Selective Harmonic Elimination via Optimal Control Theory

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Abstract

In this document, we study the selective harmonic elimination problem from view of point of a control theory.

Contents

1 Introduction

In this document, we propose a optimal control perspective of selective harmonic elimination problem (SHE) with symmetry of quarter wave and symmetry of half wave. In mathematical point of view, SHE problem can be seen as search of a square wave function $f(\omega t) \mid \omega t \in (0, 2\pi)$ which have fixed a few Fourier coefficients.

In this way, the $f(\omega t)$ can be written in Fourier series as follows:

$$f(\omega t) = \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \tag{1.1}$$

Where a_n and b_n coefficients are:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\omega t) \cos(n\omega t) d(\omega t) \tag{1.2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\omega t) \sin(n\omega t) d(\omega t) \tag{1.3}$$

Problem 1.1 (SHE two levels) Given $\boldsymbol{b}_T = [b_T^1 \ , \ b_T^3 \ , \ b_T^5 \ , \ \dots \ , \ b_T^{N_b/2}] \in \mathbb{R}^{N/2}$ and $\boldsymbol{a}_T = [a_T^1 \ , \ a_T^3 \ , \ a_T^5 \ , \ \dots \ , \ a_T^{N_a/2}] \in \mathbb{R}^{N/2}$, we search a wave form $f(\omega t) \mid \omega t \in (0, \pi/2)$ such that f only can take values $\{-1, 1\}$ and its Fourier coefficients b_n satisfies $b_n = b_T^n$ and $a_n = a_T^n \mid \forall n \in \{1, 3, \dots, N/2\}$.

In the typical formulation of this problem, the function $f(\omega t)$ can be represented by locations where the function $f(\omega t)$ changes its value, this locations are named switching angles. Given a some vectors $\boldsymbol{a}^T$ and $\boldsymbol{b}^T$, the number of switching angles M is *a priori* unknown, so it's necessary fixed it. If we name switching angles as $\boldsymbol{\phi} = [\phi_1, \phi_2, \dots, \phi_M] \in \mathbb{R}^M$, we can simplify the Fourier coefficients as follows:

$$a_n(\boldsymbol{\phi}) = \dots \mid \forall n \ odd \tag{1.4}$$

$$b_n(\boldsymbol{\phi}) = \frac{4}{n\pi} \left[-1 + 2 \sum_{i=1}^{M} (-1)^{i+1} \cos(n\phi_i) \right] \mid \forall n \ odd \tag{1.5}$$

With this expression, we can formulate the problem (1.1) as the next minimization problem:

Problem 1.2 (Minimization problem for SHE)

$$\min_{\boldsymbol{\phi} \in \mathbb{R}^m} \sum_{n \ odd}^{N/2} \left[(a_n(\boldsymbol{\phi}) - a_T^n)^2 + (b_n(\boldsymbol{\phi}) - b_T^n)^2 \right] \tag{1.6}$$

$$\text{subject to: } \begin{cases} 0 < \phi_1 \\ \phi_n < \phi_n + 2 \quad \forall n \in \{3, 5, \dots, N/2 - 2\} \\ \phi_{N/2} < \pi/2 \end{cases} \tag{1.7}$$

This formulation don't give a clearly procedure to choose a number of angles.

We propose consider a search of a function $f(\omega t)$ directly. In this way, instead of looking for the switching angles $\boldsymbol{\phi} \in \mathbb{R}^M$, we look for a function $f(\omega t) \in \{g(\omega t) \in L^\infty([0, \pi/2]) \;/\; |g(\omega t)| < 1\}$.

