HW 8 Problem 1

Austin Marga

```
from integrals import *
from scipy import integrate
%pylab inline
```

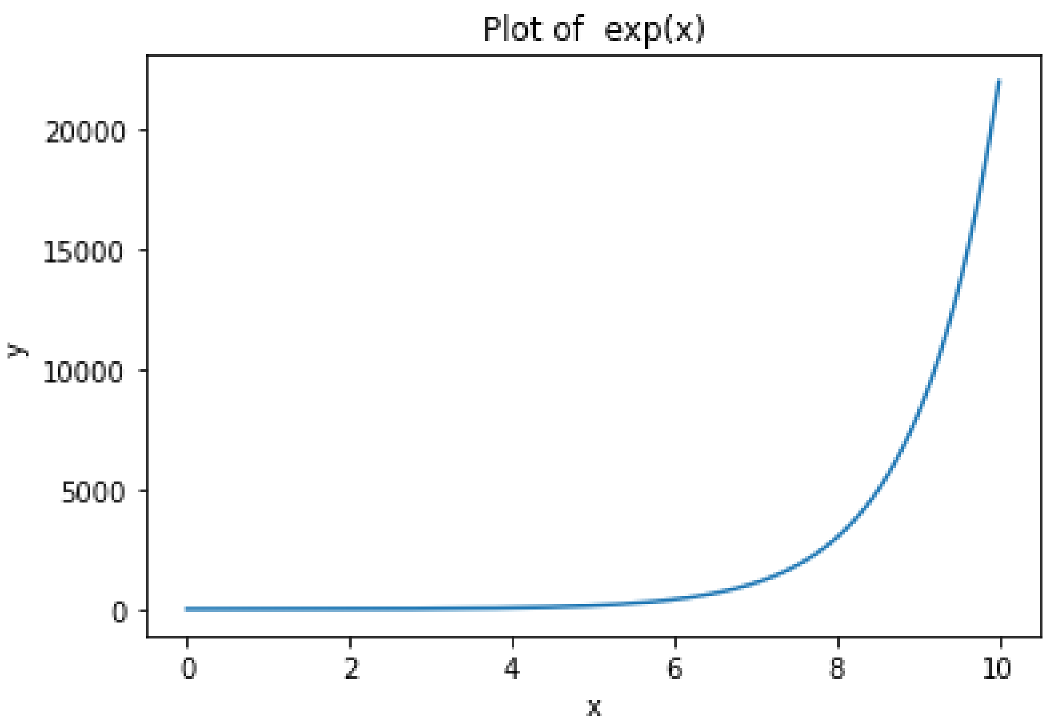
```
Populating the interactive namespace from numpy and matplotlib
```

I could have been more pythonic by introducing some `if` statements to group the 6 functions into 2, but I had some problems with this for implementation.

$f(x) = e^x \quad [0, 10]$

First, let's plot the function $f(x) = e^x$.

```
a = 0
b = 10
x = np.linspace(a,b,1000)
y = np.exp(x)
plt.xlabel("x")
plt.ylabel("y")
plt.title("Plot of  exp(x)")
plt.plot(x,y)
plt.show()
```



In order to plot the antiderivative of some function, we can use the Fundamental Theorem of Calculus.

$$\int_a^b f(x)dx = F(b) - F(a)$$

We can calculate the integral of some function numerically by approximating the area of the curve, we only need to know the value of the antiderivative at some point a. The antiderivative is a family of functions varying by some constant, so we can specify one by specifying one point (a,F(a)).

All other points can be calculated from this.

$$F(b) = \int_a^b f(x)dx + F(a)$$

For these examples, let a be the left boundary of the point, and let b be some point x at which we want the antiderivative.

$F(x) = \int e^x dx = e^x + C$. If we drop the constant, the integral of e^x is just e^x. Therefore, the integral should look the same as the original plot.

The real answer for $F = \int_0^{10} e^x dx \approx 22025.46579480672$. Let's see how close our integration methods get.

In this example, $F(a) = F(0) = 1$, therefore, for any point after this, we can calculate the antiderivative via the following:

$$F(x) = \int_0^x e^x dx + 1$$

```
res = simpson(np.exp,0,10,50)
print(res)
```

```
22025.66064832975
```

```
print(type(x))
```

```
<class 'numpy.ndarray'>
```

```
def integrate_simp(f,a,b,n,C):
    """
    Generates a list of x and y values of the antiderivative using the Simpson Rule.

    args:
    f: some (lambda) function
    a: left end point
    b: right end point
    n: number of partitions along the x-axis
    C: The value of the antiderivative at the left end point to uniquely describe the curve.

    output:
    returns a list of lists, with the first entry being the x values
    and the second entry being the y values for the antiderivative plot


    """
    y = []
    x = []
    step_size = (b-a) / n
    m = 0
    for step in arange(0,n):
        step_left = a+m*step_size
        step_right = a + (m+1)*step_size
        val = simpson(f,a,step_right,200) + C
        y.append(val)
```

```
        x.append(step_right)
        m = m+1
    x = numpy.array(x)
    y = numpy.array(y)
    return [x,y]
```

