CHAPTER 6 Calculus

The word 'calculus' quickly brings to mind $\frac{dy}{dx}$ (this makes most students break out in sweat), and this is the basic building block of differential calculus (commonly called differentiation). However, before using the tool $\left(\frac{dy}{dx}\right)$ at the workshop, it is important to have a good grasp of it. Honestly, there is no big deal about it.

The gradient of any straight line is a measure of how steep the straight line is. Gradient is quantified as the ratio of change in y to change in x, and the gradient of any **straight** line is mathematically expressed as $\frac{dy}{dx}$

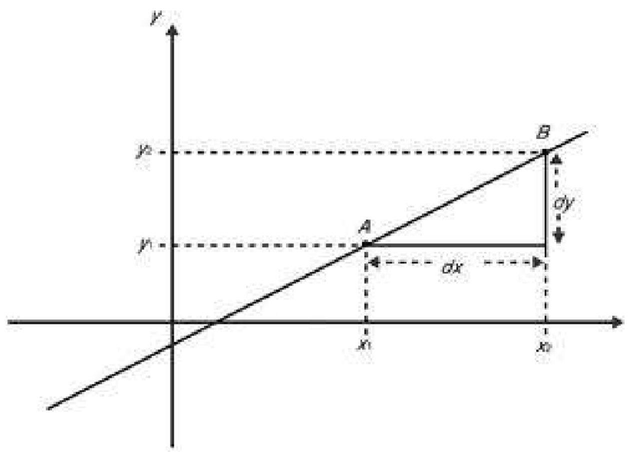

Fig 6.1

The gradient of the line in Figure 6.1 is expressed as $\frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$

How can the gradient of curves be calculated, bearing in mind that unlike straight lines, the gradient of a curve keeps changing as it stretches on? The curve of the equation $x^2 + 2$ can be made a point of reference.

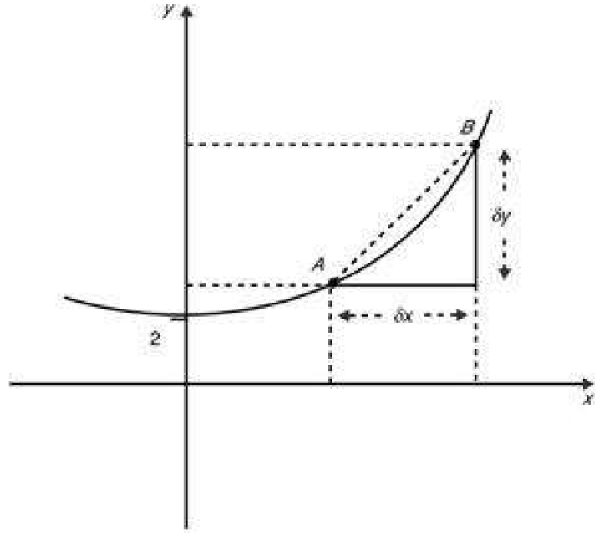

Fig 6.2

Figure 6.2 shows the right side view of the curve of $x^2 + 2$. δy and the δx are being used to represent the vertical and horizontal distances between points A and B on the curve. dy and dx are **not** used because we are dealing with a curve here and **NOT** a straight line (However, bear in mind that one can always find the gradient of this curve at **each point** by drawing a tangent to the curve at that point). The gradient of a curve at a point P on a curve can be found by drawing a tangent to the curve (a straight line that touches the curve at that point P alone); the gradient of this tangent is the gradient of the curve at point P.

Imagine that point A, in Figure 6.2, is fixed (for example, a supermarket along a street), and think of point B approaching point A gradually (see point B as a person walking towards the supermarket).

But we can write the gradient of the broken line AB as $\frac{\delta y}{\delta x}$, the only difference is that here, we name difference in y as δy and the change in x as δx because of the curve.

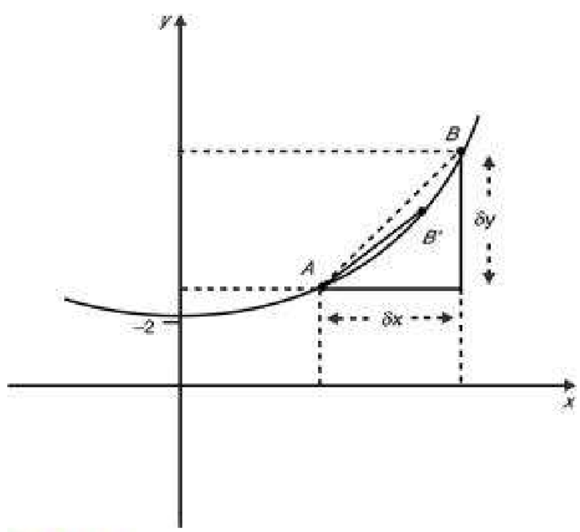

Fig 6.3a

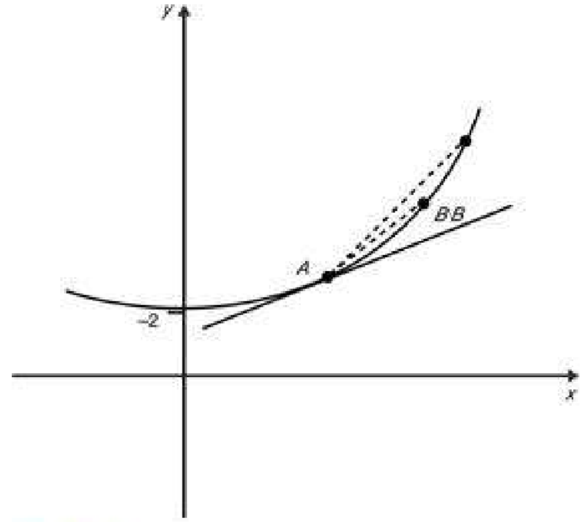

Fig 6.3b

As point B approaches the fixed point A, it gets to a new point B' and, as you can see in Figure 6.3a, the steepness (gradient) of line AB' has reduced, compared to the gradient of AB. Take a careful look at Figure 6.3b to see that as point B approaches the fixed point A, the steepness (gradient) of successive lines will be reducing so that the gradient of line AB will be gradually getting closer (approaching) to the gradient of the tangent drawn at point A as shown in Figure 6.3b.

A close study of Figures 6.3*a* and *b* shows that as point B moves towards the fixed point A, distance δx is gradually reducing (and if point B gets to A, then $\delta x = 0$). Thus, as B approaches A, δx tends to zero. Recall that we have already established that as point B approaches the fixed point A, the gradient of line AB approaches the gradient of

the tangent to the curve at point A. Therefore, all these findings can be put together as follows: "the least value of the **steepness of line *AB*** (in other words $\frac{\delta y}{\delta x}$) as **point *B* approaches the fixed point *A*** (that is, as δx approaches zero) will be the **gradient of the tangent to the curve at *A*** (which is $\frac{\delta y}{\delta x}$).

