$f''(x) = ?$

(1) $f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)(\Delta x)^2 + \frac{1}{3!} f'''(x)(\Delta x)^3 + \frac{1}{4!} f^{(4)}(x)(\Delta x)^4 +$ até o $f^{(6)}$

(2) $f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2} f''(x)(\Delta x)^2 - \frac{1}{3!} f'''(x)(\Delta x)^3 + \frac{1}{4!} f^{(4)}(x)(\Delta x)^4 -$ até $f^{(6)}$

$$f(x+\Delta x) + f(x-\Delta x) = 2f(x) + f''(x)(\Delta x)^2 + \frac{2}{4!} f^{4}(x)(\Delta x)^4 + \frac{2}{6!} f^{6}(x)(\Delta x)^6$$

$$f''(x) = \frac{1}{(\Delta x)^2}\left(f(x+\Delta x) - 2f(x) + f(x-\Delta x) \right) - \frac{2}{24} f^{(4)}(x)(\Delta x^2)$$

Ordem Quadrática

Para chegarmos ao erro de ordem $(\Delta x)^4$ precisamos expandir $f(x-\Delta x)$ até o termo $f^{(6)}$. Para isso vamos expandir para $(x+2\Delta x)$ e $(x-2\Delta x)$

(3) $f(x+2\Delta x) = f(x) + f'(x)(2\Delta x) + \frac{1}{2} f''(x)(2\Delta x)^2 + \frac{1}{3!} f'''(x)(2\Delta x)^3 + \dots + \frac{1}{6!} f^{6}(x)(2\Delta x)^6$

(4) $f(x-2\Delta x) = f(x) - f'(x)(2\Delta x) + \frac{1}{2} f''(x)(2\Delta x)^2 - \frac{1}{3!} f'''(x)(2\Delta x)^3 + \dots + \frac{1}{6!} f^{6}(x)(2\Delta x)^6$

Vamos combinar as expressões para eliminar os termos da f', f''', f^{4},

$(1) + \alpha(2) + \beta(3) + \gamma(4)$

(1): $f'(x)\Delta x\,(\underbrace{1 - \alpha + 2\beta - 2\gamma}_{=0})$

(3): $\frac{1}{4!} f^{4}(x)(\Delta x)^4\,(1 - \alpha + 16\beta - 16\gamma)$

(2): $\frac{1}{3!} f'''(x)(\Delta x)^3\,(1 - \alpha + 8\beta - 8\gamma)$

Resolvendo o sistema

$$\begin{bmatrix} -1 & 2 & -2 \\ -1 & 8 & -8 \\ -1 & 16 & -16 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Teremos: $\gamma = -1/16$, $\beta = -1/16$, $\alpha = 1$

$$(1) + (1)(2) + \left(-\tfrac{1}{16}\right)(3) + \left(-\tfrac{1}{16}\right)(4)$$

$$f(x+\Delta x) + f(x-\Delta x) - \frac{1}{16} f(x+2\Delta x) - \frac{1}{16} f(x-2\Delta x) =$$

$$f(x)\underbrace{\left(1+1-\tfrac{1}{16}-\tfrac{1}{16}\right)}_{15/8} + \frac{1}{2} f''(x)(\Delta x)^2 \underbrace{\left(1+1-\tfrac{1}{16}\cdot 4-\tfrac{1}{16}\cdot 4\right)}_{3/2} +$$

$$\frac{1}{5!} f^{(5)}(x)(\Delta x)^5\left(1-1-\tfrac{1}{16}\cdot 32+\tfrac{1}{16}\cdot 32\right) + \frac{1}{6!} f^{(6)}(x)(\Delta x)^6 \underbrace{\left(1+1-\tfrac{1}{16}\cdot 64-\tfrac{1}{16}\cdot 6\right.}_{-6}$$

$$f''(x) = \frac{4}{(\Delta x)^2 \cdot 3}\left(-\frac{1}{16} f(x+2\Delta x) + f(x+\Delta x) + f(x-\Delta x) - \frac{1}{16} f(x-2\Delta x) - \frac{15}{8} f(x\right.$$

$$\underbrace{+6 \cdot \frac{1}{6!} f^{(6)}(x)\cdot(\Delta x)^6 \cdot \left(\frac{4}{3(\Delta x)^2}\right)}$$

$$\frac{8}{6!} f^{(6)}(x)(\Delta x)^4$$

erro

$$\rightarrow \frac{1}{(\Delta x)^2}\left(-\frac{1}{12} f(x+2\Delta x) + \frac{4}{3} f(x+\Delta x) + \frac{4}{3} f(x-\Delta x) - \frac{1}{12} f(x-2\Delta x) - \frac{5}{2} f(x)\right)$$

