CHAPTER 1 Set

A set is a collection of things, and each collection is usually enclosed in curly brackets {}. The word 'things' in defining a set is used in a general sense; it can refer to people, names, animals, objects, among other things. The individual things that make up a set are called elements of that set; elements are separated by commas.

A collection of things (which can be called the universal set) can be regrouped into smaller groups (that can be called a subset of the universal set), and all elements in each of the subgroups are also found in the established universal set. The topic "set" is birthed from our day to day activities of trying to group and classify things. Below, a list is presented of the things (living and non-living) to be found in Abadina College.

Given that the set of things in Abadina College is a universal set ε, then the elements of the universal set are as follows:

ε = {Mr Akano, Mr Subar, Mrs Binuyo, Mr Oyebode, Mrs Okoruwa, Mrs Bolarinwa, Mrs Nasamu, Mr Clement, other teachers, Paul, Chineme, Seye, David, Jide, Victor, Ayo, Okhoya, Hope, Seyi, other students, chair, table, blackboard, chalk, classroom, office, grass, sand, grasshopper, goalpost, cane, book, bag, pen}

Please note that elements of a set must be listed in a curly bracket like the one above.

Other basic features of set can be explained using the universal set ε that has just been defined. The subset of set ε is another set (for example set A) that has all its elements contained in the universal set ε. If set A is a set of teachers in Abadina College, then set A will be expressed as

A = {Mr Akano, Mr Subar, Mrs Binuyo, Mr Oyebode, Mrs Okoruwa, Mrs Bolarinwa, Mrs Nasamu, Mr Clement, other teachers}

Since all elements in set A is found in the universal set ε, then set A is a subset of set ε. The following sets are also subsets of ε: Set B, a set of students of Abadina College:

B = {Paul, Chineme, Seye, David, Jide, Victor, Ayo, Okhoya, Hope, Seyi, other students}.

Set C is a set of non-living things in Abadina College:

C = {chair, table, blackboard, chalk, classroom, office, grass grasshopper sand, goalpost, cane, book, bag, pen}.

The union of two sets A and B (written as $A \cup B$) is a set of all elements in sets A and B combined.

From the example, $A \cup B$ will be a set of students

and teachers combined.

Thus,

$A \cup B$ = {Mr Akano, Mr Subar, Mrs Binuyo, Mr Oyebode, Mrs Okoruwa, Mrs Bolarinwa, Mrs Nasamu, Mr Clement, other teachers, Paul, Chineme, Seye, David, Jide, Victor, Ayo, Okhoya, Hope, Seyi, other students}.

If we name a set D as a set of other living things in Abadina College besides human beings, the elements of set D will be expressed as follows:

D = {grass, grasshopper}. (Grasses are plants, which are living things.)

Hence, if we define set L as a set of living things in Abadina College, set $L = A \cup B \cup D$, so,

L = {Mr Akano, Mr Subar, Mrs Binuyo, Mr Oyebode, Mrs Okoruwa, Mrs Bolarinwa, Mrs Nasamu, Mr Clement, other teachers, Paul, Chineme, Seye, David, Jide, Victor, Ayo, Okhoya, Hope, Seyi, other students, grass, grasshopper}.

The intersection of two sets B and L (written as $B \cap L$) is a set of elements that is common to sets B and L. Again from the example,

$B \cap L$ = {Paul, Chineme, Seye, David, Jide, Victor, Ayo, Okhoya, Hope, Seyi, other students}. So these are the elements common to set B and L.

Given a universal set ε, having a subset A, the compliment of a set A (written as A' or A^C) is a set of elements that are **NOT** in A but are found in the universal set. From the example above,

A' = {Paul, Chineme, Seye, David, Jide, Victor, Ayo, Okhoya, Hope, Seyi, other students, chair, table, blackboard, chalk, classroom, office, grass, sand, grasshopper, goalpost, cane, book, bag, pen}.

A null set is an empty set, it is a set **without** an element; a null set is usually represented as φ or { } (that is an empty curly bracket).

