Phy 505

20/02/2024

Complex Analysis

* Partial differential equation and how it can be solved.
* complex Analysis (Analytic function)

Complex analysis has many application in heat conduction, fluid flow, electrostatic, etc. It is concerned with the theory and application of analytic Function.

Knowledge of real calculus can be applied to complex analysis. In complex analysis, the case is seen with analytic function, i.e, the functions that are differentiable in some domain and have derivative of all order. This differs from real function ~~whereas~~ whose derivative only have a certain order. So more difficult problem in real calculus can be much easier in the real analysis.

A complex function $f(z)$ in a domain E is a set of instructions or rule that assigned to every z in the domain. A complex number, let call the value of $f(z)$ here z exists and it is called a complex variable.

$$\begin{pmatrix} 1+2i \\ 2+3i \\ 3+2i \end{pmatrix}$$

Ex: $f(z) = z^2 + 3z$, $z = x + iy$

$f(z) = (x+iy)^2 + 3(x+iy)$

$f(z) = x^2 - y^2 + 3x + i(2xy + 3y)$

$u(x,y) + iv(x,y)$

$\text{let } (1+2i) = 1^2 - 2^2 + 3(1) + i(2(1)(2) + 3(2))$

$\text{let} = 0 + i(10)$

$\text{let} = 10i$

If for every z in a domain D, there exist only one $f(z)$, then $f(z)$ is said to be single-valued. A function is said to be continuous at $z = z_0$ if $f(z_0)$ is defined and the limit must exist. $f(z)$ is continuous in a domain if it is continuous at each point in the domain

Elementary Complex Functions

Some elementary complex functions are the exponential functions, trigonometric functions, logarithm functions, hyperbolic functions, etc. These complex functions have some interesting properties not shared by their real counterparts and they reduce to the real counterpart when z is made up of only the real variable. The exponential function which is an important analytic function is written as:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos x + i\sin y) \quad \text{---} \quad (2)$$

e^x is analytic for all $z \Rightarrow$ It is an entire function

$$e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$e^{-i\pi} = \cos\pi - i\sin\pi = -1$$

$$e^{i\pi/2} = \cos \pi/2 + i\sin \pi/2 = i$$

$$e^{-i\pi/2} = \cos \pi/2 - i\sin \pi/2 = -i$$

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \quad \text{---} \quad (3)$$

NB: Finding the exponential function of a purely imaginary value is always 1

Trigonometric function:

$$e^{ix} = \cos x + i\sin x \quad \text{---} \quad (4)$$

$$e^{-ix} = \cos x - i\sin x \quad \text{---} \quad (5)$$

$$(4)+(5) = 2\cos x = e^{ix} + e^{-ix}$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \quad \text{---} \quad (6)$$

$$(4)-(5) = \sin x = \frac{e^{ix}-e^{-ix}}{2} \quad \text{---} \quad (7)$$

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ $\quad$ $\sec z = 1/\cos z$

$\sin z = 1/2(e^{iz} - e^{-iz})$ $\quad$ $\operatorname{cosec} z = 1/\sin z$

$\tan z = \sin z / \cos z$

$\cot z = \cos z / \sin z$

Since e^z is an entire function, then $\cos z$ and $\sin z$ are entire functions. Tan z and Sec z are not entire functions because they are not analytic when $\cos z$ is equals to 0. Cot z and cosec z are not entire functions because they are not analytic when $\sin z$ is equals to 0.

Hyperbolic functions:

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z}) \quad --- (8)$$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z}) \quad --- (9)$$

Also $\tanh z = \frac{\sinh z}{\cosh z}$ $\quad$ $\coth z = \frac{\cosh z}{\sinh z}$

$\operatorname{Sech} z = \frac{1}{\cosh z}$ $\quad$ $\operatorname{Cosech} z = \frac{1}{\sinh z}$

Derivative of a complex function:

Using the first principle, the derivative of a complex function $f(z)$ at a point z_0 is given by:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad --- (10)$$

$$\text{or; } f'(z_0) = \lim_{\Delta z \to z_0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad --- (11)$$

If the limit of either (10) or (11) exist, then $f(z)$ is said to be differentiable at z_0 irrespective of whatever path taken by z to approach z_0.

Ex: Show that the function $f(z) = z^3$ is differentiable

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^3 - z^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^3} + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - \cancel{z^3}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^{\cancel{3}2}}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} 3z^2 + 3z(\Delta z) + (\Delta z)^2$$

$$\therefore f'(z) = 3z^2$$

Ex 2: Show that the function $f(z) = 3\bar{z}$ is differentiable using first principle.

$$f'(z) = \lim_{\Delta z \to 0} \frac{3(\bar{z} + \Delta\bar{z}) - 3\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{3\bar{z}} + 3\Delta\bar{z} - \cancel{3\bar{z}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3\Delta\bar{z}}{\Delta z} \qquad \begin{pmatrix} \Delta z = \Delta x + i\Delta y \\ \Delta\bar{z} = \Delta x - i\Delta y \end{pmatrix}$$

$$= \lim_{\Delta z \to 0} \frac{3(\Delta x - i\Delta y)}{\Delta x + i\Delta y}$$

$\Delta x \to 0$; $= -3$

$\Delta y \to 0$; $= 3$

$\therefore$ The function $f(z) = 3\bar{z}$ is not differentiable.

Analytic function:

A function $f(z)$ is said to be analytic or regular or holomorphic at a point z_0 in a domain D if $f(z)$ is defined and differentiable at z_0 and in the neighbourhood of z_0 or everywhere around z_0.

Characteristic of Analytic Function

(i) An analytic function $f(z)$ has a unique derivative at every point in some neighbourhood of z_0 including at z_0

2. When a function is analytic, it can be expressed in an infinite series in the power of z

3. They have derivative of all ~~other~~ order.

The point or points in a domain where a complex function is not analytic is called the singular point or singularity of that function.

A complex function that is analytic in the entire finite plane is said to be an entire function (e.g, e^x, $\sin x$, $\cos x$, polynomial of non-negative ~~integer~~ powers, $z^0, z^1, z^2, \ldots$ etc). --- (12)

A rational function given by $f(z) = g(z)/h(z)$, where $g(z)$ and $h(z)$ are polynomial is an analytic function except at $h(z) = 0$

Cauchy - Riemann Equation.

The C-R equation provides a means of testing for the analyticity of a function. A function is analytic in some domain of the partial derivative of U and V satisfy C-R equation. The derivative of a complex function $f(z)$ is given by:

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y} \quad --- (13)$$

Equ (13) will yield the same result irrespective of the path taken by $\Delta z \to 0$

If $\Delta y \to 0$; $$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x,y) + iv(x,y)]}{\Delta x} \quad --- (14)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x} \quad --- (15)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad --- (16)$$

If $\Delta x \to 0$,

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y) - [u(x,y) + iv(x,y)]}{i\Delta y} \quad \text{--- (17)}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y} \quad \text{--- (18)}$$

$$f'(z) = \frac{-i\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{--- (19)}$$

For the function to be analytic, eqn (16) must be equal to (19)

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}} \quad \text{--- (20)}$$

The polar coordinate where $z = r(\cos\theta + i\sin\theta)$

$$f(z) = u(r,\theta) + iv(r,\theta)$$

C-R equations are $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$ --- (21)

Ex: Determine if these equations are analytic using the C-R equation: (a) $f(z) = z^2 - 4 = (x+iy)^2 - 4$

$$f(z) = x^2 - y^2 - 4 + i2xy$$

$$u(x,y) = x^2 - y^2 - 4$$

$$v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial x} = 2y$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$, the function is analytic

(b) $f(z) = z^2 - 4$

(c) $f(z) = z\bar{z} = (x + i\Delta y)(x - i\Delta y)$

$$= x^2 + y^2$$

$$\Rightarrow u(x,y) = x^2 + y^2, \quad v(x,y) = 0$$

$\partial u/\partial x = 2x, \quad \partial v/\partial y = 0$

Harmonic functions:

A function of two variables $u(x,y)$ and $v(x,y)$ is said to be harmonic if it satisfy laplace's equation. The theory of the function being harmonic is called potential theory. The real and imaginary part of ~~any~~ any ~~harmonic~~ analytic function are harmonic and are reffered to as harmonic conjugate function of each other.

If $F(z) = u(x,y) + iv(x,y)$ is analytic, then u and v satisfy Laplace's equation given by

$$\boxed{\begin{aligned} \nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \nabla^2 v &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \end{aligned}} \quad --- (22)$$

Also know that the derivative of any analytic function is also analytic. If a function satisfy laplace's equation, the function is harmonic

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Ex (1) $F(z) = z^2 - 4$

$= x^2 - y^2 - 4 + i2xy$

$u(x,y) = x^2 - y^2 - 4, \quad v(x,y) = 2xy$

$\partial u/\partial x = 2x \quad \frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial v}{\partial x} = 2y \quad \frac{\partial^2 v}{\partial x^2} = 0$

$\frac{\partial u}{\partial y} = -2y \quad \frac{\partial^2 u}{\partial y^2} = -2 \qquad \partial v/\partial y = 2x \quad \partial^2 v/\partial y^2 = 0$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$0 + 0 = 0$

$2 - 2 = 0$

(2) Is the given function $v = e^{-x}\sin 2y$ harmonic? If yes, find its harmonic conjugate

Solution

$v = e^{-x}\sin 2y$

$\partial v/\partial x = -e^{-x}\sin 2y \qquad \partial v/\partial y = 2e^{-x}\cos 2y$

$\partial^2 v/\partial x^2 = e^{-x}\sin 2y \qquad \partial^2 v/\partial y^2 = -4e^{-x}\sin 2y$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$e^{-x}\sin 2y + (-4e^{-x}\sin 2y) = 0$

$= -3e^{-x}\sin 2y \neq 0$

Since $\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \neq 0$, V is not harmonic.