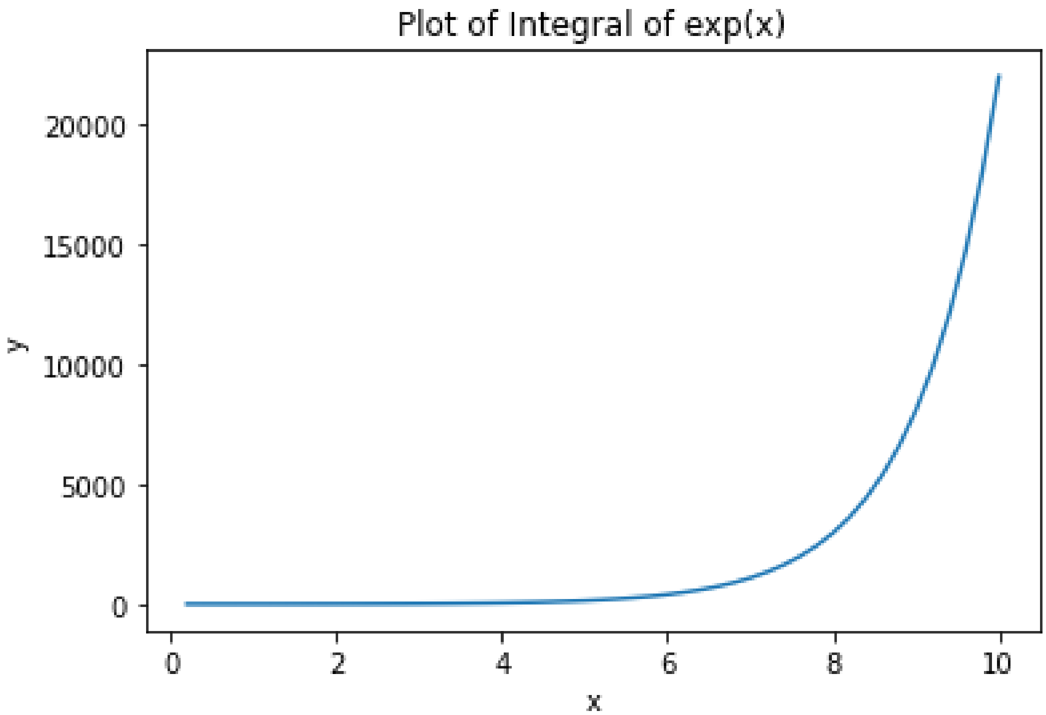
Here, in order to calculate accuracy, I use the `scipy.integrate` function. This has to be outside of the function because of the difference in syntax between `numpy` and `scipy` for e^x, being `np.exp` and `np.exp(x)` respectively.

```
real = integrate.quad(lambda x: np.exp(x), 0, 10)
print(real)
```

```
(22025.465794806725, 6.239389118119916e-10)
```

```
def integrate_simp_and_plot(f,a,b,n,name,C):
    """
    Generates a list of x and y values of the antiderivative using the Simpson Rule and plots the antid

    args:
    f: some (lambda) function
    a: left end point
    b: right end point
    n: number of partitions along the x-axis
    name: The name of the function as a string.
    C: The value of the antiderivative at the left end point to uniquely describe the curve.

    output:
    returns a numerical approximation for the integration.


    """
    val = integrate_simp(f,a,b,n,C)
    x = val[0]
    y = val[1]
    plt.xlabel("x")
    plt.ylabel("y")
    plt.title("Plot of Integral of " + str(name))
    plt.plot(x,y)
    plt.show()
    integral = simpson(f,a,b,n)
    print("The integration of {} by the Simpson Rule with {} steps is {}".format(str(f),n,integral))
    return integral
```

```
a = 0
b = 10
n = 50
integral = integrate_simp_and_plot(np.exp,a,b,n,"exp(x)",1)
```




```
The integration of <ufunc 'exp'> by the Simpson Rule with 50 steps is 22025.66064832975
```

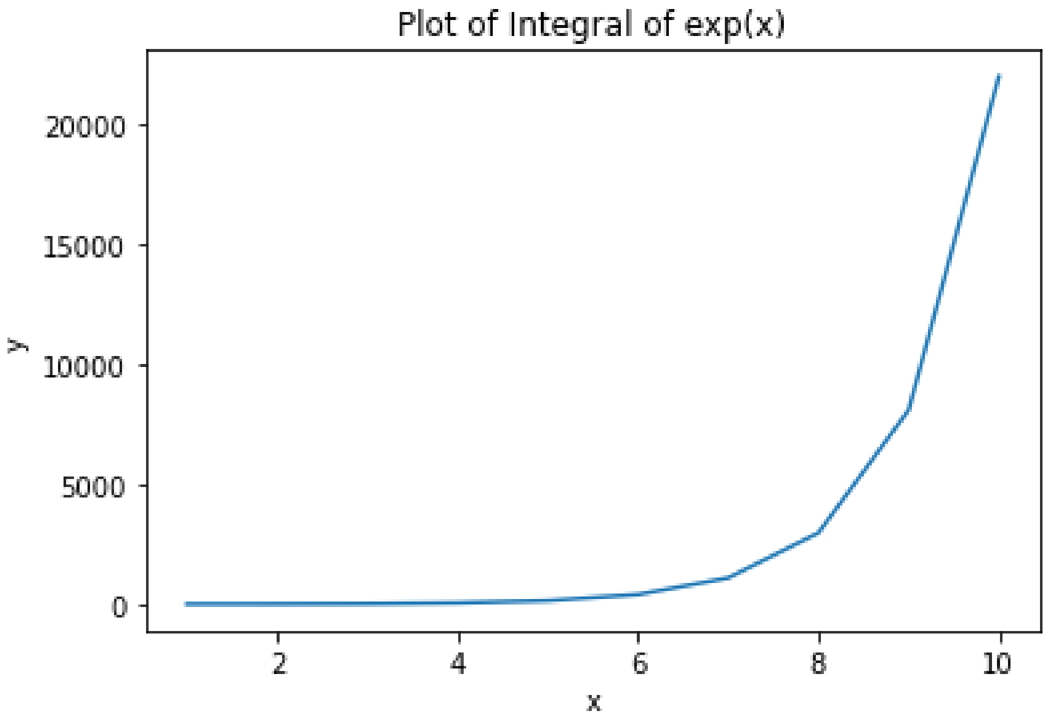
```
real = integrate.quad(lambda x: np.exp(x), a, b)
acc = 1 - np.abs((integral - real[0])) / real[0]
perc = str(100*acc)+"%"
print("The real value of the integration is {} with an error of {}.".format(real[0],real[1]))
print("The accuracy of the Simpson Rule with {} steps is {}.".format(n,perc))
```

```
The real value of the integration is 22025.465794806725 with an error of 6.239389118119916e-10.
The accuracy of the Simpson Rule with 50 steps is 99.99911532620995%.
```

Here, we define accuracy to be

$$\text{acc} = 1 - \frac{|\text{theoretical} - \text{experimental}|}{\text{theoretical}}$$

```
a = 0
b = 10
n = 10
integral = integrate_simp_and_plot(np.exp,a,b,n,"exp(x)",1)
real = integrate.quad(lambda x: np.exp(x), a, b)
acc = 1 - np.abs((integral - real[0])) / real[0]
perc = str(100*acc)+"%"
print("The real value of the integration is {} with an error of {}.".format(real[0],real[1]))
print("The accuracy of the Simpson Rule with {} steps is {}.".format(n,perc))
```