Kindly note carefully that set $H = \{0\}$ is NOT a null set because it has one element, which is element zero.

The Venn Diagram

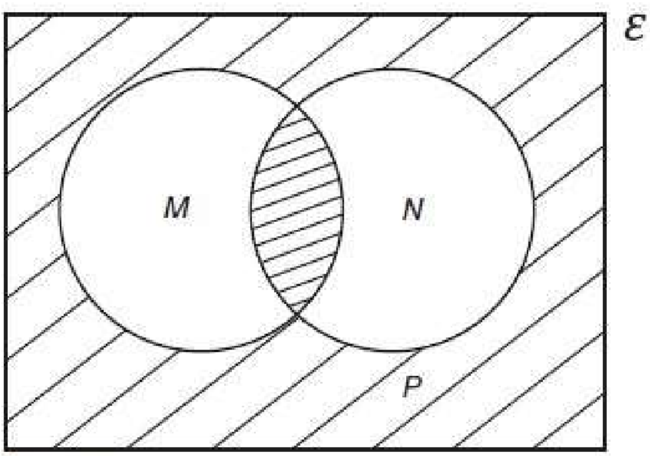

Fig. 1.1

The Venn diagram represents the subsets of a universal set as circles; just as shown in Figure 1.1, sets M and N are the two subsets of the universal set. The shaded area in between M and N where M and N intersects contains elements common to sets M and N, while the shaded region outside sets M and N is a set P of all elements that are found in the universal set, but are not in sets M and N. From the Venn diagram, $\varepsilon = M \cup N \cup P$.

Sets

1. If $A = \{(x, y): y^2 = 4x\}$ *and* $B = \{(x, y): y^x = 16\}$, find the elements of $A \cap B$.

(WAEC)

Workshop

A = {(x, y): $y2$ = $4x$} simply means that set A is made up of elements in the form (x, y), such that y^2 is always equal to 4^x, for each element (x, y) in the set. $A \cap B$ is a set of elements common to sets A and B. It means these sets of element will satisfy equations $y^2 = 4x$ (for set A) and $y^x = 16$ *(for set B)*; these elements can be known by solving the two equations simultaneously.

$y^2 = 4^x$ .... *(i)*

$y^x = 16$ ....*(ii)*

From equation (*i*), $y = \sqrt{4^x} = (4^x)^{\frac{1}{2}} = 4^{x \times \frac{1}{2}}$

$= 4^{\frac{x}{2}} = (2^2)^{\frac{x}{2}} \therefore y = 2^{2 \times \frac{x}{2}} = 2^x$.

Put $y = 2^x$ into equation (*ii*) to get

$y^x = (2^x)^x = 16; \quad 2^{x \times x} = 16 = 2^4; \quad 2^{x^2} = 2^4$.

Because the bases are equal, therefore, the powers are also equal, hence,

$x^2 = 4; \quad x = \pm\sqrt{4} = \pm 2$.

When $x = +2$, $y = 2^x = 2^2 = 4$ and when $x = -2$

$y = 2^x$

$= 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$. The elements (x, y) that satisfy the two sets are $(2, 4)$ and $\left(-2, \frac{1}{4}\right)$. Therefore,

$A \cap B = \left\{(2, 4), \left(-2, \frac{1}{4}\right)\right\}$.

Be sure to list the elements of a set in curly bracket "{ }", so that you may obtain good marks.

Therefore, the two elements of set $A \cap B$ are

$(2, 4)$ and $\left(-2, \frac{1}{4}\right)$.

2. Given the sets P and Q defined by the ordered pair:

   $P = \{(x, y) : 0 < x < 4, 0 < y < 4\}$,

   $Q = \{(x, y) : 3 \leq x \leq 4, 3 \leq y \leq 4\}$,

   where x and y are integers, find:

   (a) P (b) Q (c) $P \cap Q$.

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Workshop

(a) $P = \{(x, y): 0 < x < 4, 0 < y < 4\}$.

