The Transfer Matrix Method in Linear Dielectrics

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Outline

Electromagnetic Plane-Waves in Dielectric Media

As always, our starting point is the Maxwell equations:

$$\nabla \cdot \mathsf{D} = \rho_f \tag{1}$$

$$\nabla \cdot \mathsf{B} = 0 \tag{2}$$

$$\nabla \times \mathsf{E} = -\frac{\partial \mathsf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathsf{H} = \frac{\partial \mathsf{D}}{\partial t} + \mathsf{J}_f \tag{4}$$

In the case of linear and isotropic media we have the following relations for our fields:

$$\mathsf{D} = \epsilon \mathsf{E} \tag{5}$$

$$\mathsf{H} = \frac{1}{\mu} \mathsf{B} \tag{6}$$

In the absence of any sources, the Maxwell equations can be combined to obtain the Helmholtz equations:

$$(\nabla^2 + \mu\epsilon\omega^2)\mathrm{E} = 0 \tag{7}$$

$$(\nabla^2 + \mu\epsilon\omega^2)\mathrm{B} = 0 \tag{8}$$

The solutions? Plane-waves, of course!

$$\mathrm{E}(\mathrm{r}, t) = \mathrm{E}_0 e^{i(k\mathrm{n}\cdot\mathrm{r} - \omega t)} \tag{9}$$

$$\mathrm{B}(\mathrm{r}, t) = \mathrm{B}_0 e^{i(k\mathrm{n}\cdot\mathrm{r} - \omega t)} \tag{10}$$

The dispersion relation and index of refraction can also be obtained now:

$$k^2 = \mu\epsilon\omega^2 \tag{11}$$

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \tag{12}$$

Reflection and Transmission at Dielectric Interfaces

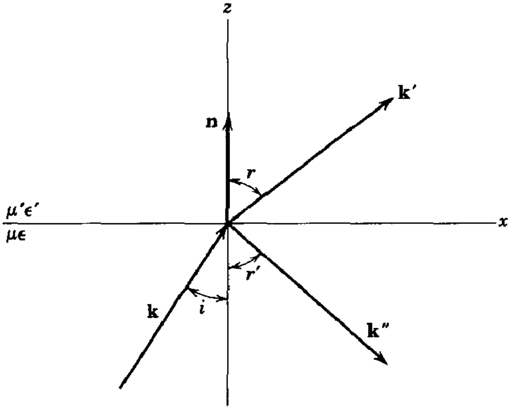

Figure: Propagation of a wave into a medium with a different index of refraction, which gives rise to reflected and refracted waves. The y-axis points into the page for this geometry [**1**].

We can join all three waves at the interface and use the superposition principle:

$$E_0 e^{i(\mathrm{k}\cdot\mathrm{r}-\omega t)} + E_0'' e^{i(\mathrm{k}''\cdot\mathrm{r}-\omega t)} = E_0' e^{i(\mathrm{k}'\cdot\mathrm{r}-\omega t)} \tag{13}$$

$$|\mathrm{k}| = |\mathrm{k}''| = \omega\sqrt{\mu\epsilon} \tag{14}$$

$$|\mathrm{k}'| = \omega\sqrt{\mu'\epsilon'} \tag{15}$$

$$\mathrm{k}\cdot\mathrm{r} = \mathrm{k}''\cdot\mathrm{r} = \mathrm{k}'\cdot\mathrm{r} \tag{16}$$

The laws of reflection and refraction then come out as natural consequences:

$$|k||r|\cos\theta_i = |k''||r|\cos\theta_{r'} \qquad \rightarrow \qquad \theta_i = \theta_{r'} \tag{17}$$

$$k\sin\theta_i = k'\sin\theta_r \tag{18}$$

The dynamic properties are a direct result of the boundary conditions at the dielectric interface:

$$[\epsilon(\mathsf{E}_0 + \mathsf{E}_0'') - \epsilon' \mathsf{E}_0'] \cdot \mathsf{n} = 0 \tag{19}$$

$$[\mathsf{k} \times \mathsf{E}_0 + \mathsf{k}'' \times \mathsf{E}_0'' - \mathsf{k}' \times \mathsf{E}_0'] \cdot \mathsf{n} = 0 \tag{20}$$

$$(\mathsf{E}_0 + \mathsf{E}_0'' - \mathsf{E}_0') \times \mathsf{n} = 0 \tag{21}$$

$$\left[\frac{1}{\mu}(\mathsf{k} \times \mathsf{E}_0 + \mathsf{k}'' \times \mathsf{E}_0'') - \frac{1}{\mu'}(\mathsf{k}' \times \mathsf{E}_0')\right] \times \mathsf{n} = 0 \tag{22}$$

The the key outcome is the acquisition of the Fresnel equations:

$$r_p = \frac{n_f \cos\theta_i - n_i \cos\theta_f}{n_f \cos\theta_i + n_i \cos\theta_f} \tag{23}$$

$$t_p = \frac{2n_i \cos\theta_i}{n_f \cos\theta_i + n_i \cos\theta_f} \tag{24}$$

$$r_s = \frac{n_i \cos\theta_i - n_f \cos\theta_f}{n_i \cos\theta_i + n_f \cos\theta_f} \tag{25}$$

$$t_s = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_f \cos\theta_f} \tag{26}$$

The Transfer Matrix Method

We imagine a stacked layer of dielectric slabs with different properties:

$$
\begin{aligned}
n(z) = & n_0, && z < z_0 \\
& n_1, && z_0 < z < z_1 \\
& n_2, && z_1 < z < z_2 \\
& \vdots && \\
& n_N, && z_{N-1} < z
\end{aligned}
\tag{27}
$$

Note that n is a function of z.

Lets examine the relation between forward and backward E-fields in two different layers:

$$E_F' = E_F e^{ik_{zj}z}$$
$$E_B' = E_B e^{-ik_{zj}z} \tag{28}$$

Looks a lot like a matrix equation...

$$E' = T_j E \tag{29}$$

The two transfer matrices are then:

$$T_j = \begin{pmatrix} \exp i\Phi_j & 0 \\ 0 & \exp -i\Phi_j \end{pmatrix} \tag{30}$$

$$T_{ji} = \frac{1}{t_{ji}} \begin{pmatrix} 1 & r_{ji} \\ r_{ji} & 1 \end{pmatrix} \tag{31}$$

Whereas the *full* transfer matrix becomes:

$$T = T_{N(N-1)} T_{N-1} \dots T_{32}\ T_2\ T_{21}\ T_1\ T_{10} \tag{32}$$

Finally, our desired results!

$$R = |r|^2 \tag{33}$$

$$T_p = |t|^2 \frac{\mathrm{Re}(n_f \cos\theta_f^*)}{\mathrm{Re}(n_i \cos\theta_i^*)} \tag{34}$$

$$T_s = |t|^2 \frac{\mathrm{Re}(n_f \cos\theta_f)}{\mathrm{Re}(n_i \cos\theta_i)} \tag{35}$$

References

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- E. Hecht, *Optics* (Pearson, 2002).
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