Chapter 10

Chapter 10

Coordinate Geometry of Straight Lines

OBJECTIVES

At the end of the chapter, students should be able to:

1. identify the Cartesian rectangular coordinate.
2. draw and interpret linear graphs.
3. determine the distance between two coordinate points.
4. find the mid-points of the line joining two points.
5. apply the concept to real life situation.

I. Cartesian Rectangular Coordinate

(i) Introduction

The coordinates of a point referred to as the perpendicular axes, are written as an ordered pair with the x-coordinates always being written first followed after a comma by the y-coordinate. Thus, the point (3, 2) represents the point whose x-coordinate or abscissa is 3 and whose y-coordinate or ordinate is 2. The x coordinate is the distance of the point from the y-axis or the distance measured parallel to the axis of x; the y-coordinate is the distance from the x-axis or the distance measured parallel to the axis of y. It is conventional to vertical and the point of intersection of these axes is called the origin O.

Either or both of the coordinates may be negative depending on the quadrant in which the point lies. In the second and third quadrants, the abscissa is negative; in the third and fourth quadrants, the ordinate is negative. When we deal with a general point instead of a particular point such as (−1, 3), we need letters for the coordinates. It is perfectly reasonable to call a point (h, k) but there is considerable advantage in naming it (x_1, y_1). Firstly, x_1 obviously stands for the x-coordinate and y_1 for the y-coordinate and secondly, further points may be named$(x2, y2)$, $(x3, y3)$, etc. There is no connection between $x1$ and $x2$ except that they both represent abscissae; the suffix in no way corresponds to an index.

II. Distance between Two Coordinate Points

Suppose we wish to find the distance between the points A(1, 2) and B(4, 6) as
shown in Figure 10.1. Through A, draw a line parallel to the axis of x to meet the
line through B parallel to the axis of y at N.
Then, AN = 4 − 1 = 3
BN = 6 − 2 = 4
∴ AB2 = AN2 + BN2 = 32 + 42 = 25

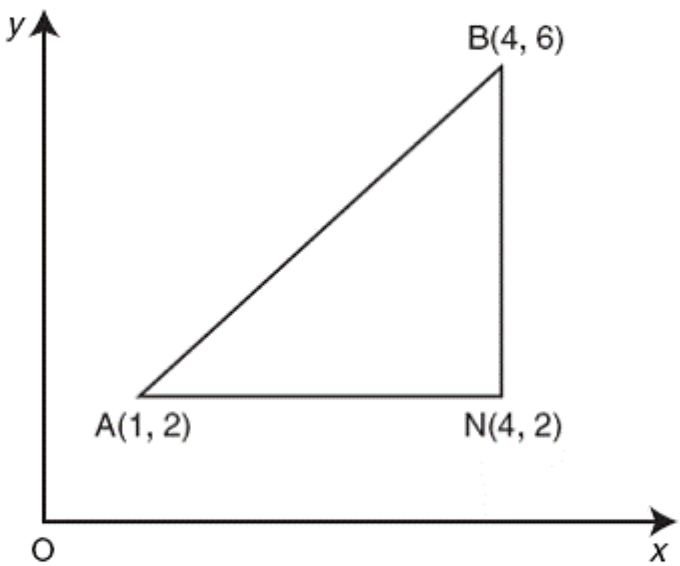


Figure 10.1

Therefore AB = 5. That is, the distance between the points is 5. Now let us generalise this example by finding the distance between the points
P1$(x1, x1)$ and P2$(x2, y2)$.
In Figure 10.2, P1NP2 is a right angle.
P1N = $x2 - x1$ and P2N = $y2 - y1$
By Pythagoras' theorem,

$$|P_1P_2|^2 = |P_1N|^2 + |P_2N|^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that this formula still holds if any one of the coordinates is negative.