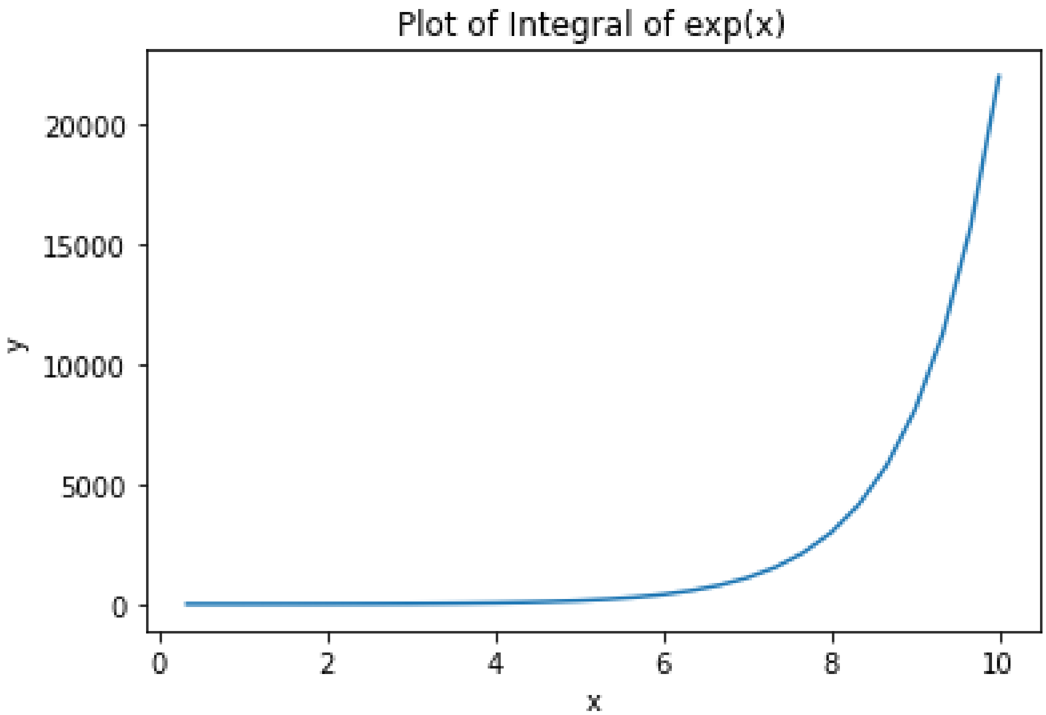

```
The integration of <ufunc 'exp'> by the Simpson Rule with 10 steps is 22134.650007857348
The real value of the integration is 22025.465794806725 with an error of 6.239389118119916e-10.
The accuracy of the Simpson Rule with 10 steps is 99.50428193406758%.
```

We can see that with 10 steps, we can get an acccuracy of greater than 99%.

Let's try now with the Trapezoid Rule.

```
def integrate_trap(f,a,b,n,C):
    """
    Generates a list of x and y values of the antiderivative using the Trapezoidal Rule.

    args:
    f: some (lambda) function
    a: left end point
    b: right end point
    n: number of partitions along the x-axis
    C: The value of the antiderivative at the left end point to uniquely describe the curve.

    output:
    returns a list of lists, with the first entry being the x values
    and the second entry being the y values for the antiderivative plot


    """
    y = []
    x = []
    step_size = (b-a) / n
    m = 0
    for step in arange(0,n):
        step_left = a+m*step_size
```

```
        step_right = a + (m+1)*step_size
        val = trapezoid(f,a,step_right,200) + C
        y.append(val)
        x.append(step_right)
        m = m+1
    x = numpy.array(x)
    y = numpy.array(y)
    return [x,y]

def integrate_trap_and_plot(f,a,b,n,name,C):
    """
    Generates a list of x and y values of the antiderivative using the Trapezoidal Rule and plots the a

    args:
    f: some (lambda) function
    a: left end point
    b: right end point
    n: number of partitions along the x-axis
    name: The name of the function as a string.
    C: The value of the antiderivative at the left end point to uniquely describe the curve.

    output:
    returns a numerical approximation for the integration.


    """
    val = integrate_trap(f,a,b,n,C)
    x = val[0]
    y = val[1]
    plt.xlabel("x")
    plt.ylabel("y")
    plt.title("Plot of Integral of " + str(name))
    plt.plot(x,y)
    plt.show()
    integral = trapezoid(f,a,b,n)
    print("The integration of {} by the Trapezoidal Rule with {} steps is {}".format(str(f),n,integral)
    return integral

a = 0
b = 10
n = 30
integral = integrate_trap_and_plot(np.exp,a,b,n,"exp(x)",1)
```




```
The integration of <ufunc 'exp'> by the Trapezoidal Rule with 30 steps is 22229.028623517694
```

```python
acc = 1 - np.abs((integral - real[0])) / real[0]
perc = str(100*acc)+"%"
print("The real value of the integration is {} with an error of {}.".format(real[0],real[1]))
print("The accuracy of the Trapezoidal Rule with {} steps is {}.".format(n,perc))
```

```
The real value of the integration is 22025.465794806725 with an error of 6.239389118119916e-10.
The accuracy of the Trapezoidal Rule with 30 steps is 99.0757842280958%.
```

```python
def integrate_adapt_trap(f,a,b,n,C,acc):
    """
    Generates a list of x and y values of the antiderivative using the Adaptive Trapezoidal Rule.

    args:
    f: some (lambda) function
    a: left end point
    b: right end point
    n: number of partitions along the x-axis
    C: The value of the antiderivative at the left end point to uniquely describe the curve.
    acc: some accuracy requirement for each partition. This is the same that is used in the adaptive_tr

    output:
    returns a list of lists, with the first entry being the x values
    and the second entry being the y values for the antiderivative plot


    """
    y = []
    x = []
    step_size = (b-a) / n
```

```
    m = 0
    for step in arange(0,n):
        step_left = a+m*step_size
        step_right = a + (m+1)*step_size
        val = adaptive_trapezoid(f,a,step_right,200) + C
        y.append(val)
        x.append(step_right)
        m = m+1
    x = numpy.array(x)
    y = numpy.array(y)
    return [x,y]
```