Thanks to fundamental calculus theorem, we can say:

$$\alpha_n(\tau) = \frac{4}{\pi} \int_0^\tau f(\omega t) \sin(n\omega t) d(\omega t) \Rightarrow \begin{cases} \frac{\partial \alpha_n}{d\tau} & = \frac{4}{\pi} f(\tau) \cos(n\tau) \\ \alpha_n(0) & = 0 \end{cases} \tag{1.8}$$

$$\beta_n(\tau) = \frac{4}{\pi} \int_0^\tau f(\omega t) \sin(n\omega t) d(\omega t) \Rightarrow \begin{cases} \frac{\partial \beta}{d\tau} & = \frac{4}{\pi} f(\tau) \sin(n\tau) \\ \beta(0) & = 0 \end{cases} \tag{1.9}$$

If we solve the ODE (1.9) from $\tau = 0$ to $\tau = \pi/2$. So, this ODE can see as control system, where $\alpha_n(\tau)$ and $\beta_n(\tau) \mid \forall n$ is the states and $f(\tau)$ is the control variable. In this way, the problem (1.1) can be solve via optimal control problem.

Part I

Open Loop Optimal Control

2 Optimal control formulation

In this section, we show the optimal control formulation for selective harmonic elimination for a two-level converter and three-level converter.

2.1 OC SHE in two levels for symmetry of half-wave

Problem 2.1 Given $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we define a cost functional in this way:

$$J[f(\tau)] = \left[||\boldsymbol{a}_T - \boldsymbol{\alpha}(T)||^2 + ||\boldsymbol{b}_T - \boldsymbol{\beta}(T)||^2 - \epsilon \int_0^{\pi/2} ||f(\tau)||^2 d\tau \right] \tag{2.1}$$

where:

$$\boldsymbol{\alpha}(\tau) = [\alpha_1(\tau)\ \alpha_3(\tau)\ ...\ \alpha_{N_a/2}(\tau)]^T \tag{2.2}$$

$$\boldsymbol{\beta}(\tau) = [\beta_1(\tau)\ \beta_3(\tau)\ ...\ \beta_{N_b/2}(\tau)]^T \tag{2.3}$$

the $||.||$ is a euclidean norm and ϵ is a parameter to maximized the norm of control $f(\tau)$.

So, the optimal control problem can be write:

$$\min_{|f(\tau)|<1} J[f(\tau)] \tag{2.4}$$

suject to:

$$\begin{aligned} \forall n \in \{1, 3, 5, \dots, N_a/2\} \quad & \begin{cases} \frac{d\alpha}{d\tau} = (4/\pi)\cos(n\tau) f(\tau) & \tau \in [0, \pi/2] \\ \alpha(0) = 0 \end{cases} \\ \forall n \in \{1, 3, 5, \dots, N_b/2\} \quad & \begin{cases} \frac{d\beta_n}{d\tau} = (4/\pi)\sin(n\tau) f(\tau) & \tau \in [0, \pi/2] \\ \beta_n(0) = 0 \end{cases} \end{aligned} \tag{2.5}$$

2.2 OC SHE in two levels for symmetry of quarter-wave

The symmetry of quarter wave implies:

$$f(\omega t + \pi) = -f(\omega t) \quad t \in (0, \pi) \tag{2.6}$$

$$f(\omega t + \pi/2) = +f(\omega) \quad t \in (0, \pi/2) \tag{2.7}$$

This two conditions simplify the expressions (1.2) and (1.3), in this way:

$$a_n = 0 \ \mid \ \forall n \in \mathbb{Z} \tag{2.8}$$

$$b_n = \begin{cases} (4/\pi) \int_0^{\pi/2} f(\omega t) \sin(n\omega t) d(\omega t) & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases} \tag{2.9}$$

So, in summary $f(\omega t)$ can be written as follows:

$$f(\omega t) = \sum_{n\ odd}^{\infty} b_n \sin(n\omega t) \tag{2.10}$$

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} f(\omega t)\sin(n\omega t)d(\omega t) \quad | \quad n\ odd \tag{2.11}$$

Now in this context, we can define a SHE problem as follows:

Problem 2.2 Given $\boldsymbol{b}_T \in \mathbb{R}^{n_b}$, we define a cost functional in this way:

$$J[f(\tau)] = \left[||\boldsymbol{b}_T - \boldsymbol{\beta}(T)||^2 - \epsilon \int_0^{\pi/2} ||f(\tau)||^2 d\tau \right] \tag{2.12}$$

where $\boldsymbol{\beta}(\tau) = [\beta_1(\tau)\ \beta_3(\tau)\ ...\ \beta_{N/2}(\tau)]^T$, the $||.||$ is a euclidean norm and ϵ is a penalization parameter to maximized the norm of control $f(\tau)$.