These findings can be expressed mathematically as "limit of $\frac{\delta y}{\delta x}$ as δx tends to (approaches) zero is equal to $\frac{\delta y}{\delta x}$. That is $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$

Please note carefully that $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$ are NOT the same, they are different. $\frac{\delta y}{\delta x}$ gives the ratio of the vertical distance to the horizontal distance between two specified points on a curve, while $\frac{dy}{dx}$ the gradient of a straight line, or the gradient of the tangent to a curve at a point, and this is also always equal to the gradient of the curve at that same point.

Find, from the first principles, the derivatives, with respect to x, of $y = 3x^2$. **(WAEC)**

Workshop

Know that y is a function of x since $y = 3x^2$, so that $y = f(x) = 3x^2$.

The differential, $\frac{dy}{dx}$, of any function of x (that is, $y = f(x)$), from the first principle of differentiation, is given by $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$, where δy is the change in y and δx is the change in x.

> *Note carefully that Δ is the greek capital letter, while δ is its corresponding small letter. So any of the two can be used, but remember that both of them are not the same as the d in* $\frac{dy}{dx}$.

Thus, $\frac{dy}{dx}$ is the limit of $\frac{\delta y}{\delta x}$ as δx tends to zero.

Since $y = f(x)$, then $y + \delta y = f(x + \delta x)$. Therefore,

$y + \delta y = f(x + \delta x)$ recall that $y = f(x) = 3x^2$; so,

$y + \delta y = f(x + \delta x) = 3(x + \delta x)^2 = 3(x2 + 2x\delta x + (\delta x)^2)$; $\delta y + 6x\delta x + 3(\delta x)^2$;

$$\therefore \frac{\delta y}{\delta x} = \frac{6x\delta x + 3(\delta x)_2}{\delta x} = 6x + 3\delta x \cdot \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$$

$$= \lim_{\Delta x \to 0}(6x + 3\delta x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}(6x + 3\delta x) = 6x + 3(0) = 6x.$$

Therefore, the derivative, with respect to x, of $y = 3x^2$ is $6x$.

In case this basic concept of calculus is yet to be fully grasped, read through one more time. Before we start out to the workshop, let us refresh our memory on some standard differentials and integrals.

Integration provides the means to work from the differentials gotten from differentiation back to the original function. Find below some standard differentials and integrals.

Standard Differentials

$y = f(x)$	$\frac{dy}{dx}$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\text{cosec} x$	$-\text{cosec } x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\text{cosec}^2 x$
$\text{In } x$	$\frac{1}{x}\ (x > 0)$
a^x	$a^x \text{ In } a$

Standard Integrals

$f(x)$	$\int f(x)$
x^n	$\frac{x^{n+1}}{n+1}$ $(n \neq -1)$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\frac{1}{x}$	$\text{In}\,\lvert x \rvert$
e^x	e^x
a^x	$\frac{a^x}{\text{In}\, a}$ $(a > 0)$

Differentiation

1. Find the derivative with respect to *x* of $y = \frac{2x}{x^2+3}$ **(WAEC)**

Workshop

Given that $y = \frac{u}{v}$, where u and v are functions of x, $\frac{dy}{dx}$ is calculated as $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. In this case, $y = \frac{2x}{x^2+3}$, thus, $u = 2x$ and $v = x^2 + 3$.

As $u = 2x$; $\frac{du}{dx} = 2$; $v = x^2 + 3$; $\frac{dv}{dx} = 2x$. Having known u, $\frac{du}{dx}$, v and $\frac{dv}{dx}$, then, $\frac{dy}{dx}$ can be calculated as follows:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^2+3)(2) - 2x(2x)}{(x^2+3)^2}$$

$$= \frac{2x^2 + 6 - 4x^2}{(x^2+3)^2} = \frac{6 - 2x^2}{(x^2+3)^2}.$$

Therefore, the derivative with respect to *x* of $y = \frac{2x}{x^2+3}$ is $\frac{6-2x^2}{(x^2+3)^2}$.

2. Differentiate with respect to *x*, the function $y = \frac{4x^3 - 3}{x^2 + 1}$. **(WAEC)**

Workshop

$y = \dfrac{4x_3 - 3}{x_2 + 1}$, let $4x_3 - 3 = u$, and $x_2 + 1 = v$; given a function

$y = \dfrac{f(x)}{g(x)} = \dfrac{u}{v}$, $\dfrac{dy}{dx}$ can be calculated by the quotient rule that states thus: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$. In the case of this problem, $u = 4x_3 - 3\ \dfrac{du}{dx} = 12x_2$, while $v = x^2 + 1$ and $\dfrac{dv}{dx} = 2x$.

Hence,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v_2} = \frac{(x_2 + 1)(12x_2) - (4x_3 - 3)\,(2x)}{(x_2 + 1)^2};$$

$$\frac{dy}{dx} = \frac{12x_4 + 12x_2 - 8x_4 + 6x}{x_4 + 2x_2 + 1} = \frac{4x_4 + 12x_2 + 6x}{x_4 + 2x_2 + 1}$$

3. If $y = px^2 + qx, \frac{dy}{dx} = 7$ and $\frac{d^2y}{dx^2} = 6$, find the values of p and q. **(WAEC)**

Workshop

$y = px_2 + qx, \dfrac{dy}{dx} = 7$ and $\dfrac{d_2y}{dx^2} = 6; y = px_2 + qx;$

$\dfrac{dy}{dx} = 2px + q = 7; \dfrac{d_2y}{dx^2} = 2\,p = 6; p = \dfrac{6}{2} = 3.$

$\dfrac{dy}{dx} = 2px + q = 2(3)x + q = 7; \dfrac{dy}{dx} = 6x + q = 7;$

$6x + q = 7; q = 7 - 6x.$

Therefore, the values of p and q are 3 and $7 - 6x$ respectively.

Integration

1. Using the substitution $U = 5 - x^2$, evaluate

$\int_1^2 \frac{x}{\sqrt{5 - x^2}}\, dx.$ **(WAEC)**

Workshop

Using the substitution,

$U = 5 - x_2,\ \int_1^2 \dfrac{x}{\sqrt{5 - x_2}} \cdot dx = \int_1^2 \dfrac{x}{\sqrt{U}} \cdot dx$

We cannot find these integral because x is not the same as U: in other words, to evaluate the integral, the integral should either be all in U or all in x.

