{aligned}
&T_0(\cos(\alpha_i)) = 1,\\
&T_1(\cos(\alpha_i)) = \cos(\alpha_i),\\
&T_2(\cos(\alpha_i)) = \cos(2\alpha_i) = 2\cos(\alpha_i)T_1(\cos(\alpha_i)) - T_0(\cos(\alpha_i)) = 2\cos^2(\alpha_i) - 1,\\
&T_3(\cos(\alpha_i)) = \cos(3\alpha_i) = 2\cos(\alpha_i)T_2(\cos(\alpha_i)) - T_1(\cos(\alpha_i)) = 4\cos^3(\alpha_i) - 3\cos(\alpha_i).
\end{aligned}
$$

Hence, from the second equation in (2.1) we get

$$\frac{1}{2} + \frac{3\pi b_3}{8V_{dc}} = \sum_{i=1}^{n}(-1)^{i+1}\cos(3\alpha_i) = 4\sum_{i=1}^{n}(-1)^{i+1}\cos^3(\alpha_i) - 3\sum_{i=1}^{n}(-1)^{i+1}\cos(\alpha_i).$$

Using again the change of variables (2.2), and noticing that

$$x_i^3 = (-1)^{3i+3}\cos^3(\alpha_i) = (-1)^{i+1}\cos^3(\alpha_i), \quad i = 1, 2, \ldots, n,$$

we obtain

$$\frac{1}{2} + \frac{3\pi b_3}{8V_{dc}} = 4(x_1^3 + x_2^3 + \ldots + x_n^3) - 3(x_1 + x_2 + \ldots + x_n) = 4(x_1^3 + x_2^3 + \ldots + x_n^3) - 3s_1.$$

This yields

$$x_1^3 + x_2^3 + \ldots + x_n^3 = \frac{1}{4}\left(\frac{1}{2} + \frac{3\pi b_3}{4V_{dc}} + 3s_1\right) = \frac{1}{2}\left(1 + \frac{3}{4}\frac{\pi b_3}{4V_{dc}} + \frac{3}{4}\frac{\pi b_1}{4V_{dc}}\right) := s_3.$$

Step 3. By employing once again the Chebyshev polynomials, we can easily compute $\cos(5\alpha_i)$

$$T_5(\cos(\alpha_i)) = \cos(5\alpha_i) = 16\cos^5(\alpha_i) - 20\cos^3(\alpha_i) + 5\cos(\alpha_i).$$

Hence, from the third equation in (2.1) we get

$$
\begin{aligned}
\frac{1}{2} + \frac{5\pi b_5}{8V_{dc}} &= \sum_{i=1}^{n}(-1)^{i+1}\cos(5\alpha_i)\\
&= 16\sum_{i=1}^{n}(-1)^{i+1}\cos^5(\alpha_i) - 20\sum_{i=1}^{n}(-1)^{i+1}\cos^3(\alpha_i) + 5\sum_{i=1}^{n}(-1)^{i+1}\cos(\alpha_i).
\end{aligned}
$$

Using again the change of variables (2.2), and noticing that

$$x_i^5 = (-1)^{5i+5}\cos^5(\alpha_i) = (-1)^{i+1}\cos^5(\alpha_i), \quad i = 1, 2, \ldots, n,$$

we obtain

$$
\begin{aligned}
\frac{1}{2} + \frac{5\pi b_5}{8V_{dc}} &= 16(x_1^5 + x_2^5 + \ldots + x_n^5) - 20(x_1^3 + x_2^3 + \ldots + x_n^3) + 5(x_1 + x_2 + \ldots + x_n)\\
&= 16(x_1^5 + x_2^5 + \ldots + x_n^5) - 20s_3 + 5s_1.
\end{aligned}
$$

This yields

$$x_1^5 + x_2^5 + \ldots + x_n^5 = \frac{1}{16}\left(\frac{1}{2} + \frac{5\pi b_5}{8V_{dc}} + 20s_3 - 5s_1\right) = \frac{1}{16}\left(8 + \frac{5}{2}\frac{\pi b_5}{4V_{dc}} + \frac{15}{2}\frac{\pi b_3}{4V_{dc}} - \frac{5}{2}\frac{\pi b_1}{4V_{dc}}\right) := s_5.$$