(3) $u = x^3 - 3xy^2$

$\partial u/\partial x = 3x^2 - 3y^2 \quad \partial u/\partial y = -6xy$

$\partial^2 u/\partial x^2 = 6x \quad \partial^2 u/\partial y^2 = -6x$

$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 6x - 6x = 0$

$\therefore$ U is harmonic.

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$ (C-R equation)

$\partial v = (3x^2 - 3y^2)\partial y$

$\therefore V = 3x^2 y - y^3 + K(x)$

$\partial v/\partial x = 6xy + K'(x) = -\frac{\partial u}{\partial y} = -6xy$

$6xy + K'(x) = +6xy$

$K'(x) = 0$

$\therefore K = 0$, ~~analytic function is therefore given~~

$\therefore V = 3x^2 y - y^3$

The analytic function is therefore given by:

$x^3 - 3xy^2 + i(3x^2 y - y^3) = f(z)$

(4) $V = 2xy$

$\partial v/\partial x = 2y \quad \partial v/\partial y = 2x$

$\partial^2 v/\partial x^2 = 0 \quad \partial^2 v/\partial y^2 = 0$

$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0$

$\therefore$ V is harmonic.

$\partial v/\partial y = 2x = \partial u/\partial x$

$\partial u = 2x\,\partial x$

$u = \cancel{2}\cdot x^2 + k(y)$

$\partial u/\partial y = 0 + k'(y) = -\partial v/\partial x = -2y$

$\therefore k'(y) = -2y$

$k(y) = -y^2$

$u = x^2 - y^2$

$\therefore f(z) = x^2 - y^2 + i\,2xy$

Complex Integration (Contour Integration)

Cauchy Integral theorem:

Complex integration is simply integration on the complex plane. It is similar to integration of real functions but with some unique features that make it more flexible than the integration of real functions. It is centered around cauchy integral theorem which is executed through the cauchy integral formula, i.e, the evaluation of integrals having analytic integrands. The complex number z being a function of two variables (x, y) and the complex function F being a function of two real variables (u, v) makes complex integration to be similar to the integration of real functions. If z moves along a curve C at each point on the complex plane, z will have a value $f(z)$. Summing on the curve, the complex number up all these values along the curve, on the complex plane is a line integration in the z-plane. In the complex plane, such integration is referred to as contour integration.

$$F(z) = u + iv$$

$$dz = dx + i\,dy$$

$$\int f(z)\,dz = \int (u+iv)(dx + i\,dy) \quad \text{--- (1)}$$

$$= \int u\,dx - v\,dy + i\int v\,dx + u\,dy \quad \text{--- (2)}$$

Each part of the eqn (2) can be in the form $\int P\,dx + Q\,dy$

used for general line integral. Recall that the value obtained for a line integral is dependent on the path of integration but in a simply connected domain and with $f(z)$ being analytic, the line integral of $Pdx + Qdy$ is not dependent on the path. In a simply connected region, we have this relationship everywhere in the region: $\partial P/\partial y = \partial Q/\partial x$. If the integration path with a simply connected closed path, then

$$\oint_c f(z)dz = \oint_c (udx - vdy) + \oint_c (vdx + udy) \quad \text{--- (3)}$$

Applying Green's theorem to each part of eqn (3) given

$$\oint_c (udx - vdy) = \iint_s \left[\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dxdy \quad \text{--- (4)}$$

$$\oint_c vdx + udy = \iint_s \left[\partial u/\partial x - \partial v/\partial y\right] dxdy \quad \text{--- (5)}$$

Assuming that $f(z)$ is analytic everywhere inside and on the curve, then by applying C-R equation to eqn (4) and (5) reduces the integrations to zero, thus, Cauchy's theorem states that If $f(z)$ is analytic or holomorphic or regular everywhere inside and on a simply connected contour, then $\oint_c f(z)dz = 0$ --- (6) applies. If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ on any closed path is independent of the path.

Basic properties of Cauchy's theorem:

1 Linearity: This implies that the integration of sums or differences can be performed term by term.

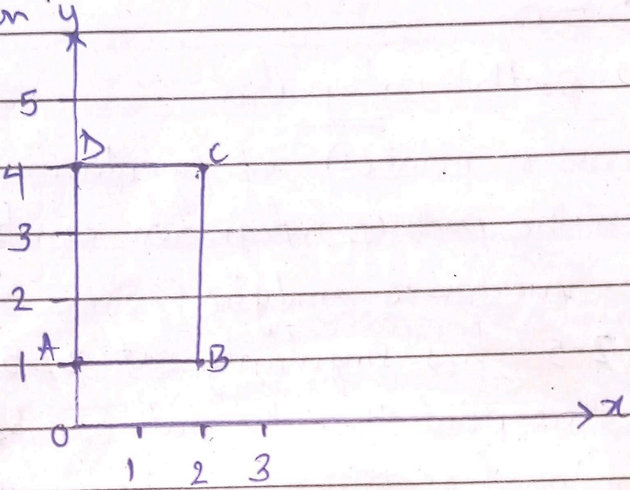
$$\oint_c [k_1 f_1(z) + k_2 f_2(z)]dz$$
$$= k_1 \oint_c f_1(z)dz + k_2 \oint_c f_2(z)dz$$

2 Reversal of path: $\int_A^B f(z)dz = -\int_B^A f(z)dz$

3 Partitioning of path: $\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz$

Ex(1): Verify Cauchy's theorem by evaluating $f(z)=z^2$ round the rectangle formed by joining the points $z=2+i$, $z=2+4i$, $z=4i$, $z=i$.

Solution



$$f(z) = z^2 = (x+iy)^2 = x^2-y^2+i\,2xy$$

$$dz = dx + i\,dy$$

$$f(z)dz = (x^2-y^2+i2xy)(dx+idy)$$

$$f(z)dz = (x^2-y^2)dx - 2xydy + i\left[2xy\,dx + (x^2-y^2)dy\right]$$

$$\oint f(z)dz = \oint (x^2-y^2)dx - 2xydy + i\oint 2xy\,dx + (x^2-y^2)dy \quad \text{--- (1)}$$

Along AB: $y=1$, $dy=0$, eqn (1) reduces to;

$$\int_{AB} f(z)dz = \int_0^2 (x^2-1)dx + i\int_0^2 2x\,dx$$

$$= \left[\frac{x^3}{3} - x + ix^2\right]_0^2 = \frac{2}{3} + 4i$$

Along BC: $x=2$, $dx=0$, eqn (1) reduces to;

$$\int_{BC} f(z) = \int_1^4 -4y\,dy + i\int_1^4 (4-y^2)dy$$

$$= \left[-2y^2 + i\left(4y - \tfrac{1}{3}y^3\right)\right]_1^4 = -30-9i$$

Along CD: $y=4$, $dy=0$

$$\int_{CD} f(z) = \int_2^0 (x^2-16)dx + i\int_2^0 8x\,dx$$

$$= \left[\frac{x^3}{3} - 16x + i\,4x^2\right]_2^0 = \frac{88}{3} - 16i$$

Along DA: $x=0$, $dx=0$

$$\int_{DA} f(z) = i\int_4^1 -y^2\,dy$$

$$= i\left[-\tfrac{1}{3}y^3\right]_4^1 = i21$$

$\therefore \oint f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$

$= \left(\frac{2}{3}+4i\right) + \left(-30-9i\right) + \left(\frac{88}{3}-16i\right) + \left(21i\right) = 0$

$0 = 0$

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Deformation of path integration

If a contour encloses a point(s) of singularities of a given complex function, then the path of integration can be deformed into parts for which the function is analytic.

For example, $f(z) = z/z-5$ has singularity at $z=5$. If the path of integration encloses this point then the path can be deformed ~~the~~ so to cut out the point of singularity. Assuming another contour C_1 is created close to the singular point, with Cauchy's theorem it can be shown that:

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz \quad --- (8)$$

If the contour encloses multiple singular point, small contours could be created close to the singular point such that

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \ldots \quad (9)$$

Consider the integral $\oint f(z)dz$ evaluated round a closed contour given that $f(z) = 1/z$, at $z=0$, $f(z)$ ceases to be regular, thus, $z=0$ is a singular point for this function. If the closed contour is specified and it did not enclose the singular point then $\oint \frac{1}{z}dz = 0$ will hold. If the contour encloses the singular point, the contour can then be deformed into a smaller contour and evaluated as given below according to eqn (8)

$z = re^{i\theta}$

$dz = ire^{i\theta}d\theta$

$\oint f(z)dz = \oint \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{re^{i\theta}}(ire^{i\theta}d\theta)$

$= \int_0^{2\pi} i\,d\theta = 2\pi i \quad --- (10)$

$\oint f(z)\,dz = \oint \frac{1}{z}\,dz = \begin{cases} 0 & \text{if the contour does not enclose the singular point.} \\ 2\pi i & \text{if the contour encloses the singular point.} \end{cases}$

Assignment: Prove, If $f(z) = \frac{1}{z^n}$, $\oint f(z)\,dz = \oint \frac{1}{z^n}\,dz = 0$ [whether the singular point is enclosed or not]

Example: $f(z) = \dfrac{3z-6-i}{(z-i)(z-3)}$, contour: $|z| = 2$ [the contour is a circle with a center of 0 and radius of 2]

Solution.