But $U = 5 - x^2$, thus, $\frac{dU}{dx} = -2x$;

$$dU = -2x \cdot dx;\ dx = \frac{dU}{-2x}.$$

Put $dx = \frac{dU}{-2x}$ into the integral to get $\int_1^2 \frac{x}{\sqrt{U}} \cdot dx$

$$= \int_1^2 \frac{x}{\sqrt{U}} \cdot \frac{dU}{-2x} = \int_1^2 \frac{x}{-2x} \cdot \frac{dU}{\sqrt{U}} = \int_1^2 \frac{1}{-2} \cdot \frac{dU}{\sqrt{U}}$$

$$= \int_2^1 \frac{1}{-2\sqrt{U}} \cdot dU = -\frac{1}{2}\int_1^2 \frac{1}{\sqrt{U}} \cdot dU.$$

Note that $\int -\frac{1}{2\sqrt{U}} \cdot dU = -\frac{1}{2}\int \frac{1}{\sqrt{U}}$.

while $\int\left(\frac{1}{\sqrt{U}} - \frac{1}{2}\right) \cdot dU$

$$= \left(\int \frac{1}{\sqrt{U}} \cdot dU\right) - \left(\int \frac{1}{2} . dU\right)$$

$$= \left(\int \frac{1}{\sqrt{U}} \cdot dU\right) - \left(\frac{1}{2}\int dU\right).$$ *Also.*

Hence, $-\frac{1}{2}\int_1^2 \frac{1}{\sqrt{U}} \cdot$

$$dU = -\frac{1}{2}\int \frac{1}{\sqrt{U}} \cdot dU = -\frac{1}{2}\int_1^2 U^{-\frac{1}{2}} \cdot dU;$$

$$= -\frac{1}{2}\left[\frac{U^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^2 = -\frac{1}{2}\left[\frac{U^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^2 = -\frac{1}{2}\left[2U^{\frac{1}{2}}\right]_1^2$$

$$= -\frac{1}{2}\left[(2\sqrt{U})\right]_1^2.$$

Remember to write U as a function of x before using the boundary values of x, which are 2 and 1.

Recall that, $U = 5 - x^2$, therefore,

$$-\frac{1}{2}\left[2\sqrt{U}\right]_1^2 = -\frac{1}{2}\left[2\sqrt{5 - x^2}\right]_1^2$$

$$\frac{1}{2}\left[(2\sqrt{5 - 2^2}) - (2\sqrt{5 - 1^2})\right]$$

$$\frac{1}{2}\left[(2\sqrt{5 - 4}) - (2\sqrt{5 - 1})\right] = -\frac{1}{2}\left[2\sqrt{1} - 2\sqrt{4}\right]$$

$$\frac{1}{2}\left[2(1) - 2(2)\right]$$

$$\frac{1}{2}\left[2 - 4\right] = -\frac{1}{2}\left[-2\right] = 1.$$

Thus, $\int_1^2 \frac{x}{\sqrt{5 - x^2}} \cdot dx = 1$.

2. Using the substitution, $U^2 = (4 - x^2)$,

evaluate $\int_0^1 \frac{x}{\sqrt{4 - x^2}}\,dx$. **(WAEC)**

Workshop

If $U^2 = (4 - x^2)$; then $2U\dfrac{dU}{dx} = 0 - 2x$.

Carefully note that if U is a function of x (i.e, $U = f(x)$), to differentiate any term in U with respect to x, $\dfrac{dU}{dx}$ must be included in the result. For example, if you want to differentiate U^3 with respect to x, the answer will be $3U^2\dfrac{dU}{dx}$. This is what is called implicit differentiation. Also, if $U^4 + x^2 + 1 = 0$ is differentiated with respect to x, the answer will be $4U^3\dfrac{dU}{dx} + 2x^1 + 0 = 0$; $4U^3\dfrac{dU}{dx} + 2x = 0$; $4U^3\dfrac{dU}{dx} = -2x$; $\dfrac{dU}{dx} = \dfrac{-2x}{4U^3}$.

$2U\dfrac{dU}{dx} = 0 - 2x$; $2U\dfrac{dU}{dx} = -2x$; $2U\,dU = -2x\,dx$;

$dx = \dfrac{2U \cdot dU}{-2x}$.

Since $U^2 = (4 - x2) \therefore U = \sqrt{4 - x^2}$;

hence, $\int_0^1 \dfrac{x}{\sqrt{4 - x^2}} \cdot dx = \int_0^1 \dfrac{x}{U} \cdot dx$

Put $dx = \dfrac{2U \cdot dU}{-2x}$ into the integral, to get

$$\int_0^1 \left(\frac{x}{U}\right) \cdot dx = \int_0^1 \left(\frac{x}{U}\right) \cdot \left(\frac{2U \cdot dU}{-2x}\right) = \int_0^1 \frac{x \cdot 2U \cdot dU}{U(-2x)}$$

$$= \int_0^1 \frac{x}{-2x} \cdot \frac{2U}{U} \cdot dU = \int_0 +1 \cdot dU = -1\int_0^1 dU$$

$$= -1\int_0^1 U^0 \cdot dU;$$

Recall that $U^0 = 1$, then,

$dU = 1 \times dU = U^0 \times du$, therefore,

$$-1\int_0^1 U^0 . dU = -1\left[\frac{U^{0+1}}{0+1}\right]_0^1 = [-\mathrm{U}]\Big|_0^1.$$

Also, recall that $U = \sqrt{4 - x^2}$,

then $[-U]\Big|_0^1$

$= -\sqrt{4 - x^2}\,\Big|_0^1$.

Please note that $\int_0^1 \frac{x}{\sqrt{4-x^2}} \cdot dx$ *means* $\int_{x=0}^{x=1} \frac{x}{\sqrt{4-x^2}} . dx$*; hence,* $-U\Big|_0^1$ *means* $-U\Big|_{x=0}^{x=1}$ *then to evaluate* $-U\Big|_0^1$, *you* ***must first*** *rewrite* $-U$ *as a function of x, before evaluating* $-U\Big|_0^1$, *as we will do shortly. Also, know that to evaluate* $\int f(x)\, du$, *you either try to rewrite* $f(x)$ *in terms of u, or you rewrite du in terms of x, before you go ahead to find the integral.*

Thus, $-\sqrt{4-x^2}\Big|_0^1 = -\sqrt{4-(1)^2} - \left(-\sqrt{4-0^2}\right)$

$$= -\sqrt{3} + \sqrt{4} = 2 - \sqrt{3}.$$

Therefore, $\int_0^1 \frac{x}{\sqrt{4-x^2}} \cdot dx = 2 - \sqrt{3}.$

3. Using the substitution $U = x^2 + x + 4$, Find;

$\int \frac{2x+1}{\sqrt{x^2+x+4}} \cdot dx.$ **(WAEC)**

Workshop

If $U = x^2 + x + 4$, $\frac{dU}{dx} = 2x + 1$. Substituting $U = x^2 + x + 4$, and $2x + 1 = \frac{dU}{dx}$ into the integral gives

$$\int \left(\frac{2x+1}{\sqrt{x^2+x+4}}\right) \cdot dx = \int \left(\frac{\frac{dU}{dx}}{\sqrt{U}}\right) \cdot dx = \int \frac{dU}{dx} \div \sqrt{U} \cdot dx$$

$$= \int \frac{dU}{dx} \times \frac{1}{\sqrt{U}} \cdot dx = \int \frac{dU}{dx} \times \frac{dx}{\sqrt{U}} = \int \frac{du}{\sqrt{U}} = \int \frac{1}{\sqrt{U}} \cdot dU$$

$$= \int \frac{1}{\sqrt{U}} \cdot dU$$

is an arbitrary constant.

$$\frac{U^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{U^{\frac{1}{2}}}{\frac{1}{2}} + C = 2U^{\frac{1}{2}} + C = 2\sqrt{U} + C.$$

Recall that $U = x^2 + x + 4$, thus, $2\sqrt{U} + C = 2\left(\sqrt{x^2+x+4}\right) + C$, where C is an arbitrary onstant.

