

Chapter 11

Chapter 11

Coordinate Geometry 2

OBJECTIVES

At the end of the chapter, students should be able to:

1. define and determine the gradient and intercepts of a line.
2. determine the equation of a line.
3. find the angle between two intersecting straight lines.

I. Gradient and Intercept of a Straight Line

(i) Gradient of a straight line

The gradient of a straight line is a measure of its shape with respect to the x-axis and is defined as:

$$\frac{\text{increase in } y - \text{coordinate}}{\text{increase in } x - \text{coordinate}}$$

Consider the line $y = 2x + 3$. When $x = 0$, $y = 3$, the point $(0, 3)$ lies on the line.

The coordinates of a point are written as an ordered pair with the x-coordinate being written first. Hence, we say that the line passes through the point $(0, 3)$. When $x = 1$, $y = 5$, the line passes through the point $(1, 5)$. The line is shown in Figure 11.1.

The increase in y between these points is $5 - 3$ or 2. The increase in x between these points is $1 - 0$ or 1. The gradient of the line is 2.

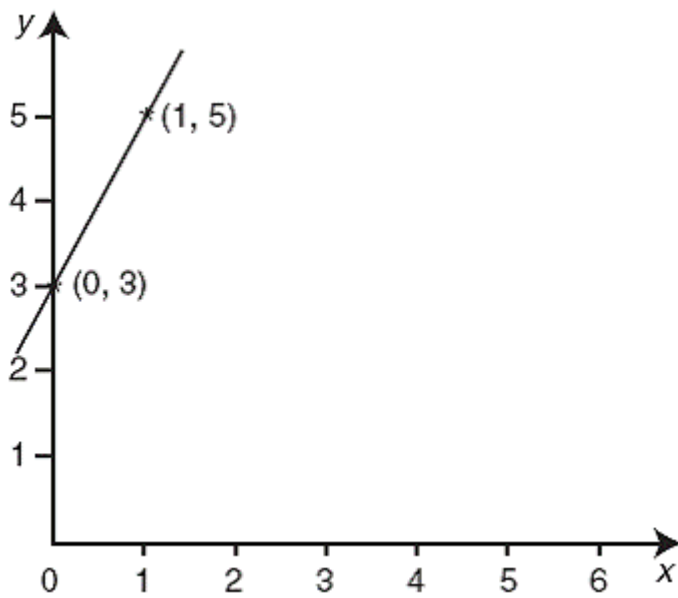


Figure 11.1

Note that the gradient is independent of the position on the line using similar triangles. Now, consider the line $5y + x = 1$ when $x = 1$, $y = 2$. Therefore, the point $(1, 2)$ lies on the line. When $y = 0$, $x = 11$, the point $(11, 0)$ lies on the line. The line is shown in Figure 11.2.

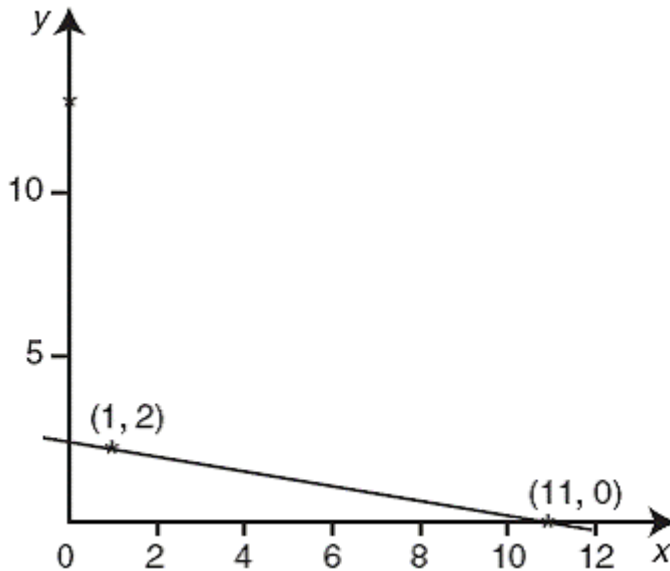


Figure 11.2

The increase in x between these two points is $11 - 1$ or 10 .

