Chapter 6: Gradient of a Curve

OBJECTIVES

At the end of the chapter, students should be able to:

1. Define and identify the **x**-intercept and **y**-intercept of any linear graph.
2. Draw the graph of linear equation **ax** + **by** + **c** = 0.
3. Find the gradient of a line

(a) From the two given points.

(b) From the **y**-intercept only.

(c) From the intercepts.

4. Draw the graph of a linear equation by using the gradient method.
5. Find the equation of a line using:

(a) Point quadratic equation.

(b) Two point equation.

6. Draw tangents to a curve at a given point.

I. Linear Graphs

In coordinate geometry, we make use of points in a plane. A point consists of the **x**-coordinate called **abscissa** and the **y**-coordinate known as **ordinate**.

In locating a point on the **x–y** plane, **x**-coordinate is first written and then the **y**-coordinate. For example, in a given point (**a**, **b**), the value of **x** is **a** and that of **y** is **b.** Similarly, in a point (3, 5), the value of **x** is 3 and that of **y** is 5. A linear graph gives a straight line graph from any given straight line equation which is in the general form mx + c = y or ax + by + c = 0.

Worked Example 1

Draw the graph of equation 4x + 2y = 5.

SOLUTION

Choose any three values of x and find their corresponding values in y.

4x + 2y = 5

2y = 5 âˆ’ 4x

$$y = \frac{5 - 4x}{2}$$

Table 6.1

x	0	1	2
y	$2\frac{1}{2}$	$\frac{1}{2}$	$-1\frac{1}{2}$

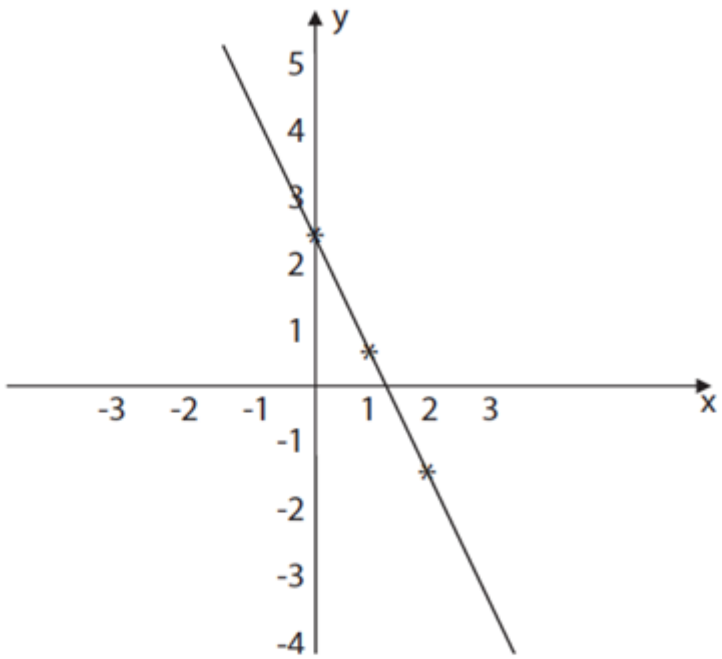


Figure 6.1

Note:

1. The coordinates form an ordered pair with the abscissa written first.
2. Coordinates of general points are often taken as (**x**, **y**) or (**a**, **b**) which implies ($\mathbf{x_1}$, $\mathbf{y_1}$), ($\mathbf{x_2}$, $\mathbf{y_2}$), â€¦, ($\mathbf{x_n}$, $\mathbf{y_n}$).

(i) Point of intersection of two linear equations

Two lines $\mathbf{y_1}$ = **ax** + **b** and $\mathbf{y_2}$ = **cx** + **d** intercept when **ax** + **b** = **cx** + **d** (**a** â‰ **c** ; **b** â‰ **d**)
In general, the point of intersection of the linear equations **ax** + **by** + $\mathbf{c_1}$ = 0 and **cx** + **dy** + $\mathbf{c_2}$ = 0 can be obtained by solving the two equations simultaneously.