Therefore, $\int \frac{2x+1}{\sqrt{x^2+x+4}} \cdot dx \cdot = 2\left(\sqrt{x^2+x+4}\right) + C.$

*Note carefully that, since additional information is not provided, we **cannot** know the value of C, so we will leave the answer as written above. You should also note that for bounded integrals like $\int_a y.dx$ you do **not** need to include any arbitrary constant C, when the integral is evaluated. However, the integral in question is **not** a bounded integral; therefore we **must** include the arbitrary constant C to get maximum (full) mark.*

Let us see an example to show why it is necessary to include the arbitrary constant *C* in evaluating any unbounded integral. If, $y = 2x2 + 5$, $\frac{dy}{dx} = 4x$. This means that if we integrate 4*x*, with respect to *x* (*that is* $\int 4xdx$)we should get $2x^2 + 5$; again if $y = 2x^2 - 3$, $\frac{dy}{dx} = 4x$, *and so*, if we integrate 4x with respect to x, we should get $y = 2x^2 - 3$.

Now, assuming we were to evaluate $\int 4xdx$, without having a prior knowledge that

$$\frac{d(2x_2 + 5)}{dx} = 4x; \int 4xdx = \int 4x_1 dx$$

$$= 4\frac{x_{1+1}}{1+1} + C = \frac{4x_2}{2} + C = 2x_2 + C.$$

Therefore, $\int 4xdx$ could be $= 2x2 + 5$, that is *C* can be 5 going by what we started with, or *C* can be −3, as also explained, or any other number.

Thus, *C* can only be known if more information is given in the question.

4. Using the trapezium rule with seven ordinates, evaluate $\int_0^3 \frac{dx}{x^2+1}$ correct to two decimal places. **(WAEC)**

Workshop

$$\int_0^3 \frac{dx}{x_2+1} = \int_0^3 \frac{1}{x_2+1} \cdot dx, \text{ so, } f(x) = \frac{1}{x^2+1}.$$

The trapezium rule states thus:

$\int_{x_1}^{x_n} f(x).\, dx = \frac{1}{2}h[f(x_1) + f(x_n) + 2[f(x_2) + f(x_3) + f(x_4)] + \ldots + f(x_{n-1})]]$. Thus, to evaluate $\int_0^3 \frac{dx}{x^2+1}$,

since we are told to use seven ordinates (ordinates means *x* values), then, $n = 7$. So interval width,

$$h = \frac{x_n - x_1}{n-1} = \frac{x_7 - x_1}{7-1} = \frac{3-0}{7-1} = \frac{3}{6} = \frac{1}{2} = 0.5.$$

The interval width $h = 0.5$, and $x1 = 0$. Therefore

$x_2 = x_1 + h = 0 + 0.5 = 0.5;$

$x_3 = x_2 + h = 0.5 + 0.5 = 1; x_4 = x_3 + h = 1 + 0.5 = 1.5;$

$x_5 = x_4 + h = 1.5 + 0.5 = 2; x_6 = x_5 + h = 2 + 0.5 = 2.5;$

$x_7 = x_6 + h = 2.5 + 0.5 = 3.$

Now we have seven ordinates $x1$ to $x7$, which are having equal interval width of 0.5. We can now find $f(x_1), f(x_2), \ldots\ldots f(x_7)$.

Note that h depends on the number of ordinates n that you want to make use of. For example, if we want to use 13 ordinates, then $h = \frac{3-0}{13-1} = \frac{3}{12} = \frac{1}{4} = 0.25$. So, the interval width in this case will be 0.25 and the 13 ordinates will be $x = 0, 0.25, 0.5, 0.75, 1.0, \ldots, 2.5, 2.75, 3.0$.

x	0	0.5	1	1.5	2	2.5	3
x^2	0	0.250	1.000	2.250	4.000	6.250	9.000
x^2+1	1.000	1.25	2.000	3.250	5.000	7.250	10.000
$\frac{1}{x^2+1}$	1.000	0.800	0.500	0.308	0.200	0.138	0.100

It is good to calculate the components of $f(x)$ (in this case x^2 and x^3+1) to 3 decimal places and also the values of $f(x)$ $\left(\text{in this case } \frac{1}{x^2+1}\right)$ to 3decimal places as explained above to make our final answer $(\int f(x)dx)$ more accurate.

Let us see how one of the $f(x)$ in the boxes above was calculated.

In column 2, $x = 0.5$; $x^2 = (0.5)^2 = 0.250$; $x^2 + 1 = 0.250 + 1 = 1.250$;

$f(x) = \frac{1}{x^2+1} = \frac{1}{1.250} = 0.800$ And the same method applies to others, so,

$h = 0.5$; $f(x_1) = 1.00$; $f(x_2) = 0.800$; $f(x_3) = 0.500$; $f(x_4) = 0.308$; $f(x_5) = 0.200$

$f(x_6) = 0.138$; $f(x_7)$; $= 0.100$ Recall the trapezium rule:

$$\int_{x_1} f(x).dx = \frac{1}{2} h\, f(x_1) + f(x_n) + 2$$

$$[f(x_2) + f(x_3\, f(x_4) + \ldots + f(x_{n-1}))];$$

$$\int_0^3 \frac{1}{x^2+1} \cdot dx = \frac{1}{2}h\,[f(x_1) + f(x_7) + 2(f(x_2) + f(x_3) +$$

$$f(x_4) + f(x_5) + f(x_6)];$$

$$\int_0^3 \frac{1}{x^2+1} \cdot dx = \frac{1}{2}\,(0.500)$$

$$[1 + 0.1 + 2\,[0.8 + 0.5 + 0.308 + 0.2 + 0.138]];$$

$$= \frac{0.500}{2}\,(1.1 + 2(1.946)) = 0.250(1.1 + 3.892)$$

$$= 0.25 \times 4.992 = 1.248.$$

Therefore, $\int_0^3 \frac{1}{x^2+1} .dx$ correct to 2 decimal places is 1.25.