$$\frac{3z-6-i}{(z-i)(z-3)} = \frac{A}{z-i} + \frac{B}{z-3}$$

$$\frac{3z-6-i}{(z-i)(z-3)} = \frac{A(z-3) + B(z-i)}{(z-i)(z-3)}$$

At $z=3$; $9-6-i = B(3-i)$

$B = 1$

at $z=i$; $3i-6-i = A(i-3)$

$A = 2$

$$\therefore f(z) = \frac{3z-6-i}{(z-i)(z-3)} = \frac{2}{z-i} + \frac{1}{z-3}$$

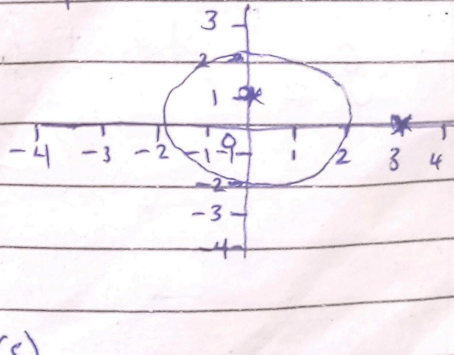
Recall: $\oint \frac{1}{z-a}\,dz = \begin{cases} 2\pi i, & \text{if the singular point is enclosed} \\ 0, & \text{If the singular point is not enclosed.} \end{cases}$

$$\Rightarrow \oint f(z)\,dz = \oint \frac{2}{z-i}\,dz + \oint \frac{1}{z-3}\,dz$$

$$= 2\oint \frac{1}{z-i}\,dz + 1\oint \frac{1}{z-3}\,dz$$

$$= 2(2\pi i) + 1(0)$$

$$= 4\pi i$$



$\oint \frac{1}{(z-a)^n}\,dz = \begin{cases} 0, & \text{whether or not the singular point(s) is/are enclosed.} \end{cases}$

Cauchy's Integral Formula

18/03/24 Example: $f(z) = \frac{5z-2-3i}{(z-i)(z-1)}$ (a) $|z-1| = 1$ (b) $|z| = 2$

Solution

$$\frac{5z-2-3i}{(z-i)(z-1)} = \frac{A}{z-i} + \frac{B}{z-1}$$

$$\frac{5z-2-3i}{(z-i)(z-1)} = \frac{A(z-1)+B(z-i)}{(z-i)(z-1)}$$

when $z=1$

$5-2-3i = 0 + B(1-i)$

$\Rightarrow B = 3$

when $z = i$

$5i-2-3i = A(i-1) + 0$

$\Rightarrow A = 2$

$$\frac{5z-2-3i}{(z-i)(z-1)} = \frac{2}{z-i} + \frac{3}{z-1}$$

$$\oint \frac{5z-2-3i}{(z-i)(z-1)}dz = \oint \frac{2}{z-i}dz + \oint \frac{3}{z-1}dz$$

(a) $|z-1| = 1$

$$\oint \frac{2}{z-i}dz + \oint \frac{3}{z-1}dz = 0 + 3(2\pi i)$$
$$= \underline{6\pi i}$$

(b) $|z| = 2$

$$\oint \frac{2}{z-i}dz + \oint \frac{3}{z-1}dz = 2\pi i(2) + 2\pi i(3)$$
$$= 4\pi i + 6\pi i$$
$$= \underline{10\pi i}$$

Cauchy's Integral formula

It is derived from cauchy's theorem and used in evaluating complex integrals. It can be used to show that any analytic function has derivative of all others and that analytic functions can be expressed in taylor's series. Cauchy's integral formula is given by

$$\oint \frac{F(z)}{z-z_0} dz = 2\pi i f(z_0) \quad --- \quad (11)$$

NOTE: The singular point must be enclosed.

Ex 1: $\oint \frac{z+2}{z-2} dz$, $c: |z\cancel{+1}| = 2$

Since the singular point is enclosed; Cauchy's integral formula can be used.

$\oint \frac{z+2}{z-2} dz$, $z_0 = 2$, $f(z_0) = 2+2 = 4$

$= 2\pi i (4) = 8\pi i$

(2) $\oint \frac{e^z}{ze^z - 2iz} dz$ $\quad c: |z| = 0.6$

$\oint \frac{e^z}{z(e^z-2i)} dz$, $F(z) = \frac{e^z}{e^z-2i}$, $z=0$ (Singular point).

Since the singular point is enclosed; CIF can be used.

$$\oint \frac{e^z}{z(e^z-2i)} dz = 2\pi i \left(\frac{e^z}{e^z-2i}\right)_{z=0}, \quad e^0 = 1$$

$$= \frac{2\pi i}{1-2i} \quad \text{(Rationalize)}$$

$$= \frac{2\pi}{5}(-2+i)$$

Equation (11) can be re-arranged to get

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0} dz \quad --- \quad (12)$$

$$f'(z_0) = \frac{1!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2} dz \quad --- \quad (13)$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^3} dz \quad ---- \quad (14)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad --- \quad (15)$$

(3) (a) $\oint \frac{\sin 2z}{z^4}\,dz$ (b) $\oint \frac{e^{-z}\sin z}{z^2}\,dz$, the contour is unity.

Solution.

(a) $f'''(z_0) = \frac{3!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^4}\,dz$

$f'''(z_0) = \frac{3!}{2\pi i}\oint \frac{\sin 2z}{z^4}\,dz = \cancel{\frac{3!}{2\pi i} f'''(z_0)}$

$f(z_0) = \sin 2z$, $f'(z_0) = 2\cos 2z$, $f''(z_0) = -4\sin 2z$

$f'''(z_0) = -8\cos 2z$

$\Rightarrow \oint \frac{\sin 2z}{z^4} = \frac{2\pi i}{3!} f'''(z_0)$

$= \frac{2\pi i}{3\times2\times1} \times (-8\cos 2z)_{z=0}$

$= \frac{2\pi i}{3\times2\times1} \times -8\cos 0$

$= \frac{2\pi i}{3\times2\times1} \times -8 = \frac{-8\pi i}{3}$

(b) $f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}\,dz$

$\oint \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0)$

$f(z_0) = e^{-z}\sin z$, $f'(z_0) = -e^{-z}\sin z + e^{-z}\cos z$

$\oint \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i \times (-e^{-z}\sin z + e^{-z}\cos z)_{z=0}$

$= 2\pi i \times (-e^{-0}\sin 0 + e^{-0}\cos 0)$

$= 2\pi i(-0+1) = 2\pi i(1) = 2\pi i$

Maclaurin Series

The maclaurin series expansion of any complex function is an infinite series expansion of the function about the point $z=0$. If $f(z)$ is a complex analytic function at $z=0$ in the complex plane, its maclaurin series expansion is given by:

$$f(z) = f(0) + zf'(0) + \frac{z^2 f''(0)}{2!} + \frac{z^3 f'''(0)}{3!} + \dots + \frac{z^n f^n(0)}{n!} \quad \text{---} (1)$$

Ex: $f(z) = \cos z$

$f(0) = \cos 0 = 1 \quad f'(0) = -\sin 0 = 0 \quad f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0 \quad f^{iv}(0) = \cos 0 = 1$

$$\cos z = 1 + z(0) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(0) + \frac{z^4}{4!}(1) + \dots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} + \dots \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \dots \frac{(-1)^{2n+1} z^{2n+1}}{(2n+1)!}$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \dots + \frac{(-1)^{n+1} z^n}{n}$$

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots + \frac{z^n}{n!}$$

NOTE: Power series helps to get rid of singular point.

$1/1-z = 1 + z + z^2 + z^3 + \dots + z^n$

$1/1+z = 1 - z + z^2 - z^3 + \dots + (-1)^n z^n$

Tutorial Questions:

1. At what point(s) (if any) will these equations have singular point
(a) $f(z) = z^2$ (b) $f(z) = \bar{z}z$, (c) $f(z) = 4/z-5$ (d) $f(z) = 1/(z-4)(z-1)$
2. Verify Cauchy's theorem for the closed path C consisting of three straight lines joining $A(1+i)$, $B(3+3i)$, $C(-1+3i)$ where $f(z) = z+1+i$
3. Evaluate $\oint_C f(z)\,dz$ where $f(z) = 5z-2-3i/(z-i)(z-1)$ round the contour C of radius (a) $|z| = 2$ (b) $|z-1| = 1$
4. If $f(z) = 5z+i/(z+i)(z+2i)$, evaluate $\oint_C f(z)\,dz$ along the contour

(a) $|z-1|=1$ (b) $|z| = 3/2$ (c) $|z|=3$

(5) Evaluate the following functions over a unit circle :
(a) $f(z) = \frac{2z+3i}{z^2+\frac{1}{4}}$ (b) $f(z) = \frac{z+1}{z^2+2z}$ (c) $f(z) = \frac{1}{5z-1}$

(6) Integrate the following integral around the unit circle
(a) $\oint_c \frac{z^6}{(2z-1)^6} dz$ (b) $\oint_c \frac{\sinh 2z}{(z-\frac{1}{2})^4} dz$ (c) $\oint_c \frac{e^{-z}\cos z}{(z-\frac{\pi}{4})^3} dz$
(d) $\oint_c \frac{e^{-z}\sin z}{z} dz$

(9) Integrate $\oint_c \frac{z^3+\sin z}{(z-i)^3} dz$ around a contour C given by the boundary of a square with vertices $\pm 2, \pm 2i$.

Complex Analysis (Analytic Function) Assignment

1. Find the real and imaginary parts of $f(z)$. Also find their values and W at the given z points
(a) $f(z) = 5z^2-12z+3+2i$ at $4-3i$
(b) $f(z) = 3z^2-6z+3i$ at $z=2+i$
(c) $f(z) = z/z+1$ at $4-5i$
(d) $f(z) = 1/1-z$ at $\frac{1}{2}+\frac{1}{4}i$
(e) $f(z) = 1/z^2$ at $z=1-i$
2. Find $|e^z|$
3. Using the First principle show if the following functions are analytic
(a) $2y+ix$ (b) $(z^2+1)^2$
4. Are the following functions holomorphic?
(a) $f(z) = e^x(\cos y - i\sin y)$
(b) $f(z) = \sin x \cosh y + i\cos x \sinh y$
5. Using C-R equation, derive the Laplace equation given in equation.
6. Are these functions harmonic? If yes, find their corresponding conjugates of the analytic function (a) $v=xy$ (b) $u=\sin x\cosh y$ (c) $u=x^3-3xy^2$
(d) $v=e^{-x}\sin 2y$.

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Partial Differential Equation.

A PDE is one involving one or more partial derivatives of an unknown function F of two or more independent variables $(x, y, t, z, \ldots)$. These variables may be time t and one or several coordinates in space e.g $F(x, y, z, t, u, \partial u/\partial x, \partial u/\partial y, \partial u/\partial z, \ldots)$.