(ii) Intersection of a line with the x- or y-axis

The point of intersection of a line with the **x**-axis can be obtained by putting y = 0 to find the corresponding value of **x** = **a,** say. The required point of intersection gives (**a**, 0).
Similarly, for the point of intersection of a line with the **y**-axis, put **x** = 0 to find the corresponding value of **y.** If the corresponding value of **y** is **b,** the required point of intersection is (0, **b**).

Worked Example 2

Find the point of intersection of the line 2**x** + 3**y** + 2 = 0 with the (a) **x**-axis (b) **y**-axis.

SOLUTION

Given the line 2**x** + 3**y** + 2 = 0.
(a) At the point of intersection of the line with the **x**-axis, **y** = 0. Substituting **y** = 0 in the equation, we obtain
2**x** + 3(0) + 2 = 0, that is,
2**x** = âˆ’2
â‡’ **x** = âˆ’1
Hence, the required point of intersection is (âˆ’1, 0).

(b) On the y-axis, x = 0. Substituting x = 0 in the given equation, we obtain
2(0) + 3y + 2 = 0, that is,
3y = âˆ’2

$y = -\frac{2}{3}$

The required point is $\left(0, \frac{-2}{3}\right)$

Worked Example 3

Find the point of intersection of the lines y = 3x + 2 and y = 2x + 5.

SOLUTION

y = 3x + 2 .... (1)
y = 2x + 5 .... (2)
At the point of intersection,
3x + 2 = 2x + 5
3x âˆ’ 2x = 5 âˆ’ 2
âˆ´x = 3
Substituting 3 for x in equation (1), we obtain y = 3(3) + 2 = 11.
Hence, the point of intersection is (3, 11).

Exercise 1

Draw the graph of the following lines and in each case, find the gradient by taking measurements.
1. 5x âˆ’ 2y = 5
2. 7x + 4y âˆ’ 8 = 0
3. y = 3x + 1
4. y = 3x âˆ’ 2
5. y = âˆ’2x + 3
6. 4x âˆ’ 2y + 1 = 0
7. 4x âˆ’ 3y = 5
8. 2x âˆ’ 5y = 6
9. 2x + 3y = 0
10. 2x + 3y = 6
11. Find the point of intersection of the line 3x + 2y âˆ’ 6 = 0 with the:
(a) y-axis
(b) x-axis.
12. Find the point of intersection of the lines 2y âˆ’ x âˆ’ 10 = 0 and 2y + 3x âˆ’ 6 = 0.
13. Find the point of intersection of the lines y = 5x + 3 and y = âˆ’4x âˆ’ 7.
14. Find the point of intersection of the line âˆ’5x + 3y âˆ’ 7 = 0 with the:
(a) y-axis
(b) x-axis.
15. Find the point of intersection of the line y âˆ’ 7x + 3 = 0 with the:
(a) y-axis
(b) x-axis.

II. Gradient of a Straight Line

The gradient of a straight line is defined as the ratio
change in y
change in x
in moving from one point to another on the line.

In Figure 6.2, the gradient of the line AB measured from A to B is $\frac{20}{10}$ = 2. If we go from B to A, the gradient = $\frac{-20}{-10}$ = 2.
Hence, the gradient of the line is the same in either direction. The gradient of a straight line is

always constant.

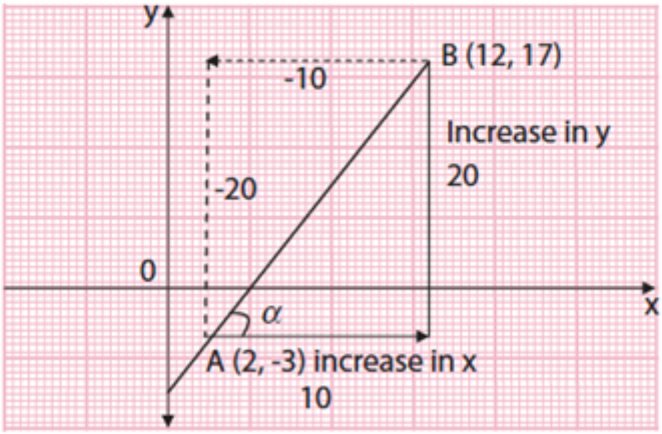


Figure 6.2

Note: $\frac{\text{change in } y}{\text{change in } x}$ from A to $B = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{17-(-3)}{12-2}$$

From A to $B = \frac{17+3}{10} = \frac{20}{10} = 2$

since y increases as x increases, the gradient is positive. AB makes an acute angle a with the positive direction of the x-axis and tan a is positive.