Item b) Desenvolvendo fórmula usando polinômio de Newton:

- Nossa avaliação preliminar da série de Taylor mostrou que precisaremos de um polinômio de Newton com grau 4

→ Construindo polinômio de interpolação:

$$g(s) = \sum_{k}^{6} \binom{s}{k} \Delta^k f_0 \qquad \text{onde: } \binom{s}{k} = \frac{s!}{k!\,(s-k)!}$$

e

$$\Delta^k f_i = \begin{cases} f_i & \text{se, } k=0 \\ \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i & \text{se } k>0 \end{cases}$$

Logo:

$$g(s) = \Delta^0 f^0 + s\Delta^1 f_0 + \frac{1}{2}(s^2-s)\Delta^2 f_0 + \frac{1}{6}(s^3-3s^2+2s)\Delta^3 f_0$$
$$+ \frac{1}{24}(s^4-6s^3+11s^2-6s)\Delta^4 f_0$$

e temos que $\dfrac{d^2 g}{ds^2} = \frac{1}{\cancel{2}}\Delta^2 f_0 \cancel{2} + \frac{1}{6}\Delta^3 f_0(6s-6) + \frac{1}{24}\Delta^4 f_0(12s^2-36s+22)$

Então: usamos $s=2$, pois a filosofia é central.

$$\frac{d^2 f}{dx^2} = \frac{1}{(\Delta x^2)}\left(\Delta^2 f_0 + \frac{1}{6}\Delta^3 f_0(12-6) + \frac{1}{24}\Delta^4 f_0(48-72+22)\right.$$

$$\frac{d^2}{dx^2} = \frac{1}{(\Delta x^2)}\left(\Delta^2 f_0 + \Delta^3 f_0 - \frac{1}{12}\Delta^4 f_0\right)$$

Calculando Δ's:

- $\Delta^2 f_0 = [\Delta^1 f_1] - [\Delta^1 f_0]$
  $= [(f_2-f_1) - (f_1-f_0)]$
  $= f_2 - 2f_1 + f_0$

- $\Delta^3 f_0 = [\Delta^3 f_1 - \Delta^3 f_0]$
  $= [(\Delta^1 f_2) - (\Delta^1 f_1)] - [(\Delta^1 f_1) - (\Delta^1 f_0)]$
  $= \dots\ f_3 - 3f_2 + 3f_1 - f_0$

- $\Delta^4 f_0 = [\Delta^3 f_1] - [\Delta^3 f_0]$

$$= \cdots f_4 - 4f_3 + 6f_2 - 4f_1 + f_0$$

Substituindo na equação geral temos:

$$\frac{d^2f}{dx^2} = \frac{1}{\Delta x^2}\left(f_2 - 2f_1 + f_0 + f_3 - 3f_2 + 3f_1 - f_0 - \left(\frac{1}{12} \cdot (f_4 - 4f_3 + 6f_2 - 4f_1 + f_0)\right)\right)$$

$$\frac{d^2f}{dx^2} = \frac{1}{\Delta x^2}\left(-\frac{1 f_4}{12} + \frac{4f_3}{3} - \frac{5 f_2}{2} - \frac{4f_1}{3} - \frac{1 f_0}{12} \right)$$

que é equivalente à fórmula encontrada pelas expansões de Taylor. $\blacksquare$

O erro é equivalente à formula do polinômio de Taylor!

data / /

S T Q Q S S D

Passando a limpo e simplificando

$$f''(x) = \frac{-f(x-2\Delta x) + 16f(x-\Delta x) - 30f(x) + 16f(x+\Delta x) - f(x+2\Delta x)}{(12 \cdot \Delta x^2)}$$

$$\text{erro} = \frac{8 f^{(6)}(x)(\Delta x)^4}{6!}$$

Preencher a tabela

erro < 0,00001

$\Delta^{(k)}$	$f(x)$	$f''(x)$	$e(x) = \left\| \frac{f''(\Delta^{(k)}) - f''(\Delta^{(k-1)})}{f''(\Delta^{(k)})} \right\|$
0,5		44,9043600441	1
0,25		45,063360841	0,00352
0,125		45,072914361	0,000211957
0,0625		45,0735 0557	0,000013116
0,03125		45,0735424322	0,00000081782