So, the optimal control problem can be write:

$$\min_{|f(\tau)|<1} J[f(\tau)] \tag{2.13}$$

$$\text{suject to: } \begin{cases} \frac{d\beta_n}{d\tau} = (4/\pi)\sin(n\tau)f(\tau) & \tau \in [0, \pi/2] \\ \beta_n(0) = 0 \end{cases} \tag{2.14}$$

$$\forall n \in \{1, 3, 5, \ldots, N/2\}$$

In SHE problem, we search a function $f(\tau)|\tau \in [0, \pi/2]$ whose only take a two values, $\{-1, 1\}$. This is the reason to add L^2-penalization in control. This maximization of L^2-norm of control and the constraint $|f(\tau)| < 1$ produce a *bang-bang* control. In other words, the optimal control of this problem only take the values $\{-1, 1\}$.

Now, we show *Bang-Bang* property in optimal control of problem (1.1). For this, we define a Hamiltonian H as:

$$H(\tau, \boldsymbol{p}(\tau), f(\tau)) = -\epsilon||f(\tau)||^2 + \frac{4}{\pi}\sum_n p_n(\tau)\sin(n\tau)f(\tau) \tag{2.15}$$

where $\boldsymbol{p}(\tau) = [p_1(\tau), p_3(\tau), \ldots, p_{N/2}(\tau)]^T$. Now, we can use follow condition of the minimum principle of Pontryagin:

$$H(\tau, \boldsymbol{p}^*(\tau), f^*(\tau)) \leq H(\tau, \boldsymbol{p}^*(\tau), f(\tau)) \tag{2.16}$$

Vemos que el el Hamiltoniano H es un parábola invertida cuando nos fijamos en la variable de control $f(\tau)$. Por lo tanto el óptimo siempre será los valores extremos $\{-1, 1\}$

Entonces

$$f^*(\tau) = \arg\min_{f \in \{-1,1\}} H(\tau, \boldsymbol{p}^*(\tau), f) \tag{2.17}$$

We can obtain the optimal control as:

$$f^*(\tau) = \begin{cases} +1 \text{ if } \sum_n p_n^*(\tau)\sin(n\tau) < 0 \\ -1 \text{ if } \sum_n p_n^*(\tau)\sin(n\tau) > 0 \end{cases} \tag{2.18}$$

So optimal control of problem (1.1) is *bang-bang*.

2.3 SHE in three levels

For the SHE in three levels problem the before discussion can be used. The idea is change the constraint $|f(\tau)| < 1$ by $0 < f(\tau) < 1$. Thank to a symmetry of quarter-wave, if the function $f(\tau) \mid \tau \in [0, \pi/2]$ whose take a two values $\{0, 1\}$, his representation in full wave $f(\tau) \mid \tau \in [0, 2\pi]$ takes only three values: $\{-1, 0, 1\}$.

So, We can obtain the optimal control of problem (2.2) but with control constraints $\{0 < f(\tau) < 1 \mid \tau \in [0, \pi/2]\}$ as:

$$f^*(\tau) = \begin{cases} 1 \text{ if } \sum_n p_n^*(\tau)\sin(n\tau) < 0 \\ 0 \text{ if } \sum_n p_n^*(\tau)\sin(n\tau) > 0 \end{cases} \tag{2.19}$$

3 Numerical experiments

To solve the optimal control problem, we use a direct method. This method considered a time discretization. In this way, if we have a partition $\{\tau_0, \tau_1, \dots, \tau_{N_t}\}$ of interval $[0, \pi/2]$, we can represent a function $\{f(\tau) \mid \tau \in [0, \pi/2]\}$ as a vector $\boldsymbol{f} \in \mathbb{R}^{N_t}$ where component $f_i = f(t_i)$. Then the optimal control problem (1.1) can be written as optimization problem with variable $\boldsymbol{f} \in \mathbb{R}^{N_t}$.