The `adaptive_trapezoid` function takes different arguments than the other integral functions. Here, I let each adaptive trapezoid have an accuracy of $acc = 0.99$.

```
def integrate_adapt_trap_and_plot(f,a,b,n,acc,name,C):
    """
    Generates a list of x and y values of the antiderivative using the Adaptive Trapezoidal Rule and pl

    args:
    f: some (lambda) function
    a: left end point
    b: right end point
    n: number of partitions along the x-axis
    acc: The accuracy required for one specific partition, as used in the adaptive_trapezoid function.
    name: The name of the function as a string.
    C: The value of the antiderivative at the left end point to uniquely describe the curve.

    output:
    returns a numerical approximation for the integration.


    """
    val = integrate_adapt_trap(f,a,b,n,C,acc)
    x = val[0]
    y = val[1]
    plt.xlabel("x")
    plt.ylabel("y")
    plt.title("Plot of Integral of " + str(name))
    plt.plot(x,y)
    plt.show()
    integral = trapezoid(f,a,b,n)
    print("The integration of {} by the Adaptive Trapezoidal Rule with {} steps is {}".format(str(f),n,i
    return integral
```

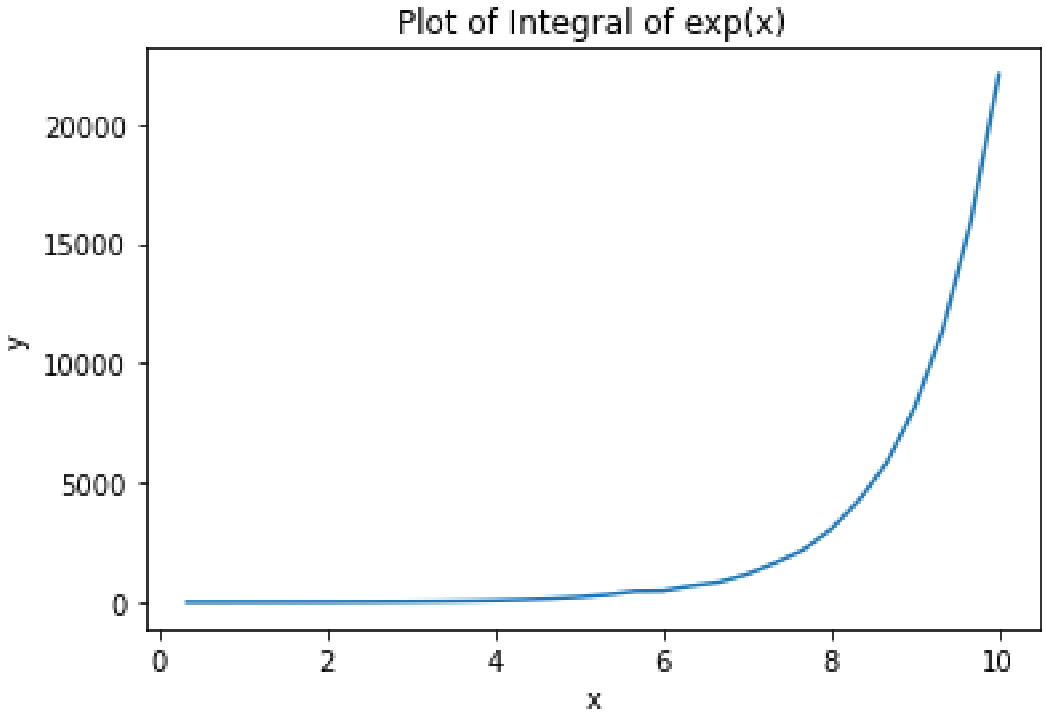
```
z_1 = adaptive_trapezoid(np.exp,0,10,1,0.99)
z_2 = adaptive_trapezoid(np.exp,0,10,1,.01)
print(z_1,z_2)
```

```
22025.64083720381 22025.64083720381
```

We can see that the `acc` argument does not change the integration value to any serious degree. This is due to the `adaptive_trapezoid` function from `integrals.py`, not from its application here.

```
a = 0
b = 10
n = 30
```

```
integral = integrate_adapt_trap_and_plot(np.exp,a,b,n,0.99,"exp(x)",1)
```




```
The integration of <ufunc 'exp'> by the Adaptive Trapezoidal Rule with 30 steps is 22229.028623517694
```

```
acc = 1 - np.abs((integral - real[0])) / real[0]
perc = str(100*acc)+"%"
print("The real value of the integration is {} with an error of {}.".format(real[0],real[1]))
print("The accuracy of the Adaptive Trapezoidal Rule with {} steps is {}.".format(n,perc))
```

```
The real value of the integration is 22025.465794806725 with an error of 6.239389118119916e-10.
The accuracy of the Adaptive Trapezoidal Rule with 30 steps is 99.0757842280958%.
```

In summary for $f(x) = e^x$

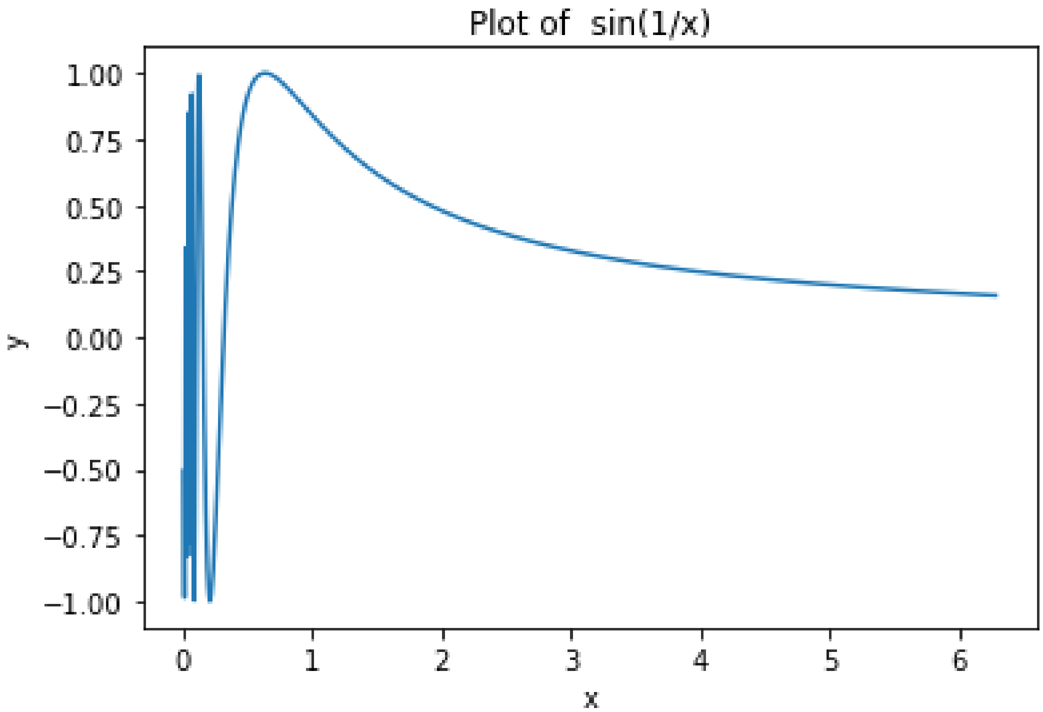
- The Simpson Rule can get an accuracy of over 99% in only 10 steps.
- The Trapezoid Rule can get an accuracy of over 99% in 30 steps.
- The Adaptive Trapezoid Rule can get an accuracy of over 99% in 30 steps given a step accuracy of 99%.