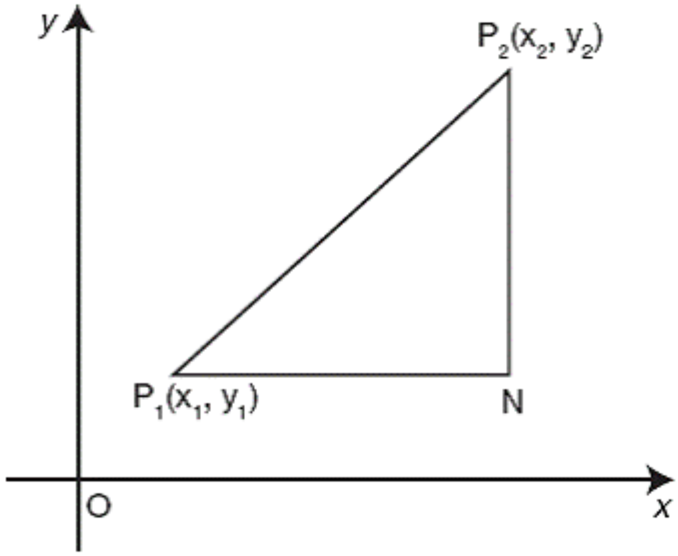


Figure 10.2

If the points given are (−1, 5) and (11, 10), the distance between the points measured parallel to the axis of x is 11 − (−1) or 12, which is the difference between the abscissae. The distance between the points measured parallel to the axis of y is 10 − 5 or 5 and the distance between the points is $\sqrt{12^2+5^2}$ or 13.

The distance between P1 and P2 measured parallel to the x-axis is called the projection of P1P2 on the x-axis.

Worked Example 1

Find the distance between the points (4, −3) and (−6, 7).

SOLUTION

$$\text{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\text{where } x_1 = 4;\ y_1 = -3$$

$$x_2 = -6;\ y_2 = 7$$

$$\therefore \quad \text{Distance} = \sqrt{(4+6)^2 + (-3-7)^2}$$

$$= \sqrt{(10)^2 + (-10)^2}$$

$$= \sqrt{100 + 100}$$

$$= \sqrt{200}$$

$$= 10\sqrt{2}$$

Worked Example 2

Find the distance between the points whose coordinates are as follows:
(a) P (3, −2) and Q(2, 5)
(b) A (−2, −3) and B(3, 1)

SOLUTION

(a) The distance between P and Q is

$$|PQ| = \sqrt{(2-3)^2 + (5-(-2))^2}$$

$$= \sqrt{(2-3)^2 + (5+2)^2}$$

$$= \sqrt{(-1)^2 + 7^2}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2} \text{ units.}$$

(b) The distance between A and B is

$$|AB| = \sqrt{(3-(-2))^2 + (1-(-3))^2}$$

$$= \sqrt{(3+2)^2 + (1+3)^2}$$

$$= \sqrt{5^2 + 4^2}$$

$$= \sqrt{25 + 16}$$

$$= \sqrt{41}$$

$$= 6.403 \text{ units.}$$

Worked Example 3

Find the distance between the point A(3, 4) and the point of intersection of the lines $2x - y = 4$ and $x + y = 2$.

SOLUTION

To get the point of intersection of the two lines, we solve the two equations simultaneously. Solving the equations $2x - y = 4$ and $x + y = 2$

simultaneously, as previously discussed, we obtain $x = 2$, $y = 0$.

∴ The point of intersection of the two lines is (2, 0).

Let B(2, 0) represent the point of intersection.

The distance between A(3, 4) and B(2, 0) is given by

$$|AB| = \sqrt{(2-3)^2 + (0-4)^2}$$

$$= \sqrt{(-1)^2 + (-4)^2}$$

$$= \sqrt{1 + 16}$$

$$= \sqrt{17}$$

Exercise 1

1. Find the distance between the points whose coordinates are P(3, −3) and Q(−2, 3).
2. Find the distance between the points A(3, 5) and the point of intersection of the lines $x + y = 5$, $2x - y = 1$.
3. What is the distance between the points whose coordinates are (3, −5) and (−3, 3)?
4. Find the distance between the points (3, −4) and the points common to the lines whose equations are $x - y = 3$ and $2x + y = 6$.
5. Find the distance between the points (3, −2) and (−4, 5). Find the distance between the pairs of points in Questions 6–10:
6. (3, 4) and (2, 7)
7. (5, 1) and (−3, −4)
8. (−3, 1) and (2, −3)
9. (−2, −3) and (3, −1)
10. (1, 2) and (−1, 4)

Find the distance between the points in Questions 11–20:

11. (2, 5) and (5, 9)
12. (−4, 5) and (−3, 2)
13. (3, −6) and (2, 0)
14. (−1, 5) and (7, −3)
15. (0, −2) and (−4, 0)
16. $(0, a)$ and $(a, 0)$
17. $(p, p + 1)$ and $(p - 1, 2p + 1)$
18. $(p2, 2p)$ and $(q2, 2q)$
19. (0, 2) and (6, 0)
20. $(2a, b)$ and $(a, 2b)$

III. Mid-Point of the Line Joining Two Points

To find the mid-point of the line joining A(1, 3) and B(5, 7), draw parallels to the axes to meet at N as shown in Figure 10.3.