The values of integer x, in the range $0 < x < 4$ are $x = 1$, $x = 2$ and $x = 3$, while the values of integer y in the range $0 < y < 4$ are $y = 1$, $y = 2$ and $y = 3$ (*recall that x and y are integers*).

So, x: 1, 2, 3

y: 1, 2, 3

The possible elements (x, y), of P, that can be formed from the values of x and y written above are (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3).

Thus, $P = \{(x, y): 0 < x < 4, 0 < y < 4\}$

*Remember that all the elements of a **set** are enclosed in **curly** brackets as shown above.*

(b) $Q = \{(x, y): 3 \le x \le 4, 3 \le y \le 4\}$.

The values of integer, x, in the range $3 \le x \le 4$ are $x = 3$ and $x = 4$. The values of integer y in the range $3 \le y \le 4$ are $y = 3$ and $y = 4$ (*recall that x and y are integers*).

So, x: 3

y: 3, 4

The possible elements (x, y), of set Q, that can be formed from the values of x and y written above are (3, 3), (3, 4), (4, 3), (4, 4).

Hence, $Q = \{(x, y): 3 \le x \le 4, 3 \le y \le 4\}. = \{(3,3), (3, 4), (4,3), (4, 4)\}$.

(c) $P \cap Q$ is a set (i.e collection) of elements common to set P and set Q. The element common to sets P and Q is (3, 3).

Therefore, $P \cap Q = \{(3, 3)\}$.

*Note that $P \cap Q$ is also a **set**, so you have to put the element(s) of the set in **curly** brackets as shown above to get maximum mark.*

3. In a group of 50 students, 24 study Government and 33 Economics. If 5 of them do not study either of the subjects,

(a) how many study both?

(b) what is the probability that a student chosen at random from the group studies Economics only?

(WAEC)

Workshop

(a) Let the number of students that study both subject be x. From the question, the total number of students that study Government is 24. The number of students that study Government only (without Economics) will be $(24 - x)$. Also, the total number of students studying Economics is 33. The number of students that study Economics only (without Government) will be $(33 - x)$. 5 out of the total 50 students **did not** do any of the two subjects. We can represent all these information in a Venn diagram as shown in Figure 1.2 below.

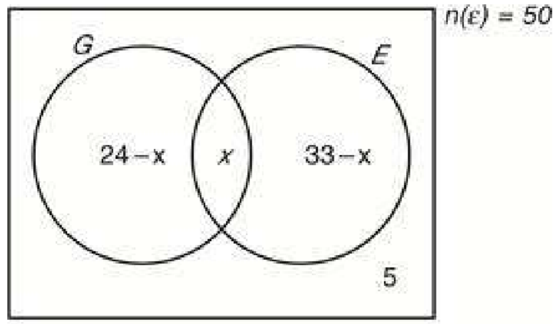

Fig. 1.2

$n(\varepsilon)$ is the number of elements in the universal set which is 50 elements (50 students). From the Venn diagram in Figure 1.2, if we add the number of students that study Government only, to the number that study both subjects, to the number that study Economics only and to the number that did not study any of the two subjects, the answer should be equal to the total number of students, which is the number of elements in the universal set.

Hence, $(24 - x) + x + (33 - x) + 5 = 50$;

$24 - x + x + 33 - x + 5 = 50$;

$24 + 33 + 5 - x = 50$;

$62 - x = 50$;

$62 - 50 = x$; $x = 12$.

Therefore, 12 students study both subjects.

(b) Recall that, $\Pr(\textit{that an event E will occur}) = \frac{\textit{number of elements in event space}}{\textit{number of elements in sample space}}$

$$\Pr\left(\begin{matrix}\textit{that a student will be chosen at random} \\ \textit{study economics only}\end{matrix}\right) = \frac{\textit{number of elements in event space}}{\textit{number of elements in sample space}}$$

$$\frac{33-x}{50} = \frac{33-12}{50} = \frac{21}{50}$$

4. In a survey, out of 100 out-patients who reported at a clinic in a week, it was found out that 70 complained of fever, 50 had stomach ache and 30 had injuries. All the 100 patients had at least one of the complaints and 44 had exactly two of the complaints. How many patients had all the three complaints?

