CHAPTER 5 Algebraic Equations and Polynomials

A polynomial is an expression that combines constants and variables having positive whole number exponents. These combinations are **strictly** joined by addition, subtraction and multiplication signs and **never** with the division sign.

The following are examples of polynomials: $2x,\ 3x^2 - 8,\ 5x^3 + 9x,\ \frac{1}{2}x^4$ while these are not examples of polynomials $\frac{1}{x^2},\ 5x^{-4},\ 8x^{\frac{1}{2}}$, and so on.

> *Note carefully that $\frac{1}{2}x^4$ can be further expressed as $0.5x^4$ which has got no division sign; therefore, $\frac{1}{2}x^4$ is a polynomial.*

A variable is an alphabet (or symbol) that can be assigned any value. For instance, given a variable *x*, Ali can assign 2 to *x* (i.e. $x = 2$), while Charles can assign −8 to *x*. So, the reason why *x* is a variable is that anyone can chose to assign any value to it, as explained above.

A constant is any number that you can think of. As its name implies, a constant does not change, it is always the same. Examples of constants are 2, −8.5, $\sqrt{2}$, 1.9999, 1000, etc.

An exponent is the power to which a constant or variable is raised; the exponent of $2x^5$ is 5 , while the exponent of 8^3 is 3.

The degree of a polynomial is the value of its greatest variable exponent. The polynomial

$x^2 + x - 5x^4$ is of degree 4, while the polynomial $x^3 + x - 1$ is of degree 3.

> *Note that some textbooks prefer to use the degree of a polynomial instead of the order of a polynomial; both are right.*

Polynomials can be added, subtracted, multiplied and divided by each other; the division of polynomials is not also a big deal,an example is presented below.

Unlike numbers, polynomials **DO NOT** have decimal parts; therefore, one can **only** divide a polynomial of a high degree by a polynomial of an **equal or lower degree**, and **NOT** the other way round. Let us divide $6x^2 + 13x^2 + x - 2$ by $2x + 1$.

$$
\begin{array}{r|l}
 & 3x^2 + 5x \quad 2 \\
\hline
2x+1 & 6x^3 + 13x^2 + x - 2 \\
- & 6x^3 + 3x^2 \\
\hline
 & \quad 10x^2 + x \\
- & \quad 10x \;\; + 5x \\
\hline
 & \qquad -4x - 2 \\
- & \qquad -4x \quad 2 \\
\hline
 & \qquad\; 0 \quad 0
\end{array}
$$

The first step in dividing a polynomial by another polynomial is to identify the term having the highest degree in each polynomial. In the case of the example above, the term with the highest degree for the numerator is $6x^3$, while the term with the highest degree in the denominator is $2x$. The next step is to divide $6x^3$ by $2x$ and this gives $3x^2$. The result, $3x^2$ is written directly on top of the term $6x^3$. Then $3x^2$ is used to multiply each of the terms of the denominator ($2x + 1$) and the result is written as $6x^3 + 3x^2$, and it is directly written under $6x^3 + 13x^2$. $6x_3 + 3x^2$ is then subtracted from $6x^3 + 13x^2$ to get $10x^2$. The term + x, which is yet to be touched in the numerator, is now brought straight down beside $10x^2$ (go back to the long division). $10x^2$ is also divided by the $2x$ (which is the term of highest degree in the denominator) , and the result ($+5x$) is written on top, immediately after $3x^2$. The $5x$ is also used to multiply the denominator ($2x + 1$) to get $10x^2 + 5x$, and this result is written under $10x^2 + x$. $10x^2 + 5x$ is also subtracted from $10x^2 + x$ and the result ($-4x$) is written directly under $+5x$.

The final term in the numerator (-2), is then brought down beside $-4x$. $-4x$ is also divided by $2x$ (which is the term of highest degree in the denominator) to get -2. This result (-2) is written just after $+5x$. Just like $3x^2$ and $+5x$, -2 is also used to multiply $2x$ (which is the term of highest degree in the denominator) to get $-4x - 2$ and this result is written under $-4x - 2$. Subtracting these two gives zero showing that polynomial $2x + 1$ is a factor of the other polynomial $6x^3 + 13x^2 + x - 2$.

Zeroes of a Polynomial

When a number (a constant) is substituted in place of a variable in a polynomial, and the evaluated result gives zero, such number is referred to as 'a zero of the polynomial.' For example, substitute $x = -2$ into the numerator of the example above; this gives $6(-2)^3 + 13(-2)^2 + (-2) - 2 = 6(-8) + 13(4) - 2 - 2 = -48 + 52 - 4 = 0$. Therefore, -2 is a zero of the polynomial $6x^3 + 13x^2 + x - 2$. And, if $x = -2$, taking -2 to the other side of the equality sign gives $x + 2 = 0$, so, $x + 2$ will be a factor of $6x^3 + 13x^2 + x - 2$. Thus, if $6x^3 + 13x^2 + x - 2$ is divided by $x + 2$, the remainder will be zero.

Now, try to divide $6x^3 + 13x^2 + x - 2$ by $x - 1$ using the long division method. What would the remainder be? The remainder is 18. But, there is a short method to get this remainder though. Just find the value of x for which $x - 1$ is equal to zero as follows: $x - 1 = 0$; $x = +1$. Then substitute $x = 1$ into $6x3 + 13x2 + x - 2$ to get

$6(1)^3 + 13(1)^2 + (1) - 2 = 6 + 13 + 1 - 2 = 18$.

Hence, if $6x3 + 13x2 + x - 2$ is divided by $x - 1$ its remainder will be 18.

Polynomials

1. If $x + 2$ and $x - 1$ are factors of $f(x) = 6x4 + mx3 - 13x2 + nx + 14$, find the:

   (a) values of m and n;

   (b) remainder when $f(x)$ is divided by $(x + 1)$.

(WAEC)

Workshop

(a) By the remainder theorem, when $f(x)$ is divided by $(x - r)$, the remainder is $f(r)$ and when $f(x)$ is divided by $(x + r)$, the remainder is $f(-r)$. Moreover, by the remainder theorem, if $f(x)$ is divided by $(x + 2)$, the remainder is $f(-2)$. But from the question, we were told $(x + 2)$ is a factor of $f(x)$: this means if $f(x)$ is divided by $(x + 2)$ the remainder will be zero; therefore, $f(-2) = 0$. $f(x) = 6x4 + mx3 - 13x2 + nx + 14$;

$f(-2) = 0; f(-2) = 6(-2)^4 + m(-2)^3 - 13(-2)^2 + n(-2) + 14 = 0;$

$6(16) + m(-8) - 13(4) + n(-2) + 14 = 0;$

$96 - 8m - 52 - 2n + 14 = 0;$

$96 - 52 + 14 = 8m + 2n; 8m + 2n = 58.$

Divide through the equation by 2 to get

$4m + n = 29$.....(*i*). Also, since $(x - 1)$ is also a factor of $f(x)$, then $f(+1) = 0$, hence,

$f(+1) = 6(+1)4 + m(m(+1)3 - 13(+1)2 + n(+1) + 14 = 0;$

$6 + m - 13 + n + 14 = 0;$

$m + n = 13 - 14 - 6 = -7; m + n = -7$...(*ii*)

$4m + n = 29$ ....(*i*)

$m + n = -7$ ....(*ii*).

Subtract equation (*ii*) from (*i*) to get

$4m + n - (m + n) = 29 - (-7);$

$4m + n - m - n = 29 + 7; 4m - m + n - n = 36;$

$3m = 36; m = \frac{33}{3} = 12.$

Put $m = 12$ into equation (*ii*) to get $12 + n = -7; n = -7 - 12 = -19$.

Therefore the values of m and n are respectively 12 and -19.

(b) Also by the remainder theorem, when $f(x)$ is divided by $(x + 1)$ the remainder is $f(-1)$.