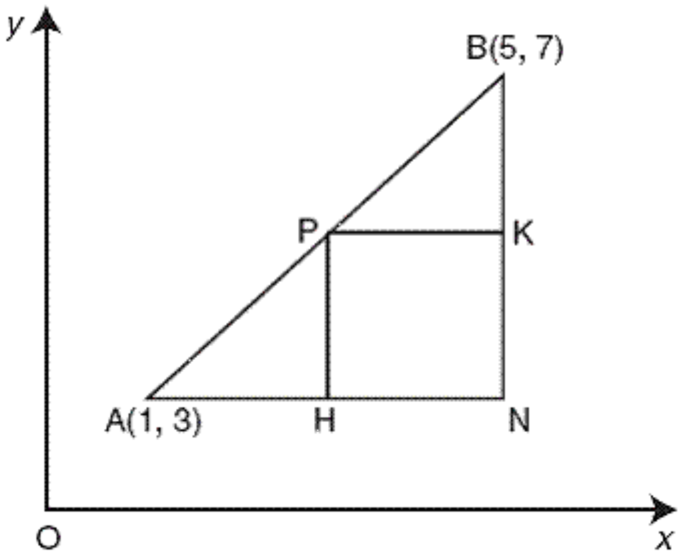


Figure 10.3

If P is the mid-point of the line, let the line through P parallel to the *x*-axis meet BN at K and the line through P parallel to the *y*-axis meet AN at H. Then H and K are the mid-points of AN and BN, respectively (mid-point theorem).

AN = 5 − 1 = 4,

∴ AH = 2

The *x*-coordinate of P is, therefore, 3 + 2 or 5.

∴ P is the point (3, 5)

We shall now generalise to find the midpoint of the line joining P1(*x*1, *y*1) to P2(*x*2, *y*2).

In Figure 10.4, let the mid-point of P1P3 be Q and let the parallels through Q to the axes meet AN and BN at H and K, respectively, as shown. Then H and K are the mid-points of P1N and P2N, respectively.

$$P_1N = x_2 - x_1,$$

$$\therefore \quad P_1H = \frac{1}{2}(x_2 - x_1)$$

$$\text{The } x\text{-coordinate of } Q = x_1 + P_1H$$

$$= x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$$

Similarly, the *y*-coordinate of Q is: $\frac{1}{2}(y_1 + y_2)$ and the coordinates of Q are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Thus, the mid-point of a line is found by taking the average of the coordinates separately.

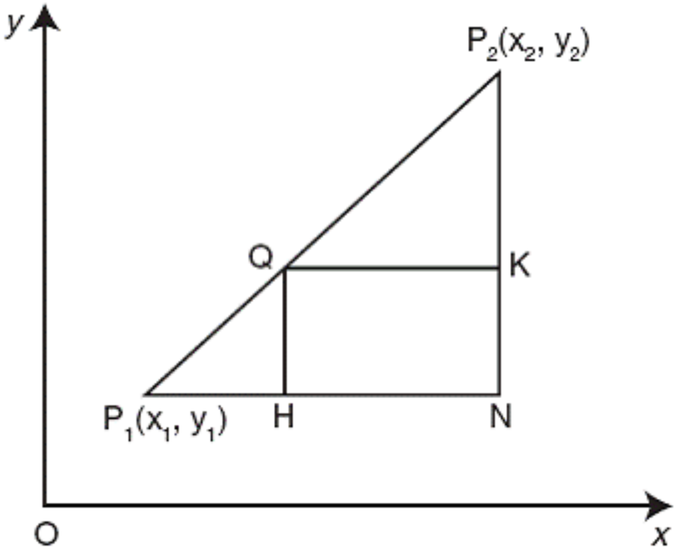


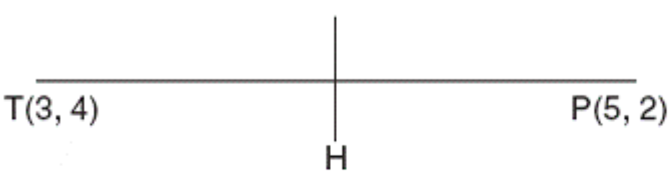
Figure 10.4

Worked Example 4

Find the mid-point of the line joining the following pairs of points:
(a) T(3, 4) and P(5, 2)
(b) P(3, −5) and Q(−7, 5)