The decrease in y between these two points is $2 - 0$ or 2 .

The increase in y therefore is -2 and the gradient is:

$$\frac{-2}{10} = \frac{-1}{5} *$$

In general, a line which makes an acute angle with the positive direction has a positive gradient; a line which makes an obtuse angle with the positive x direction has a negative gradient.

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Generally, the gradient of the line joining the points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$. The order in which the points are taken is immaterial provided the same order is chosen for both the numerator and the denominator, that is, the gradient is also equal to

$$\frac{y_2 - y_1}{x_2 - x_1}$$

From *, the gradient of a line may be defined as the tangent of the angle that it makes with the positive direction of the x -axis, where the angle is measured in an anticlockwise direction.

In general, the gradient of a line passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\frac{\text{increase in } y}{\text{increase in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta = m$$

(ii) Intercept of a straight line

In finding the point of intersection of a line with y -axis, substitute $x = 0$ to find the corresponding value of y . If the corresponding value of y is b , the required point of intersection is $(0, b)$

Similarly, the point of intersection of a line with x -axis can be obtained by substituting $y = 0$ to find the corresponding value of $x = a$ (say).

The required point of intersection is $(a, 0)$.

Worked Example 1

Find the graph of the line joining the point $(3, -8)$ to $(-3, 2)$.

SOLUTION

The selection of points as (x_1, y_1) or (x_2, y_2) does not matter as we have seen that the slope measure in either direction of the line joining two points is the same.

Let $(x_1, y_1) = (3, -8)$ and

$$(x_2, y_2) = (-3, 2)$$

$$\begin{aligned} \text{Then } m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-8)}{-3 - 3} \\ &= \frac{-10}{6} = \frac{-5}{3} \therefore m = \frac{-5}{3} \end{aligned}$$

Worked Example 2

Find the gradient of the line passing through $(1, 3)$ and $(-4, 2)$.

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SOLUTION
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The gradient of the line passing through points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Here $x_1 = 1; y_1 = 3; x_2 = -4, y_2 = 2$

$$\therefore m = \frac{2 - 3}{-4 - 1} = \frac{-1}{-5} = \frac{1}{5}$$

$\therefore m = \frac{1}{5}$ which gives the required gradient.

Worked Example 3

Find the point of intersection of the line $2x + 3y + 2 = 0$ with the
(a) x -axis (b) y -axis.

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SOLUTION
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(a) The line $2x + 3y + 2 = 0$

At the point of intersection of the line and the x -axis, $y = 0$.

Substituting $y = 0$ for y in the given equation, we obtain

$$2x + 3(0) + 2 = 0$$

$$\text{That is } 2x = -2$$

$$\Rightarrow x = -1$$

Hence, the required point of intersection is $(-1, 5)$.

(b) On the y -axis, $x = 0$. Substituting $x = 0$ for x in the given equation, we obtain

$$2(0) + 3y + 2 = 0$$

$$\therefore 3y + 2 = 0$$

$$y = \frac{-2}{3}$$

The required point is $(-1, 5)$. (b) On the y -axis, $x = 0$. Substituting $x = 0$ for x in the given equation, we obtain

$$2(0) + 3y + 2 = 0$$

$$\therefore 3y + 2 = 0$$

$$y = \frac{-2}{3}$$

The required point is $(0, \frac{-2}{3})$.

Worked Example 4

Find the point of intersection of the lines $y = 3x + 2$ and $y = 2x + 5$.

SOLUTION

$$y = 3x + 2 \dots\dots\dots (1)$$

$$y = 2x + 5 \dots\dots\dots (2)$$

At the point of intersection

$$3x + 2 = 2x + 5$$

$$3x - 2x = 5 - 2$$

$$\therefore x = 3$$

Substituting 3 for x in (1), we obtain $y = (3 \times 3) + 2 = 11$

Hence, the point of intersection is $(3, 11)$.