$f(-1) = 6(-1)^4 + m(-1)^3 - 13(-1)^2 + n(-1) + 14$ recall that $m = 12$ and $n = -19$ thus

$f(-1) = 6(-1)^4 + 12(-1)^3 - 13(-1)^2 + (-19)(-1) + 14$

$= 6(1) + 12(-1) - 13(1) - 19(-1) + 14$

$= 6 - 12 - 13 + 19 + 14$

$= 14; f(-1) = 14.$

Therefore, the remainder when $f(x)$ is divided by $(x + 1)$ is 14.

2. (a) Sketch the curve $y = x^2 - 4$.

(b) Find the area of the finite region enclosed by the curve $y = x^2 - 4$, the x-axis and the ordinates $x = -2, x = 3$.

(WAEC)

Workshop

(a) To sketch the curve $y = x2 - 4$, we must know the intercepts of the curve on the x-axis and the y-axis, the coordinates of its turning **point**, and we also have to know if the turning point is a maximum or a minimum turning point. At the points where the curve (*or any other curve*) intercept the x-axis, the value of $y = 0$; therefore, $y = x^2 - 4 = 0$; $x^2 = 4$; $x = \pm\sqrt{4} = \pm 2$.

Hence, the graph $y = x^2 - 4$ cuts the x-axis at $x = +2$ *and* $x = -2$.

At the point where the curve (*or any other curve*) intercepts the y-axis, the value of $x = 0$. Then, $y = x^2 - 4 = 02 - 4 = -4$. So, the value of y at the point where the curve $y = x^2 - 4$ cuts the y-axis is $y = 4$.

Recall that at the turning point of any curve, $\frac{dy}{dx} = 0$;

$y = x2 - 4, \frac{dy}{dx} = 2x = 0, x = \frac{0}{2} = 0$ when $x = 0$,

$y = x2 - 4 = 0^2 - 4 = -4$. So, the coordinate of the turning point of the curve is $(0, -4)$.

$\frac{dy}{dx} = 2x$; $\frac{d\,2y}{dx2} = 2$. Since $\frac{d\,2y}{dx2}$ is greater than zero $\left(\frac{d^2y}{dx^2} > 0\right)$, then the turning point $(0, -4)$

is a minimum turning point. From the calculations above, the graph $y = x^2 - 4$ cuts the x-axis at $x = +2$ *and* $x = -2$. The same curve cuts the y-axis at $y = 4$ and has a minimum turning point $(0, -4)$. Thus, the curve of the equation $y = x^2 - 4$ can be sketched as shown in Figure 5.1 below.

(i)

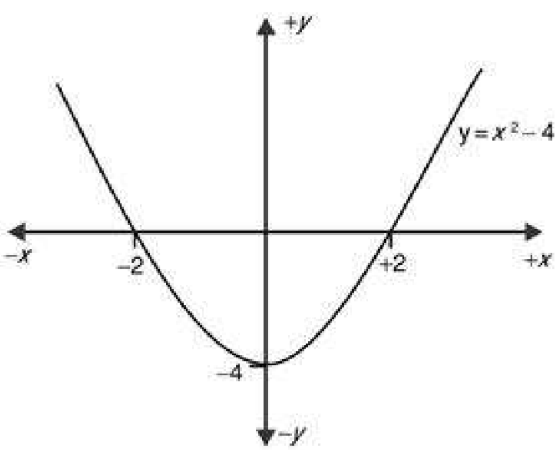

Fig. 5.1

(b)

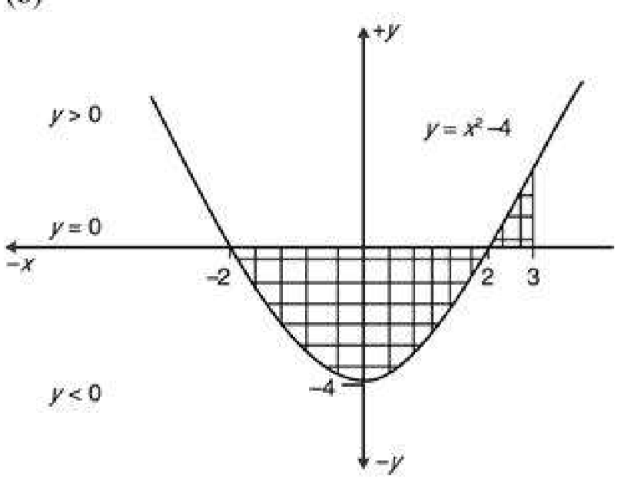

Fig. 5.2

The shaded area in Figure 5.2, is the area of the finite region enclosed by the curve $y = x^2 - 4$, the x-axis and the ordinates $x = -2$, $x = 3$. This simply means the area enclosed by the curve and x-axis between the interval $x = -2$ *and* $x = 3$.

From Figure 5.2, the shaded region between $x = -2$ *and* $x = 2$ will have an area bearing a negative sign as the area is on the negative side (where $y < 0$) of the graph.

The area A enclosed by a curve $y = f(x)$, the x-axis and the ordinates $x1$ *and* $x2$ is expressed as

$A = \int_{x_1}^{x_2} f(x).dx$. Hence, the area A enclosed by the curve $y = x_2 - 4$ and the x-axis and the ordinates -2 and 2 will be expressed as

$$\int_{-2}^{2} (x_2 - 4)\,.\,dx = \int_{-2}^{2} (x_2 - 4x_0).$$

$$dx = \left[\frac{x^{2+1}}{2+1} - 4\left(\frac{x_{0+1}}{0+1}\right)\right]_{-2}^{2} = \left[\frac{x_3}{3} - 4x\right]_{-2}^{2};$$

$$A = \left[\frac{2_3}{3} - 4(2)\right] - \left[\frac{(-2)_3}{3} - 4(-2)\right]$$

$$= \left(\frac{8}{3} - 8\right) - \left[\frac{-8}{3} + 8\right];$$

$$A = \frac{8}{3} - 8 + \frac{8}{3} - 8 = \frac{16}{3} - 16 = \frac{-32}{3}.$$

The area enclosed by the curve, the x-axis and the ordinates -2 and 2 is $\frac{-32}{3}$ *square unit i.e*(*unit*)2.

The negative sign shows that this area is in the negative half of the graph as can be seen from the diagram. However the area shaded between $x = 2$ *and* $x = 3$ will bear a positive sign since it is in the positive half of the graph. This area can be calculated as

$$A = \int_{2} (x_2 - 4).\,dx = \left[\frac{x_3}{3} - 4x\right]_{2}^{3}$$

$$= \left[\frac{3_3}{3} - 4\,(3)\right] - \left[\frac{2_3}{3} - 4(2)\right] = \left[\frac{27}{3} - 12\right] - \left[\frac{8}{3} - 8\right]$$

$$A = (9 - 12) - \left(\frac{8}{3} - 8\right)$$

$$= -3 - \frac{8}{3} + 8 = 5 - \frac{8}{3} = \frac{7}{3} \textit{ square units.}$$

The area enclosed by the curve $x^2 - 4 = y$, the x-axis and the ordinates $x = 2$ *and* $x = 3$ is $\frac{7}{3}$ square units.

So, the area enclosed by the curve $x2 - 4 = y$, the x-axis and the ordinates $x = -2$ *and* $x = 3$ will be $\frac{32}{3} + \frac{7}{3} = \frac{30}{3} = 13$ *square units.*

*Note that in calculating areas that fall in different segment of the curves as was the case in this question, it is necessary to calculate separately the area for each segment, then the negative sign on the value(s) bearing a negative sign **must** be removed before adding the areas for each segment together. Also note that because the areas of shapes are scalar quantities, the negative sign of any area calculated can be ignored.*

Algebraic Equation

1. If α and β are the roots of the equation $3x^2 - 5x + 4 = 0$, find the value of

$\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$. **(WAEC)**

Workshop

Given the general equation $ax^2 + bx + c = 0$, having roots α and β, the sum of roots $\alpha + \beta =$

$\frac{-a}{b}$., while the product of roots $\alpha\beta = \frac{c}{a}$. Hence, if α and β are the roots of the equation $3x^2 - 5x + 4 = 0$;

$$\alpha + \beta = -\frac{b}{a} = -\frac{(-5)}{3} = \frac{5}{3};$$

$$\alpha + \beta = \frac{5}{3} \quad \text{..}(i)$$

and

$$\alpha\beta = \frac{c}{a} = \frac{4}{3};\ \alpha\beta = \frac{4}{3};\ \text{........................}(ii).$$

But, $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$; and $(\alpha + \beta)^2$

$= \alpha^2 + 2\alpha\beta + \beta^2;\ \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta.$

Therefore, $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} \text{............ }(iii).$$

Put the values of $\alpha + \beta$ and $\alpha\beta$ in equations *(i)* and *(ii)* into *(iii)* to get

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{\left(\frac{5}{3}\right)^2 - 2\left(\frac{4}{3}\right)}{\frac{4}{3}}$$

$$= \frac{\frac{25}{9} - \frac{8}{3}}{\frac{4}{3}} = \frac{\frac{25 - 24}{9}}{\frac{4}{3}} = \frac{1}{9} \div \frac{4}{3} = \frac{1}{9} \times \frac{3}{4} = \frac{1}{12}.$$

Therefore, the value of $\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$. $= \frac{1}{12}$.