Note that if you are told to give an answer to a particular decimal place, you must always approximate the values you obtain in-between calculation to a decimal place higher than that in which you are told to leave your final answer. For example, in this solution, since the question requires you to leave the final answer correct to 2 decimal place, after evaluating 0.8 + 0.5 + 0.308 + 0.2 + 0.138, efforts should be made to obtain the value of this addition to at least 3 decimal places (that is 1.946). Also in multiplying 1.946 by 2 and also 0.25 by 4.992, you should leave your answer to at least 3 decimal place as explained so that the final approximation will be more accurate to 2 decimal place as explained above. Note that this applies to all other problems in this book.

Applications of Calculus: Area Under Curves, Maximum Points and Minimum Points

1. A curve, $y = px^2 + qx + 3$, where p and q are constants, has a turning point at (1, 2).

   (a) Find the values of p and q.

   (b) Determine whether the turning point is maximum or minimum. **(WAEC)**

Workshop

(a) At the turning point of **any** curve with equation $y = f(x)$, $\frac{dy}{dx} = 0$.

If $y = px^2 + qx + 3$, $\frac{dy}{dx} = 2px + q$, which is equal to zero at the turning point i.e $\frac{dy}{dx} = 2px + q = 0$. From the question, the turning point is at (1.2); therefore,

$\frac{dy}{dx} = 2px + q = 2p(1) + q = 0$; $2p + q = 0$ ......(1)

Since point (1, 2) is a point on the curve, $y = px2 + qx + 3$, the equation of the curve at this point will be $y = 2\ p(1)^2 + q(1) + 3$;

(*Recall that at point* (1, 2), $y = 2$ while $x = 1$) Hence,

$2 = p(1)^2 + q(1) + 3$; $2 = p + q = -1$; ....(2).

Solving equations (1) and (2) simultaneously we get $p = 1$ and $q = -2$.

(b) For any point to be a minimum turning point, $\frac{d^2y}{dx^2}$ at that point, must be greater than zero (that is $\frac{d^2y}{dx^2}$, at that point, must be positive). From (a) above, $\frac{dy}{dx} = 2px + q = 2(1)\,x + (-2) = 2x - 2$; $\frac{d^2y}{dx^2} = 2$.

Since $\frac{d^2y}{dx^2}$ is positive, the point (1, 2) is a **minimum** turning point.

2. Find the finite area enclosed by the curve $y^2 = 4x$ and the line $y + x = 0$. (WAEC)

Workshop

To find the finite area enclosed by the curve $y^2 = 4x$ and the line $y + x = 0$, let us first make rough sketches of the curve and the line $y^2 = 4x$, $\frac{y^2}{4} = x$, so, $x = \frac{y^2}{4}$.

In this case $x = f(y)$. To sketch the curve $x = \frac{y^2}{4}$, we need to know the values of y when $x = 0$ and also the turning point of this quadratic equation, ($x = f(y)$).

When, $x = 0$, $\frac{y^2}{4} = = 0$; $\frac{y^2}{4} = 0$; $y\ 2 = 0$; $y = \pm\sqrt{0} = \pm\ 0 = 0$. So, when $x = 0$, $y = 0$. Equation $x = \frac{y^2}{4} = \frac{y^2}{4}$ is a quadratic equation since the highest power of y is 2. So, it is going to have a turning point, and at its turning point, gradient of the curve = 0 .

However, recall that in this case, $x = f(y)$. So, gradient $= \frac{dy}{dx} = 0$

> *Note that since $x = f(y)$, gradient of the curve $x = f(y)$ will be $\frac{dx}{dy}$ and **not** $\frac{dy}{dx}$.*

Hence, at turning point, $\frac{dx}{dy} = \frac{d\left(\frac{y^2}{4}\right)}{dy} = 0$; $\frac{2y}{4} = 0$,
$\frac{y}{2} = 0$, $y = 0$.

Therefore, at the turning point of the curve, $y = 0$.

When $y = 0$; $\frac{y\,2}{4} = x$, *so* , $\frac{0_2}{4} = x$; $0 = x$; $x = 0$.

Therefore, the coordinate of the turning point of the curve is at $x = 0$, $y = 0$, that is, point (0,0). Having known the value of y when $x = 0$ and the co-ordinates of the turning point of the curve, we can make a sketch of the curve $\frac{y^2}{4} = x$ as shown in Figure 6.4 below.

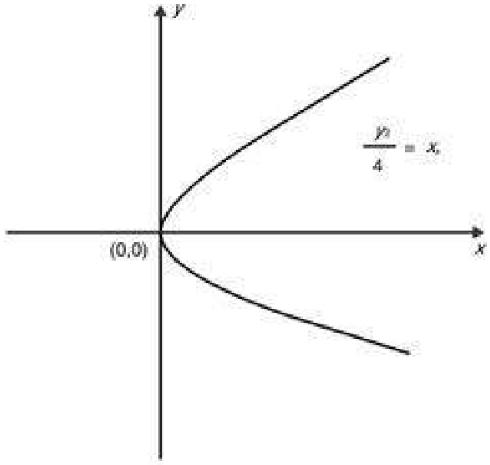

Fig. 6.4

Note that this curve of equation $x = \frac{y^2}{4}$ is also the curve of equation $y^2 = 4x$. we have only rearranged the equation, and they mean the same thing. Also note that if co-efficient of y^2 is negative, for example $x = \frac{-y^2}{4}$ the graph of $x = \frac{-y^2}{4}$ will be as shown in Figure 6.5.

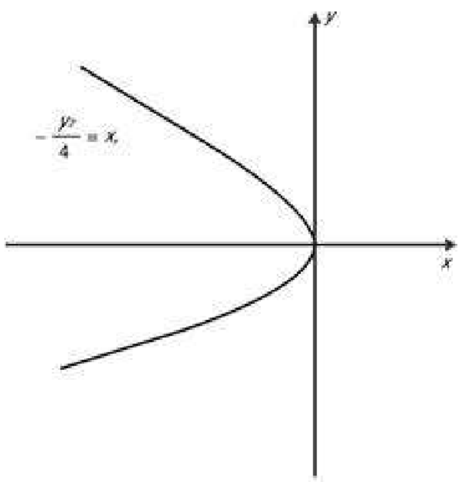

Fig. 6.5

Let us also try to sketch the graph of $y + x = 0$; we can rewrite this equation in the gradient intercept form ($y = mx + c$) as $y = -x + 0$. So, the intercept of the line on the y-axis is $c = 0$ and the gradient of the line is $m = -1$ (negative). With these two parameters known, we can construct the graph of the line as shown in Figure 6.6.