SOLUTION

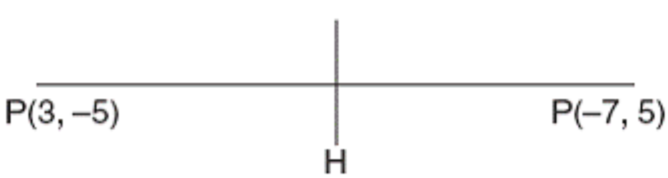
(a) Let H be the mid-point of line TP.



$$\text{The mid-point of TP} = H\left(\frac{3+5}{2}, \frac{4+2}{2}\right)$$

$$= H(4, 3)$$

(b) Let R be the mid-point of PQ.



The mid-point of PQ

$$= R\left(\frac{3+(-7)}{2}, \frac{-5+5}{2}\right)$$

$$= R\left(\frac{3-7}{2}, \frac{0}{2}\right)$$

$$= R(-2, 0)$$

Worked Example 5

Show that the points A(1, 3), B(2, 5), C(5, 8) and D(4, 6) form the vertices of a parallelogram. The points are shown in Figure 10.5.

SOLUTION

The mid-point of AC is $\left(\frac{1+5}{2}, \frac{3+8}{2}\right)$ or $\left(3, 5\frac{1}{2}\right)$.

The mid-point of BD is $\left(\frac{2+4}{2}, \frac{5+6}{2}\right)$ or $\left(3, 5\frac{1}{2}\right)$.

Since AC and BD have a common mid-point, these lines must bisect each other. Therefore, ABCD is a parallelogram (diagonals bisect).

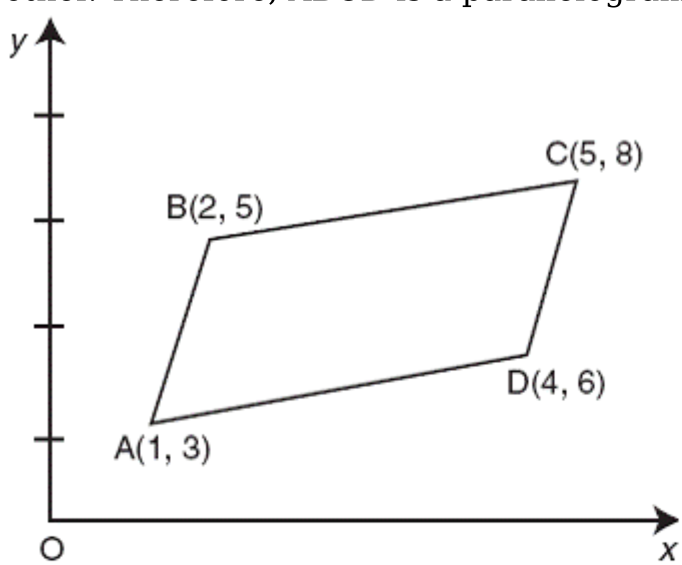


Figure 10.5

Worked Example 6

Prove that the points A(−1, 2), B(3, 4) and C(2, −4) form a right-angled triangle and find its area.

SOLUTION

$AB^2 = (3 + 1)^2 + (4 - 2)^2 = 20$

$AC^2 = (2 + 1)^2 + (-4 - 2)^2 = 45$

$BC^2 = (3 - 2)^2 + (4 + 4)^2 = 65$

∴ BC2 = AB2 + AC2 and hence angle A is a right angle, that is A = 90°.
Therefore, the area of the triangle:

$= \frac{1}{2}\sqrt{20} \times \sqrt{45}$

$= \frac{1}{2}\sqrt{900}$

$= 15$ sq units $\quad = \frac{1}{2} AB \cdot AC$

Exercise 2

Write down the mid-points of the line joining the points given in Questions 1–5:

1. (1, 1) and (4, 5)
2. (5, 1) and (3, −1)
3. (4, 2) and (−5, 0)
4. (6, 1) and (8, −2)
5. (−1, −6) and (−3, −2)