2. If the roots of $4x^2 + 8x - m = 0$ differ by 3, find the value of the constant m. **(WAEC)**

Workshop

Let the roots of the equation be α *and* β; $\alpha - \beta = 3$; $\alpha = 3 + \beta$. Given the general quadratic equation, $ax^2 + bx + c = 0$. If the roots of this quadratic equation are α and β, then, sum of roots, $\alpha + \beta = \frac{-b}{a}$ while the product of roots, $\alpha\beta = \frac{c}{a}$.

Thus, for this problem, the equation in question can be rewritten in the general form as $4x^2 + 8x + (-m) = 0$, so that the sum of the roots will be $\alpha + \beta = \frac{-b}{a} = \frac{-8}{4} = -2$.

Recall that $\alpha = 3 + \beta$, so that, $\alpha + \beta = (3 + \beta) + \beta = -2$.

$3 + \beta + \beta = -2$; $3 + 2\beta = -2$; $2\beta = -5$; $\beta = \frac{-5}{2}$.

The product of root, $\alpha\beta = \frac{c}{a} = \frac{-m}{4}$. recall that $\alpha = 3 + \beta$, hence,

$\alpha\beta = (3+\beta)\beta = \frac{-m}{4}; \left[3 + \left(\frac{-5}{2}\right)\right]\left(\frac{-5}{2}\right) = \frac{-m}{4};$

$\left(3 - \frac{5}{2}\right)\left(-\frac{5}{2}\right) = -\frac{m}{4}; \frac{1}{2} \times -\frac{5}{2} = -\frac{m}{4}; -\frac{5}{4} = -\frac{m}{4};$

$$-4m = -5 \times 4;\ m = \frac{-5 \times 4}{-4} = 5.$$

Therefore, the value of the constant, m, is 5.

3. Find the possible values of m with which $x2 + (m - 2)x + m + 1 = 0$, has two equal roots.

(WAEC)

Workshop

For any quadratic equation, $ax^2 + bx + c = 0$, to have equal roots, $b^2 - 4ac$ **must** be equal to zero ($b^2 - 4ac = 0$). In the case of

$x^2 + (m - 2)x + m + 1 = 0$, $a = 1$, $b = m - 2$ and $c = m + 1$. Thus, for $x^2 + (m - 2)x + m + 1 = 0$ to have equal roots,

$b^2 - 4ac = (m - 2)2 - 4(1)(m + 1) = 0$; $m2 - 4m + 4 - 4m - 4 = 0$;

$m^2 - 8m = 0$; $m(m - 8) = 0$; $m = 0$ *or* $m - 8 = 0$;

$\therefore m = 0$ *or* $m = 8$.

Therefore, the possible values of m for which the equation will have equal roots are 0 and 8.

4. The roots of the equation $x2 - Px + 8 = 0$ are α and β. If the roots differ by 2,
   (a) calculate the possible values of P
   (b) Hence, find the possible values of $\alpha 3 + \beta 3$.

Workshop

a) $x^2 - Px + 8 = 0$; the roots of this equation are α and β. Let the greater of the two roots be α, so that

$\alpha - \beta = 2$; $\alpha = 2 + \beta$. For any quadratic equation, $ax^2 + bx + c = 0$, the sum of its roots, $\alpha + \beta = -\left(\frac{b}{a}\right)$,

and the product of its roots $\alpha\beta = \left(\frac{c}{a}\right)$

In the case of the equation, $x^2 - Px + 8 = (1)x^2 + (-P)x + 8 = 0$, $a = 1$, $b = -P$ and $c = +8$.

Hence, $\alpha + \beta = -\frac{b}{a}$; $(2 + \beta) + \beta$

$$= -\left(\frac{-P}{1}\right) = P;$$

$2 + \beta + \beta = 2 + 2\beta = P$ ....(i)

$\alpha\beta = \frac{c}{a}$; $(2 + \beta)\,\beta = \frac{+8}{1} = 8$; therefore,

$2\beta + \beta^2 = 8$ ....(ii).

$2 + 2\beta = P$ ....(i)

$2\beta + \beta^2 = 8$..(ii).

From equation (*ii*), $\beta2 + 2\beta - 8 = 0$;

$\beta^2 + 4\beta - 2\beta - 8 = 0$;

$\beta(\beta + 4) - 2(\beta + 4) = 0$; $(\beta + 4)(\beta - 2) = 0$;

$\beta + 4 = 0$ *or* $\beta - 2 = 0$; $\beta = -4$ *or* $\beta = 2$.

Put the values of β into equation (*i*), so that,

when $\beta = 2$, $P = 2 + 2(2) = 2 + 4 = 6$,

when $\beta = -4$, $P = 2 + 2(-4) = 2 - 8 = -6$.

Therefore, the possible values of P are + 6 and − 6.

(b) The clause 'hence, find the possible values of $\alpha^3 + \beta^3$', means that from the previous answers, find the possible values of $\alpha^3 + \beta^3$. Thus, we must look for a means to evaluate $\alpha^3 + \beta^3$, from the previous answer(s).

Recall that $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$; $(\alpha + \beta)^3$;

$$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta);$$

$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = \alpha^3 + \beta^3$.

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ ....(*).

Recall from the previous solutions that $\alpha + \beta = P$ and $\alpha\beta = 8$. Also, recall that $P = +6$ or -6; therefore, from equation(*), when $P = +6$,

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = (+6)^3 - 3(8)(+6) = 216 - 144 = 72$.

when $P = -6$, $\alpha^3 + \beta^3 = (-6)^3 - 3(8)(-6) = -216 + 144 = -72$.

Therefore, the possible values of $\alpha^3 + \beta^3$ are + 72 and −72.

5. The roots of the equation $x^2 + mx + 11 = 0$ are α and β, where m is a constant. If $\alpha^2 + \beta^2 = 27$, find the values of m. **(WAEC)**

Workshop

The roots of $x^2 + mx + 11 = 0$ are α and β we were told to find m if $\alpha^2 + \beta^2 = 27$. From the general quadratic equation $ax^2 + bx + c = 0$; sum of roots $= \frac{-b}{a}$ while product of roots $= \frac{c}{a}$. Comparing the quadratic equation in question with the general quadratic equation; $a = 1$, $b = m$ and $c = 11$. Hence, the sum of roots $= \alpha + \beta = \frac{-b}{a} = \frac{-m}{1}$. Therefore, $\alpha + \beta = \frac{-m}{1} = -m$, while the product of roots, $\alpha\beta = \frac{c}{a} = \frac{11}{1} = 11$.

From the question, we were told that $\alpha^2 + \beta^2 = 27$, recall that, $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$; $(\alpha + \beta)^2 - 2\alpha\beta = \alpha^2 + \beta 2$; thus, $\alpha^2 + \beta^2 = (a + \beta)^2 - 2\alpha\beta = 27$; also, recall that $\alpha + \beta = -m$ and $\alpha\beta = 11$, then,

$\alpha^2 + \beta^2 = (a + \beta)^2 - 2\alpha\beta = (-m)^2 - 2(11) = 27$;

$(-m)^2 - 22 = 27$; $(-m)^2 = 27 + 22 = 49$; $(-m)^2 = 49$;

$-m = \pm\sqrt{49}$. ; $-m = \}7$, *so*, $-m = +7$ *or* $-m = -7$; $m = -7$ *or* $m = +7$.