Aliter

By solving the two equations simultaneously, we can get the point of intersection.

From (1),

$$y - 3x = 2 \dots\dots\dots (3)$$

From (2)

$$y - 2x = 5 \dots\dots\dots (4)$$

(4) - (3) gives

$$x = 3$$

Substituting 3 for x in (3), we have

$$y - 3(3) = 2$$

$$y = 9 + 2$$

$$\therefore y = 11$$

Hence, the point of intersection of the two lines is $(3, 11)$.

Exercise 1

Find the gradient of the line through each pair of the following points:

1. (1, 2) and (5, 6)
2. (-4, -9) and (-2, 1)
3. (0, 0) and $(-\sqrt{3}, 3)$
4. (10, 4) and (2, -3)
5. (-1, -8) and (5, 7)
6. $(at, 2at)$ and $(at_2^2, 2at_2)$

Find the point of intersection of the line $3x + 2y - 6 = 0$ with the following:

7. y-axis
8. x-axis
9. Show that the gradient of the line which makes intercepts of 2 and -5

on the x- and y-axis, respectively, is $\frac{5}{2}$.

10. Find the point of intersection of the line $5x - y = -3$ with the following:
 - (a) x-axis
 - (b) y-axis

II. Determination of the Equation of a Straight Line (Revision)

We have earlier learnt in Book 2 (Chapter 6) the five different ways of determining the equation of a straight line, which are as follows:

1. Equation of a line with gradient m and intercept on the y-axis is given by $y = mx + c$.
2. The general equation of a line passing through a point and whose

$$m = \frac{y - y_1}{x - x_1}.$$

gradient is known is given by

3. The general equation of a line passing through two given points is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

given by

4. Intercept form of the equation of a line: The equation of a line which has an intercept 'a' on the x-axis and intercept 'b' on the y-axis is given

$$\frac{x}{a} + \frac{y}{b} = 1.$$

by

5. Equation of a line passing through a point and making an angle θ with the horizontal axis: The general equation of a line passing through the point (x_1, y_1) and making an angle θ with the horizontal axis is

$$\frac{y - y_1}{x - x_1} = \tan \theta$$

$$\text{or } y - y_1 = (x - x_1) \tan \theta.$$

Worked Example 5

Find an equation of the line which has the y -intercept equal to $-3/4$ and is perpendicular to $3x - 4y + 11 = 0$.

SOLUTION

Slope of $3x - 4y + 11 = 0$

$$3x - 4y = -11$$

$$-4y = -11 - 3x$$

$$y = \frac{-3x}{-4} - \frac{11}{-4}$$

$$y = \frac{3x}{4} + \frac{11}{4}$$

$$\therefore \text{Slope} = \frac{3}{4}$$

Slope of a perpendicular line is $-\frac{4}{3}$. The equation required is

$$y = -\frac{4}{3}x + \frac{4}{3} \text{ or } 4x + 3y = 4.$$

Worked Example 6

Find the equation of a line whose gradient is 4 and intercept on y -axis is -3 .

SOLUTION

The equation of a line with gradient m and intercept c on the y -axis is $y = mx + c$.

Here $m = 4$ and $c = -3$. The required equation is $y = 4x - 3$.

Worked Example 7

Find the equation of the line with gradient 3 which passes through the point $(-3, 5)$.

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SOLUTION
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The equation of a line with known gradient m and passing through the point (x_1, y_1) is given by $\frac{y - y_1}{x - x_1} = m$.