Step 4. By iterating the above procedure we get the general algebraic equation

$$x_1^{2n-1} + x_2^{2n-1} + \ldots + x_n^{2n-1} = s_{2n-1}, \tag{A.1}$$

where the coefficients s_{2n-1} are obtained through the recursive algorithm

$$
\begin{aligned}
&T_1(x)|_{x=s_1} = s_1 = \frac{1}{2} + \frac{\pi b_1}{8V_{dc}},\\
&T_{2n-1}(x)|_{x^{2n-1}=s_{2n-1}} = \frac{1}{2} + \frac{(2n-1)\pi b_{2n-1}}{8V_{dc}}, \quad n \geq 2.
\end{aligned} \tag{A.2}
$$

The procedure (A.2) to compute the coefficients $\{s_{2\ell-1}\}_{\ell=1}^{n}$ is implemented through the Matlab functions presented in Algorithms 1 and 2.

Algorithm 1 Computation of the coefficients $\{s_{2\ell-1}\}_{\ell=1}^{n}$ in (2.3)

```
function S = coeffSHE(b,Vdc)

n = length(b);
s = 0.5 + (pi*b(1)/(8*Vdc));
s = [0 s 0];
rhs = 0.5 + (1/(4*Vdc))*[1:2:2*n].*b;
j = 2;

for i = 3:2:2*n
        c = ChebPoly(i);
        aux = (rhs(j)-c(2:end)*s')/c(1);
        s = [0 aux s];
        j = j+1;
end

s = fliplr(double(s));
S = s(s~=0);
```

Algorithm 2 Coefficients of the n-th Chebyshev polynomial

```
function c = ChebPoly(n)

PolyOrder = 1:2:2*n-1;
L = length(PolyOrder);
C = zeros(L+1,L+1);

C(1,end) = 1;
C(2,end-1) = 1;

for i = 3:L+1
        aux = 2*circshift(C(i-1,:),-1);
        C(i,:) = aux-C(i-2,:);
end

c = C(end,:);
```

B Resolution of the algebraic equations

Let us now discuss the resolution of (2.3). To this end, let us introduce the function

$$G(x) = \exp\left(-\sum_{\substack{\ell\geq 1\\ \ell\ odd}} \left(\sum_{i=1}^{n} \frac{x_i^\ell}{\ell} \right) x^{-\ell} \right). \tag{B.1}$$

Then, according to [1, Theorem 1], the solution $\{x_i\}_{i=1}^n$ of (2.3) is given by the roots of the numerator

$$p(x) = \prod_{i=1}^{n} (x - x_i) = \sum_{m=0}^{n} p_m x^{n-m}.$$

in the Padé approximation of order (n, n) of G:

$$G(x) = \frac{p(x)}{q(x)}, \tag{B.2}$$

with $p(x)$ and $q(x)$ two polynomials of degree n. Hence, to solve (2.3) we need to determine $p(x)$ and compute its roots. Moreover, let us recall that the Padé approximation of a function is unique. Hence, also the solution $\{x_i\}_{i=1}^n$ of (2.3) will be unique.

Notice that $G(-x) = (G(x))^{-1}$. Then, the Padé approximant $q(x)$ has the property $q(x) = (-1)^n p(-x)$ and, from (B.2), we easily obtain the identity

$$p(x) = (-1)^n p(-x) G(x). \tag{B.3}$$

Let us now introduce the functions

$$v(x) = -2 \sum_{\substack{\ell\geq 1\\ \ell\ odd}} \left(\sum_{i=1}^{n} \frac{x_i^\ell}{\ell} \right) x^\ell, \qquad g(x) = \exp\big(v(x)\big). \tag{B.4}$$

Then, $G(x) = g(1/x)$ and (B.3) can be rewritten as

$$p(x) = (-1)^n p(-x) g\left(\frac{1}{x}\right). \tag{B.5}$$

We can now use (B.5) to obtain explicitly the coefficients $\{p_m\}_{m=0}^n$ through the following procedure.