Note:

1. If 0 < q < 90Â°, tan q is positive and the line has a positive gradient.
2. If q = 90Â°, the line has no gradient but it is perpendicular to the x-axis.
3. If 90Â° < q < 180Â°, tan q is negative and the line has a negative gradient. Gradient of a line that passes through points (x_1, y_1) and (x_2, y_2) is given as gradient

$m = \frac{y_2 - y_1}{x_2 - x_1}$ â€¦â€¦â€¦...................... (1)

In other words, the gradient of the line is the ratio of increase in y to increase in x.

(i) Tangent of angle of slope

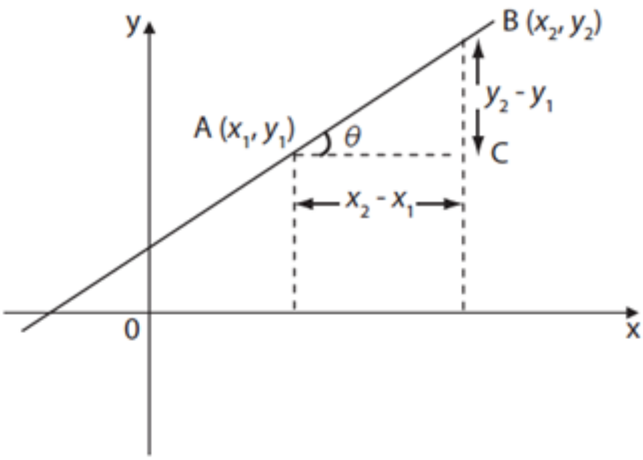


Figure 6.3

In Figure 6.3, θ is the angle of slope.

$$\tan\theta = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$

But from equation (1),

$$\frac{y_2 - y_1}{x_2 - x_1} = m \text{ .... } (2)$$

$$\tan\theta = m$$

It then follows from equation (2) that the gradient of a line can be defined as tangent of angle of slope.

Worked Example 4

Find the gradient and the angle of slope of the line passing through (1, 3) and (**âˆ’**4, 2).

SOLUTION

The gradient of the line passing through points (**x**1, **y**1) and (**x**2, **y**2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Here, **x**1 = 1, **y**1 = 3 and **x**2 = âˆ’4, **y**2 = 2.

$$m = \frac{2-3}{-4-1} = \frac{-1}{-5} = \frac{1}{5}$$

âˆ´ $m = \frac{1}{5}$ which gives the required gradient.

Let q be the angle of slope

âˆ´m = tan q

â‡’ $\frac{1}{5}$ = tan q

âˆ´tan q = 0.2

q = tanâˆ’1(0.2)

q = 11.31Â°

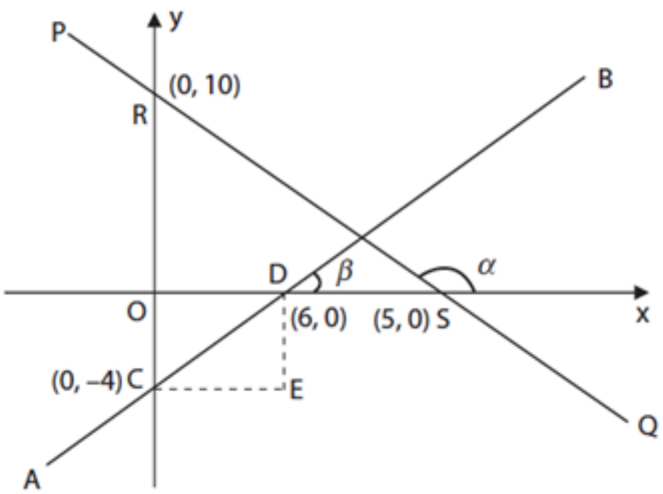


Figure 6.4

Worked Example 5

In Figure 6.4, calculate the gradient of (a) line AB (b) line PQ and angles of slope of the line

passing through the points AB and PQ respectively.
SOLUTION
(a) Consider points C and D on line AB and **Î”**CDE.