Find the mid-point of the following line segments in Questions 6–8:

6. A(2, 4) and B(−3, 2)
7. P(−2, 3) and Q(5, −5)
8. X(−4, 5) and Y(−3, 7)

State the coordinates of the midpoint of the line joining the points in Questions 9–13:

9. (−1, 3) and (5, −4)
10. (0, 2) and (6, 0)
11. (2*a*, *b*) and (*a*, 2*b*)
12. (−5, −5) and (−7, −5)
13. (2, 3) and (4, 7)
14. Find the sizes of the angles of a triangle ABC if A is (−1, −1), B is (2, −4) and C is (4, 1).
15. Show that the points (−1, 4), (2, 2), (2, 5) and (5, 3) are the vertices of a

parallelogram.

16. Show that the points (2, 0), (−2, 0), (0, 3) and (0, −3) are the vertices of a rhombus.

17. Using Pythagoras' theorem, show that the triangle ABC is a right angled triangle where A is (−4, −2), B is (4, 2) and C is (2, 6).

18. The points A(3, 4), B(4, −3) and C(8, 6) form a triangle ABC. Give the coordinates of M , the midpoint of BC.

19. The vertices of a triangle ABC are A(2, 6), B(2, 3) and C(4, 1).

    (a) Find the coordinates of *D,* the mid-point of BC.

    (b) Show that AB2 + AC2 = 2BD2 + 2AD2.

20. Show that the points A(1, 4), B(2, 7) and C(5, 6) form a right-angled triangle and find its area.

IV. Practical Application of Coordinate Geometry

In a system of pulleys, an eff ort of *P*1 N is required to lift a mass of *M* kg. It is thought that a law of the form $P = aM + b$, where *a* and *b* are constants, exists between *a* and *b.* From the given experimental values, fi nd if such a law does hold and determine suitable values for *a* and *b.*

Table 10.1

P	12	24	32	43	52
M	2	4	6	8	10

It is possible to take two pairs of values and substitute them in the equation $P = aM + b$. The values of *a* and *b* can then be found by solving the resulting simultaneous equations for *a* and *b.* The disadvantage of this method is that one of the measurements taken may be a faulty one, and the best method is one which utilises all the information given. Plot the points on a graph taking *P* as the vertical axis and M as the horizontal axis. The axes are chosen such that the equation $P = aM + b$ corresponds to $y = mx + c$.

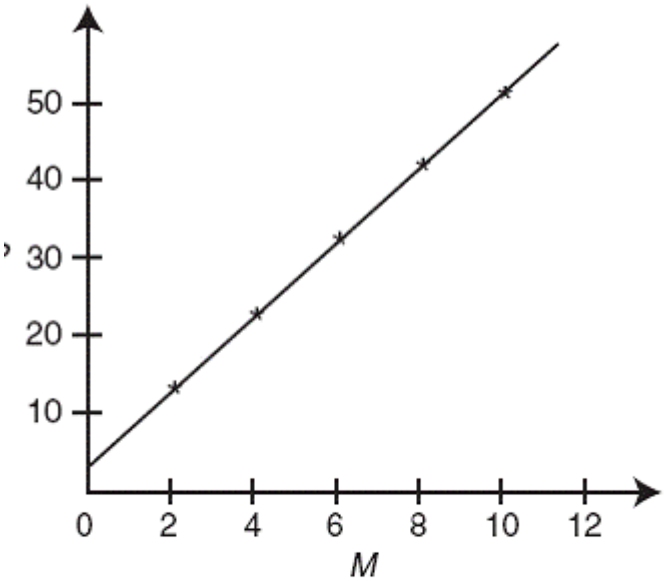


Figure 10.6

If the points are approximately on a straight line, a law such as $P = aM + b$ does hold. Draw a line which most nearly fits the points. If one of the points is clearly not on the best line suggested by other points, disregard it and consider the discrepancy to be due to an error in measurement. Produce the line to meet the axis at P as shown in Figure 10.6. The value of '*a*' is the gradient of the line and *b* is equal to the intercept on the axis of P which is 2.

$$\text{The gradient} = \frac{52-2}{10} = 5$$

$$\therefore \quad a = 5, b = 2$$

Consider the relation $y = axn$. The same principle may also be used to find the values of *a* and *n* when it is known that variables *x* and *y* satisfy the equation $y = axn$, where *a* and *n* are constants.