(WAEC)

Workshop

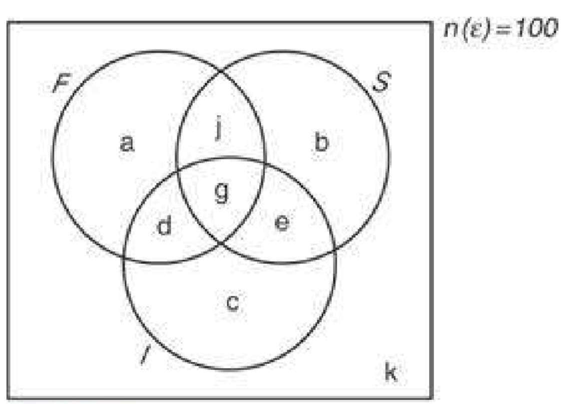

Fig. 1.3

Where $n(\varepsilon)$ is the total number of patients that reported at the clinic. On the Venn diagram, $n(F)$ is the total number of patients that complained of fever; $n(F) = 70$. $n(S)$ is the total number of patients that complained of stomach ache; $n(S) = 50$. $n(I)$ is the total number of patients that had injuries; $n(I) = 30$.

From the venn diagram in Figure 1.3, $n(F) = a + d + g + j = 70$.

The number of patients that complained of fever alone is a;

$a = 70 - d - g - j$ ....*(i)*

$n(S) = b + e + g + j = 50$.

The number of patients that complained of stomach ache alone is b;

$b = 50 - e - g - j$ .... *(ii)*

$n(I) = c + e + g + d = 30$.

The number of patients that had injury alone is c;

$c = 30 - e - g - d$. ....*(iii)*

From the venn diagram, it can be seen that

$n(\varepsilon) = a + b + c + j + d + e + g + k = 100$.

Where k is the number of patients that reported at the clinic but did **not** complain of any of the three ailments. From the question, we were told that all of the 100 patients complained of at least one ailment so that $k = 0$.

Hence, $n(\varepsilon) = a + b + c + j + d + e + g + 0 = 100$ ....*(iv)*

Put equations (*i*), (*ii*) and (*iii*) into (*iv*) so that a, b and c can be expressed in terms of d, e, g and j.

$n(\varepsilon) = (70 - d - g - j) + (50 - e - g - j) + (30 - e - g - d) + j + d + e + g = 100;$

$70 - d - g - j + 50 - e - g - j + 30 - e - g - d + j + d + e + g = 100;$

$70 + 50 + 30 - d - d + d - g - g - g + g - j - j + j - e - e + e = 100:$

$150 - 3g + g - d - j - e = 100;$

$150 - 2g - (d + j + e) = 100.$

In the question, it is stated that 44 complained of exactly two ailments. From the Venn diagram, it can be seen that

$$\left(\begin{matrix} \textit{the number of patient that} \\ \textit{complained of exactly 2 ailments} \end{matrix}\right) = d + j + e = 44;$$

hence, $150 - 2g - (d + j + e) = 150 - 2g - 44 = 100; 106 - 2g = 100;$

$2g = 106 - 100; 2g = 6; g = \frac{6}{2} = 3$

Looking at the Venn diagram, g is the number of patients that complained of fever, stomach ache and injury. From the calculation, $g = 3$; therefore, the number of patients that had all the three ailments is 3.

Binary Operation

1. A binary operation * is defined over R (the set of real numbers) by

$x * y = xy + x2 + y2$ for all $x, y \in R$.

(a) Determine whether or not * is commutative.

(b) If $x * (x + 2) = 49$, find x. **(WAEC)**

Workshop

(a) For the operation * to be commutative, $x * y$ must be equal to $y * x$ (for all $x, y \in R$)

$x * y = xy + x^2 + y^2 = yx + y^2 + x^2 = y * x$, since $x * y = y * x$, then, the operation * is commutative for all x, y being elements of set R.

Note that R stands for real numbers, so R is a set of real numbers.