Gradient of AB = $\tan\beta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-0}{0-6} = \frac{-4}{-6} = \frac{4}{6}$
To find the angle of slope, let β be the angle.
âˆ´m = tan β
tan β = **2**/**3** = 0.67
β = tanâˆ’1 (0.67)
âˆ´ β = 33.82Â°
(b) Consider points R and S on line PQ and Î”ROS.
Gradient of PQ = tan α = âˆ’tan QSR

tan α = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{OR}{OS} = \frac{-10}{5}$ = âˆ’2
tan α = âˆ’2
âˆ´ α = tan âˆ’1 (âˆ’2)
or α = 116.57Â°
Note: Gradient is also known as slope and is denoted by m.

Exercise 2
Find the gradient of the lines joining the following pairs of points:
1. (2, 5), (5, 9)
2. (3, âˆ’6), (2, 0)
3. (0, âˆ’2), (âˆ’4, 0)
4. (p, p + 1), (p âˆ’ 1, 2p + 1)
5. (p^2, 2p), (q^2, 2q)
6. (âˆ’4, 5), (âˆ’3, 2)
7. (âˆ’1, 5), (7, âˆ’ 3)
8. (0, a), (a, 0)
9. (2, 3), (4, 7)
10. (âˆ’5, âˆ’5), (âˆ’7, âˆ’5)
11. (2a, b), (a, 2b)
12. (âˆ’1, 3), (5, âˆ’4)
13. (0, 2), (6, 0)
14. (1.5, 0.5), (2.5, +5)
15. (a, b), (c, d)

III. Equation of a Straight Line
(i) Equation of a line with gradient m **and** y**-intercept** c
Equation of a line with gradient m and y-intercept c is given as y = mx + c.

Worked Example 6
Find the equation of a line whose gradient is 2 and intercept on the y-axis is âˆ’5.
SOLUTION
The equation of a line with gradient m and intercept c on the y-axis is y = mx + c.
Here, m = 2 and c = âˆ’5.
The required equation is y = 2x âˆ’ 5

(ii) Equation of a line passing through the point (x_1, y_1) with gradient

The general equation of a line with known gradient m and which passes through the point (x_1, y_1) is given as

$$m = \frac{y - y_1}{x - x_1}$$

Worked Example 7

Find the equation of the line with gradient 2 and which passes through the point (âˆ’3, 2).

SOLUTION

The equation of a line with known gradient and passing through the point (**x_1**, **y_1**) is given by

$$\frac{y - y_1}{x - x_1} = m$$

Here, m = 2, (x_1, y_1) = (âˆ’3, 2).

âˆ´The required equation of the line is

$$\frac{y-2}{x-(-3)} = 2$$

$$\Rightarrow \frac{y-2}{x+3} = 2$$

$$\Rightarrow y - 2 = 2(x+3)$$

$$y = 2x + 6 + 2 = 2x + 8$$

$$\therefore y = 2x + 8$$

(iii) Equation of a line passing through two given points

The equation of a line passing through two given points (**x_1**, **y_1**) and (**x_2**, **y_2**) is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Worked Example 8

Find the equation of the line passing through points A (3, 1) and B (2, **âˆ’**3).

SOLUTION

The given equation is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Here, (x_1, y_1) = (3, 1) and (x_2, y_2) = (2, **âˆ’**3).

The required equation of the given line is

$$\frac{y-1}{x-3} = \frac{-3-1}{2-1} = -4$$

That is

$$\frac{y-1}{x-3} = -4$$

âˆ´ y âˆ’ 1 = âˆ’4(x âˆ’ 3)

y âˆ’ 1 = âˆ’4x + 12

y = âˆ’4x + 13

Hence, the required equation is y = âˆ’4x + 13.