Here $m = 3$, $(x_1, y_1) = (-3, 5)$

\therefore The required equation of the line is

$$\frac{y - 5}{x - (-3)} = 3$$

$$\frac{y - 5}{x + 3} = 3$$

that is, $y - 5 = 3(x + 3)$

$$y - 5 = 3x + 9$$

$$y = 3x + 9 + 5$$

$$y = 3x + 14$$

Worked Example 8

Obtain the equation of the line through the point $(4, -3)$ and (a) parallel and (b) perpendicular to the line $3x + 4y + 5 = 0$.

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SOLUTION
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By comparing $3x + 4y + 5 = 0$ with $y = mx + c$, the slope of the line is $-\frac{3}{4}$.

(a) Slope of the line parallel to $3x + 4y + 5 = 0$ is $-\frac{3}{4}$.

\therefore The equation of the line passing through the point $(4, -3)$ and parallel

to $3x + 4y + 5 = 0$ is $y + 3 = \frac{-3}{4}(x - 4)$

or $4(y + 3) = -3(x - 4)$

or $4y + 12 = -3x + 12$

$4y + 3x = 0$

(b) Slope of the line perpendicular to $3x + 4y + 5 = 0$ and through the point $(4, -3)$ is:

$y + 3 = \frac{4}{3}(x - 4)$

or $3(y + 3) = 4(x - 4)$

$3y + 9 = 4x - 16$

$3y - 4x + 9 + 16 = 0$

$3y - 4x + 25 = 0$

Worked Example 9

Find the equation of the line passing through point $A(5, 3)$ and $B(3, -7)$.

SOLUTION

The general equation of a line passing through two given points (x_1, y_1) and (x_2, y_2)

is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$.

Here $(x_1, y_1) = (5, 3)$; $(x_2, y_2) = (3, -7)$.

The required equation of the given line is

$\frac{y - 3}{x - 5} = \frac{-7 - 3}{3 - 5} = \frac{-10}{-2} = 5$

$$\begin{aligned}
 \text{i.e., } \frac{y-3}{x-5} &= 5 \\
 5(x-5) &= y-3 \\
 5x-25 &= y-3 \\
 5x-25+3 &= y \\
 5x-22 &= y \\
 y &= 5x-22
 \end{aligned}$$

Hence, the required equation is $y = 5x - 22$.

Worked Example 10

Find the intercept form of the equation of a line which makes intercept of 3 and 4 with x - and y -axis, respectively. Write the equation in the form $y = mx + c$. Hence, find the gradient of the line.

SOLUTION

The general equation of a line which makes intercept of ' a ' and ' b ' with the x and y -axis respectively is given by

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Since $a = 3$ and $b = 4$, the required equation in the intercept form is

$$\frac{x}{3} + \frac{y}{4} = 1 \quad \dots\dots\dots (1)$$

Multiplying both sides of (1) by 12, we obtain

$$4x + 3y = 12$$

$$\therefore 3y = 12 - 4x$$

$$y = \frac{12}{3} - \frac{4x}{3}$$

$$\therefore y = \frac{-4x}{3} + 4$$

which is the required equation of the line. Its gradient $m = \frac{-4}{3}$.

Worked Example 11

The angle between the positive horizontal axis and a given line is 135° . Find the equation of the line if it passes through the point $(3, 5)$.

SOLUTION

The general equation of a line passing through the point (x_1, y_1) and making an angle θ with the horizontal axis is $y - y_1 = (x - x_1) \tan \theta$. Now $(x_1, y_1) = (3, 5)$ and