Step 1. First of all, let us introduce the series expansion of $v(x)$:

$$v(x) = \sum_{\ell\geq 0} v_\ell x^\ell.$$

Comparing this with (B.4) we get

$$v(x) = -2 \sum_{\substack{\ell\geq 1\\ \ell\ odd}} \left(\sum_{i=1}^{n} \frac{x_i^\ell}{\ell} \right) x^\ell = \sum_{\ell\geq 0} v_\ell x^\ell$$

and, equating the coefficients of the same order, we have

$$\begin{cases} v_\ell = -2 \displaystyle\sum_{i=1}^{n} \frac{x_i^\ell}{\ell}, & \text{for } \ell \text{ odd} \\ v_\ell = 0, & \text{for } \ell \text{ even.} \end{cases} \tag{B.6}$$

Notice that, according to (2.3), we have

$$\sum_{i=1}^{n} x_i^\ell = s_\ell \qquad \text{for } \ell = 1, 3, 5, \ldots 2n-1.$$

Hence, from (B.6) we can compute the coefficients v_ℓ up to $\ell = 2n$ as

$$\begin{cases} v_\ell = -\dfrac{2s_\ell}{\ell}, & \text{for } \ell = 1, \dots, 2n, \quad \ell \text{ odd} \\ v_\ell = 0, & \text{for } \ell \text{ even.} \end{cases} \tag{B.7}$$

This is done with the Matlab function presented in Algorithm 3. Nevertheless, for $\ell \geq 2n+1$, we can only know the even coefficients (which are all zero) while the odd coefficients cannot be computed.

Algorithm 3 Computation of the coefficients $\{v_\ell\}_{\ell=1}^{2n}$ using (B.7)

```
function v = coefficients_v(s)

[lMa,n] = size(s);

% The i-th row of the matrix v contains the 2n coefficients v_i
% corresponding to the i-th value of the modulation index

v = zeros(lMa,2*n);

for l = 1:n
        v(:,2*l-1) = -(2/(2*l-1))*s(:,l);
end
```

Step 2. Let us now introduce the series expansion of $g(x)$

$$g(x) = \sum_{\ell \geq 0} g_\ell x^\ell$$

and notice that, from the expression $g(x) = \exp\big(v(x)\big)$, we obtain

$$\frac{dg(x)}{dx} = \frac{dv(x)}{dx} g(x).$$

Expanding both sides of the identity above, we get

$$\begin{aligned} g_1 + 2g_2x + 3g_3x^2 + \dots &= (v_1 + 2v_2x + 3v_3x^2 + \dots)(g_0 + g_1x + g_2x^2 + \dots) \\ &= (v_1g_0) + (2v_2g_0 + v_1g_1)x + (3v_3g_0 + 2v_2g_1 + v_1g_2)x^2 + \dots \end{aligned}$$

By setting $g_0 = 1$ (which yields $v_0 = \ln(g_0) = 0$) and equating the coefficients of the left and right-hand side, we thus find that

$$g_0 = 1, \quad g_\ell = \sum_{k=1}^{\ell} \frac{k}{\ell} v_k g_{\ell-k} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\ell} \frac{k}{\ell} v_k g_{\ell-k} \quad \text{for } \ell \geq 1. \tag{B.8}$$

Recall that (B.7) allows to obtain the coefficients $\{v_\ell\}_{\ell=1}^{2n}$. Hence, we can use (B.8) to compute $\{g_\ell\}_{\ell=1}^{2n}$ through the Matlab function presented in Algorithm 4. Nevertheless, also in this case, for $\ell \geq 2n+1$ the coefficients g_ℓ cannot be computed.

Step 3. From (B.5) we have

$$p(x) = (-1)^n p(-x) g\left(\frac{1}{x}\right) = (-1)^n p(-x)\left(g_0 + \frac{g_1}{x} + \frac{g_2}{x^2} + \dots\right),$$

which gives

$$\begin{aligned} \sum_{m=0}^{n} p_m x^{n-m} = (-1)^n \left(\sum_{m=0}^{n} p_m(-x)^{n-m}\right)\left(\sum_{m\geq 0} g_m x^{-m}\right) &= \left(\sum_{m=0}^{n} (-1)^{2n-m} p_m x^{n-m}\right)\left(\sum_{m\geq 0} g_m x^{-m}\right) \\ &= \left(\sum_{m=0}^{n} (-1)^{m} p_m x^{n-m}\right)\left(\sum_{m\geq 0} g_m x^{-m}\right). \end{aligned}$$