(iv) Double intercept form of the equation of a line

The equation of a line which has an intercept â€˜aâ€™ on the x-axis and intercept â€˜bâ€™ on the y-axis is given by $\frac{x}{a} + \frac{y}{b} = 1$

Worked Example 9

Find the equation of a line which makes intercepts of 2 and 5 on the **x-** and **y**-axes respectively. Write the equation in the form **y** = **mx** + **c.** Hence, fi nd the gradient of the line.

SOLUTION

The equation of a line which makes intercepts of â€˜**a**â€™ and â€˜**b**â€™ on the **x-** and **y**-axes,

respectively, is given by $\frac{x}{a}+\frac{y}{b}=1$

Since **a** = 2 and **b** = 5, the required equation in the intercept form is $\frac{x}{2}+\frac{y}{5}=1$

Multiplying both sides of the equation by 10, we obtain

5x + 2y = 10.

â‡’ **2**y = âˆ’**5**x + **10**

âˆ´ **y** = $-\frac{5x}{2}$ +5

which is the required equation of the line. Its gradient (m)= $-\frac{5}{2}$

(v) Equation of a line passing through a point and making an angle q **with the horizontal axis**

The equation of a line passing through the point (**x**1, **y**1) and making an angle q with the horizontal

axis is $\frac{y-y_1}{x-x_1}$ = tan q or **y** âˆ’ **y**1 = (**x** âˆ’ **x**1) tan q.

Worked Example 10

The angle between the positive horizontal axis and a given line is 135Â°. Find the equation of the line, if it passes through the point (3, âˆ’2).

SOLUTION

The general equation of a line passing through the point (**x**1, **y**1) and making an angle q with the horizontal axis is **y** âˆ’ **y**1 = (**x** âˆ’ **x**1) tan q**.**

Hence, the required equation of the line is **y** âˆ’ (âˆ’2) = (**x** âˆ’ 3) tan 135Â°.

y + 2 = (**x** âˆ’ 3) âˆ’1 (since tan 135Â° = âˆ’1)

y + 2 = âˆ’**x** + 3

Hence, **y** = âˆ’**x** + 1 = 1 âˆ’ **x**

The required equation of the line is **y** = 1 âˆ’ **x.**

Exercise 3

1. Write the equation of the following lines in the form y = mx + c and sketch each line.

Gradient y-intercept

(a) 2, 3

(b) 4, âˆ’2

(c) âˆ’2, 3

(d) $-\frac{1}{2}$, âˆ’4

(e) $-\frac{3}{2}$, 2.5

2. Rewrite the equations of the lines in Question 1 in the form ax + by = c.

3. Find the equations of the following lines given the facts stated. Gradient Passes through

(a) 3 (2, 1)

(b) âˆ’2 (âˆ’2, âˆ’3)

(c) 1.25 (1, âˆ’3)

(d) âˆ’1.5 (0, âˆ’4)
(e) âˆ’5 (âˆ’3, âˆ’5)
4. Find the equation of the lines which pass through the following pairs of points.
(a) (2, 4) and (5, 6)
(b) (âˆ’1, âˆ’2) and (3, 0)
(c) (âˆ’4, 3) and (2, âˆ’5)
(d) (3, 2) and (âˆ’5, âˆ’2)
(e) (1, 2) and (1, âˆ’4)
(f) (âˆ’2, âˆ’6) and (5, âˆ’4)
(g) (5, 6) and (âˆ’3, âˆ’5)
(h) (âˆ’10, 2) and (12, âˆ’6)
5. State the gradient and y-intercept of the following lines:
(a) x + y = 4
(b) y + 2x = 3
(c) 2x âˆ’ y = 3
(d) x + 3y = 6
(e) 3x âˆ’ 4y âˆ’12 = 0
(f) x âˆ’ y = 3
(g) 2x = y + 3
(h) 2x + 3y + 1 = 0
(i) 4x + 5y = 10
(j) x + y = 0

(k) $\frac{x}{3} + \frac{y}{2} = 1$
(l) y = x âˆ’ 1
6. Find the gradient of a line whose equation is 5x + 2y âˆ’ 3 = 0.
7. (a) Find the equation of a line whose gradient is âˆ’2 and has intercept 3 on the y-axis.
(b) Find the gradient and the intercept on the y-axis of the equation 3x + 2y = 1.