$\theta = 135^\circ$. Hence, the required equation of the line is

$$(y - 5) = (x - 3) \tan 135^\circ$$

$$\therefore y - 5 = (x - 3)(-1)$$

$$y - 5 = -x + 3$$

$$y - 5 = 3 - x$$

Hence $y = 3 + 5 - x$

$$y = 8 - x$$

The required equation of the line is

$$y = 8 - x.$$

Exercise 2

1. Write the equation of the line with the following data:
 - (a) y-intercept = -5 and gradient = $+4$.
 - (b) y-intercept = 2 and gradient = 7 .
 - (c) y-intercept = -1 and is parallel to $y = 5x - 3$.
 - (d) y-intercept = 2 and is inclined at 45° to the x-axis.
2. What will be the value of m and c if the straight line $y = mx + c$ passes through the points $(3, -4)$ and $(-1, 27)$?
3. Obtain the equation of the lines joining the given points:
 - (a) $X(1, 1)$ and $Y(2, 3)$
 - (b) $A(a, b)$ and $B(b, a)$
 - (c) $P(3, 3)$ and $Q(1, 1)$
 - (d) $M(2, -3)$ and $N(5, -3)$
4. Write the equation of the straight line through the given point P and having the given slope m if
 - (a) $P(2, 2)$; $m = 1$
 - (b) $P(4, 2)$; $m = 0$
 - (c) $P(-8, 3)$; $m = \frac{34}{-}$
 - (d) $P(-4, 7)$; $m = -\sqrt{3}$
 - (e) $P(-1, -5)$; $m = \frac{-1}{16}$
5. Find the equation of the line joining the origin to the point of intersection of
 $4x + 3y = 8$ and $x + y = 1$.
6. Determine the x-intercept ' a ' and the y-intercept ' b ' of the following lines:
 - (a) $5x + 3y - 15 = 0$
 - (b) $x - 2y - 9 = 0$
7. Find the equation of the line which makes equal intercept on the axes and passes through the point $(2, 3)$.

8. Write down the equation of the line which makes an intercept of $2a$ on the x -axis and $3a$ on the y -axis. Given that the line passes through the point $(14, -9)$, find the value of a .
9. Find the gradient of a line whose equation is $7x - 2y + 9 = 0$.
10. (a) Find the equation of a line whose gradient is -5 and has intercept 5 on the y -axis.
(b) Find the gradient and the intercept on the y -axis of the equation $5x + 3y = 3$.
11. Find the equation of a line which passes through $(1, -5)$ and has gradient $\frac{3}{5}$.
12. Find the equation of the line passing through points $P(5, 3)$ and $Q(2, 7)$.
13. (a) Find the equation and the gradient of a line which makes intercepts of 7 and -3 on the x - and y -axis, respectively.
(b) Find the intercept on y -axis of the line whose equation is $-3x + 7y = -3$.
14. The angle between the positive horizontal axis and a given line is 45° . Find the equation of the line if it passes through the point $(3, -5)$.
15. Given that the equation of a line passing through point $(a, 5)$ is $y + 2x - 1 = 0$. Find the value of a .

III. Angle between Two Intersecting Straight Lines

If two lines make angles of θ and ϕ with the x axis, the angle α between the lines is given by

$$\alpha = \theta - \phi \text{ (See Figure 11.3)}$$

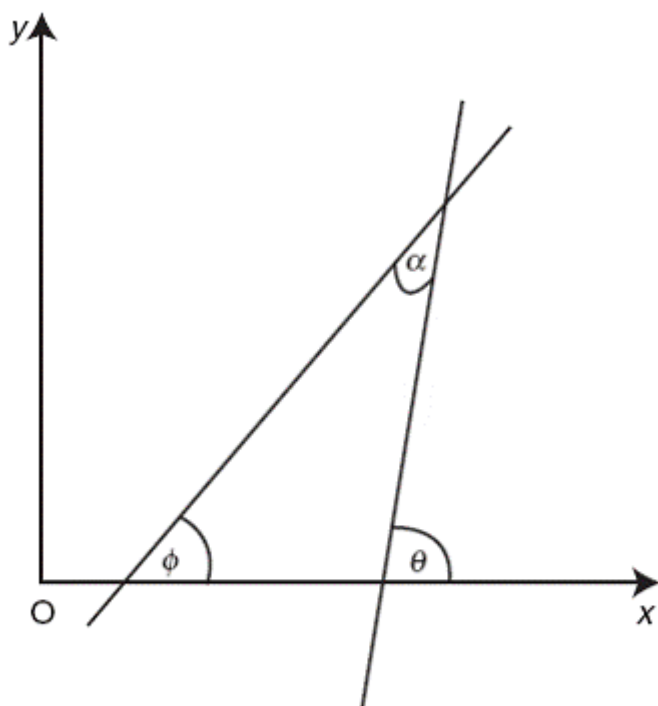


Figure 11.3

If the gradients of the lines are m and n so that $\tan \theta = m$ and $\tan \phi = n$, then