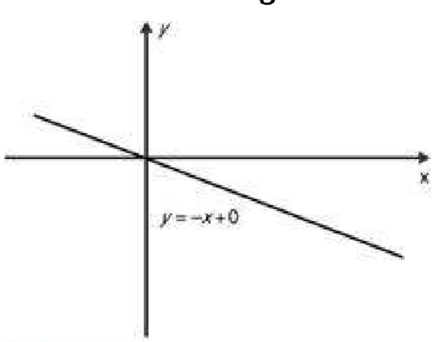

Fig. 6.6

Please note that the graph of line $y+x=0$ slopes down from left to right as in Figure 6.6 because the slope of the line is negative ($m=-1$). If its slope had been positive, the line would have sloped down from right to left.

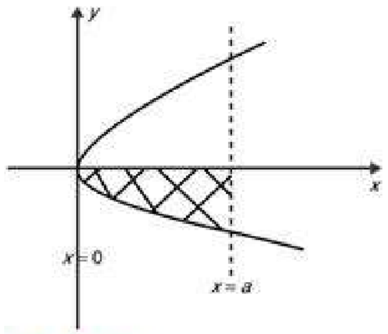

Fig. 6.7(a)

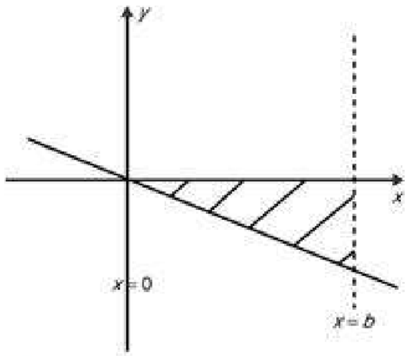

Fig. 6.7(b)

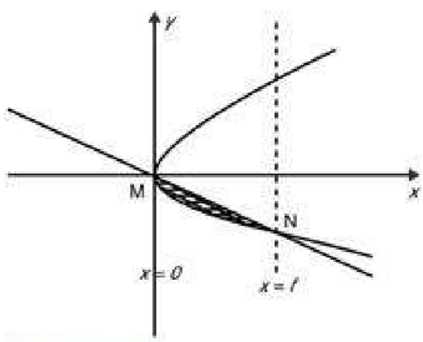

Fig. 6.7(c)

Figure 6.7(a) above, shows the area enclosed by curve $y^2 = 4x$, line $x = 0$, line $x = a$ and the *x-axis*. Figure 6.7(b) shows the area enclosed by line $y + x = 0$, line $x = 0$, line $x = b$ and the *x-axis*. Figure 6.7(c) shows the finite area enclosed by the curve $y^2 = 4x$ and line $y + x = 0$.

If you look carefully at Figure 6.7(c) above, you will notice that:

$$\begin{pmatrix}\text{Area enclosed by} \\ \text{curve } y^2 = 4x \text{ and} \\ \text{line } y+x=0.\end{pmatrix} =$$

$$\begin{pmatrix}\text{Area enclosed by} \\ \text{curve } y^2 = 4x\text{, line} \\ x=0\text{, line } x=f \text{ and} \\ \text{the } x-\text{axis.}\end{pmatrix} - \begin{pmatrix}\text{Area enclosed by} \\ \text{line } y+x=0\text{, line} \\ x=0\text{, line } x=f \\ \text{and the } x-\text{axis.}\end{pmatrix}$$

To know the finite area enclosed by the curve and the line, we need to know the *x* values of the co-ordinates of points *M* and *N* which are the points of intersection of the curve and the line as these are the points showing the boundaries of the area enclosed by the curve and line and this is shown in Figure 6.7(c). These points of intersection lie on both the curve and the line. So, the values of x and y for points M and N will satisfy the equations of the curve and line. Therefore, we can get the co-ordinates of the two points common to the line and the curve by solving the equations of the curve and line simultaneously.

$y^2 = 4x; y^2 - 4x = 0;$

$y^2 - 4x = 0$ ...(*i*)

$y + x = 0$ ... (*ii*)

From equation (*ii*), $y = 0 - x; y = -x;$ so put $y = -x$ into equation (*i*) to get

$(-x)^2 - 4x = 0; (-x \times -x) - 4x = 0; x2 - 4x = 0; x(x - 4) = 0; x = 0$ or $x - 4 = 0; x = 0$ or $x = 4$. We do not need the *y-components* of points M and N for our calculation. So, we can now redraw the curve and the line as shown in Figure 6.8 below.

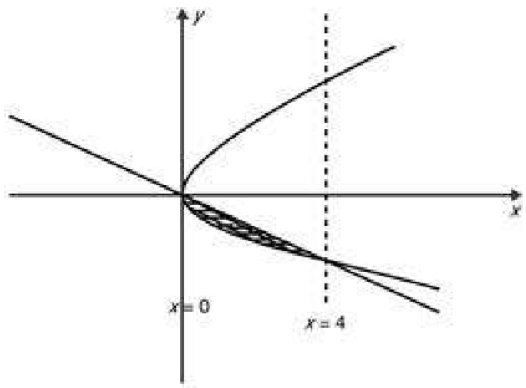

Fig. 6.8

The area of finite region enclosed by a curve $y = f(x)$, line $x = a$, line $x = b$ and *x-axis* is expressed

as Area $= \int_a^b f(x)\,.\,dx$.

Note that since this integral bears dx, a and b must be values of x and **not** *values of y, and f(x) must be written in terms of x to be able to simplify the integral. However, if the integral bears dy, a and b must be values of y (for example, instead of using $x = 0$ and $x = 4$ as calculated above, we will find the values of y when $x = 0$ and when $x = 4$, then we use these values of y as a and b in the integral $\int_a f(y).\,dy$. But $\int_{y=a}^{y=b} f(y)\,.\,dy$ will be the area bounded by curve $x = f(x)$ and line $y = b$, line $y = a$ and y-axis. Also, if the integral bears dy, f(y) must be written in terms of y before simplifying the integral.*

So, since Area $= \int_{x=a}^{x=b} f(x) \,.\, dx$, we need to write y as a function of x to be able to simplify this integral as the integral bears dx and **not** dy.

Hence, $\frac{y2}{4} = x$, $y2 = 4x$, $y = \sqrt{4x} = 2\sqrt{x}$; therefore, $y = f(x) = 2\sqrt{x}$. Thus, the area enclosed by curve $y = 2\sqrt{3}$ and *line x = 0, line x = 4 and the x-axis*

$= \int_0^4 2\sqrt{x} \,.\, dx.$

Note that the curve of $y = 2\sqrt{x}$ is the same as the curve of $\frac{y2}{4} = x$ that was drawn, they are one and the same equation. We only made y the subject of the formula to arrive at the second equation.