Therefore, the values of m are -7 *and* $+ 7$.

Binomial Theorem

1. (a) Write down the first three terms of the binomial expansion of $(1 + 2x)^5$ in ascending powers of x.

(b) Use your result in **1**(a) evaluate $(1.02)^5$. **(WAEC)**

Workshop

(a) $(a + b)^n$, can be expanded using the binomial expansion as follows:

$(a + b)^n = {}^nC_0 a^{n-0} b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2$

$+ \ldots + {}^nC_r a^{n-r} b^r + \ldots + {}^nC_n a^{n-n} b^n$, where n is a positive integer (that is, $n = 1, 2, 3, 4, \ldots$).

$(1 + 2x)^5 = {}^5C_0 1^{5-0} (2x)^0 + {}^5C_1 1^{5-1} (2x)^1 + {}^5C_2 1^{5-2} (2x)^2 + \ldots$, since we were told to write the first three terms only.

$$(1 + 2x)^5 = \frac{5!}{(5-0)!\,0!} 1^5 + \frac{5!}{(5-1)!\,1!} 1^4(2x) + \frac{5!}{(5-2)!\,2!} 1^3(2x)^2 + \ldots.$$

$$(1 + 2x)^5 = 1 + \frac{5 \times 4!}{4! \times 1!}(2x) + \frac{5 \times 4 \times 3!}{3! \times 2!}(2x)^2 \ldots;$$

$$(1 + 2x)^5 = 1 + 10x + 10(4x^2) + \ldots;$$

$$(1 + 2x)^5 = 1 + 10x + 40x^2 + \ldots;$$

Therefore, the first three terms in the binomial expansion of $(1 + 2x)^5$ in ascending powers of x are 1, $10x$ *and* $40x^2$.

(b) To be able to use our result in **1**(a) to evaluate(1.02)5, we have to re-write $(1.02)^5$ to look like $(1 + 2x)^5$ as we are going to do shortly.

$(1.02)^5 = (1 + 0.02)^5 = (1 + 2(0.01))^5$; compare this with $(1 + 2x)^5$ to see that $x = 0.01$ in this case. And so, using the result, $(1+2x)^5 = 1 + 10x + 40x^2$ in 9(a) above,

$(1 + 2(0.01))^5 = 1 + 10(0.01) + 40(0.01)^2 = 1 + 0.1 + 0.004 = 1.104.$

2. (a) Write down the binomial expansion of $(1\ x)^4$.

(b) Using the linear expansion of $(1 - 2x)^{\frac{1}{3}}$, find, correct to 4 significant figures, the value of $(1.05)^{\frac{1}{3}}$.
(WAEC)

Workshop

(a) The binomial expansion of $(a + b)n$ is expressed as

$$(a + b)^n = {}^nC_0a^{n-0}b^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \dots + {}^nC_ra^{n-r}b^r + \dots + {}^nC_na^{n-n}b_n,$$

where n is a positive whole number, that is $n = 1, 2, 3, 4, 5,\dots$

$$(1 + x)^4 = {}^4C_01^{4-0}x^0 + {}^4C_11^{4-1}x^1 + {}^4C_21^{4-2}x^2 + {}^4C_31^{4-3}x^3 + {}^4C_41^{4-4}x^4;$$

$$(1+x)_4 = \frac{4!}{(4-0)!\,0!} \times 1_4 + \frac{4!}{(4-1)!\,1!} \times 1_3 x_1 + \frac{4!}{(4-2)!\,2!} 1_2 x_2 + \frac{4!}{(4-3)!\,3!} 1 \cdot x_3 + \frac{4!}{(4-4)!\,4!} x_4;$$

$$(1+x)_4 = 1 + \frac{4\times 3!}{3!\,1!} x + \frac{4\times 3\times 2!}{2!\,2!} x_2 + \frac{4\times 3!}{1!3!} x_3 + \frac{4!}{0!\,4!} x_4$$

$$= 1 + 4x + 6x_2 + 4x_3 + x_4;\ (1+x)_4$$

$$= 1 + 4x + 6x_2 + 4x_3 + x_4.$$

(b) The binomial expansion for $(1 + a)^n$, where n can be any rational number, is expressed as

$$(1+a)_n = 1 + \frac{n}{1!} a + \frac{n(n-1)}{2!} a_2 + \frac{n(n-1)(n-2)}{3!} a_3 + \frac{n(n-1)(n-2)(n-3)}{4!} a_4 + \dots$$

Note that this form of binomial expansion – as shown above – can also be used in expanding $(1+x)_4$, as it is also in the form $(1+a)_n$.

$(1-2x)^{\frac{1}{3}} = (1+(-2x))^{\frac{1}{3}}$; so that, in this case,
$a = -2x,\ n = \frac{1}{3}$.

$$(1+(-2x))^{\frac{1}{3}} = 1 + \frac{\frac{1}{3}}{1!}(-2x) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(-2x)_2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-2x)_3 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!}(-2x)_4 + \dots$$

From the expression above, the linear expansion of, $(1 + (-2x))^{\frac{1}{3}} = 1 + \frac{1}{3}(-2x) = 1 - \frac{2x}{3}$.

Note that the word linear means that the highest power of the terms in x in the expansion must be 1.

Hence, the linear expansion of, $(1 - 2x)^{\frac{1}{3}} = 1 - \frac{2x}{3}$. To use this linear expansion to solve $(1.05)^{\frac{1}{3}}$, , there is need to express $(1.05)^{\frac{1}{3}}$, in the form, $(1 - 2x)^{\frac{1}{3}}$. $(1.05)^{\frac{1}{3}} = , (2 - 0.95)^{\frac{1}{3}}$

$$= \left(2\left(1 - \frac{0.95}{2}\right)\right)^{\frac{1}{3}} = 2^{\frac{1}{3}}\left(1 - \frac{0.95}{2}\right)^{\frac{1}{3}};$$

$$(1.05)^{\frac{1}{3}} = 2^{\frac{1}{3}}(1 - 0.475)^{\frac{1}{3}}$$

$$= 2^{\frac{1}{3}}\left(1 - 2\left(\frac{0.475}{2}\right)\right)^{\frac{1}{3}}$$

$= (2)^{\frac{1}{3}}(1 - 2(0.2375))^{\frac{1}{3}}$. We can now use the linear expansion of $(1 - 2x)^{\frac{1}{3}}$, which is $1 - \frac{2x}{3}$, to solve the problem. Compare

$(1 - 2(0.2375))^{\frac{1}{3}}$ with $(1 - 2x)^{\frac{1}{3}}$ to see that,

$x = 0.2375$.

So, $(1 - 2(0.2375))^{\frac{1}{3}} = 1 - \frac{2x}{3} = 1 - \frac{2(0.2375)}{3}$

$= 1 - 0.1583 = 0.8417;$

$2^{\frac{1}{3}}(1 - 2(0.2375))^{\frac{1}{3}} = 2^{\frac{1}{3}}(0.8417)$

$= 1.26(0.8417) =$ 1.061 correct to four significant figures. Therefore, the value of $(1.05)^{\frac{1}{3}}$ using the linear expansion of $(1 - 2x)^{\frac{1}{3}}$ is 1.061, correct to 4 significant figures.

3. (a) Write down the binomial expansion of $(1 + y)^8$, simplifying all the terms.

(b) Using the substitution $y = x - x^2$ in (a), deduce the expansion of $(1 + x - x^2)^8$ in ascending powers of x, as far as the term in $x4$.

(c) Find, by inspection, the value of x such that $1 + x - x^2 = 1.09$.