Algorithm 4 Computation of the coefficients $\{g_\ell\}_{\ell=1}^{2n}$ using (B.8)

```
function g = coefficients_g(v)

[lMa,n] = size(v);

% The i-th row of the matrix g contains the 2n coefficients g_i
% corresponding to the i-th value of the modulation index

g = zeros(lMa,n+1);
g(:,1) = 1;
for l = 2:n+1
        G = 0;
        for k = 1:l-1
                G = G + (k/(l-1))*v(:,k).*g(:,l-k);
        end
        g(:,l) = G;
end
```

By developing the products on the right-hand side of the above identity, and taking into account that $p_0 = 1 = g_0$, we get

$$\begin{aligned}
\sum_{m=0}^{n} p_m x^{n-m} &= \sum_{m\geq 0} g_m x^{n-m} - \sum_{m\geq 0} p_1 g_m x^{n-1-m} + \sum_{m\geq 0} p_2 g_m x^{n-2-m} + \ldots + (-1)^n \sum_{m\geq 0} p_n g_m x^{-m} \\
&= x^n + g_1 x^{n-1} + g_2 x^{n-2} + \ldots + g_{n-1}x + g_n + \sum_{m\geq n+1} g_m x^{n-m} \\
&\quad - p_1 x^{n-1} - p_1 g_1 x^{n-2} - p_1 g_2 x^{n-3} - \ldots - p_1 g_{n-2} x - p_1 g_{n-1} - \sum_{m\geq n} p_1 g_m x^{n-1-m} \\
&\quad + p_2 x^{n-2} + p_2 g_1 x^{n-3} + p_2 g_2 x^{n-4} + \ldots + p_2 g_{n-3} x + p_2 g_{n-2} + \sum_{m\geq n-1} p_2 g_m x^{n-2-m} \\
&\quad + \ldots + (-1)^n p_n + (-1)^n \sum_{m\geq 1} p_n g_m x^{-m} \\
&= x^n + (g_1 - p_1)x^{n-1} + (g_2 - p_1 g_1 + p_2)x^{n-2} \\
&\quad + \ldots + (g_{n-1} - p_1 g_{n-2} + p_2 g_{n-3} - \ldots - (-1)^n p_{n-1})x \\
&\quad + (g_n - p_1 g_{n-1} + p_2 g_{n-2} + \ldots + (-1)^n p_n) \\
&\quad + \sum_{m\geq 1} R_m x^{-m},
\end{aligned}$$

where the coefficients R_m are given by

$$R_m := g_{m+n} - p_1 g_{m+n-1} + p_2 g_{m+n-2} + \ldots + (-1)^{n-1} p_{n-1} g_{m+1} + (-1)^n p_n g_m.$$

This leads to the following identity

$$\begin{aligned}
x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1}x + p_n &= x^n + (g_1 - p_1)x^{n-1} + (g_2 - p_1 g_1 + p_2)x^{n-2} \\
&\quad + \ldots + (g_{n-1} - p_1 g_{n-2} + p_2 g_{n-3} - \ldots - (-1)^n p_{n-1})x \\
&\quad + (g_n - p_1 g_{n-1} + p_2 g_{n-2} + \ldots + (-1)^n p_n) \\
&\quad + \sum_{m\geq 1} R_m x^{-m}, \qquad \text{(B.9)}
\end{aligned}$$