(b)

$$x * (x+2) = 49;$$
$$x * (x+2) = x(x+2) + x^2 + (x+2)^2 = 49;$$
$$x^2 + 2x + x^2 + x^2 + 4x + 4 = 49;$$
$$3x^2 + 6x + 4 = 49;\ 3x^2 + 6x - 45 = 0;$$

divide through by 3, since 3, 6 and 45 are divisible by 3, to get $x2 + 2x - 15 = 0$;

$x2 + 5x - 3x - 15 = 0$;

$x(x + 5) - 3(x + 5) = 0$;

$(x + 5)(x - 3) = 0$; $(x + 5) = 0$ *or* ;

$(x - 3) = 0$; $x = -5$ *or* $x = 3$.

Therefore, the possible values of x are 3 and -5.

2. (a) A binary operation * is defined on the set R of real numbers by

$x * y = x^2 + y^2 - 1$, for all $x, y \in R$.

Determine whether or not the operation is

(i) closed;

(ii) commutative;

(iii) associative.

(b) Find, under the operation * in (a) above, the identity element. **(WAEC)**

Workshop

(a) (i) Let a and b be elements of set R (real numbers), so that $a * b = a^2 + b^2 - 1$.

Since a and b are real numbers, then $a2$ and $b2$ are also real numbers. Then, $a^2 + b^2 - 1$ will also be a real number which means that $a^2 + b^2 - 1$ is also an element of R. Because $a^2 + b^2 - 1$ is an element of R (real numbers), for all $a, b \in R$ (*that is a and b can be any set of two real numbers*), then, the operation * is closed.

(ii) For operation * to be commutative,

$x * y = y * x$;

$x * y = x^2 + y^2 - 1 = y^2 + x^2 - 1 = y * x$.

Since $x * y = y * x$, the operation * is commutative.

(iii) Let z also be an element of R, for * to be associative, $x * (y * z) = (x * y) * z$;

$$x * (y * z) = x * (y^2 + z^2 - 1)$$
$$= x^2 + (y^2 + z^2 - 1)^2 - 1$$
$$= x^2 + y^4 + y^2z^2 - y^2 + y^2z^2$$
$$+ z^4 - z^2 - y^2 - z^2 + 1 - 1$$
$$= y^4 + z^4 + x^2 - 2y^2 - 2z^2$$
$$+ 2y^2z^2 + 1 - 1;$$
$$x * (y * z) = y^4 + z^4 + 2y^2z^2$$
$$- 2y^2 - 2z^2 + x^2;$$
$$(x * y) * z = (x^2 + y^2 - 1) * z$$
$$= (x^2 + y^2 - 1)^2 + z^2 - 1;$$
$$= x^4 + x^2y^2 - x^2 + y^2x^2 + y^4$$
$$- y^2 - x^2 - y^2 + 1 + z^2 - 1$$
$$= x^4 + y^4 + 2x^2y^2 - 2y^2 - 2x^2$$
$$+ z^2 + 1 - 1$$
$$= x^4 + y^4 + 2x^2y^2 - 2y^2$$
$$- 2x^2 + z^2.$$

From the expressions, $y^4 + z^4 + 2y^2z^2 - 2y^2 - 2z^2 + x^2$ is **not equal to**

$x^4 + y^4 + 2x^2 y2 - 2y2 - 2x2 + z^2$ for all $x, y, z \in R$ then $x *(y *z) \neq (x * y) *z$,

so, the operation * is **not** associative for all $x, y, z \in R$ (note that $\neq$ is the symbol for **not equal to**).

(b) Let e be the identity element of the operation *; because * is closed,

then $a *e = e * a = a$; that is,

$a^2 + e^2 - 1 = e^2 + a^2 - 1 = a;$

$e^2 + a^2 - 1 = a; e^2 = a - a^2 + 1;$

$e = \sqrt{a - a^2 + 1}$. Thus, the identity element under the operation * is $e = \sqrt{a - a^2 + 1}$.