8. Find the equation of a line which passes through (0, âˆ’3) and has gradient $\frac{1}{4}$.
9. Find the equation of the line passing through points A(3, âˆ’2) and B(2, âˆ’1).
10. Find the equation and the gradient of a line which makes intercepts of 2 and 5 on the x- and y-axes
respectively.

IV. Drawing Tangents to a Curve

The gradient at any particular point on a curve is defi ned as being the gradient of the tangent to the curve at that point.
In Figure 6.5, the gradient of the curve at point A is the gradient of the tangent BA, that is, tan **q**.
The tangent is drawn by placing a ruler against the curve at A and drawing a line considering that the angles between the line and the curve are equal.

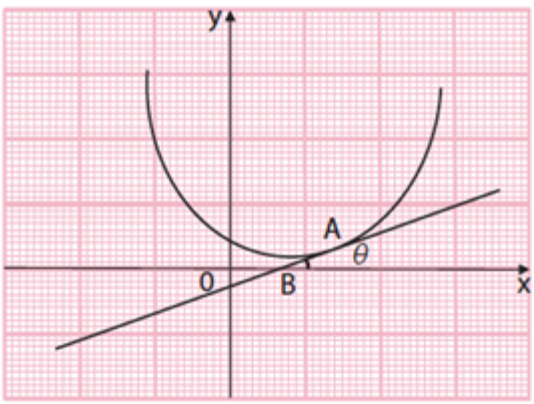


Figure 6.5

Worked Example 11

Figure 6.6 below is the graph of the curve y = 5 + x âˆ’ 4x2 for values of x from âˆ’2 to 3. Use the given tangents to fi nd the gradient of the curve at (a) P and (b) Q.

SOLUTION

(a) Gradient of the curve at P = Gradient of the tangent TP.

$$\frac{|OP|}{|TO|} = \frac{0-5}{-3-0} = \frac{-5}{-3} = \frac{5}{3} = 1\frac{2}{3}$$

(b) Gradient of the curve at Q = Gradient of the tangent QR.

$$\frac{|MR|}{|QM|} = \frac{-28}{1}$$

$$= -28$$

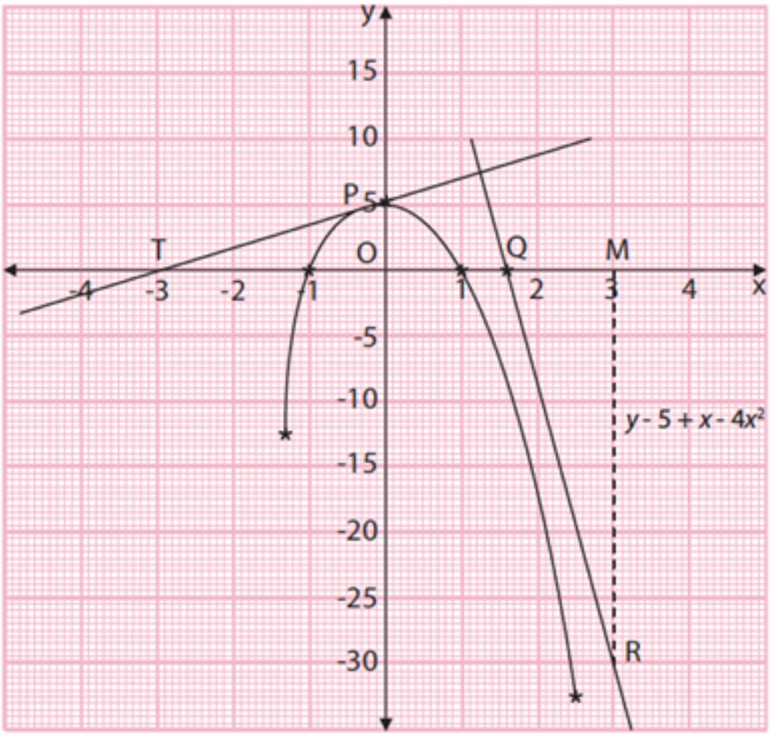


Figure 6.6

Worked Example 12

A (5, 3) and B (8, 0) are points on a straight line PQ.