If $y = ax^n$, then

$$\log y = \log a + n \log x$$

So log *y* (vertically) is plotted against log *x* (horizontally). The points should lie on a straight line for which the gradient is *n*.
The intercept made by the line on the log *y*-axis is equal to log *a*, from which *a* may be found. See the graph in Figure 10.7.

Worked Example 8

If *x* and *y* obey the law $y = axn$. Find the values of *a* and *n* from the corresponding values of *x* and *y* given in Table: 10.2 below.

Table 10.2

y	178	617	1 400	2 820	4 900
x	4	6	8	10	12

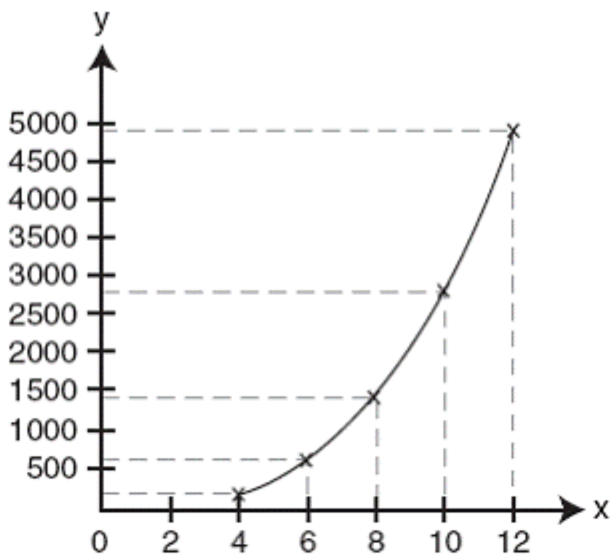


Figure 10.7

SOLUTION

Find the values of log x and log y.

Table 10.3

log y	2.2504	2.7903	3.1461	3.4502	2.6902
log x	0.6021	0.7782	0.9031	1.0000	1.0792

Plot log y against log x. The points lie on a straight line and this line when produced meets the axis of log y at the point where log y = 0.45.

$\therefore$ log $a = 0.45$ and $a = 2.82$

The gradient of the line = 3.45 − 0.45 = 3

$y = 2.82xn$ (approximately)

Exercise 3

1. It is thought that the values of P and W given in the table are connected by an equation of the form $P = a\text{W} + b$. Find suitable values for a and b

Table 10.4

P	2.5	3.8	5.0	6.5	7.5
W	6	10	14	18	22

2. The following values of x and y are thought to satisfy $y = axn$. Find suitable values for a and n.

Table 10.5

y	0.4	0.9	1.6	2.5	3.6
x	2	3	4	5	6

3. If the corresponding values of p and v are given as in Table 10.6, show that p and v satisfy $p = av + b$, and fi nd the values of a and b.

Table 10.6

p	3.3	4.0	4.7	5.4	6.0
v	12	15	18	22	25

4. A law such as $p = avn$ is thought to connect p and v. If experimental values are given as in Table 10.7,find suitable values for a and n.

Table 10.7

p	6.32	8.94	10	12.7	14.1
v	10	20	25	40	50

5. It is known that the quantities x and y are connected by a law such as $y = axn$. There is a misprint in the value of y given in the table. Find which value is wrong and state what it should be.

Table 10.8

y	3.162	2.632	1.826	1.581	1.414
x	10	20	30	40	50

6. It is predicted from theory that two variables P and T are related by the equation

$$P = a + \frac{b}{\sqrt{T} - 3},$$

where the constants a and b have to be found. Some values of P and T are found as in Table 10.9.

Table 10.9

T	10	20	30	40	50
P	13.3	10.6	9.4	8.9	8.7

By drawing a graph of P against $\frac{1}{\sqrt{T} - 3}$, confirm that the equation is approximately correct and find the values of a and b.

7. Two variables T and R are believed to be related by the equation $T = aRb$ where a and b are constants. The values in Table 10.10 were obtained by experiment.

Table 10.10

R	2	5	8	10	12
T	24	167	450	700	1050

Plot a graph of log T against log R and show that the law is approximately true for these values. From your graph, deduce the values of a and b, and fi nd the value of T when $R = 6$.

8. A graph of log y against log x, for a set of experimental results, gives

a straight line. The gradient of the line is 2 and the intercept is 1.7.