$f(x) = \sin(\frac{1}{x}) \quad (0, 2\pi]$

We can clearly see that $\sin(1/x)$ as x tends to 0 is not defined, so I shall omit this point and start at $a = 0.0001$.

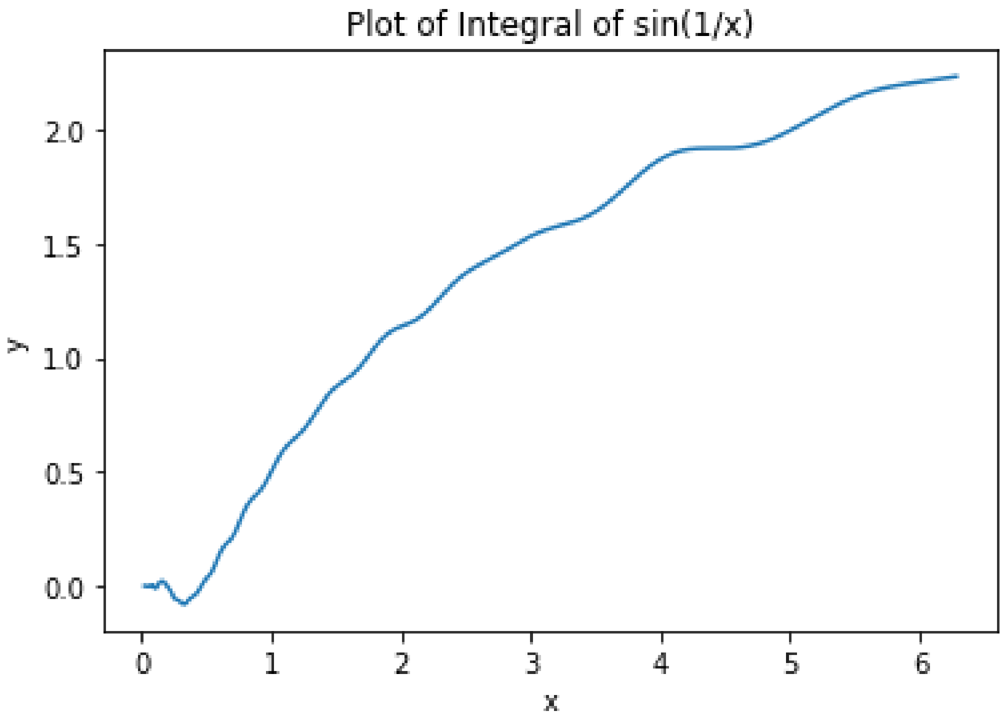
```
a = 0.01
b = 2*np.pi
x = np.linspace(a,b,1000)
y = np.sin(1/x)
plt.xlabel("x")
plt.ylabel("y")
plt.title("Plot of  sin(1/x)")
```

```
plt.plot(x,y)
plt.show()
```



I don't know what the antiderivative of $sin(1/x)$ at 0.01 is, so I will just pick the antiderivative curve where such value is 0.

```
n = 500
integral = integrate_simp_and_plot(lambda x: np.sin(1/x),a,b,n,"sin(1/x)",0)
```




```
The integration of <function <lambda> at 0x7fec175c0af0> by the Simpson Rule with 500 steps is 2.281012
```

We can see that even when plotting the antiderivative, it seems to taper off, consistent with the trends we see from $sin(1/x)$, the limit does not appear to be approaching some value, but the rate of change of the antiderivative is decreasing.

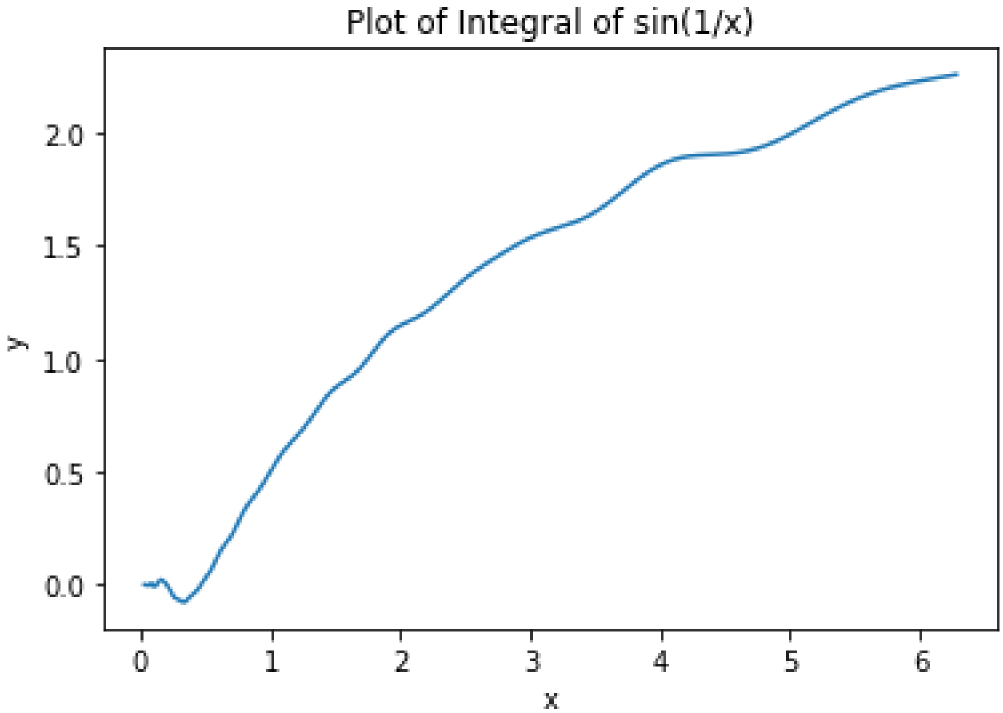
Apparently, `str()` does not like to work for lambda functions.

```
real = integrate.quad(lambda x: np.sin(1/x), a, b)
acc = 1 - np.abs((integral - real[0])) / real[0]
perc = str(100*acc)+"%"
print("The real value of the integration is {} with an error of {}.".format(real[0],real[1]))
print("The accuracy of the Simpson Rule with {} steps is {}.".format(n,perc))
```

```
The real value of the integration is 2.2626857545897185 with an error of 1.474769539892536e-08.
The accuracy of the Simpson Rule with 500 steps is 99.19005404746054%.
```