(a) Calculate the gradient of AB.

(b) Find the equation of PQ.

SOLUTION

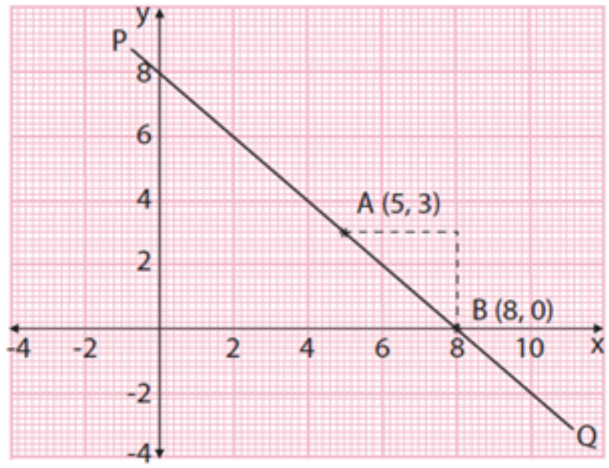


Figure 6.7

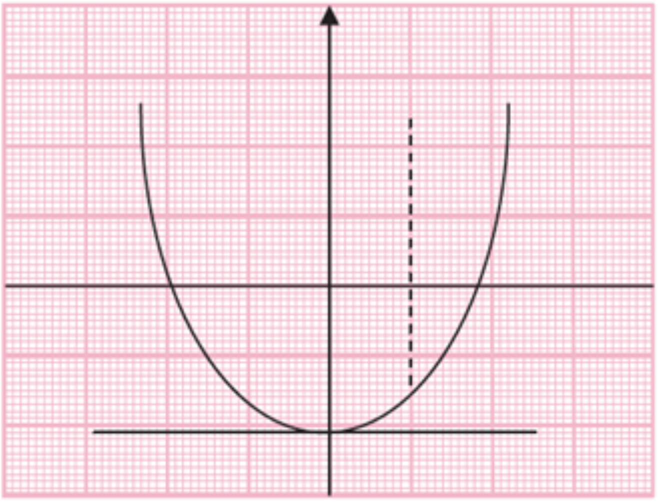
(a) From Figure 6.7,
Gradient of AB = Î”y/Î”x_

= $\frac{3}{-3}$ = âˆ’**1**
(b) Equation of PQ **is given by**

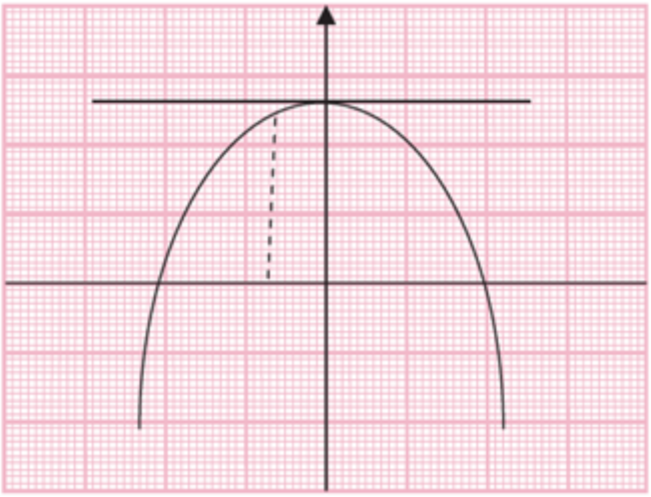
_ $\frac{y-3}{x-5} = -1$
y âˆ’ **3** = âˆ’**1(**x âˆ’ **5)**
y âˆ’ **3** = âˆ’x + **5**
y + x âˆ’ **8** = **0**

Note:
1. The gradient of a straight line is the same at any point on the line. That is the gradient of any straight line is always constant
2. The gradient of a curve changes from point to point.
3. The gradient at any turning point is zero.