Hence, evaluate $(1.09)^8$ correct to three decimal places. **(WAEC)**

Workshop

(a) Using binomial expansion,

$(a + b)^n = {}^nC_0a^{n-0}b^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + ... + {}^nC_ra^{n-r}b^r + ... + {}^nC_na^{n-n}b_n$, where n is a positive whole number, i.e. 1, 2, 3, 4, ...

$(1 + y)^8 = {}^8C_0 1^8 y^0 + {}^8C_1 1^{8-1} y^1 + {}^8C_2 1^{8-2} y^2 + {}^8C_3 1^{8-3} y^3 + {}^8C_4 1^{8-4} y^4 + {}^8C_5 1^{8-5} y^5 + {}^8C_6 1^{8-6} y^6 + {}^8C_7 1^{8-7} y^7 + {}^8C_8 1^{8-8} y^8;$

$$(1+y)_8 = 1^8 + \frac{8}{1!}(1^7)\,y_1 + \frac{8\times 7}{2!}\,1^6 y_2$$

$$+\frac{8\times 7\times 6}{3!}1^5 y_3 + \frac{8\times 7\times 6\times 5}{4!}1^4 y_4$$

$$+\frac{8\times 7\times 6\times 5\times 4}{5!}1^3 y_5$$

$$+\frac{8\times 7\times 6\times 5\times 4\times 3}{6!}1^2 y_6$$

$$+\frac{8\times 7\times 6\times 5\times 4\times 3\times 2}{7!}1^1 y7 + y_8;$$

$$(1+y)_8 = 1_8 + 8y + \frac{8\times 7}{2\times 1}y_2$$

$$+\frac{8\times 7\times 6}{3\times 2\times 1}y_3 + \frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1}y_4 +$$

$$\frac{8\times 7\times 6\times 5\times 4}{5\times 4\times 3\times 2\times 1}y_5$$

$$+\frac{8\times 7\times 6\times 5\times 4\times 3}{6\times 5\times 4\times 3\times 2\times 1}y_6$$

$$+\frac{8\times 7\times 6\times 5\times 4\times 3\times 2}{7\times 6\times 5\times 4\times 3\times 2\times 1}y_7 + y_8;$$

$(1 + y)^8 = 1 + 8y + 28y^2 + 56y^3 + 70y^4 + 56y^5 + 28y^6 + 8y^7 + y^8$

Therefore, the binomial expansion of $(1 + y)^8$ is $1 + 8y + 28y^2 + 56y^3 + 70y^4 + 56y^5 + 28y^6 + 8y^7 + y^8$

(b) Put $y = x - x^2$ in the expansion in (a) to get $(1 + y)^8 = [1 + (x - x^2)]^8$

Recall that $(1 + y)^8 = 1 + 8y + 28y^2 + ... + y^8$ then,

$[1 + (x - x^2)]^8 = 1 + 8(x - x^2) + 28(x - x^2)^2 + 56(x - x^2)^3 + 70(x - x^2)^4$

(*because we are to expand as far as the term in* x^4)

$= 1 + 8x - 8x^2 + 28(x^2 - 2x^3 + x^4) + 56\,(x^3 + 3x^2(-x^2) + 3x(-x^2)^2 + (-x^2)^3) + 70(x^4 + 4x^3(-x^2) + 6x^2(-x^2)^2 + 4x(-x^2)^3 + (-x^2)^4)$

$= 1+ 8x - 8x^2 + 28x^2 - 56x^3 + 28x^4 + 56x^3 + 168x^2(-x^2) + 168x(-x^2)^2 + 56(-x^2)^3 + 70x^4 + 280x^3(-x^2) + 420x^2(-x^2)^2 + 280x(-x^2)^3 + 70\,(-x^2)^4$

$= 1 + 8x - 8x^2 + 28x^2 - 56x^3 + 28x^4 + 56x^3 - 168x^4 + 168x^5 - 56x^6 + 70x^4 - 280x^5 + 420x^6 - 280x^7 + 70x^8.$

But, recall that we are to expand $[\,1 + (x - x^2)\,]^{\,8}$ as far as the term in $x4$; therefore, we must eliminate **all** the terms having power of x that is greater than 4 as we will do shortly:

$[1 + (x - x^2)]^8 = 1 + 8x - 8x^2 + 28x^2 - 56x^3 + 28x^4 + 56x^3 - 168x^4 + 70x^4;$

$[1 + (x - x^2)]^8 = 1 + 8x + 20x^2 - 70x^4.$

Therefore, the expansion of $(1 + x - x^2)^8$ as far as the term in $x4$ is $1 + 8x + 20x^2 - 70x^4$.

(c) By inspection (*that is, merely looking at the equation*), x is approximately equal to 0.09. This is because

$1 + x - x^2 = 1.09;$

$x - x^2 = 1.09 - 1;$

$x - x^2 = 0.09;\ x = 0.09 + x^2.$

The equation $x = 0.09 + x^2$ explains that x is greater than x^2 because you need to add 0.09 to x^2 to get x, (i.e $x > x^2$) and the range of values within which x is greater than x^2 is between 0 and 1. That is, $x > x^2$ in the range $0 < x < 1$. For example, $x = 0.4$ is within the range; if $x = 0.4$,

$= 1 + 8x - 8x2 + 28(x2 - 2x3 + x4) + 56$

$(x3 + 3x2(-x2) + 3x(-x2)2 + (-x2)3)$

$+ 70(x4 + 4x3(-x2) + 6x2(-x2)2$

$+ 4x(-x2)3 + (-x2)4)$

$= 1 + 8x - 8x2 + 28x2 - 56x3 + 28x4$

$+ 56x3 + 168x2(-x2)$

$+ 168x(-x2)2 + 56(-x2)3 + 70x4$

$+ 280x3(-x2) + 420x2(-x2)2$

$+ 280x(-x2)3 + 70 (-x2)4$

From these two examples, you can see that in the range $0 < x < 1$, x^2 is much less than x. Then, if we want to solve the equation above by inspection, since x is much greater than $x^2 (x >> x^2)$ we can ignore the value of x^2 and simply write $1 + x - x^2 \approx 1 + x$. Recall that $1 + x - x^2 = 1.09$. Hence, $1 + x - x^2 = 1.09 \approx 1 + x$; (the symbol $\approx$ means approximately equal to).

$1 + x = 1.09$; $x = 1.09 - 1 = 0.09$

Therefore, by inspection, $x = 0.09$.

*Note that if you are **not** told to find x by inspection, then you are to solve $1 + x - x^2 = 1.09$, using any appropriate method for solving quadratic equations.*

The word 'hence evaluate $(1.09)^8$' means that from the previous solution(s), find

$(1.09)^8$. Now, we have to evaluate $(1.09)^8$ from the last solution. From 3(*c*), recall that we found out that $1 + x - x^2 = 1.09 \approx 1 + x$; therefore, $1 + x - x^2 \approx 1 + x$.

Also, recall from our previous result that

$(1 + x - x^2)^8 = 1 + 8x + 20x^2 - 56x^3 + 28x^4 \approx (1 + x)^8$

(since $1 + x - x^2 \approx 1 + x$.)

$(1 + x)^8 = 1 + 8x + 20x^2 - 56x^2 + 28x^4$. To evaluate $(1.09)^8$, using previous answers, we will write $(1.09)^8$ in the form $(1+ x)^8$;

$(1.09)^8 = (1 + 0.09)^8$.

Comparing $(1 + 0.009)^8$ with $(1 + x)^8$, we see that $x = 0.09$.

Therefore, $(1 + 0.09)^8 = 1 + 8(0.09) + 20(0.09)^2 - 56(0.09)^3 + 28(0.09)^4$

$= 1 + 0.72 + 0.162 - 0.0408 + 0.00184$

$= 1.84304$.

Therefore, the value of $(1.09)^8$, using our previous results is 1.843 (to 3 decimal places).