Problem 3.1 Given $\boldsymbol{b}_T \in \mathbb{R}^{n_b}$, the optimization problem can be write:

$$\min_{\boldsymbol{f} \in \mathbb{R}^{N_t}} \left[||\boldsymbol{b}_T - \boldsymbol{\beta}^{N_t}||^2 - \epsilon\Delta\tau \sum_{i=0}^{N_t} f_{\tau_i}^2 \right] \tag{3.1}$$

$$\text{suject to: } \begin{cases} \beta_n^{i+1} = \beta_n^i + \Delta\tau(4/\pi)\sin(n\tau_i)f_\tau \quad \tau \in [0, \pi/2] \\ \beta_n^0 = 0 \end{cases} \tag{3.2}$$
$$\forall n \in \{1, 3, 5, \dots, N/2\}$$

Where we use a euler scheme to discretizate the dynamics. This problem is a Nonlinear programming, for this we use CasADi software to solve.

3.1 Numerical result for SHE of two levels

We considered the harmonics $[\beta_1, \beta_5, \beta_7, \beta_{11}]$. We want waveform that $\boldsymbol{b}_T = [b_T^1, b_T^5, b_T^7, b_T^{11}] = [m_a, 0, 0, 0, 0]$, where m_a is a parameter . We will compare the different solutions of problem (1.2). Compararemos el problema mediante soluciones obtenidas del problema (1.2), mediante algoritmos genéticos.

El problema de control óptimo (2.2) reproduce una de las soluciones obtenidas

3.2 Numerical result for SHE of three levels

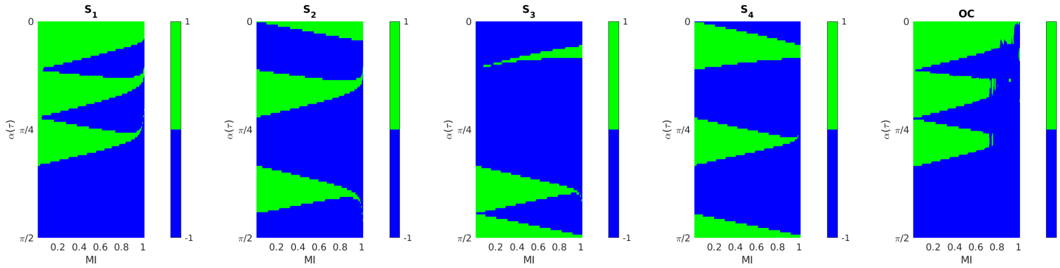

Figure 1: Comparison of solutions for different values of m_a. Solutions S_1, S_2, S_3 correspond to problem (1.2) where the number of switching angles is prefixed, while OC solution correspond to optimal control problem.