Example



(a) Minimum value



(b) Maximum value

Figure 6.9

Exercise 4

Determine the gradient and **y**-intercept of the following straight lines:

1. 4**x** + 3**y** = 6
2. âˆ’2**y** + 6**x** = 4
3. âˆ’7 + 4**y** âˆ’ 3**x** = 0
4. 3**x** + 2**y** + 5 = 0
5. 3**x** + 5**y** âˆ’ 2 = 0

Find the equation of the line passing through:

6. (5, âˆ’2) with gradient $\frac{7}{2}$.
7. (âˆ’4, 8) with gradient $\frac{1}{4}$.
8. (âˆ’3, âˆ’5) with gradient 2.
9. (5, 3) with gradient âˆ’0.5.

Determine the equation of the straight line passing through the points (2, âˆ’2) and (3, 4).

SUMMARY

In this chapter, we have learnt the following:

â– The gradient of a straight line is given as:

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = m$$

(rate of change of y compared with x).

â– The general form of a straight line is y = mx + c, where m = gradient of the line and c is the y-intercept.

â– Other forms of the equation of a straight line are:

(a) $m = \frac{y - y_1}{x - x_1}$

(b) $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

(c) $\frac{x}{a} + \frac{y}{b} = 1$ or $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ and

(d) $\frac{y - y_1}{x - x_1} = \tan\theta$

â– A line parallel to the **x**-axis has its gradient as zero and the equation is of the form **y** = **c,** while the equation of a line parallel to the **y**-axis is of the form **x** = **a.**

â– The gradient of a curve at a point is given by the gradient of the tangent at that point.

â– The gradient at a turning point of any quadratic equation equals zero.

GRADUATED EXERCISES

1. Draw the graph of y = 1 + x âˆ’ x$_2$ from x = âˆ’2 to x = 3. Find the gradient at the point where x has the value:

(a) 2.5

(b) 1.5

(c) âˆ’1

(d) 0.5.

2. Draw the graph of y = x + _ 2x_5_ for 1 â‰¤ x â‰¤ 13. From the graph, find the gradient of the curve at x = 3.

3. Given that A (1, 3), B (2, 4) and C (a, 1) are three points on the same straight line, find:

(a) The gradient of AB and BC.

(b) The numerical value of a.

(c) The equation of the line. (WAEC)

4. A ball is thrown up. The height in metres attained by the ball in t seconds is given by the relation h = 40t âˆ’ 10h$_2$.

(a) Prepare a table of values for h for 0 â‰¤ t â‰¤ 4.

(b) Using a scale of 2 cm to 1 second on the horizontal axis for time, and 2 cm to 10 m on the vertical axis for height, draw the graph of h = 40t âˆ’ 10t$_2$ for 0 â‰¤ t â‰¤ 4.

(c) Use your graph to determine the slope when t = 3 secs. (WAEC)

5. Draw the graph of y = x$_2$. From your graph, obtain the gradient of the tangent to the curve at x =

$\frac{1}{2}$

Calculate from the graph, the gradient of the straight line given by:

6. y + 3 = 5x

7. 3y + 7x âˆ’ 21 = 0

Determine the gradient of the straight line joining the following points:

8. A (3, 6), B (1, 8)

9. P (**âˆ’**2, **âˆ’**9), Q (**âˆ’**7, 4)

10. D (5, **âˆ’**3), E (8, **âˆ’**1)

11. A (**âˆ’**4, 2), B (**âˆ’**6, 9)

By drawing the tangent to the graph, find the gradient of the curve at the given points:

12. y = **âˆ’**x$_2$ **âˆ’** 4x **âˆ’** 5 (i) A (0, **âˆ’**5) (ii) B (3, 2)

13. y = 2x_2 **âˆ’** x **âˆ’** 3 (i) P (1, **âˆ’**2) (ii) Q (**âˆ’**2, 7)
14. y = 10 + 3x **âˆ’** x_2 (i) A (**âˆ’**1, 6) (ii) B (6, **âˆ’**8)
15. y = x_2 **âˆ’** 3x + 4 (i) P (0, 4) (ii) Q (4, 8)