$$\begin{aligned}\tan \alpha &= \tan (\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi} \\ &= \frac{m - n}{1 + mn}\end{aligned}$$

Hence, the angle between the lines is

$$\tan^{-1} \frac{m - n}{1 + mn}.$$

(a) If the lines are parallel, the angle between them is zero.

$$\therefore \frac{m - n}{1 + mn} = 0 \text{ or } m = n$$

(ii) If the lines are perpendicular, the angle between them is 90° , and $\tan 90^\circ$ is infinite. Therefore, the denominator of the fraction must be zero and so mn

Worked Example 12

Find the angle between the lines $3x - 4y = 2$ and $5x + 2y = 7$.

SOLUTION

The gradients of the lines are $\frac{3}{4}$ and $\frac{-5}{2}$.

$$\tan \alpha = \frac{m-n}{1+mn} = \frac{\frac{3}{4} - \left(\frac{-5}{2}\right)}{1 + \frac{3}{4}\left(\frac{-5}{2}\right)} = \frac{6+20}{8-15} = \frac{-26}{7}$$

The angle is $\tan^{-1}\left(\frac{-26}{7}\right)$ or $\pi - \tan^{-1}\frac{26}{7}$

Worked Example 13

Find the angle between the lines whose slopes are -2 and $\frac{-1}{4}$.

SOLUTION

Given: $m = 2$ and $n = \frac{-1}{4}$

Substituting in the formula

$$\begin{aligned}\tan \alpha &= \frac{m-n}{1+mn} = \frac{-2 - \left(\frac{-1}{4}\right)}{1 + (-2)\left(\frac{-1}{4}\right)} \\ &= \frac{\frac{-7}{4}}{\frac{3}{2}} = \frac{-7}{4} \times \frac{2}{3} = \frac{-7}{6}\end{aligned}$$

The angle is $\tan^{-1}\left(\frac{-7}{6}\right)$ or $\pi - \tan^{-1}\frac{7}{6}$.

(i) Collinear points

When the gradients between adjacent pairs of points are the same, we say that they are collinear, that is, they lie on the same line.

Worked Example 14

Are the points $(2, 5)$, $(-1, -1)$ and $(-4, -7)$ collinear?

SOLUTION

Let the points be X, Y, Z respectively.

$$\text{Gradient } XY = \frac{-1-5}{-1-2} = \frac{-6}{-3} = 2$$

$$\text{Gradient } YZ = \frac{-7+1}{-4+1} = \frac{-6}{-3} = 2$$

∴ The gradients are the same, hence the points are collinear.