Find the relationship between x and y.

9. The pairs of values of x and y in Table 10.11 have been found.

Table 10.11

X	2	5	8	10	15
Y	16	14	13	13	12

It is expected that a law of the form $y = a + \frac{b}{x}$ connects x and y. Draw a graph of y against _ 1x_ toverify if this law is approximately valid.

10. It is thought that two variables $\lambda V2 = bV + s$, where b and s are constants. The series of experimental results shown in Table 10.12 was taken.

Table 10.12

V	1	2	5	8
λ	12	3.5	0.8	0.406

Plot the values of $\lambda V2$ on the y-axis against the values of V on the x-axis to verify that the variables do obey the equation. From your graph, deduce the approximate value of the constants b and s, and also find the value of λ when $V = 4$.

SUMMARY

In this chapter, we have learnt the following:

- v The position of a point in a plane can be given as its coordinates, that is, the assigned distances of the point from two perpendicular axes Ox and Oy.
- v The x-coordinate is called the abscissa and the y-coordinate is called theordinate.
- v The coordinates form an ordered pair with the abscissa written fi rst.
- v Distance between (x_1, y_1) and (x_2, y_2) $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
- v Mid-point of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- v Coordinate geometry can be used to solve a system of linear equations by plotting a linear graph and then reading from the graph.
- v If $y = ax^b$, then $\log y = b \log x + \log a$, that is $Y = bX + \log a$, a linear equation between $Y(\log y)$ and $X(\log x)$, with gradient b and intercept $\log a$.

GRADUATED EXERCISES

1. Find the distance between points (6, −8), (2, −5).
2. Show that (3, 4), (−2, −2) and (3, −8) are the vertices of an isosceles triangle.

Find the distance between the given points in Questions 3–10:

3. $(7, 9), (4, 5)$

4. $(15, 11), (3, 6)$

5. $(4, -5), (0, 0)$

6. $(2, -11), (-4, -3)$

7. $(\sqrt{2}, 1), (2\sqrt{2}, 3)$

8. $(\sqrt{3}, -\sqrt{2}), (3\sqrt{3}, \sqrt{2})$

9. $\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{-4}{3}, \frac{4}{3}\right)$

10. $(1, -3), (3, 5)$

Find the unknown values in Questions 11–14:

11. $A = (1, 5), B = (x, 2), |AB| = 5$

12. $A = (-3, y), B = (9, 2), |AB| = 13$

13. $P = (x, x), Q = (1, 4), |PQ| = \sqrt{5}$

14. $M = (x, 2x), N\ (2x, 1), |MN| = \sqrt{2}$

15. Show that $(-2, 4)$, $(2, 0)$, $(2, 8)$ and $(6, 4)$ are the vertices of a square

16. Show that $(1, 1)$, $(4, 1)$, $(3, -2)$ and $(0, -2)$ are the vertices of a parallelogram

In Questions 17–21, find the mid-point of the line joining the given points:

17. $(5, 8), (9, 11)$
18. $(0, 0), (8, -15)$
19. $(-7, 0), (0, 10)$
20. $(-4, 3), (6, 7)$
21. $(4, -1), (3, 3)$

22. Find the mid-point of the sides of a triangle whose vertices are $A(1, -1)$, $B(4, 1)$, $C(4, 3)$.

23. The coordinates of three points are given. By finding the distance between each pair of the three points, state the type of triangle formed.

(a) $(-1, 2)$, $(3, 5)$ and $(2, 6)$
(b) $(-3, -1)$, $(-1, 0)$ and $(-2, -3)$
(c) $(-4, -2)$, $(2, 6)$ and $(4, 2)$

24. The values in Table 10.13 are obtained by experiment and

Table 10.13

x	2	3	4	5
y	113	312	640	1 118

it is suspected that y is a function of x of the form $y = axb$. By plotting log y against log x, test this theory and if correct, fi nd the approximate valuesof a and b. From your graph, find the value of x for which $y = 215$.

25. A scientific law obeys an equation $y = mnx$, where x and y are variables and

m and n are constants. Experimental results give the set of values of x and y in Table 10.14.

Table 10.14

X	1	2	3	4	5
Y	264	1 740	11 500	75 900	560 000

Plot a graph of x against log y and deduce the values of the constants m and n. For what value of x is $y = 4\ 500$? (Use your graph to determine the value of x correct to two significant figures).