It takes over 500 steps in order to get an accuracy of over 99%! We can see just how varying $\sin(1/x)$ really is.

```
integral = integrate_trap_and_plot(lambda x: np.sin(1/x),a,b,350,"sin(1/x)",0)
```

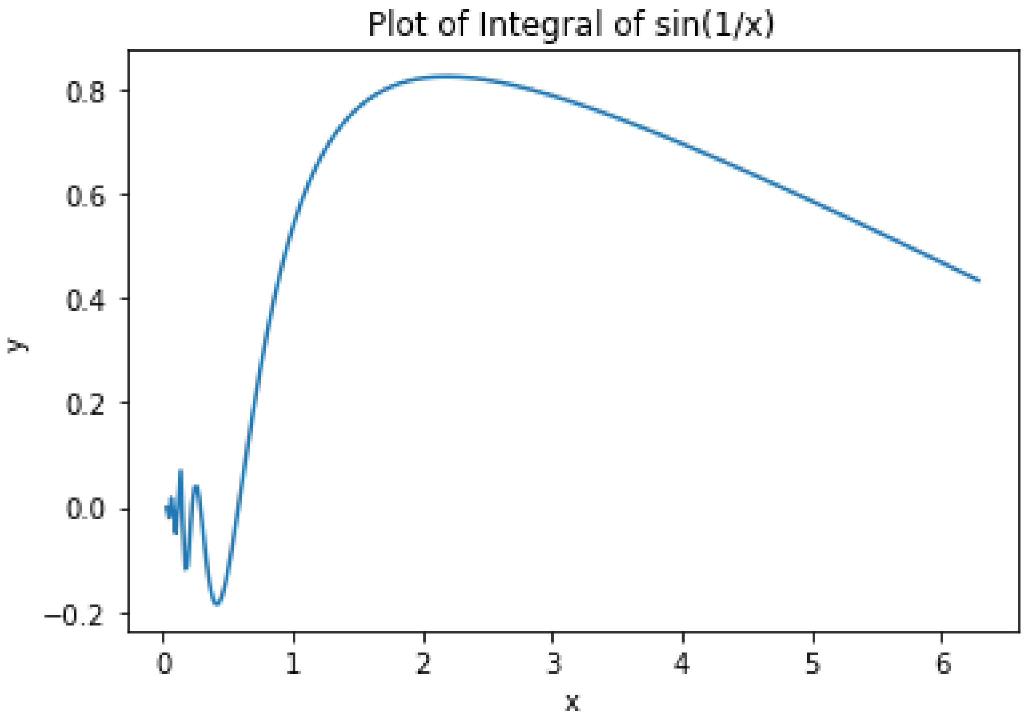



```
The integration of <function <lambda> at 0x7fec17588820> by the Trapezoidal Rule with 350 steps is 2.24
```

```
acc = 1 - np.abs((integral - real[0])) / real[0]
perc = str(100*acc)+"%"
print("The real value of the integration is {} with an error of {}.".format(real[0],real[1]))
print("The accuracy of the Trapezoidal Rule with {} steps is {}.".format(n,perc))
```

```
The real value of the integration is 2.2626857545897185 with an error of 1.474769539892536e-08.
The accuracy of the Trapezoidal Rule with 500 steps is 99.01220999372805%.
```

```
integral = integrate_adapt_trap_and_plot(lambda x: np.sin(1/x),a,b,350,0.99,"sin(1/x)",0)
```




```
The integration of <function <lambda> at 0x7fec1f7ae550> by the Adaptive Trapezoidal Rule with 350 steps
```

```
acc = 1 - np.abs((integral - real[0])) / real[0]
perc = str(100*acc)+"%"
print("The real value of the integration is {} with an error of {}.".format(real[0],real[1]))
print("The accuracy of the Adaptive Trapezoidal Rule with {} steps is {}.".format(n,perc))
```

```
The real value of the integration is 2.2626857545897185 with an error of 1.474769539892536e-08.
The accuracy of the Adaptive Trapezoidal Rule with 500 steps is 99.01220999372805%.
```