Exercise 3

- Find the slope of a line whose inclination is as follows:
 (a) 150° (b) 60°
 (c) 135° (d) 30°
 (e) 45°
- Using gradients, determine which of the following sets of three points are collinear?
 (a) $(1, -1), (-2, 4), (0, 1)$
 (b) $(5, -2), (7, 6), (0, -2)$
 (c) $(-2, 3), (8, -5), (5, 4)$
 (d) $(6, -1), (5, 0), (2, 3)$
 (e) $(-1, 5), (3, 1), (5, 7)$
- Find the angle between the lines whose slopes are
 (a) 2 and -1
 (b) 2 and $-\frac{4}{3}$
 (c) 4 and -5
 (d) $-\frac{4}{1}$ and $-\frac{12}{1}$
 (e) $-\frac{25}{1}$ and $-\frac{15}{1}$
- Find the gradient of a line parallel to the line whose gradient is
 (a) -4 (b) -7
 (c) 0 (d) 2.5
 (e) $-\frac{25}{1}$
- Find the equation of a line parallel to the line which passes through each pair of the following points:
 (a) $(0, 0)$ and $(5, 6)$
 (b) $(4, 7)$ and $(-1, 3)$
 (c) $(3, 0)$ and $(-5, -8)$
 (d) $(0, b)$ and $(-a, 0)$
- Find the gradient of a line perpendicular to the line whose slope is
 (a) $\frac{1}{3}$ (b) $-\frac{1}{5}$
 (c) 0 (d) $\frac{5}{6}$
 (e) 5
- Find the gradient of a line perpendicular to the line which passes through each pair of the following points:
 (a) $(0, 8)$ and $(-5, 2)$
 (b) $(5, 2)$ and $(1, -1)$
- The line joining $(-5, 7)$ and $(0, -2)$ is perpendicular to the line joining

(1, -3) and (4, x). Find x.

9. The line joining (-5, 7) and (0, -2) is parallel to the line joining (1, -3) and (4, x). Find x.

10. Find the tangent of the acute angle between the lines $x + 2y = 6$ and $y - 3x = 1$.

SUMMARY

In this chapter, we have learnt the following:

❖ Gradient of the line joining (x_1, y_1) , (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$

❖ Equation of a line: general linear equation is $ax + by + c = 0$.

Gradient: y-intercept form is $y = mx + c$.

Gradient: one point form is $y - y_1 = m(x - x_1)$.

Gradient: two-point form is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$.

Gradient: line passing through the point (x_1, y_1) and making an acute angle θ with the horizontal axis

is $\frac{y - y_1}{x - x_1} = \tan \theta$.

Gradient: intercept form of the equation of a line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

that is, 'a' on x-axis and 'b' on y-axis.

- ❖ Acute angle ϕ between lines of gradient m_1, m_2 is given by

$$\tan \phi = \left[\frac{m_1 - m_2}{1 + m_1 m_2} \right]$$

- ❖ For parallel lines, $m_1 = m_2$

For perpendicular lines, $m_1 m_2 = -1$

- ❖ Equation of line parallel to

$ax + by = c$ through (x_1, y_1) is $ax + by = ax_1 + by_1$.

- ❖ Equation of line perpendicular to $ax + by = c$ through (x_1, y_1) is $bx - ay = bx_1 - ay_1$.

GRADUATED EXERCISES

1. Show that the gradient of the line which makes intercept of 2 and -5 on the x- and y-axis, respectively, is $\frac{5}{2}$.
2. Given that the line $5x + 2y - 3 = 0$ is perpendicular to the line $ky + 2x - 4 = 0$
 - (a) Find the value of k for which this is possible.
 - (b) What value of the k will make the two lines parallel to each other?
3. Two perpendicular lines AB and BC have a common point at $B(2, -3)$. If the equation of the line BC is $3y = 2x - 3$, find the equation of line AB .
4. Find the equation of a line which makes an intercept of -3 on y-axis and is parallel to the line $5y - 3x - 1 = 0$.
5. For what value of k will the line $(k - 3)x + 2y = 3$ be parallel to x-axis?
6. Given that the equation of a line passing through point $(a, 5)$ is $y + 2x - 1 = 0$. Find the value of a .
7. For what value of k will a line which pass through points $(2, k - 3)$

and $(k, -2)$ and be perpendicular to the line $x - 4y = 8$?

8. Find the equation of a line which passes through $(2, 5)$ and makes angle 135° with the positive horizontal axis.

9. Find the equation of a line which passes through $(1, 3)$ and makes angle 45° with positive horizontal axis.

10. (a) Find the angle between the lines $x + y = 8$ and $x - y = 2$.

(b) Find the angle between the lines $3y - 4x = 2$ and $7y = x + 1$.