Examples are:

1. 1-Dimensional wave equation $\partial u/\partial t = c^2 \partial u/\partial x$
2. 1-Dimensional heat equation $\partial u/\partial t = c^2 \partial^2 u/\partial x^2$
3. Laplace equation $\Delta^2 \phi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0$

The order of the equation is the order of the highest derivative, e.g, the 1-Dimensional heat equation is of the 2nd order. A PDE is linear if F is a linear function of u and its derivatives, e.g,

$$\left.\begin{aligned} &\partial u/\partial x\,(x,y) = 0 \\ &\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x \end{aligned}\right\} \text{Linear}$$

$$\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)^2 = 0 \text{ and } \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u^2 = 0 \text{ [non-linear]}$$

If each term of such an equation contains either the dependent variable or one of its derivatives, then the equation is said to be homogeneous, otherwise, it is said to be non-homogeneous. A solution of a PDE is any function that satisfies the equation identically. A general solution is a solution that contains a number of arbitrary independent functions equal to the order of the equation. In general, the totality of solutions of a PDE is very large. For example, the functions $u = x^2 - y^2$, $u = e^x \cos y$ and $u = \ln(x^2 + y^2)$ which are entirely different from each other are solutions of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$u = x^2 - y^2$$

$$\partial u/\partial x = 2x, \quad \partial^2 u/\partial x^2 = 2, \quad \partial u/\partial y = -2y \quad \partial^2 u/\partial y^2 = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

The unique solution of a PDE corresponding to a given physical problem will be obtained by use of additional information arising from the physical situation. For example, in some cases the values

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The unique solution of a PDE corresponding to a given physical problem will be obtained by use of additional information arising from the physical situation. For example, in some cases the values

The motion of the string is dependent on the initial deflection (deflection at $t=0$) and on the initial velocity (velocity at $t=0$). Denoting the initial deflection by $f(x)$ and and initial velocity by $g(x)$ we thus obtain the two initial deflections $u(x,0)=f(x)$ --- (3) and $\partial u/\partial x|_{t=0} = g(x)$ --- (4). To find the solution of eqn (1) that will satisfy the conditions (2) to (4), we take the following steps:

Step 1: Apply the method of separation of variables to obtain two ODEs

Step 2: Determine solutions of those equations that satisfy the boundary conditions

Step 3: Compose those solutions so that the result will be a solution of the wave equation (1) at the same time satisfying the given initial conditions.

Solution:

Step 1: Seperating variables: The methods of separation of variables also called product method gives solutions of eqn (1) in the form $u(x,t) = F(x)G(t)$ —— (5). Eqn (5) is a product of two functions, each depending only on one of the variables x and t. Differentiating eqn (5) gives $\frac{\partial u}{\partial x} = F'(x)G(t)$, $\partial^2 u/\partial x^2 = F''(x)G(t)$, $\partial u/\partial t = F(x)\dot{G}(t)$ $\partial^2 u/\partial t^2 = F(x)\ddot{G}(t)$ —— (6). Inserting eqn (6) into eqn (1) gives $F\ddot{G} = c^2 F'' G$.

Dividing by $c^2 FG$ gives:

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}$$

Since the expression on the involves function depending on t only while that on the right involves function that depends on x only, both must be equal to a constant k, i.e,

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k, \text{ where } k \text{ is arbitrary}$$

This gives rise to the ordinary linear differential equations:

$$\ddot{G} - c^2 G k = 0 \quad --- (7)$$

$$F'' - Fk = 0 \quad --- (8)$$

Step 2: Determining solutions of these equations: The next step is to determine solution G and F of eqn (7) and (8) so that $U = FG$ satisfies eqn (2), i.e,

$$U(0,t) = F(0)G(t) = 0 \quad U(L,t) = F(L)G(t) \text{ for}$$

all t. If G is identically equals to 0, then $U \equiv 0$. Since this is of no interest, we consider $G \not\equiv 0$, this results in:

(a) $F(0) = 0$ and (b) $F(L) = 0$ --- (9)

For $k = 0$, eqn (8) becomes $F'' = 0$ whose general solution is $F = ax + b$ and applying eqn (9) we obtain $a = 0$ and $b = 0$, thus $F \equiv 0$ which gives $U \equiv 0$, again this is of no interest.

For k positive e.g $k = \mu^2$, the general solution of (8):

$$F = Ae^{\mu x} + Be^{-\mu x} \text{ and from eqn (9) } A + B = 0$$

$$Ae^{\mu L} + Be^{-\mu L} = 0, \quad A = -B$$

$$\Rightarrow -Be^{\mu L} + Be^{-\mu L} = 0$$

$$B(e^{-\mu L} - e^{\mu L}) = 0, \quad B = 0, \ A = 0$$

$$\Rightarrow F = 0$$

The only possibility we are left with is to choose k is negative, $k = -\alpha^2$ so that eqn (8) becomes $F'' + \alpha^2 F = 0$ whose general solution is $F(x) = A\cos(\alpha x) + B\sin(\alpha x)$ —— (10)

Applying eqn (9) $F(0) = A\cos 0 + B\sin 0 \Rightarrow F(0) = A = 0$

$$F(L) = A\cos(\alpha L) + B\sin(\alpha L)$$

$$F(L) = B\sin(\alpha L) = 0$$

To avoid $F \equiv 0$, we must take $B \neq 0$, therefore:

$$\sin(\alpha L) = 0$$

$$\Rightarrow \alpha L = n\pi, \quad \alpha = \frac{n\pi}{L} \quad —— (11)$$

Setting $B = 1$, we obtain infinitely many solutions $F(x) = F_n(x)$

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \dots \quad (12)$$

which satisfy eqn (9) we now restrict the value of $k = -\alpha^2 = -\left(\frac{n\pi}{L}\right)^2$

For this k eqn (7) takes the form: $\ddot{G} + c^2\left(\frac{n\pi}{L}\right)^2 G = 0$

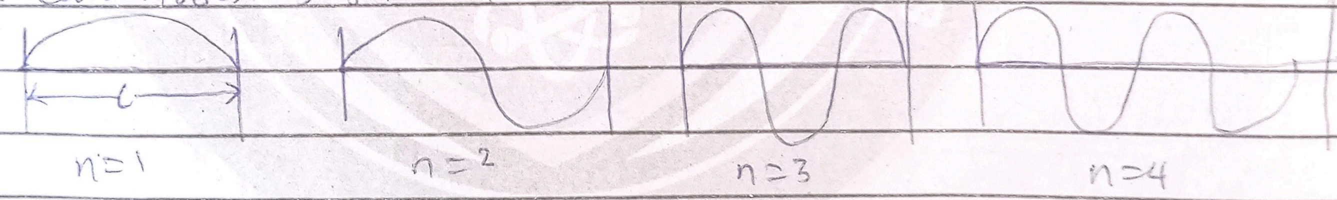
$\ddot{G} + \lambda_n^2 G = 0$, where $\lambda_n = \frac{cn\pi}{L}$ whose general solution is $G_n(t) = B_n \cos\lambda_n t + D_n \sin\lambda_n t$ — (13)

therefore, the function $U_n(x,t) = F_n(x) G_n(t)$

$$U_n(x,t) = (B_n\cos\lambda_n t + D_n \sin\lambda_n t)\sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, \dots \quad (14)$$

Eqn (14) are solutions to eqn (1) and satisfying the boundary conditions in eqn (2). These functions are called eigen or characteristic function and the values $\lambda_n = \frac{cn\pi}{\lambda}$ are called the eigen or characteristic values of the vibrating strings.

03/04/24 Each U_n represents a harmonic motion having the frequency $\lambda^1/2\pi = cn/2L$. The motion is called the end mode of the string. The first normal mode is called fundamental mode and the others are known as overtones. In eqn (14) $\sin\frac{n\pi}{L}x = 0$ at $x = L/n, 2L/n, 3L/n, \dots \frac{n-1}{n}L$. The end normal mode has $n-1$ so call nodes as shown below



Step 3: Composing Solutions: It is clear that a single solution of $U_n(x,t)$ will not satisfy the initial conditions of (3) and (4). Since eqn (1) is a linear and homogeneous equation, the sum of finitely many solutions of $U = U_1 + U_2 + U_3 + \dots + U_n$ is a solution. Therefore to obtain a solution that satisfies eqn (3) and (4), we consider the infinite series $U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty}(B_n\cos\lambda_n t + D_n\sin\lambda_n t)\sin\frac{n\pi x}{L}$ (15)

It follows from this and eqn (3) that

$$U(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x = f(x) \quad (16)$$

Also at $t = 0$ $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ for $0 \le x \le L$

Therefore for eqn (15) to satisfy (3) we must choose the coeffecient B_n so that $U(x,0)$ becomes a half-range expansion of $f(x)$ which is the fourier sin series of $f(x)$, i.e,

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \quad n = 1, 2, 3, 4 \quad \text{——— (17)}$$

By differentiating (15) wrt t and using (4) we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\left(-B_n \lambda_n \sin \lambda_n t + D_n \lambda_n \cos \lambda_n t\right) \sin \frac{n\pi}{L} x\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} D_n \lambda_n \sin \frac{n\pi}{L} x = g(x)$$

Hence for eqn (15) to satisfy (4), D_n must be chosen so that for $t=0$ $\partial u / \partial t$ becomes the fourier sin series of $g(x)$

Using $B_n = \frac{2}{L}\int_0^L f(L) \sin \frac{n\pi}{L} t\, dt$

$$D_n \lambda_n = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

Using $\lambda_n = {cn\pi}/{L}$

$$D_n = {2}/{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx \quad \text{——— (18)}$$

This shows that $u(x,t)$ given by eqn (15) with coeffecient (17) and (18) is a solution of eqn (1) that satisfy the conditions (2) to (4).