4. The first three terms of the expansion of $(1 + px)^n$ in ascending powers of x are: $1, -24x, 252x^2$. Find the values of p and n. **(WAEC)**

Workshop

(a) The rth term in the binomial expansion of $(a + b)^n$ is given by $T_{rth} = {}^nC_r a^{n-r} b^r$ where r ranges thus: 0, 1, 2, 3.... So, the first three terms in the expansion of $(a + b)^n$ will be the zeroth term (T_0), first term (T_1) and the second term (T_2). From the question, the first three terms of the expansion of $(1 + px)^n$ are 1, $-24x$ and $252x^2$; therefore, $T_0 = 1 = {}^nC_0 1^{n-0} (px)^0 = 1 \times 1^n \times 1 = 1^n$, (where, in this case, $a = 1$ and $b = px$)

$T_1 = -24x = {}_nC_1 1_{n-1} (px)_1$

$= \frac{n!}{(n-1)!1!} \times 1 \times px;$

$-24x = \frac{n!}{(n-1)!1!} \times px$ (*since* $1_{n-1} = 1$).

Recall that $5! = 5 \times 4! = 5 \times 4 \times 3!$ and so on, hence, $n! = (n- \ 1)!$. Therefore,

$-24x = \frac{n!}{(n-1)!1!} \times px = \frac{n(n-1)!}{(n-1)!\,1!} \times px$

$-24x = \frac{n\,(n-1)!}{(n-1)!\,1!} \times px; -24x = n \times px;$

$-24x = npx; -24 = np;$

$np = -24$ ....(*i*).

Also, $T_2 = 252x_2 = {}_nC_2 1_{n-2} (px)_2$

$= \frac{n!}{(n-2)!\,2!} \times p_2x_2$; (*since* $1_{n-2} = 1$)

Recall that $n! = n(n-1)(n-2)!$, so that

$252x_2 = \frac{n(n-1)\,n(n-2)}{n(n-2)!2!} \times p_2x_2$

$= \frac{n(n-1)}{2 \times 1} \times p_2\,x_2;$

$252x_2 = \frac{n(n-1) \times p_2\,x_2}{2}; 504x_2 = n(n-1)$

$\times p_2x_2$; divide through the equation by x_2 to get, $504 = n(n-1)\,p_2$; $n(n - 1)$

$p_2 = 504$ ....(*ii*)

From equation (*i*), $np = -24$; $p = \frac{-24}{n}$;

put $p = \frac{-24}{n}$ into equation (*ii*) to get

$n(n-1)\left(\frac{-24}{n}\right)^2 = 504; n(n-1)\frac{(-24)^2}{n_2} = 504.$

Note that $\left(\frac{-24}{n}\right)^2 = \frac{(-24)_2}{n_2}$ *and not* $\frac{-24_2}{n_2}$.

$n(n-1)\frac{(-24)_2}{n_2} = \frac{n(n-1) \times (-24 \times -24)}{n_2}$

$= \frac{n(n-1) \times 576}{n_2} = 504;$

$576n(n - 1) = 504n^2$; $576n^2 - 576n = 504n^2$;

$576n^2 - 504n^2 - 576n = 0$;

$72n^2 - 576n = 0$; $72n(n - 8) = 0$; $72n = 0$ or

$n - 8 = 0$; $n = \frac{0}{72}$ or $n = 8$;

$n = 0$ or $n = 8$.

When $n = 0$, $(1 + px)^n = (1 + px)^0 = 1$(any number (*except zero*) raise to the power of zero is equal to 1) but, we were told in the question that the expansion of $(1 + px^n = 1 - 24x + 252x^2 + \ldots$, thus, since $1 - 24x + 252x^2$ is **not** equal to 1, then $n = 0$ does not satisfy the binomial expansion $(1 + px)^n = 1 + 24x + 252x^2 + \ldots$.

Therefore, the value of n that satisfies the binomial expansion is $n = 8$, and when

$n = 8$, $p = \frac{-24}{n} = \frac{-24}{8} = -3$

Therefore, the values of p and n are respectively -3 and 8.

5. (a) Write down the first four terms of the binomial expansion of $(2 - \frac{1}{2}x)^5$ in ascending powers of x.

(b) Use your expansion in (a) to find, correct to two decimal places, the value of $(1.99)^5$. **(WAEC)**

Workshop

(a) The binomial expansion of the expression $(a + b)$n where n is a positive integer (that is, 1, 2, 3 ...) is expressed as follows:

$(a + b)^n = {}^nc_0\, a^{n-0}\, b^0 + {}^nc_1\, a^{n-1}\, b^1 + {}^nc_2\, a^{n-2}\, b^2 + {}^nc_3\, a^{n-3}\, b^3 + \ldots + {}^nc_n\, a^{n-n}\, b^n$.

$$\left(2 - \frac{1}{2}x\right)^5 = {}^5c_0(2^{5-0})\left(-\frac{1}{2}x\right)^0$$
$$+ {}^5c_1(2^{5-1})\left(-\frac{1}{2}x\right)^1$$
$$+ {}^5c_2(2^{5-2})\left(-\frac{1}{2}x\right)^2$$
$$+ {}^5c_3(2^{5-3})\left(-\frac{1}{2}x\right)^3 \ldots\ldots$$

As the question directed that we write the first 4 terms in ascending powers of x, we have to stop at the 4th term in ascending powers of x. However, the three dots show that the series continues.

$$\left(2 - \frac{1}{2}x\right)^5 = (1 \times (2^5) \times 1) +$$
$$\frac{5!}{(5-1)!1!}2^4\left(-\frac{1}{2}x\right) + \frac{5!}{(5-2)!2!}2^3\left(-\frac{1}{2}x\right)^2$$
$$+ \frac{5!}{(5-3)!3!}2^2\left(-\frac{1}{2}x\right)^3;$$
$$= 2^5 + \left(\frac{5 \times 4!}{4!1!}2^4\left(-\frac{1}{2}x\right)\right)$$
$$+ \left(\frac{5 \times 4 \times 3!}{3!2!}2^3\left(-\frac{1}{2}x\right)^2\right)$$
$$+ \left(\frac{5 \times 4 \times 3!}{2!3!}2^2\left(-\frac{1}{2}x\right)^3\right)$$

$= 32 + 5(2_4)\left(-\frac{1}{2}x\right) + 10(2_3)\left(\frac{1}{4}x^2\right)$

$+ 10(2_2)\left(-\frac{1}{8}x^3\right);$

$= 32 + 5(16)\left(-\frac{1}{2}x\right) + 10(8)\left(\frac{1}{4}x\right)_2$

$+ 10(4)\left(-\frac{1}{8}x^3\right);$

$= 32 + \left(-\frac{80}{2}\right)x + \left(\frac{80}{4}x^2\right) + \left(-\frac{40}{8}x^3\right)$

$= 32 - 40x + 20x_2 - 5x_3.$

Therefore, the first four terms of the binomial expansion of $(2 - \frac{1}{2}x)^5$ are $32, -40x, 20x^2$ and $-5x^3$ respectively.

(b) To use the expansion in (a) to find the value of (1.99)5, we **must** first express $(1.99)^5$ in the form $(2 - \frac{1}{2}x)^{5;}$

$(1.99)_5$ in the form $\left(2 - \frac{1}{2}x\right)_5;$

$(1.99)^5 = (2 - 0.01)_5 = \left(2 - \frac{0.02}{2}\right)^5$

$= \left(2 - \frac{1}{2}(0.02)\right)_5.$

Compare $(2 - \frac{1}{2}(00.2))^5$ with $(2 - \frac{1}{2}x)^5$, to see that $x = 0.02$ in this case.

Using the expansion in the answer to question 5(a),

$(2 - \frac{1}{2}(00.2))^5 = 32 - 40(0.02) + 20(0.02)^2 - 5(0.02)^3$

$= 32 - 0.8 + 0.008 - 0.00004$

$= 31.20796.$

Therefore, $(1.99)^5$ correct to 2 decimal places is 31.21.