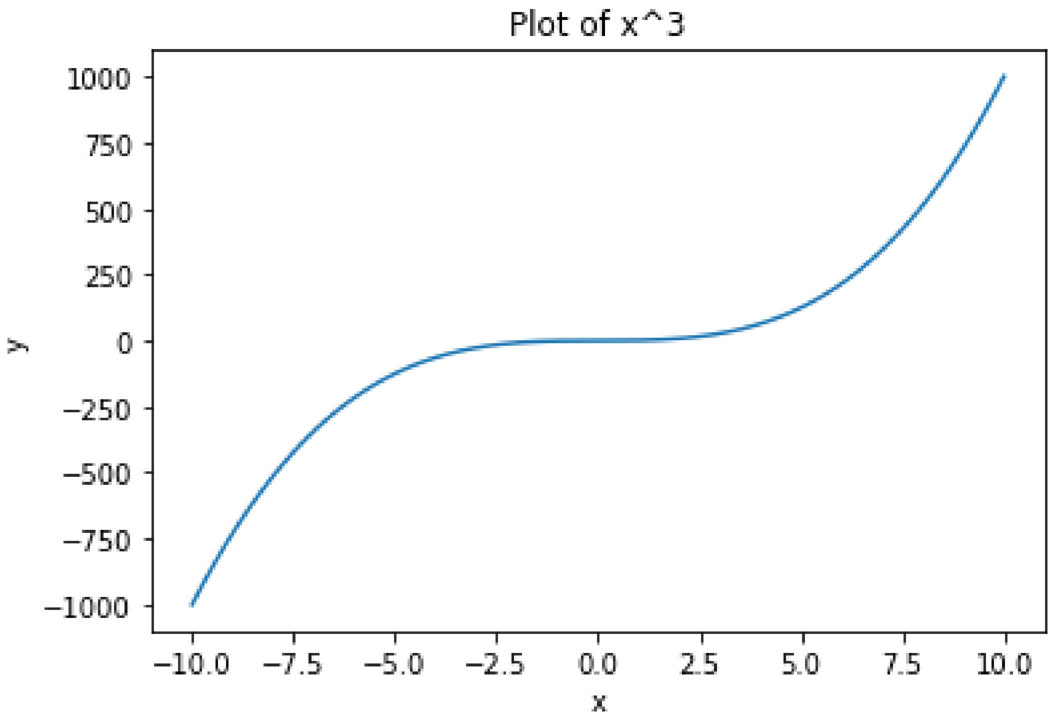
We can see that the accuracy for the Adaptive Trapezoid is much higher than the Simpson Rule when n is very large and the accuracy of each step is 0.99 to begin with.

In summary for $f(x) = \sin(\frac{1}{x})$

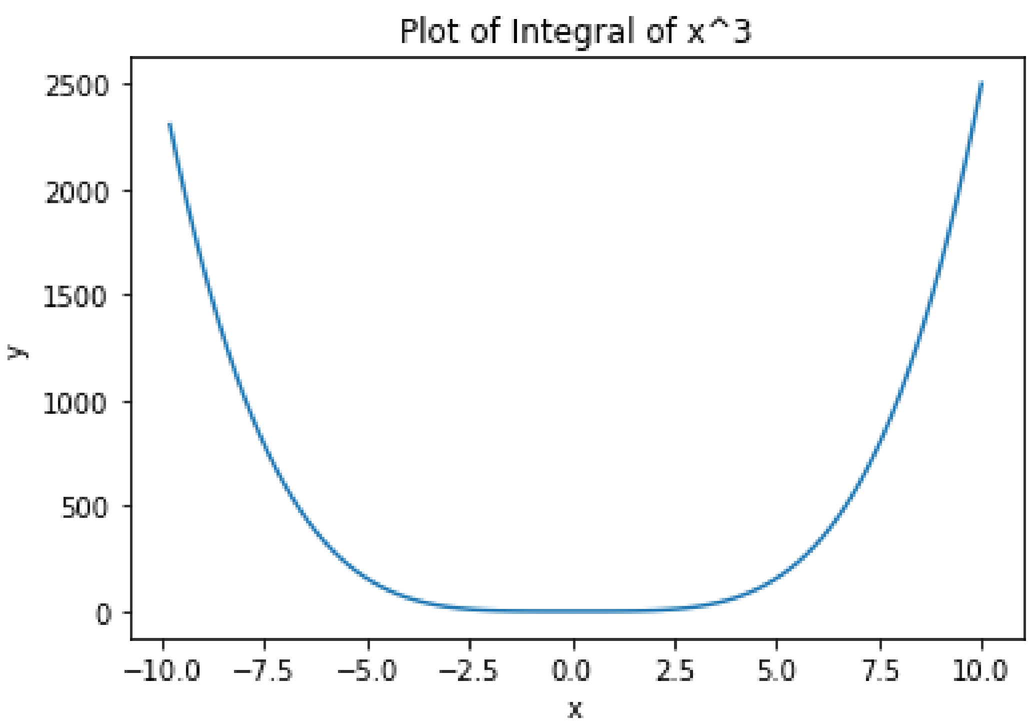
- The Simpson Rule can get an accuracy of over 99% in 500 steps.
- The Trapezoid Rule can get an accuracy of over 99% in 350 steps.
- The Adaptive Trapezoid Rule can get an accuracy of over 99% in 350 steps given a step accuracy of 99%.

$f(x) = x^3$

```
a = -10
b = 10
x = np.linspace(a,b,1000)
y = x**3
plt.xlabel("x")
plt.ylabel("y")
plt.title("Plot of x^3")
plt.plot(x,y)
plt.show()
```




```
n = 100
integral = integrate_simp_and_plot(lambda x: x**3,a,b,n,"x^3",2500)
```




```
The integration of <function <lambda> at 0x7fec17631550> by the Simpson Rule with 100 steps is 1.0762353
```

Since I only plot with the right step values, we see some missing points at the left side of the graph.

```
real = integrate.quad(lambda x: x**3, a, b)
print(np.abs((integral - real[0])))
print(real[0])
acc = 1 - np.abs((integral - real[0])) / real[0]
perc = str(100*acc)+"%"
print("The real value of the integration is {} with an error of {}.".format(real[0],real[1]))
print("The accuracy of the Simpson Rule with {} steps is {}.".format(n,perc))
```

```
1.076235397097965e-12
0.0
The real value of the integration is 0.0 with an error of 5.551208455924673e-11.
The accuracy of the Simpson Rule with 100 steps is -inf%.
```

```
/tmp/ipykernel_310/2164958878.py:4: RuntimeWarning: divide by zero encountered in double_scalars
  acc = 1 - np.abs((integral - real[0])) / real[0]
```