When the initial velocity $g(x) \equiv 0$, D_n are 0 and eqn (15) becomes

$$U(x,t) = \sum_{n=1}^{\infty} B_n \lambda_n t \sin\left(\frac{n\pi}{L} x\right) \qquad \lambda_n = \frac{cn\pi}{L} \quad \text{——— (19)}$$

Since $\cos \frac{cn\pi}{L} + \sin \frac{n\pi}{L} x = \frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x - ct)\right\} + \sin\left\{\frac{n\pi}{L}(x + ct)\right\}\right]$

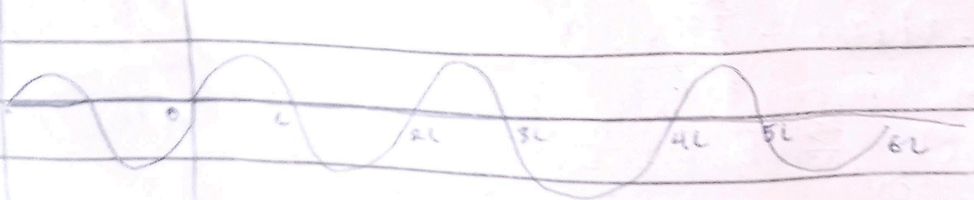
(19) may be written in the form

$$U(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x - ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \left\{\frac{n\pi}{L}(x + ct)\right\}$$

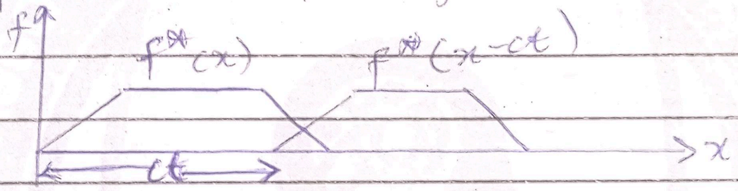
This is obtained by substituting $x - ct$ and $x + ct$ respectively for x in the fourier sin series (16) for $f(x)$. Therefore,

$$u(x,t) = \frac{1}{2}\left[f^*(x - ct) + f^*(x + ct)\right] \quad \text{——— (20)}$$

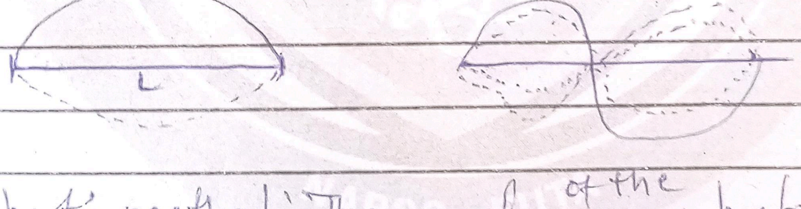
Where f^* is the odd periodic extension of f with period $2L$ as shown below



And the initial deflection $f(x)$ is continuous on the interval $0 \le x \le L$ and 0 at the end points. From eqn (20), it follows that $u(x,t)$ is a continuous function of both x and t for all their values by differentiating eqn (20), we see that $u(x,t)$ is a function of eqn (1). So long $f(x)$ is twice differentiable on the interval $0 \le x \le L$, and has one sided second derivative at $x=0$ and $x=L$ which are zero. With these conditions, $u(x,t)$ is established as a solution of eqn (1) satisfying conditions (2) to (4) — A physical interpretation of eqn (20) is that the graph of $f^*(x-ct)$ is obtained from the graph of $f^*(x)$ by shifting the graph of ct unit to the right as shown below:



This means that $f^*(x-ct)$ with $c>0$ represents a wave which is travelling to the right as t increases. In same way $f^*(x+ct)$ represents a wave which is travelling to the left and $u(x,t)$ is the superposition



(3) D'Alembert's method: The equation of the vibrating string $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ — (21) is also called D'Alembert's equation. We can transform this equation by introducing independent variables $v = x + ct$ and $w = x - ct$ — (22)

So that $\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 1 \quad \frac{\partial v}{\partial t} = c \quad \frac{\partial w}{\partial t} = -c$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial v} + \frac{\partial}{\partial w}$$

$$\therefore \partial^2 u/\partial x^2 = \partial/\partial x\left[\partial u/\partial x\right] = \left(\partial/\partial v + \partial/\partial w\right)\left(\partial u/\partial v + \partial u/\partial w\right)$$

$$= \partial^2 u/\partial v^2 + 2\,\partial^2 u/\partial v \partial w + \partial^2 u/\partial w^2 \quad \text{— (23)}$$

Similarly $\partial u/\partial t = \left(\partial u/\partial v \; \partial v/\partial t\right) + \frac{\partial u}{\partial w}\frac{\partial w}{\partial t} = c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$

$$\partial/\partial t = \partial/\partial v + \partial/\partial w \Rightarrow -\partial/\partial t = c\left(\partial/\partial v - \partial/\partial w\right)$$

$\therefore \partial^2u/\partial t^2 = \partial/\partial t(\partial u/\partial t) = c^2\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right)\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$

$= c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\partial^2u/\partial v\partial w + \partial^2 u/\partial w^2\right)$ —— (24)

By using eqn (23) and (24) in (21) we obtain

$c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\partial^2 u/\partial v\partial w + \partial^2 u/\partial w^2\right) = c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v\partial w} + \frac{\partial^2 u}{\partial w^2}\right)$

yielding $2c^2\partial^2 u/\partial v\partial w = 0$

$\Rightarrow \partial^2 u/\partial v\partial w = 0$

Integration wrt w yields:

$\partial u/\partial v = \text{constant} = f(w)$

the integration wrt v yields

$u = \int f(v)dv + \text{constant}$

$= \phi(v) + \gamma(w)$

i.e, $u(x,t) = \phi(x+ct) + \gamma(x-ct)$ —— (25)

Eqn (25) represents two plane waves travelling in opposite directions with the same period. By writing $\partial/\partial x = D_x$ and $\partial^2/\partial x^2 = D_x^2$ eqn (21) becomes: $\partial^2u/\partial t^2 = (cD_x)^2 u$

If we now treat eqn (25) as an ODE with constant coefficient

$\partial^2 u/\partial t^2 = \alpha^2 u$ —— (26)

where $\alpha = D_x$ —— (27), we obtain

$u = Ae^{\alpha t} + Be^{-\alpha t}$ —— (28), where A and B are arbitrary constants. Eqn (26) is satisfied by

$u = e^{cD_x t}\phi(x) + e^{-cD_x t}\gamma(x)$ —— (29), the arbitrary constants A and B can be replaced by arbitrary functions $\phi(x)$ and $\gamma(x)$. Since eqn (26) has been integrated wrt t

By writing $D_x^n = d^n/dx^n$, $n = 1, 2, 3, 4, \cdots$ so that $D_x^5 f(x) = \frac{d^5}{dx^5}f(x)$

the Taylor series $f(x+z) = f(x) + \frac{z}{1!}f'(x) + \frac{z^2}{2!}f''(x) + \frac{z^3}{3!}f'''(x) + \cdots + \frac{z^n}{n!}f^n(x)$

$= f(x)\left[1 + \frac{z}{1!}D_x + \frac{z^2}{2!}D_x^2 + \frac{z^3}{3!}D_x^3 + \frac{z^4}{4!}D_x^4 + \cdots + \frac{z^n}{n!}D_x^n\right]$

~~By using~~ $= e^{zD_x}f(x)$

By using eqn (30) in (21) replacing z by ct and $-ct$ respectively we obtain; $u(x,t) = \phi(x+ct) + \delta(x-ct)$ which is same as (23)

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Taylor Series

A complex function which is analytic on and within a single closed-curve can be expanded using taylor series given by eqn (2)

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + (z-z_0)^2 f''(z_0)/2! + (z-z_0)^3 f'''(z_0)/3! + \cdots + (z-z_0)^n f^n(z_0)/n! + \cdots \quad (2)$$

where z_0 is the center of circle of convergence with radius r given by $|z-z_0|$. Maclaurin series is a special case of Taylor series for $z_0 = 0$

Ex (1) $f(z) = e^z$ at $z_0 = \pi i$

$f(z_0) = e^{\pi i} = \cos\pi + i\sin\pi = -1$

$f'(z) = e^z \quad f'(z_0) = -1$

$f''(z) = e^z, \quad f''(z_0) = -1$

Taylor series for $f(z) = e^z$

$$= -1 - (z-\pi i) - (z-\pi i)^2/2! - (z-\pi i)^3/3! - (z-\pi i)^4/4! + \cdots$$

(ii) $f(z) = \cos z$ about $z_0 = \pi/6$. Solve using maclaurin series to get taylor series

Transforming the coordinate! $u = z - \pi/6$, $\Rightarrow z = u + \pi/6$

$\therefore f(z) = \cos z = \cos(u + \pi/6)$

$$= \cos u \cos \pi/6 - \sin u \sin \pi/6$$

$$= \frac{\sqrt{3}}{2}\cos u - \frac{1}{2}\sin u \quad \left[\cos\frac{\pi}{6} = \sqrt{3}/2, \ \sin\pi/6 = \frac{1}{2}\right]$$

$$= \frac{\sqrt{3}}{2}\left[1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots\right] - \frac{1}{2}\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots\right]$$

$$= \frac{1}{2}\left[\sqrt{3} - \left(z - \frac{\pi}{6}\right) - \sqrt{3}\left(z - \pi/6\right)^2/2! + \left(z - \frac{\pi}{6}\right)^3/3! + \sqrt{3}\left(z - \frac{\pi}{6}\right)^4/4! - \left(z - \frac{\pi}{6}\right)^5/5! + \cdots\right]$$

Laurent's Series

The series expansion of the complex function within a region having singular point(s) can be known using the Laurent's series:

The Laurent's series is a generalized taylor series. If that taylor series has just the positive integer power of $z-z_0$ and converges on a disk given by $|z-z_0|<R$. Laurent series has both positive and negative integer powers of $z-z_0$ and converges at an annular given by $R_1<|z-z_0|<R_2$. Laurent series is also used to classify singularities and in ~~red~~ residue integration. The Laurent series of a function that is analytic within two concentric circles C_1 and C_2 with center z_0 is given by:

$$f(z) = \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots \quad \text{—— (3)}$$

$$a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad \text{--- (3a)}$$

$$b_n = \frac{1}{2\pi i}\oint (z-z_0)^{n-1} f(z)\,dz \quad \text{--- (3b)}$$

The part of LS with the $-ve$ power of $(z-z_0)$ is called the principal part while the remaining terms make up the analytic or regular part of the series. b_1 is known as residue of $f(z)$ at $z=z_0$

Ex: (1) find the Laurent Series of $f(z) = \sin z/(z-\frac{\pi}{4})^3$ at $z=\pi/4$

Solution: Transforming the coordinate: $u = z-\frac{\pi}{4} \Rightarrow z = u+\frac{\pi}{4}$

$$\therefore f(z) = \frac{1}{u^3}\sin\left(u+\frac{\pi}{4}\right) = \frac{1}{u^3}\left(\sin u\cos\frac{\pi}{4} + \cos u\sin\frac{\pi}{4}\right)$$

$$= \frac{1}{u^3}\frac{1}{\sqrt{2}}(\sin u + \cos u)$$

$$f(z) = \frac{1}{u^3\sqrt{2}}\left[\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right) + \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots\right)\right]$$

$$= \frac{1}{\sqrt{2}u^3}\left[1 + u - \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} - \cdots\right]$$

$$= 1/\sqrt{2}\left[1/u^3 + 1/u^2 - 1/u2! - 1/3! + u/4! + u^2/5!\right]$$

$$= 1/\sqrt{2}\left[1/(z-\tfrac{\pi}{4})^3 + 1/(z-\tfrac{\pi}{4})^2 - 1/2!(z-\tfrac{\pi}{4}) - 1/3! + (z-\pi/4)/4! + (z-\pi/4)^2/5!\right]$$

Singularities and Zero

Points where a function ceases to be analytic is called a singular point(s). Classification of singularities and zeros of a function can be done using Laurent and taylor series respectively. Their are different types of singular point.

(1) Pole(s): If a function $f(z)$ has a singular point at z_0 and the laurent series of $f(z)$ has finite number of the principal part, then the singular point is called a pole of order n. Poles of first order are called single poles, poles of second order are called double poles.

E.g: $f(z) = \frac{e^z}{z^3} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + z^3/3! + z^4/4! + \cdots\right)$

$= 1/z^3 + 1/z^2 + 1/z2! + 1/3! + z/4! + \cdots$

It has a pole of order 3.

(2) Removable Singularity: e.g $f(z) = \sin z/z = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$

$= 1 - z^2/3! + z^4/5! + \cdots$

This is a singularity that disappears upon expansion of the function.

(3) Isolated Singularity

(4) Essential Singularity

A zero of an analytic function is at a complex point $z = z_0$ if $f(z_0) = 0$. A zero $f(z)$ is of order n if $f, f', f'', \cdots f^{n-1}$ are all zero at $z = z_0$ but $f^{n+1} \neq 0$. A zero of order 1 is called a simple zero.

E.g: $f(z) = z^2 + 1$ at $z_0 = \pm i$.

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Ex 1: Use separation of variables ~~to~~ on the following PDEs

(i) $\partial u/\partial t = k\,\partial^2 u/\partial x^2$, $u(x,0) = f(x)$ $u(0,t) = 0$ $u(L,t) = 0$

Solution:

$u(x,t) = F(x)G(t)$

$$\frac{\partial(F(x)G(t))}{\partial t} = k\frac{\partial^2}{\partial x^2}(F(x)G(t))$$

$$F(x)\frac{\partial G(t)}{\partial t} = k\,G(t)\frac{\partial^2}{\partial x^2}(F(x))$$

$$F\frac{\partial G}{\partial t} = kG\frac{\partial^2}{\partial x^2}F$$

Dividing both sides by FG, we obtain

$\frac{1}{G}\partial G/\partial t = k/F\,\partial^2 F/\partial x^2 = -\lambda$

$\frac{1}{G}\partial G/\partial t = -\lambda \implies \partial G/\partial t = -G\lambda$

$k/F\,\partial^2 F/\partial x^2 = -\lambda \implies k\partial^2 F/\partial x^2 = -F\lambda$

$\partial G/\partial t + G\lambda = 0$ $\quad k\,d^2F/dx^2 + F\lambda = 0$

(ii) $\partial u/\partial t = k\,\partial^2 u/\partial x^2$, $u(x,0) = f(x)$ $\partial u/\partial x(0,t) = 0$ $\partial u/\partial t(L,t) = 0$

$u(x,t) = F(x)G(t)$

$$\frac{\partial(F(x)G(t))}{\partial t} = k\frac{\partial^2}{\partial x^2}(F(x)G(t))$$

$$F(x)\frac{\partial}{\partial t}G(t) = k\,G(t)\frac{\partial^2}{\partial x^2}F(x)$$

$$\frac{F\partial G}{\partial t} = kG\frac{\partial^2 F}{\partial x^2}$$

Dividing both sides by FG, we obtain

$\frac{1}{G}\partial G/\partial t = k/F\,\partial^2 F/\partial x^2 = -\lambda$

$\frac{1}{G}\partial G/\partial t = -\lambda \implies \partial G/\partial t = -G\lambda$

$k/F\,\partial^2 F/\partial x^2 = -\lambda \implies k\partial^2 F/\partial x^2 = -F\lambda$

$\frac{dG}{dt} + G\lambda = 0$ $\quad d^2F/dx^2 + \frac{F\lambda}{k} = 0$

$\partial u/\partial x(0) = 0$ $\quad \frac{\partial u}{\partial t}(L) = 0$ $\quad \frac{\partial u}{\partial t}(0) = 0$, $\frac{\partial u}{\partial x}(t) = 0$

(iii) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

using $u(x,t) = f(x)G(t)$

$$\frac{\partial^2 (F(x)G(t))}{\partial t^2} = c^2 \frac{\partial^2 (F(x)G(t))}{\partial x^2}$$

$$F(x)\frac{\partial^2 G(t)}{\partial t^2} = c^2 G(t)\frac{\partial^2 F(x)}{\partial x^2}$$

Dividing both sides by $F(x)G(t)$

$$\frac{1}{G}\frac{\partial^2 G}{\partial t^2} = \frac{c^2}{f}\frac{\partial^2 f}{\partial x^2} = \lambda$$

$\frac{1}{G}\partial^2 G/\partial t^2 = \lambda$ $\quad\quad$ $\frac{c^2}{f}\partial^2 f/\partial x^2 = \lambda$

$\partial^2 G/\partial t^2 = G\lambda$ $\quad\quad$ $\partial^2 f/\partial x^2 = f\lambda/c^2$

$d^2G/dt^2 - G\lambda = 0$ $\quad\quad$ $\frac{d^2 f}{dx^2} - \frac{f\lambda}{c^2} = 0$

24/04/24 Laplace's Equation

Laplace's equation is one of the most important equations in physics. ~~So it is an equation~~ It corresponds to finding equilibrium solution, i.e, time-independent solution if their were no sources. So it is an equation that can arise from physical situations such as the distribution of a ~~f~~ field e.g potential, temperature, charge, etc over a plane subject to some boundary conditions. The theory of the solution of Laplace's equation is called potential theory. How to solve it will depend on the geometry of the object it is been solved on. We will begin by solving it on the rectangle given by $0 \le x \le L$, $0 \le y \le m$. The potential at a point at any point in ~~a~~ plane is a function of its position $z(x,y) = u(x,y)$, where $u(x,y)$ is the solution of the 2D Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ---- (L1)

We can determine the solution of (L1) for the rectangle bounded by lines $0 \le x \le L$ and $0 \le y \le m$ subject to the following boundary conditions: $u = 0$ when $x = 0$ $\quad 0 \le y \le m$

$u = 0$ when $x = L$ $\quad 0 \le y \le m$

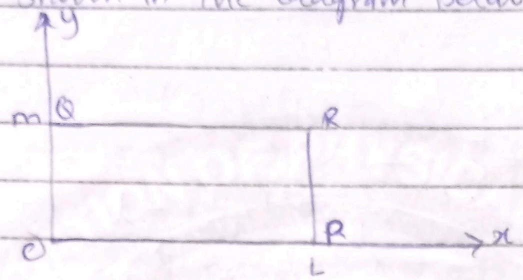
$u = 0$ when $y = m$. $0 \le x \le L$

$u = f(x)$ when $y = 0$ $0 \le x \le L$

$u(0,y) = 0$ and $u(L,y) = 0$ for $0 \le y \le m$

$u(x,m) = 0$ and $u(x,0) = f(x)$ for $0 \le x \le L$

$z = u(x,y)$ will give the potential at any point within the rectangle OPRQ shown in the diagram below.