*Note that, since you were told to use the expansion you got from question 4(a) to find the value of $(1.99)_5$, then, you **must not** use any other method other than the method you were told to use.*

Partial Fraction

1. Express $\frac{5-12x}{6x^2+5x+1}$ in partial fractions. **(WAEC)**

*Note: For a polynomial fraction like this to be a proper fraction, the degree (highest power) of x of the numerator **must** be **less** than the degree (highest power) of x at the denominator.*

Workshop

To resolve a compound fraction like this into its partial fractions, first confirm that the partial fraction is a proper fraction; second factorize its denominator into a product of its **simplest** factors.

$$\frac{5-12x}{6x \ast 5x+1} = \frac{5-12x}{(3x+1)(2x+1)}$$

$$= \frac{A}{3x+1} + \frac{B}{2x+1} = \frac{A(2x+1)+B(3x+1)}{(3x+1)(2x+1)};$$

$$\frac{5-12x}{(3x+1)(2x+1)} = \frac{A(2x+1)+B(3x+1)}{(3x+1)(2x+1)};$$

multiply through the equation by $(3x+1)(2x+1)$ to get

$$\frac{5-12x}{(3x+1)(2x+1)} \times \frac{(3x+1)(2x+1)}{1}$$

$$= \frac{A(2x+1)+B(3x+1)}{(3x+1)(2x+1)} \times \frac{(3x+1)(2x+1)}{1};$$

$5 - 12x \equiv A(2x + 1) + B(3x + 1);$

$5 - 12x \equiv 2Ax + A + 3Bx + B;$

$5 - 12x \equiv 2Ax + 3Bx + A + B;$

$5 - 12x = -12x + 5 \equiv (2A + 3B)x + A + B$

By comparing the coefficients on the two sides of this equation it will be seen that

$2A + 3B = -12$ ....*(i)*

$A + B = 5$ ....*(ii)*

Multiply through equation (*ii*) by 2 to get: $2A + 2B = 10$. ....*(iib)*.

Subtract equation (*iib*) from (*i*) to get

$2A + 3B - (2A + 2B) = -12 - 10;$

$2A + 3B - 2A - 2B = -22;$

$3B - 2B = -22; B = -22;$

from equation (*ii*), $A + B = 5$; $A = 5 - B$

$= 5 - (-22) = 5 + 22 = 27;$

$A = 27$ *and* $B = -22$.

Therefore, $\frac{5-12x}{6x^2+5x+1} = \frac{A}{3x+1} + \frac{B}{2x+1}$

$$= \frac{27}{3x+1} + \frac{-22}{2x+1} = \frac{27}{3x+1} - \frac{22}{2x+1}$$

in partial fraction.

Alternative method

Here is a shorter and more convenient method. Having gotten the **identity**

$5 - 12x \equiv A(2x + 1) + B(3x + 1)$, let us find a means to make $(2x + 1)$ equal to zero so that $A(2x + 1)$ will also be zero, then, we can evaluate B.

When $(2x+1)=0$; $2x=-1$; $x=\frac{-1}{2}$. Put $x=\frac{-1}{2}$ into the identity to get: $5-12\left(\frac{-1}{2}\right)=A\left(2\left(\frac{-1}{2}\right)+1\right)+B\left(3\left(\frac{-1}{2}\right)+1\right)$.

Please note that as soon as we give a specific value to x in the identity, $\left(\text{in this case } \frac{-1}{2}\right)$ then the equivalent sign '≡' will change to equality sign '=' as written above. This is because x has been allotted a specific number, then what we get is no more an identity, it is now an equation. Thus, the equivalent sign will change to equality sign.

$5+\frac{12}{2}=A\left(\frac{-2}{2}+1\right)B\left(\frac{-3}{2}+1\right)$; $5+6$

$=A(-1+1)+B\left(-\frac{1}{2}\right)$;

We will now try to make $(3x+1)$ equal to zero, so that $B(3x+1)$ will also be zero, then we can evaluate A. When $3x+1=0$; $3x=-1$; $x=\frac{-1}{3}$.

Also put $x=\frac{-1}{3}$ into the identity to get:

$5-12\left(\frac{-1}{3}\right)=A\left(2\left(\frac{-1}{3}\right)+1\right)+B\left(3\left(\frac{-1}{3}\right)+1\right)$;

$5+\frac{12}{3}=A\left(\frac{-2}{3}+1\right)+B\left(\frac{-3}{3}+1\right)$; $5+4$

$=A\left(\frac{1}{3}\right)+B(-1+1)$;

$9=A\left(\frac{1}{3}\right)+B\,(0)$; $9=\frac{A}{3}$; $A=27$.

So, $\frac{5-12x}{6x^2+5x+1}=\frac{A}{3x+1}+\frac{B}{2x+1}=\frac{27}{3x+1}$

$+\frac{-22}{2x+1}=\frac{27}{3x+1}-\frac{22}{2x+1}$ in partial fraction.

2. Resolve $\frac{3x^2-x+2}{(x+2)^2+(1-2x)}$ into partial fractions. **(WAEC)**

Workshop

$\frac{3x^2-x+2}{(x+2)^2(1-2x)}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}+\frac{C}{(1+2x)}$;

Recall that $\frac{1}{2}+\frac{1}{4}+\frac{1}{5}=\frac{1}{2}+\frac{1}{2_2}+\frac{1}{5}$

$=\frac{10+5+4}{20}=\frac{19}{2^2\times 5}$.

Please take note of the **denominators** in the equations below.

Then, $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{5} = \frac{19}{2^2 \times 5}$.

Therefore, if $\frac{19}{2^2 \times 5} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{5}$, then,

$$\frac{3x^2 - x + 2}{(x+2)^2\,(1-2x)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(1-2x)},$$

though we do not know the values of A, B and C in this case.

Hence, $\frac{3x^2 - x + 2}{(x+2)^2\,(1-2x)}$

$$= \frac{A\,(x+2)\,(1-2x) + B\,(1-2x) + C\,(x+2)^2}{(x+2)^2\,(1-2x)}$$

There are more than two ways to resolve the compound fraction in question into its partial fractions, however, we will only learn two of these methods while solving this problem.

The first and shortest method we will use is the substitution method:

$$\frac{3x^2 - x + 2}{(x+2)^2\,(1-2x)}$$

$$= \frac{A\,(x+2)\,(1-2x) + B\,(1-2x) + C\,(x+2)^2}{(x+2)^2\,(1-2x)}$$

Multiply both sides of the equation by; $(x + 2)^2\,(1 - 2x)$;

$$\frac{3x^2 - x + 2}{\cancel{(x+2)^2}\,\cancel{(1-2x)}} \times \frac{\cancel{(x+2)^2}\,\cancel{(1-2x)}}{1}$$

$$= \frac{A(x+2)\,(1-2x) + B\,(1-2x) + C\,(x+2)^2}{\cancel{(x+2)^2}\,\cancel{(1-2x)}}$$

$$\times \frac{\cancel{(x+2)^2}\,\cancel{(1-2x)}}{1}$$

we get $3x^2 - x + 2 \equiv A\,(x+2)(1-2x) + B\,(1-2x)$
$+ C\,(x+2)^2$.

Let us find a means to make $x + 2$ equal to zero. The value of x that will make $x + 2$ equal to zero can be evaluated, by equating $x + 2$ to zero. If $x + 2 = 0$, then $x = -2$ put $x = -2$ into the identity above to get:

$3\,(-2)^2 - (-2) + 2 = A\,(-2+2)(1 - 2(-2)) +$

$B\,(1 - 2\,(-2)) + C\,(-2+2)^2 12 +$

$2 + 2 = A\,(0)(1+4) + B\,(1+4) + C\,(0)^2;$

$16 = 5B;\ B = \frac{16}{5}.$

Having known B, let us find a way to make $1 - 2x$ equal to zero. To achieve this, we will equate $1 - 2x$ to zero as; $1 - 2x = 0$; $1 = 2x$; $x = \frac{1}{2}$. Again, put $x = \frac{1}{2}$

$3\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)+2$

$=A\left(\frac{1}{2}+2\right)\left(1-2\left(\frac{1}{2}\right)\right)+B\left(1-2\left(\frac{1}{2}\right)\right)+C\left(\frac{1}{2}+2\right)^2$

$3\left(\frac{1}{4}\right)-\frac{1}{2}+2$

$=A\left(\frac{5}{2}\right)\left(1-\frac{2}{2}\right)+B\left(1-\frac{2}{2}\right)+C\left(\frac{5}{2}\right)^2;$

$\frac{3}{4}-\frac{1}{2}+2$

$=A\left(\frac{5}{2}\right)(1-1)+B\,(1-1)+C\left(\frac{25}{4}\right);$

$\frac{9}{4}=A\left(\frac{5}{2}\right)(0)+B\,(0)+C\left(\frac{25}{4}\right);$

$\frac{9}{4}=0+0+\frac{25C}{4};$

$\frac{9}{4}=\frac{25C}{4}\quad 25C=9\,;\,C=\frac{9}{25}.$

We already know B and C, and there seems to be no other expression in the identity, that can be equated to zero so as to get the value of A. However, since we already know the values of B and C, we can find A by substituting any **real number** for x in the identity.