Moreover, if we introduce the polynomial

$$
\begin{aligned}
&r(x) = \sum_{m=0}^{n} r_m x^{n-m} \\
&r_1 = g_1 - p_1 \\
&r_2 = g_2 - p_1 g_1 + p_2 \\
&r_3 = g_3 - p_1 g_2 + p_2 g_1 - p_3 \\
&r_4 = g_4 - p_1 g_3 + p_2 g_2 - p_3 g_1 + p_4 \\
&\vdots \\
&r_{n-1} = g_{n-1} - p_1 g_{n-2} + p_2 g_{n-3} - \ldots - (-1)^n p_{n-1} \\
&r_n = g_n - p_1 g_{n-1} + p_2 g_{n-2} + \ldots + (-1)^n p_n
\end{aligned}
$$

and the remainder term

$$R(x) = \sum_{m \geq 1} R_m x^{-m},$$

the identity (B.9) becomes

$$\sum_{m=0}^{n} (p_m - r_m) x^{n-m} = R(x). \tag{B.10}$$

From (B.10), we need to obtain n equations to determine the coefficients $\{p_m\}_{m=1}^n$. To do that, we have two possibilities:

1. to equate to zero the coefficients $\{p_m - r_m\}_{m=0}^n$;
2. to equate to zero the first n coefficients coefficients $\{R_m\}_{m=1}^n$ in the remainder term R.

Following the first path, i.e. setting to zero the coefficients of $\{p_m - r_m\}_{m=0}^n$, we get the system

$$
\begin{cases}
g_1 - p_1 = p_1 \\
g_2 - p_1 g_1 + p_2 = p_2 \\
g_3 - p_1 g_2 + p_2 g_1 - p_3 = p_3 \\
g_4 - p_1 g_3 + p_2 g_2 - p_3 g_1 + p_4 = p_4 \\
\vdots \\
g_{n-1} - p_1 g_{n-2} + p_2 g_{n-3} - \ldots - (-1)^n p_{n-1} = p_{n-1} \\
g_n - p_1 g_{n-1} + p_2 g_{n-2} + \ldots + (-1)^n p_n = p_n
\end{cases} \tag{B.11}
$$

Notice that (B.11) is a cascade system, which is solved through the following n-steps procedure:

Step 1. From the first equation we obtain the value of p_1.

Step 2. Once p_1 is known, from the second equation we obtain the value of p_2.

Step 3. Once p_2 is known, from the third equation we obtain the value of p_3.

Step 4. Once p_3 is known, from the fourth equation we obtain the value of p_4.

$\vdots$

Step $n-1$. Once p_{n-2} is known, from the $(n-2)$th equation we obtain the value of p_{n-1}.

Step n. Once p_{n-1} is known, from the $(n-1)$th equation we obtain the value of p_n.

Nevertheless, the above process fails at Step 2, since the second equation is actually independent of p_2. Hence we cannot obtain the coefficients $\{p_m\}_{m=1}^n$ by solving (B.11).

Our only option is then to follow the second path and set to zero the first n coefficients $\{R_m\}_{m=1}^n$ in the remainder term $R(x)$ in (B.9) and set the coefficients to zero. Solving the equations $R_m = 0$ for $m = 1, \dots, n$, we obtain the system

$$\begin{cases} p_1 g_n - p_2 g_{n-1} + \dots + (-1)^n p_{n-1} g_2 + (-1)^{n+1} p_n g_1 = g_{n+1} \\ p_1 g_{n+1} - p_2 g_n + \dots + (-1)^n p_{n-1} g_3 + (-1)^{n+1} p_n g_2 = g_{n+2} \\ p_1 g_{n+2} - p_2 g_{n+1} + \dots + (-1)^n p_{n-1} g_4 + (-1)^{n+1} p_n g_3 = g_{n+3} \\ p_1 g_{n+3} - p_2 g_{n+2} + \dots + (-1)^n p_{n-1} g_5 + (-1)^{n+1} p_n g_4 = g_{n+4} \\ \vdots \\ p_1 g_{2n-2} - p_2 g_{2n-3} + \dots + (-1)^n p_{n-1} g_n + (-1)^{n+1} p_n g_{n-1} = g_{2n-1} \\ p_1 g_{2n-1} - p_2 g_{2n-2} + \dots + (-1)^n p_{n-1} g_{n+1} + (-1)^{n+1} p_n g_n = g_{2n} \end{cases} \tag{B.12}$$

Denote

$$\mathbf{p} := (p_1, p_2, p_3, p_4, \dots, p_{n-1}, p_n)^\top \quad \text{and} \quad \mathbf{g} := (g_{n+1}, g_{n+2}, g_{n+3}, g_{n+4}, \dots, g_{2n-1}, g_{2n})^\top.$$