3. A binary operation. * is defined on R, the set of real numbers, by $x*y = \frac{xy}{4} + x + y$ for all $x, y \in R$.

(a) Find:

(i) the identity element $e \in R$,

(ii) the inverse of an element $r \in R$.

(b) Solve the equation: $x*6 = 36$. **(WAEC)**

Workshop

(a) (i) An identity element, e, is an element in the non-empty set R, such that

$x*e = e* x = x$ *for all* $x \in R$.

$$x*e = \frac{xe}{4} + x + e = x; \quad \frac{xe}{4} + x + e = x$$

$$\frac{xe}{4} + e = x - x = 0; \quad e\left(\frac{x}{4} + 1\right) = 0;$$

$$e = \frac{0}{\left(\frac{x}{4} + 1\right)} = 0.$$

Therefore, the identity element $e \in R$ is 0.

(ii) If r−1 is the inverse of element r, which is an element of R ($r \in R$) , then $r * r^{-1} = e$. For this question, e = 0, so that $r * r^{-1} = e = 0$;

$$r * r^{-1} = \frac{r\, r^{-1}}{4} + r + r^{-1} = 0;$$

$$r^{-1}\left(\frac{r}{4} + 1\right) + r = 0; \quad r^{-1}\left(\frac{r+4}{4}\right) = -r$$

$$r^{-1} = \frac{-r}{\left(\frac{r+4}{4}\right)} = \frac{-4r}{r+4}.$$

Therefore, the inverse of an element $r \in R$ is $r^{-1} = \frac{-4r}{r+4}$

(b) $x*6 = 36; \quad x*6 = \frac{x.6}{4} + x + 6 = 36;$

$$\frac{6x}{4} + x = 30; \quad x\left(\frac{6}{4} + 1\right) = 30;$$

$$x\left(\frac{10}{4}\right) = 30; \quad \frac{10x}{4} = 30;$$

$$10x = 30 \times 4; \quad x = \frac{30 \times 4}{10} = \frac{120}{10} = 12.$$

Therefore, $x = 12$.

4. Two binary operations Δ and * are defined on the set R of real numbers by; $x \,\Delta\, y = 2(x + y) + 3xy$, $x * y = 2x + y$.

Find:

(a) $x\,(y * z)$;

(b) $(x\,y) * (x\,z)$. **(WAEC)**

Workshop

(a) $x\Delta(y * z) = x\Delta(2y + z)$; recall that

$x\Delta y = 2(x + y) + 3xy$ so that

$$x \Delta (2y + z) = 2[x + (2y + z)] + 3x(2y + z)$$
$$= 2x + 2(2y + z) + 6xy + 3xz$$
$$= 2x + 4y + 2z + 6xy + 3xz.$$

Therefore, $x\Delta (y * z) = 2x + 4y + 2z + 6xy + 3xz$.

Note that for this question, $x\Delta(y * z)$ is ***not*** *equal to $(x\Delta y)*(x\Delta z)$. This you can work out on a rough sheet. In other words, the operation Δ is not distributive over operation $*$, therefore, you* ***cannot*** *simplify $x\Delta(y*z)$ as $x\Delta(y*z) = (x\Delta y)*(x\Delta z)$ since Δ is* ***not*** *distributive over $*$. To get the correct answer, adopt the method above.*

(b) From the identities given in question,

$(x \Delta y) * (x \Delta z)$ can be simplified as;

$(x \Delta y) * (x \Delta z) = [2(x + y) + 3xy] * [2(x + z) + 3xz]$;

$(x \Delta y) * (x z) = (2x + 2y + 3xy) * (2x + 2z + 3xz)$.

Recall that $x * y = 2x + y$, so that;

$(2x + 2y + 3xy) * (2x + 2z + 3xz)$

$= 2(2x + 2y + 3xy) + (2x + 2z + 3xz)$;

$= 4x + 4y + 6xy + 2x + 2z + 3xz$

$= 6x + 4y + 2z + 6xy + 3xz$.

Therefore,

$(x \Delta y) * (x \Delta z) = 6x + 4y + 2z + 6xy + 3xz$.