Solution of L_1: We assume $u(x,y) = X(x)Y(y)$ where X is a function of x only and Y is a function of y only. Expressing the equation in terms of X and Y and seperating the variables:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 (X(x)Y(y))}{\partial x^2} + \frac{\partial^2 (X(x)Y(y))}{\partial y^2} = 0$$

$$\frac{\partial^2 (X(x)Y(y))}{\partial x^2} = -\frac{\partial^2 (X(x)Y(y))}{\partial y^2}$$

$$\frac{Y(y)\partial^2 (X(x))}{\partial x^2} = -\frac{X(x)\partial^2 Y(y)}{\partial y^2}$$

Divide both sides by $X(x)Y(y)$

$$\frac{1}{X(x)}\frac{\partial^2 (X(x))}{\partial x^2} = -\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

equating to a constant $(-\lambda^2)$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

$$\frac{X''}{X} = -\lambda^2, \quad -\frac{Y''}{Y} = -\lambda^2$$

$\Rightarrow X'' + X\lambda^2 = 0 \qquad Y'' - Y\lambda^2 = 0$

$X = A\cos\lambda x + B\sin\lambda x$

$Y = Ce^{\lambda y} + De^{-\lambda y}$

$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$

$= (A\cos\lambda x + B\sin\lambda x)(\sinh\lambda(y+\phi) \cdot E$

$= (AE\cos\lambda x + BE\sin\lambda x)\sinh\lambda(y+\phi)$

$P = AE$ and $Q = BE$

$u(x,y) = (P\cos\lambda x + Q\sin\lambda x)\sinh\lambda(y+\phi)$

when $u(0,y) = 0$, we have, $u(0,y) = P\sinh\lambda(y+\phi) = 0 \Rightarrow P = 0$

$u(x,y) = Q\sin\lambda x\sinh\lambda(y+\phi)$

Applying the second Boundary condition $u(L,y) = 0$ we have

$u(L,y) = Q\sin \lambda L \sinh\lambda(y+\phi) = 0$, $\sin\lambda L = 0$, $\lambda L = n\pi$, $\lambda = \frac{n\pi}{L}$

3rd BC, $u(x,m) = 0$, we have,

$u(x,m) = Q\sin\lambda x\sinh\lambda(m+\phi) = \sinh\lambda(m+\phi) = 0$

$\Rightarrow Q = -m \therefore u(x,y) = Q\sin\lambda x\sin\lambda(y-m)$

and since $\sinh\lambda(y-m) = -\sinh\lambda(m-y)$

we have $u(x,y) = Q\sin\lambda x\sinh\lambda(m-y)$

Since $\lambda = \frac{n\pi}{L}$, with $n = 1, 2, 3, \ldots$

There is an infinite number of solutions for $u(x,y)$ or $U = U_1 + U_2 + U_3 + U_4 + \cdots + U_\infty$ and so we can write

$u(x,y) = \sum_{n=1}^{\infty} Q_n \sin\lambda_n x \sinh\lambda_n(m-y)$

applying the 4th BC, $u(x,0) = f(x)$ we have

$u(x,y) = f(x) = \sum_{n=1}^{\infty} Q_n\sin\lambda_n x\sinh\lambda_n m$

$Q_n\sinh\lambda_n m = 2 \times$ mean value of $f(x)\sinh\lambda_n x$ from $x=0$ to $x=L$

$= \frac{2}{L}\int_0^L f(x)\sin\lambda_n x\,dx$

$= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

From which the coefficient Q_n can be found

Ex: Solve the laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u = 0$ when $x = 0$, $u = 0$ when $x = \pi$, $u = 0$ when $y = \infty$, $u = 4$ when y

Solution:

Assume the $u(x,y) = X(x)Y(y)$

$\frac{X''}{X} - \frac{Y''}{Y} = -\lambda^2$

$X'' + X\lambda^2 = 0$ and $Y'' - \lambda^2 Y = 0 \Rightarrow Ce^{\lambda y} + De^{-\lambda y}$

$X = A\cos\lambda x + B\sin\lambda x$

$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$

(i) $u(0,y) = 0$, we have

$u(0,y) = A(Ce^{\lambda y} + De^{-\lambda y}) \Rightarrow A = 0$

~~(ii) $u(\pi,y) = 0$, we have~~

$u(x,y) = B\sin\lambda x(Ce^{\lambda y} + De^{-\lambda y})$

$= \sin\lambda x(BCe^{\lambda y} + BDe^{-\lambda y})$

let $P = BC$ and $Q = BD$

$u(x,y) = \sin\lambda x(Pe^{\lambda y} + Qe^{-\lambda y})$

(ii) $u(\pi,y) = \sin\pi\lambda(Pe^{\lambda y} + Qe^{-\lambda y}) = 0$

$\sin\pi\lambda = 0$, $\pi\lambda = n\pi \Rightarrow \lambda = n$, $n = 1, 2, 3, 4, \ldots$

$\therefore u(x,y) = \sin(nx)(Pe^{\lambda y} + Qe^{-\lambda y})$

(iii) $u(x,\infty) = 0$, we have

$u(x,y) = \sin nx(Qe^{-\lambda y}) = Qe^{-ny}\sin nx$

$U_1 = Q_1 e^{-y}\sin x$, $U_2 = Q_2 e^{-2y}\sin 2x$, $U_3 = Q_3 e^{-3y}\sin 3x$

$U_r = Q_r e^{-ry}\sin rx$

(iv) $u(x,0) = 4$, we have

$u(x,y) = 4 = \sum_{r=1}^{\infty} e^{-ry}\sin rx = 0$

$Q_r = 2 \times$ mean value of $4\sin rx$ between $x = 0$ and $x = \pi$

$= \frac{2}{\pi}\int_0^{\pi} 4\sin rx\,dx = \frac{8}{\pi}\left[\frac{-\cos rx}{r}\right]_0^{\pi}$

$= 8/r\pi\,[1 - \cos r\pi]$

$Q = 0$ (r is even)

$Q = 16/r\pi$ (r is odd)

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Residue Integration

Recall the LS given by (3). If (3) is integrated around a simply closed contour, we have;

$$\oint f(z)dz = \oint \left[\cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots \right] dz$$

$$= \cdots + 0 + 2\pi b_1 + 0 + 0 + \cdots$$

$$= 2\pi b_1$$

$$\therefore b_1 = \frac{1}{2\pi i} \oint f(z)dz \quad \text{—— (4)}$$

The residue theorem can be extended to a contour having more than one singular point. If $f(z)$ is analytic inside and on a contour C except for some singular point (z_a, z_b, z_c) then:

$$\oint f(z)dz = 2\pi i (b_{1a} + b_{1b} + b_{1c})$$

$$= 2\pi i (\text{sum of the residue inside of the contour } c)$$

Three ways to calculate the residue

- Produce the LS and obtain the residue as the coefficient of $\frac{1}{z-z_0}$
- If the singularity is a simple pole then we can get the residue

$$b_1 = \underset{z \to z_0}{\text{Res}}\, f(z) = \lim_{z \to z_0} (z-z_0) f(z) \quad \text{—— (5)}$$

$$b_1 = \underset{z \to z_0}{\text{Res}}\, f(z) = \underset{z \to z_0}{\text{Res}} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} \quad \text{—— (6)}$$

where $f(z) = p(z)/q(z)$

- If the singularity is a pole of m order then we get the residue as

$$b_1 = \underset{z \to z_0}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \{(z-z_0)^m f(z)\} \right] \quad \text{—— (7)}$$

Examples: Find the singularity point and the corresponding residue of the following function:

(a) $f(z) = 1/4+z^2$

(b) $f(z) = \tan z$

(c) $f(z) = \sin 2z / z^6$

(a) $f(z) = 1/4+z^2$ has singular points at $z^2 = -4$

$z = \pm 2i$

$$b_1(2i) = \underset{z \to z_0}{Res}\, f(z) = \lim_{z \to 2i} (z - 2i) \cdot \frac{1}{(z-2i)(z+2i)}$$

$$= \frac{1}{(z+2i)}_{z=2i} = \frac{1}{4i} = \frac{-i}{4}$$

b - $F(z) = \tan z$ has singularity points at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$ and residue of $\pm\frac{\pi}{2}$

$$b_1\tfrac{\pi}{2} = \underset{z \to +\frac{\pi}{2}}{Res}\, f(z) = \underset{z \to +\frac{\pi}{2}}{Res}\, \frac{\sin z}{\cos z} = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = -1$$

$$b_1 -\tfrac{\pi}{2} = \underset{z \to -\frac{\pi}{2}}{Res}\, f(z) = \underset{z \to -\frac{\pi}{2}}{Res}\, \frac{\sin z}{\cos z} = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)} = -1$$

(c) $F(z) = \sin 2z / z^6$ has singular point at $z = 0$ and is in order 6

$$b_1 = \underset{z \to 0}{Res}\, f(z) = \frac{1}{(6-1)!} \lim_{z \to 0} \left[\frac{d^{6-1}}{dz^{6-1}}\left(z^6 \cdot \frac{\sin 2z}{z^6}\right)\right]$$

$$= \frac{1}{5!} \lim_{z \to 0} \left[\frac{d^5}{dz^5}(\sin 2z)\right]$$

$$= \frac{1}{5!} \lim_{z \to 0} \left[\frac{d^4}{dz^4}(2\cos 2z)\right]$$

$$= \frac{1}{5!} \lim_{z \to 0} \left[\frac{d^3}{dz^3}(-4\sin 2z)\right]$$

$$= \frac{1}{5!} \lim_{z \to 0} \left[\frac{d^2}{dz^2}(-8\cos 2z)\right]$$

$$= \frac{1}{5!} \lim_{z \to 0} \left[\frac{d}{dz}(16\sin 2z)\right]$$

$$= \frac{1}{5!} \lim_{z \to 0} (32\cos 2z)$$

$b_1 = \frac{1}{5!} \cdot 32[\cos 2z]_{z=0}$

$= \frac{1}{5!} \times 32 \times 1 = \frac{4}{15}$ //

(d) $\boxed{f(z) = \frac{\sin 3z}{(z^2+4)^3}}$

Ex: Evaluate $f(z) = \frac{1}{4+z^2}$ around the contour (a) $|z| = 1$ (b) $|z| = 3$ (c) $|z-1-i| = 2$

Singular points: $\pm 2i$

(a) $\oint \frac{1}{4+z^2} dz = 0$. Since the singular point isn't enclosed in the contour.

(b) $\oint \frac{1}{4+z^2} dz \quad |z| = 3$

$= 2\pi i(b_{1,2i} + b_{1,-2i})$

$= 2\pi i\left[\frac{-i}{4} + \left(\frac{+i}{4}\right)\right] = 0$

(c) $\oint \frac{1}{4+z^2} dz \quad |z-1-i| = 2 \quad |z-(1+i)| = 2$

Singular point at $2i$ is enclosed and $-2i$ isn't enclosed.