$\int_0^4 2\sqrt{x}.dx = 2\int_0^4 \sqrt{x} \,.\, dx = 2\int_0^4 x^{\frac{1}{2}}.$

Note that. $= \int 2\sqrt{x} \cdot dx \;= 2\int \sqrt{x} \cdot dx$ *while*

$\int (2 + \sqrt{x}) \cdot dx = \int 2 \cdot dx + \int \sqrt{x} \cdot dx.$

$$2\int_0^4 x^{\frac{1}{2}} \cdot dx = 2\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] = 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = 2\left[\left(\frac{4^{\frac{3}{2}}}{\frac{3}{2}}\right) - \left(\frac{0^{\frac{3}{2}}}{\frac{3}{2}}\right)\right]$$

$$= 2\left[\left(\frac{(2^2)^{\frac{3}{2}}}{\frac{3}{2}}\right) - \left(\frac{0}{\frac{3}{2}}\right)\right] = 2\left[\left(\frac{2^{2\times\frac{3}{2}}}{\frac{3}{2}}\right) - 0\right] = 2\left[\frac{2_3}{\frac{3}{2}}\right] = 2$$

$$\left[\frac{2_3}{1} \times \frac{2}{3}\right] = \frac{2 \times 2_3 \times 2}{3} = \frac{2_5}{3} = \frac{32}{3} \textit{ square units.}$$

Also, to get the area enclosed by the line $y + x = 0$ and line $x = 0$, line $x = 4$ and the *x-axis*, we will first write y as a function of x as $y + x = 0$; $y = -x = f(x)$; as the integral bears dx. Then, the area bounded by line $y = -x$ and *line* $x = 0$,

line $x = 4$ and the *x-axis* is given by $\int_{x=0}^{x=4} f(x) \cdot dx$

$= \int_0^4 -x \cdot dx = -1\int_0^4 x_1 \cdot dx.$

$$\int_{x=0}^{x=4} x_1 \cdot dx - \left[\frac{x_{1+1}}{1+1}\right]_0^4 = -1\left[\frac{x_2}{2}\right]_0^4 - 1\left[\frac{4^2}{2} - \frac{0^2}{2}\right]$$

$$= -1\left[\frac{16}{2} - 0\right]$$

*Furthermore, in calculating area, you should always find the absolute values of the areas bearing negative sign in your calculations so as to make them positive, since, area in the real sense is always positive. Or, have you heard of – 4cm3 or – 4 cm³ before? Recall that area is a scalar quantity (having magnitude alone), so it should **not** bear a negative sign.*

Therefore, the area enclosed by the line $y + x = 0$ and line $x = 0$, line $x = 0$ and the *x - axis* is 8 *square units*. Recall that

$$\begin{pmatrix}\text{Area enclosed by} \\ \text{curve } y^2 = 4x \text{ and} \\ \text{line } y + x = 0.\end{pmatrix} =$$

$$\begin{pmatrix}\text{Area enclosed by} \\ \text{curve } y^2 = 4x \text{ and} \\ \text{line } x = 0, \text{ line } x = \\ 4 \text{ and the x-axis.}\end{pmatrix} - \begin{pmatrix}\text{Area enclosed by} \\ \text{line } y + x = 0, \text{ line} \\ x = 0\text{, line } x = 4 \\ \text{and the x-axis.}\end{pmatrix}$$

$$= \frac{32}{3} - 8 = \frac{32 - 24}{3} = \frac{8}{3} \text{ square units.}$$

Therefore, the finite area enclosed by curve $y^2 = 4x$ *and line* $y + x = 0$ is $\frac{8}{3}$ *square units.*

The reason for this long explanation is to aid easy understanding of how the answer was obtained. But here is a shorter method.

Having understood the explanations in the long method, we can also adopt a shorter method. It is clear that the area enclosed by the curve, $y^2 = 4x$, and the line, $y + x = 0$, can be obtained by first calculating the co-ordinates of the two points where the curve and the line intersect, as these are the points marking the boundaries of the area we are interested in (see Figure 6.7c). You can obtain these co-ordinates by solving the two equations simultaneously. Then, the *x*-values of these co-ordinates will be your boundary lines and these are lines $x = 0$ and $x = 4$ (as drawn in Figure 6.8).

Then, find the area enclosed by curve $y^2 = 4x$, line $x = 0$ and line $x = 4$ and the *x-axis*.

Also, find the area enclosed by $y + x = 0$ and line $x = 0$, line $x = 4$ and the *x-axis*.

Make sure these areas are positive. Make any negative area positive by finding its absolute value as we did for −8. Then subtract the smaller area from the bigger area. The answer can then be obtained.

Note that this same procedure is followed to find the area enclosed by two curves. Again, note that it is not compulsory that you plot graphs to solve this question, except you are told to do so. These graphs were drawn to facilitate your understanding of the problem.

3. Given that $f: x \rightarrow x^3 - 6x^2 + 9x + 8$, find, correct to two decimal places, the:

(a) values of x at the turning points of f;

(b) set of values of x for which f is:

(i) increasing,

(ii) decreasing. **(WAEC)**

Workshop

$f: x \rightarrow x^3 - 6x^2 + 9x + 8$ means that

$f(x) = x^3 - 6x^2 + 9x + 8$.

Let $f(x) = y = x^3 - 6x^2 + 9x + 8$.

(a) At turning point of any curve,

$$y = f(x), \frac{d(f(x))}{dx} = \frac{dy}{dx} 0;$$

$$\text{so } \frac{d(f(x))}{dx} = \frac{dy}{dx} = \frac{d(x^3 - 6x^2 + 9x + 8)}{dx}$$

$$= 3x^2 - 12x + 9 = 0;\ 3x^2 - 12x + 9 = 0;$$

$$3x^2 - 9x - 3x + 9 = 0;$$

$$3x(x-3) - 3(x-3) = 0;$$

$$(x-3)(3x-3) = 0;$$

$$x - 3 = 0 \text{ or } 3x - 3 = 0;$$

$$x = 3 \text{ or } 3x = 3;$$

$$x = 3 \text{ or } x = 1.$$

Therefore, the values of x at the turning point of $f(x)$ are 1 and 3.

(b) (i) For $y = x^3 - 6x^2 + 9x + 8$ to **increase** at a given interval, $\frac{dy}{dx}$ must be **greater** than zero($\frac{dy}{dx} > 0$) at that interval $\frac{dy}{dx} = 3x^2 - 12x + 9$; *let* $3x^2 - 12x + 9 = g$, so, $\frac{dy}{dx} = g = 3x2 - 12x + 9 > 0$; factorise (see (a) above), to get $g = (3x - 3)(x - 3) > 0$. The zeros of this inequality are 1 and 3. *A* quadratic equation will have a minimum turning point (u-shape curve) if the coefficient of x^2 is positive, so that $g = 3x^2 - 12x + 9$ will have a minimum turning point. We can make a rough sketch of $g = 3x^2 - 12x + 9 > 0$ as shown in Figure 6.9 below. A close look at the graph shows that the shaded portions on the graph are the region where g is greater than zero ($g > 0$) which are the intervals $x < 1$ and $x > 3$ (because these are the region of the curve above $g = 0$). Hence, the intervals of x where g is greater than zero ($g = 3x^2 - 12x + 9 > 0$) are $x < 1$ *and* $x > 3$.