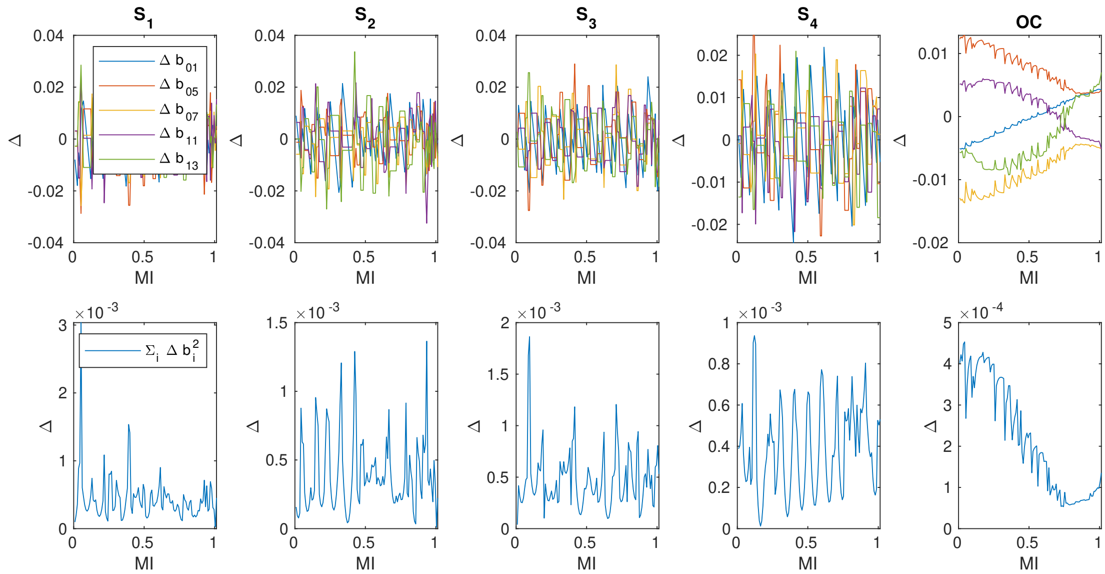

Figure 2: The order of magnitud of the square of euclidean distantes to target is the same for all solutions of figure (1)

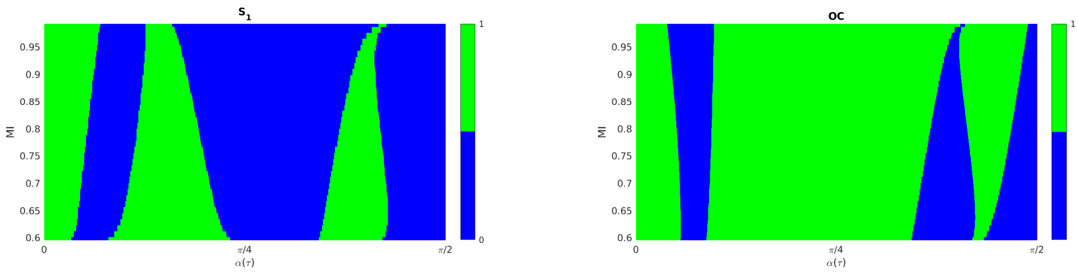

Figure 3: Solutions

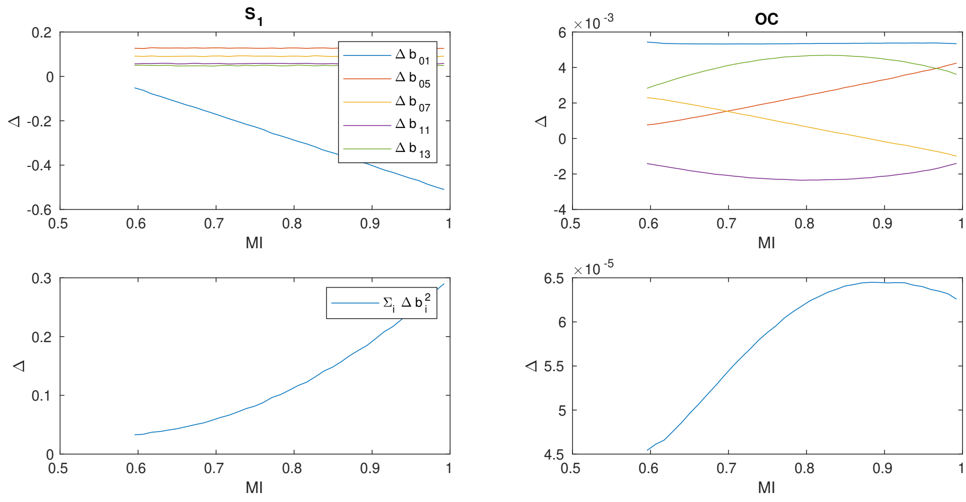

Figure 4: Errors

Part II
Feedback Optimal Control

4 SHE two levels with quarter-wave symmetry as dynamic programming problem

Con el fin de encontrar la solución al problema para distintos targets $\boldsymbol{a}_T$ y $\boldsymbol{b}_T$ consideraremos el cambio de variable $\boldsymbol{\beta}'(\tau) = \boldsymbol{\beta}(\tau) - \boldsymbol{b}_T$. De manera que, podemos plantear un problema de control donde el estado inicial el target $\boldsymbol{b}_T$ y cuyo objetivo es llevar el sistema al origen de coordenadas.