$\Rightarrow \oint \frac{1}{4+z^2} dz = 2\pi i(b_{1,2i})$

$= 2\pi i\left(\frac{-i}{4}\right) = \pi/2$ //

Integration of Real integrals using residue integration.

$I = \int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$ —— (8)

To evaluate the integral of the form given by eqn (8) where f is a real rational function in $\cos\theta$ and $\sin\theta$ and finite on the integration interval. Z is set to a unit circle given by

$z = e^{i\theta}$. $dz = ie^{i\theta} d\theta = iz\,d\theta$

$d\theta = \frac{dz}{iz}$

While the interval of integration ~~cannot~~ changes from 0 to 2π to a contour integration, also the trig functions are described as:

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1})$

$\sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2}(z - z^{-1})$

Ex: Evaluate $I = \int_0^{2\pi} 1/7+6\cos\theta \, d\theta$

Transforming to contour integration:

$$I = \oint \frac{dz}{iz\left(7+6\left(\frac{z+z^{-1}}{2}\right)\right)}$$

$$I = \oint \frac{dz}{iz\left[7+3\left(z+\frac{1}{z}\right)\right]}$$

$$= -i\oint \frac{dz}{7z+3z^2+3} = -i\oint \frac{dz}{3z^2+7z+3}$$

Using the quadratic equation formula, the following roots were obtained for the quadratic equation in the denominator

$$z = \frac{-7}{6} \pm \frac{\sqrt{13}}{6}$$

If we analyze this roots, $z_1 = -\frac{7}{6} + \frac{\sqrt{13}}{6}$, $z_2 = -\frac{7}{6} - \frac{\sqrt{13}}{6}$

$z_1 = -0.56$, $z_2 = -1.17$

Because $z_1 = -0.56$ is enclosed in the unit circle:

$$\underset{z \to -\frac{7}{6}+\frac{\sqrt{13}}{6}}{\text{Res}} f(z) = -i\left[\underset{z \to -\frac{7}{6}+\frac{\sqrt{13}}{6}}{\text{Res}} \cdot \frac{1}{3z^2+7z+3}\right]$$

$$= -i\left[\frac{1}{6z+7}\right]_{z=-\frac{7}{6}+\frac{\sqrt{13}}{6}} = \frac{-i}{\sqrt{13}}$$

$$\int_0^{2\pi} \frac{1}{7+6\cos\theta}\, d\theta = 2\pi i\left(\frac{-i}{\sqrt{13}}\right) = \frac{2\pi}{\sqrt{13}} \,//$$

$$I = \int_{-\infty}^{\infty} f(x)\,dx \quad \text{——— (9)}$$

Integral of the form given by eqn (9) is called an improper integral because the interval of integration is not finite. Such equation is solved by assuming $\oint F(z)\,dz$ on a semi-circle contour enclosing only the singular point of $F(z)$ on the upper-half of the circle

Ex 2: $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \oint \frac{dz}{z^2+1}$, Singular points at $\pm i$

Using only the $+i$;

$$\underset{z\to i}{\text{Res}}\, f(z) = \underset{z\to i}{\text{Res}} \frac{1}{z^2+1} = \frac{1}{2z}\Big|_{z=i} = \frac{1}{2i} = \frac{-i}{2}$$

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i\left(\frac{-i}{2}\right) = \underline{\underline{\pi}}$$

02/05/24

Ex 1 Use the method of separation of variables to determine the general solution to the 1-D heat equation $\partial u/\partial t = \alpha\, \partial u/\partial x$ $0 \le x \le L$, $t >$ Subject to the initial condition $u(x,0) = g(x)$, hence compute the general solution when $g(x) = 0$

Solution

let $u(x,t) = X(x)\,T(t)$

$$\frac{\partial(X(x)T(t))}{\partial t} = \alpha \frac{\partial(X(x)T(t))}{\partial x}$$

$$X(x)\frac{\partial T(t)}{\partial t} = T(t)\,\alpha \frac{\partial X(x)}{\partial x}$$

Divide both sides by $X(x)T(t)$

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{\alpha}{X(x)}\frac{\partial X(x)}{\partial x}$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = \sigma, \quad \frac{\alpha}{X}\frac{dX}{dx} = \sigma, \quad \frac{1}{X}\frac{dX}{dx} = \sigma/\alpha$$

$$\frac{dT}{dt} = \sigma T$$

Integrating both sides

$$\int \frac{dT}{dt} = \int \sigma T$$

$$T = \sigma T t$$

2. By the methods of separation of variables, solve the boundary value problem $\partial u/\partial x = 4\,\partial u/\partial y$ $u(0,y) = 8e^{3y}$

Solution.

let $u(x,y) = X(x)\,Y(y)$

$$\frac{\partial(X(x)Y(y))}{\partial x} = 4\frac{\partial(X(x)Y(y))}{\partial y}$$

$$Y(y)\frac{\partial X(x)}{\partial x} = 4X(x)\frac{\partial Y(y)}{\partial y}$$

Divide both sides by $X(x)\,Y(y)$

$$\frac{1}{X(x)}\frac{\partial X(x)}{\partial x} = \frac{4}{Y(y)}\frac{\partial Y(y)}{\partial y}$$

$$\frac{1}{X}\frac{dX}{dx} = \frac{4\,dY}{Y\,dy} = \lambda$$

$\frac{1}{X}\, dX/dx = \lambda$ — ① $\quad \frac{4}{Y}\frac{dY}{dy} = \lambda$ — ②

Solution to ①; $X = C_1 e^{\lambda x}$

Solution to ②; $Y = C_2 e^{\lambda y/4}$

$U(x,y) = C_1 e^{\lambda x} C_2 e^{\lambda y/4} = C e^{\lambda x} e^{\lambda y/4} = C e^{\lambda(x + y/4)}$

Applying the boundary condition to eqn ③ we get

$u(0,y) = C e^{\lambda y/4} = 8 e^{3y}$

$$C = \frac{8e^{3y}}{e^{\lambda y/4}} = 8e^{3y - \lambda y/4}$$

$u(x,y) = 8 e^{y(3-\lambda/4)} e^{\lambda(x+y/4)}$

3 — By the methods of separation of variables solve the boundary value problem $\frac{\partial u}{\partial x} - 5\frac{\partial u}{\partial t} = u$, $u(x,0) = e^{-2x}$.

Solution

Let $u(x,t) = X(x)T(t)$

$$\frac{\partial X(x)T(t)}{\partial x} - 5\frac{\partial X(x)T(t)}{\partial t} = X(x)T(t)$$

$$T(t)\frac{\partial X(x)}{\partial x} - 5X(x)\frac{\partial T(t)}{\partial t} = X(x)T(t)$$

Divide through by $X(x)T(t)$

$$\frac{1}{X(x)}\frac{\partial X(x)}{\partial x} - \frac{5}{T(t)}\frac{\partial T(t)}{\partial t} = 1$$

$$\frac{1}{X(x)}\frac{\partial X(x)}{\partial x} = 1 + \frac{5}{T(t)}\frac{\partial T(t)}{\partial t}$$

$\frac{1}{X}X' = \lambda$ — ①

$1 + \frac{5}{T}\dot{T} = \lambda$ — ②

Case 1: When $\lambda = 0$: $X'(x) = 0$, $X(x) = C$

$\frac{5}{T}\dot{T} = -1$

$\dot{T} = -\frac{1}{5}T$

Case 2: When $\lambda < 0$, $\lambda = -k^2$

$\frac{X'}{X} = -k^2 \Rightarrow$ ~~k~~

$\ln X = -k^2 x$

$X(x) = ce^{-k^2 x}$

$\frac{5}{T}\dot{T} = -k^2 - 1 \Rightarrow \frac{\dot{T}}{T} = \frac{-k^2-1}{5}$

$\ln T = \frac{-k^2-1}{5}t$

$T = c\exp\left(\frac{-k^2-1}{5}t\right)$

$u(x,t) = ce^{-k^2x}\,ce^{-\frac{k^2-1}{5}t} = ce^{-k^2x}e^{-\frac{k^2-1}{5}t}$

applying the initial boundary condition

$u(x,0) = ce^{-k^2x}\cancel{e} = e^{-2x}$

$c = e^{-2x+k^2x} = \cancel{e^{x}} \; e^{x(k^2-2)}$

$\Rightarrow u(x,t) = e^{x(k^2-2)}e^{-k^2x}e^{-\frac{k^2-1}{5}t}$

Case 3: When $\lambda > 0$, $\lambda = k^2$

$\frac{X'}{X} = k^2$

$\ln X = k^2x \Rightarrow X(x)\; e^{k^2x}$

$\frac{T'}{T} = \frac{k^2-1}{5} \Rightarrow \ln T = \frac{k^2-1}{5}t$

$T(t) = \exp\left(\frac{k^2-1}{5}\right)t$

Applying the $u(x,0) = e^{-2x}$,

$u(x,0) = e^{k^2x} = e^{-2x}$

$k^2 = -2$, $k = i\sqrt{2}$

$u(x,t) = e^{-2x}e^{-\frac{3}{5}t}$.

4. Show that $u(x,y) = y^2 - x^2$ $u(x,y) = e^y\sin x$ are solution to the laplace's equation $u_{xx} + u_{yy} = 0$

Solution

$u(x,y) = y^2 - x^2$

$\partial u/\partial x = -2x \qquad \partial u/\partial y = 2y$

$\partial^2 u/\partial x^2 = -2 \qquad \partial^2 u/\partial y^2 = 2$

$u_{xx} + u_{yy} = -2+2 = 0$

$u(x,y) = e^y\sin x$

$\partial u/\partial x = e^y\cos x \qquad \partial u/\partial y = e^y\sin x$

$\partial^2 u/\partial x^2 = -e^y\sin x \qquad \partial^2 u/\partial y^2 = e^y\sin x$

$u_{xx} + u_{yy} = -e^{y}\sin x + e^{y}\sin x = 0$

5 (a) List two methods for solving partial differentiate eqn
(b) List the steps involved in using the method of separation of variables to solve PDEs
(c) In which areas is PDE applicable