Therefore, the set of values of x for which $f(x)$ is increasing is the interval of x for which $\frac{dy}{dx} = g = 3x^2 - 12x + 9 > 0$ which are $x < 1.00$ and $x > 3.00$ to two decimal places.

(ii) For $f(x)$ to be **decreasing,** $\frac{dy}{dx}$ must be **less** than zero, ($\frac{dy}{dx} < 0$), Let $\frac{dy}{dx} = g = 3x^2 - 12x + 9 < 0$, a sketch of the graph of $g = 3x^2 - 12x + 9$ is as shown in Figure 6.10 below.

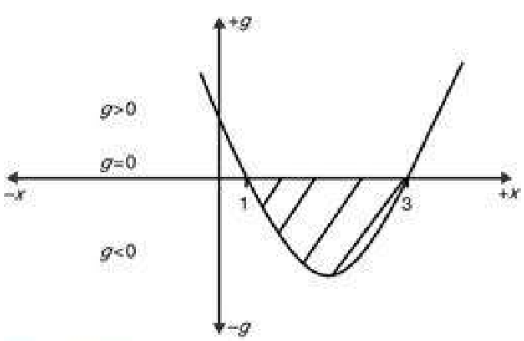

Fig. 6.10

The shaded portion on the graph, in Figure 6.10, is the region where g is less than zero ($g < 0$) and this region is between $x = 1$ and $x = 3$. As this is the region, in the curve, that is below $g = 0$; this region can be expressed as $1 < x < 3$. Hence, the interval of x where $g = 3x^2 - 12x + 9 < 0$ is $1 < x < 3$. Therefore, the set of values of x for which $f(x)$ is decreasing is the interval x for which $\frac{dy}{dx} = g = 3x^2 - 12x + 9 < 0$, which is the interval $1.00 < x < 3.00$ correct to two decimal places.

4. A rectangular metal sheet, 8 *m* by 5 *m*, is used to make an open box by removing equal squares of side *xm* from each corner of the sheet.

(a) Find, in terms of *x*, an expression for the volume (*v*) of the box.

(b) Determine the maximum possible volume of the box. **(WAEC)**

Workshop

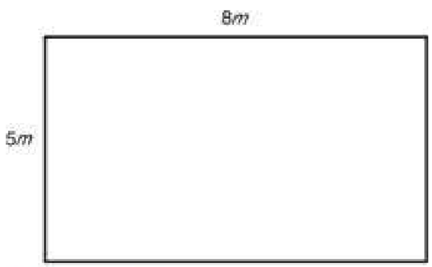

Fig. 6.11a

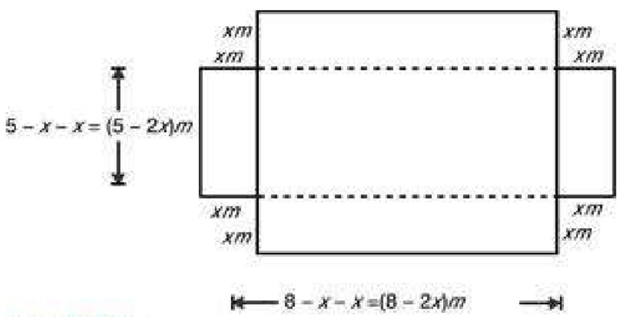

Fig. 6.11b

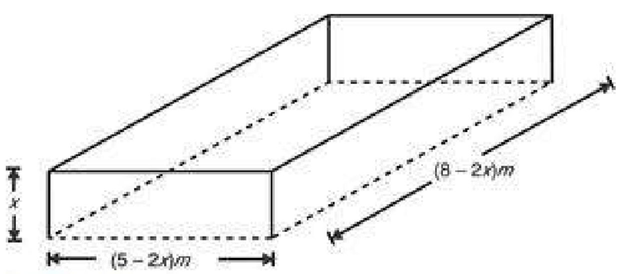

Fig. 6.12

The metal sheet, as shown in Figure 6.11b, can be folded along the broken lines into an open cuboid of length$(8 - 2x)m$, breadth $(5 - 2x)m$ and height *xm* as shown in Figure 6.12 below.

(a) The volume of a cuboid is given by

Lenght × Breath × Height.

Hence, $v = [\,(8 - 2x) \times (5 - 2x) \times x\,]\ m^3 = (4x3 - 26x2 + 40x)m^3$.

Therefore, the expression for the volume *V* of the cuboid in terms of *x* is

$v = (4x3 - 26x2 + 40x)\ \mathrm{m}^3$.

b) The equation $v = 4x^3 - 26x^2 + 40x$ is a cubic equation (*equation that has the* ***highest*** *power of* x *as* x^3), therefore, it will have two turning points, one minimum and one maximum turning point. At these turning points, the gradient $\frac{dy}{dx}$ of the curve is zero($\frac{dy}{dx} = 0$) , (where $v = f(x)$). And so, at the point where the volume of the cuboid is at maximum

$$\frac{dv}{dx} = 0; \frac{dv}{dx} = 12x^2 - 52x + 40 = 0;$$

$12x^2 - 52x + 40 = 0$; divide through out the equation by 4 to get

$$3x^2 - 13x + 10 = 0; 3x^2 - 10x - 3x + 10 = 0;$$

$$x(3x - 10) - 1\,(3x - 10) = 0;$$

$$(3x - 10)\,(x - 1) = 0; 3x - 10 = 0 \text{ or}$$

$$x - 1 = 0; 3x = 10 \text{ or } x = 1; x = \frac{10}{3} \text{ or } x = 1.$$

The points where $x = 1$ and where $x = \frac{10}{3}$ are both turning points on the curve, $v = 4x^3 - 26x^2 + 40x$, but while one is a minimum turning point, the other is a maximum turning point. At maximum turning point,

$$\frac{d^2v}{dx^2} < 0; \frac{dv}{dx} = 12x^2 - 52x + 40;$$

$$\frac{d^2v}{dx^2} = 24x - 52;$$

At point $x = 1$; $\frac{d^2v}{dx^2} = 24(1) - 52 = -28$

which is less than 0 (zero). As, $\frac{d^2v}{dx^2} < 0$

at point $x = 1$, then point $x = 1$ is a maximum turning point. This means, volume v is maximum (*highest*) at point $x = 1$. Recall that $v = 4x^3 - 26x^2 + 40x$ and v is maximum at $x = 1$; thus, $v\text{max} = 4(1)^3 - 26(1)^2 + 40(1) = 4 - 26 + 40 = 18m^2$.

Therefore, the maximum possible volume of the cuboid is 18 m^3.