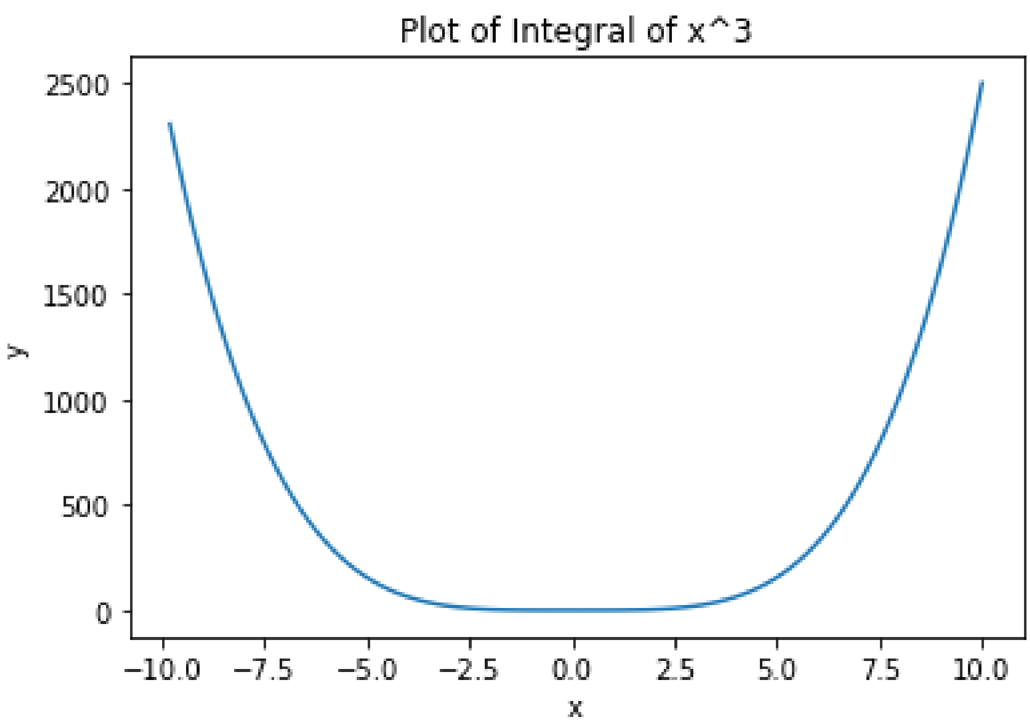
Because the expected value of the integral is 0, we will get a divide by zero error when measuring accuracy. To remedy this, let's redefine accuracy to be:

$$acc = (1 - \text{experimental}) \times 100$$

```
new_acc = (1-integral)*100
perc = str(new_acc) + "%"
print("The accuracy of the Simpson Rule with {} steps is {}.".format(n,perc))
```

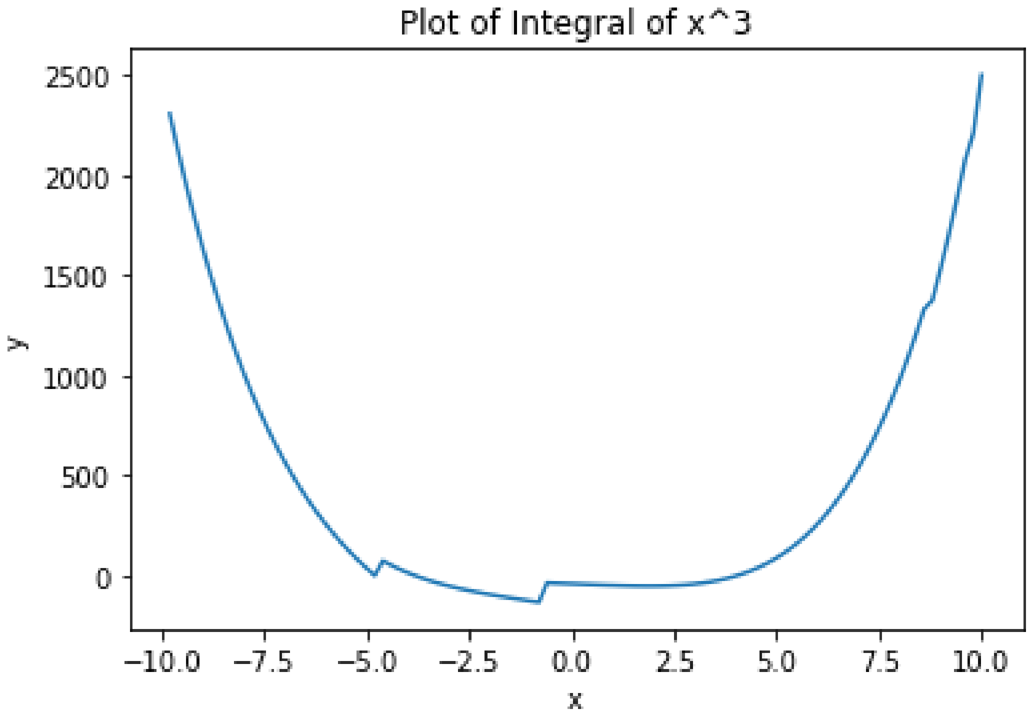
```
The accuracy of the Simpson Rule with 100 steps is 99.99999999989238%.
```

```
integral = integrate_trap_and_plot(lambda x: x**3,a,b,n,"x^3",2500)
new_acc = (1-integral)*100
perc = str(new_acc) + "%"
print("The accuracy of the Trapezoidal Rule with {} steps is {}.".format(n,perc))
```




```
The integration of <function <lambda> at 0x7fec173d4e50> by the Trapezoidal Rule with 100 steps is 9.32
The accuracy of the Trapezoidal Rule with 100 steps is 99.99999999990678%.
```

```
integral = integrate_adapt_trap_and_plot(lambda x: x**3,a,b,n,.99,"x^3",2500)
print(n)
new_acc = (1-integral)*100
perc = str(new_acc) + "%"
print("The accuracy of the Adaptive Trapezoidal Rule with {} steps is {}.".format(n,perc))
```




```
The integration of <function <lambda> at 0x7fec1734b310> by the Adaptive Trapezoidal Rule with 100 steps
100
The accuracy of the Adaptive Trapezoidal Rule with 100 steps is 99.99999999990678%.
```

This measurement for accuracy isn't truly meaningful when comparing to other functions, but it gives us some baseline when comparing integration methods for one function.

In summary for $f(x) = x^3$

- The trapezoidal rules are ever so slightly more accurate than the simpson rule, but the difference is of the order 10^{-12}.
- The plot for the adaptive trapezoid is contorted, but I do not know why.
- All accuracies with 100 steps are of the order 10^{-11}.