Problem 4.1 Given $\boldsymbol{b}_T \in \mathbb{R}^{n_b}$, we define a cost functional in this way:

$$J[f(\tau)] = \frac{1}{2}||\boldsymbol{\beta}(T)||^2 \tag{4.1}$$

So, the optimization problem can be write:

$$\min_{f \in \{-1,1\}} J[f(\tau)]) \tag{4.2}$$

$$\text{suject to: } \begin{cases} \frac{d\beta_n(\tau)}{d\tau} = -(4/\pi)\sin(n\tau)f(\tau) \quad \tau \in [0, \pi/2] \\ \beta_n(0) = b_T^n \end{cases} \tag{4.3}$$

$$\forall n \in \{1, 3, 5, \ldots, N_b/2\}$$

Tomamo un discretización $\{\tau_0, \tau_1, \ldots, \tau_{N_t}\}$ del intervalo $[0, \pi/2]$. Entonces podemos discretizar el problema anterior:

Problem 4.2 Given $\boldsymbol{b}_T \in \mathbb{R}^{n_b}$, the optimization problem can be write:

$$\min_{\boldsymbol{f}\in\mathbb{R}^{N_t}} \frac{1}{2}||\boldsymbol{\beta}^{N_t}||^2 \tag{4.4}$$

$$\text{suject to: } \begin{cases} \beta_n^{i+1} = \beta_n^i - \Delta\tau(4/\pi)\sin(n\tau_i)f_\tau \quad \tau \in [0,\pi/2] \\ \beta_n^0 = b_T^n \end{cases} \tag{4.5}$$

$$\forall n \in \{1,3,5,\ldots,N/2\}$$

4.1 Dynamic programming

Función valor

$$v_t(\boldsymbol{b}_T) = \min_{\boldsymbol{f}\in\mathbb{R}^{N_t-t}} \frac{1}{2}||\boldsymbol{\beta}^t||^2 \tag{4.6}$$

$$\text{suject to: } \begin{cases} \beta_n^{i+1} = \beta_n^i - \Delta\tau(4/\pi)\sin(n\tau_i)f_\tau \quad \tau \in [0,\pi/2] \\ \beta_n^0 = b_T^n \end{cases} \tag{4.7}$$

$$\forall n \in \{1,3,5,\ldots,N/2\}$$

Entonces la función valor de estado cumple la ecuación de Bellman, que en este caso se puede escribir como:

$$v_t(\boldsymbol{\beta}^t) = \min_{f\in\{-1,1\}} v_{t+1}(\boldsymbol{\beta}^{t+1}) \tag{4.8}$$

$$v^{N_t}(\boldsymbol{\beta}^{N_t}) = \frac{1}{2}||\boldsymbol{\beta}^{N_t}||^2 \tag{4.9}$$

La función $v_t(\boldsymbol{\beta}^t)$ representa el mejor coste que se puede alcanzar desde el estado $\boldsymbol{\beta}^t$ en $(N_t - t)$ pasos. Es por ello que en $t = Nt$ el mejor coste que se puede alcanzar desde el punto $\boldsymbol{\beta}^t$ es el valor del coste final $\frac{1}{2}||\boldsymbol{\beta}^{N_t}||^2$

Luego el control óptimo se puede calcular mediante la siguiente expresión

$$f^*(\tau,\boldsymbol{\beta}_t) = \min_{f\in\{-1,1\}} v_t(\boldsymbol{\beta}_t) \tag{4.10}$$

References