Then, (B.12) is equivalent to $G\mathbf{p} = \mathbf{g}$ with

$$G = \begin{pmatrix} g_n & -g_{n-1} & \dots & (-1)^n g_2 & (-1)^{n+1} g_1 \\ g_{n+1} & -g_n & \dots & (-1)^n g_3 & (-1)^{n+1} g_2 \\ g_{n+2} & -g_{n+1} & \dots & (-1)^n g_4 & (-1)^{n+1} g_3 \\ g_{n+3} & -g_{n+2} & \dots & (-1)^n g_5 & (-1)^{n+1} g_4 \\ \vdots & \vdots & & \vdots & \vdots \\ g_{2n-2} & -g_{2n-3} & \dots & (-1)^n g_n & (-1)^{n+1} g_{n-1} \\ g_{2n-1} & -g_{2n-2} & \dots & (-1)^n g_{n+1} & (-1)^{n+1} g_n \end{pmatrix}. \tag{B.13}$$

Hence, if the matrix G is invertible, we obtain

$$\mathbf{p} = G^{-1}\mathbf{g}. \tag{B.14}$$

The computation of $\{p_m\}_{m=1}^n$ is then carried through the Matlab functions presented in Algorithms 5 and 6.

Algorithm 5 Computation of the coefficients $\{p_m\}_{m=1}^n$ solving (B.14)

```
function p = coefficients_p(g)

[gMatrix,gVector] = constructionMatrix_G(g);
p = gMatrix\gVector;
```

Algorithm 6 Construction of the matrix G in (B.13)

```
function [gMatrix,gVector] = constructionMatrix_G(g)

n = 0.5*length(g);
gMatrix = zeros(n,n);

for i = n:-1:1
        gMatrix(:,n+1-i) = ((-1)^i)*g(i:n+i-1)';
end

gVector = g(n+1:end)';
```

This gives us the coefficients $\{p_m\}_{m=1}^n$, hence the analytic expression of the polynomial $p(x)$. The solutions to (2.3) will be the roots of this polynomial, which can be approximated through the Newton method. The switching angles $\{\alpha_i\}_{i=1}^n$ are then determined as

$$\alpha_i = \arccos\left((-1)^i x_i\right), \quad i = 1, \dots, n.$$

Remark 2. For completeness, let us briefly discuss the link between the function G and the polynomial p. To this end, let us notice that we can rewrite

$$\begin{aligned} p(x) &= \prod_{i=1}^{n}(x-x_i) = x^n \prod_{i=1}^{n}\Big(1-\frac{x_i}{x}\Big) \\ &= x^n \exp\left[\log\left(\prod_{i=1}^{n}\Big(1-\frac{x_i}{x}\Big)\right)\right] = x^n \exp\left[\sum_{i=1}^{n}\log\Big(1-\frac{x_i}{x}\Big)\right]. \end{aligned} \tag{B.15}$$

Moreover, if we introduce the Taylor expansions

$$\log\Big(1-\frac{x_i}{x}\Big) = -\sum_{\ell\geq 1}\frac{x_i^\ell}{\ell x^\ell}, \quad i=1,\dots,n$$

we get from (B.15) that

$$p(x) = x^n \exp\left(-\sum_{\ell\geq 1}\sum_{i=1}^{n}\frac{x_i^\ell}{\ell x^\ell}\right).$$

Finally, we can get rid of the multiplicative factor x^n by noticing that

$$p(-x) = (-1)^n x^n \exp\left(-\sum_{\ell\geq 1}\sum_{i=1}^{n}(-1)^\ell\frac{x_i^\ell}{\ell x^\ell}\right),$$

which gives

$$\frac{p(x)}{p(-x)} = (-1)^n \exp\left(-\sum_{\ell\geq 1}\sum_{i=1}^{n}\big(1-(-1)^\ell\big)\frac{x_i^\ell}{\ell x^\ell}\right) = (-1)^n \exp\left(-2\sum_{\substack{\ell\geq 1\\ \ell\ odd}}\sum_{i=1}^{n}\frac{x_i^\ell}{\ell x^\ell}\right). \tag{B.16}$$

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