> *Note: The difference between identities and equations is that, while an identity holds for all real numbers,* ***not*** *all equations hold for all real numbers. For example, given the equation* $y=\frac{3x+5}{x-3}$, *when* $x=3$, $y=\frac{3(3)+5}{3-3}=\frac{14}{0}=\infty$. *It is observed, that the value of y does not exist when* $x=3$, *so,* $y=\frac{3x+5}{x-3}$ *is an equation; it is not an identity, because the equation does not hold, for all real numbers.*

Let us put $x = 1$ into the identity in question to get

3(1)2 − (1) + 2 = A (1 + 2)(1 − 2(1)) + B (1 − 2(1)) + C (1 + 2)2

3 − 1 + 2 = A (3)(1 − 2) + B(1 − 2) + C(3)2;

4 = −3A − B + 9C ...(*).

Recall that $B=\frac{16}{5}$ and $C=\frac{9}{25}$. Put the values of B and C into (*) to get:

$4=-3A-\left(\frac{16}{5}\right)+9\left(\frac{9}{25}\right);\ 4=-\,3A-\frac{16}{5}+\frac{81}{25};$

$4=-3A+\frac{1}{25};$

$$4 - \frac{1}{25} = -3A;\ \frac{99}{5} = -3A;\ \frac{99}{25 \times (-3)} = A;$$

$$A = -\frac{99}{75} = -\frac{33}{25}.$$

$$A = -\frac{33}{25},\ B = \frac{16}{5} \text{ and } C = \frac{9}{25}.$$

Therefore, $\frac{3x^2 - x + 2}{(x+2)^2 + (1-2x)}$ resolved into partial fraction is

$$\frac{-33}{25(x+2)} + \frac{16}{5(x+2)^2} + \frac{9}{25(1-2x)}.$$

The second method is called comparing coefficient. Now, let us go ahead to use this method to resolve the compound fraction in question into its respective partial fractions.

$$\frac{3x^2 - x + 2}{(x+2)^2 (1-2x)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(1-2x)};$$

$$\frac{3x^2 - x + 2}{(x+2)^2 (1-2x)}$$

$$= \frac{A(x+2)(1-2x) + B(1-2x) + C(x+2)^2}{(x+2)^2 (1-2x)};$$

$$\frac{3x^2 - x + 2}{(x+2)^2 (1-2x)}$$

$$= \frac{A(x - 2x^2 + 2 - 4x) + B - 2Bx + C(x^2 + 4x + 4)}{(x+2)^2 (1-2x)};$$

$$\frac{3x^2 - x + 2}{(x+2)^2 (1-2x)}$$

$$= \frac{A(2 - 2x^2 - 3x) + B - 2Bx + Cx^2 + 4Cx + 4C}{(x+2)^2 (1-2x)};$$

$$\frac{3x^2 - x + 2}{(x+2)^2 (1-2x)}$$

$$= \frac{2A - 2Ax^2 - 3Ax + B - 2Bx + Cx^2 + 4Cx + 4C}{(x+2)^2 (1-2x)};$$

$$\frac{3x^2 - x + 2}{(x+2)^2 (1-2x)}$$

$$= \frac{Cx^2 - 2Ax^2 + 4Cx - 2Bx - 3Ax + 2A + B + 4C}{(x+2)^2 (1-2x)};$$

$$\frac{3x^2 - x + 2}{(x+2)^2 (1-2x)}$$

$$= \frac{(C - 2A)x^2 + (4C - 2B - 3A)x + 2A + B + 4C}{(x+2)^2 (1-2x)};$$

multiply both sides of the equation by $(x+2)^2 (1-2x)$ to get

$$\frac{3x^2 - x + 2}{(x+2)^2 (1-2x)} \times \frac{(x+2)^2 (1-2x)}{1}$$

$$= \frac{(C - 2A)x^2 + (4C - 2B - 3A)x + 2A + B + 4C}{(x + 2)^2 (1 - 2x)}$$

$$\times \frac{(x + 2)^2 (1 - 2x)}{1};$$

$$3x^2 - x + 2 \equiv (C - 2A)x^2 + (4C - 2B - 3A)\, x + 2A + B + 4C;$$

$$(3)x^2 + (-1)\, x + 2 \equiv (C - 2A)x^2 + (4C - 2B - 3A)\, x + 2A + B + 4C.$$

By comparing coefficients, $C - 2A = 3$ ... *(i)*;

$4C - 2B - 3A = -1$ .... *(ii)*;

$2A + B + 4C = 2$ .... *(iii)*;

To solve for A, B and C, let us try to eliminate C from equations *i* and *ii*. To achieve this, we will multiply both sides of equation *i* by 4 to get

$4C - 8A = 12$ ... *(i*)*; then, we subtract equation *(ii)* from *(i*)* to get

$4C - 8A = (4C - 2B - 3A) = 12 - (-1)$; $4C - 8A - 4C + 2B + 3A = 12 + 1$;

$4C - 4C - 8A + 3A + 2B = 13$; $-5A + 2B = 13$... (a).

We can further eliminate C by subtracting equation *(iii)* from *(ii)* (since the coefficient of C are equal in both equation) to get

$4C - 2B - 3A - (2A + B + 4C) = -1 - (2)$;

$4C - 2B - 3A - 2A - B - 4C = -3$

$4C - 4C - 2B - B - 3A - 2A = -3$;

$-3B - 5A = -3$ ... (b).

Hence, $-5A + 2B = 13$ ... (a)

$-3B - 5A = -3$ ... (b)

We have been able to eliminate C from the 3 simultaneous equations. Also, we have reduced the 3 equations (i.e equations *(i)*, *(ii)* and *(iii)*) containing three unknowns (i.e A, B and C), to 2 equations (i.e equations (a) and (b)) with 2 unknowns (i.e $A$and B). Solve equations (a) and (b) simultaneously to get:

$A = -\frac{33}{25}$ and $B = \frac{16}{5}$. You can put $A = -\frac{33}{25}$ and $B = \frac{16}{5}$ into **any one of** equation *(i)*, *(ii)* or *(iii)*, to get the value of C. If we insert the values of A and B into euqation *(i)*, we get $C - 2\left(-\frac{33}{25}\right) = 3$;

$C + \frac{66}{25} = 3$; $C = 3 - \frac{66}{25}$; $= \frac{75 - 66}{25} = \frac{9}{25}$.

Therefore, $\frac{3x^2 - x + 2}{(x + 2)^2 (1 - 2x)}$, resolved into partial fractions, is $\frac{-33}{25(x + 2)} + \frac{16}{5(x + 2)^2} + \frac{9}{25(1 - 2x)}$.

Mapping and Function

1. (a) If $g: x \rightarrow \frac{2x-1}{5x+3}$, for all $x \in R$, the set of real numbers, find the:

(i) domain of g;

(ii) g^{-1};

(iii) value of x for which g^{-1}; does not exist.

(b) Determine whether or not g in **1**(a) is onto.

(c) Find the range of values of x for which $2x^2 + x - 6 < 0$. **(WAEC)**

Workshop

(i) $g: x \rightarrow \frac{2x-1}{5x+3}$ simply means g is a function of x, such that $g(x) = \frac{2x-1}{5x+3}$

From the question, the domain of g is made up of all real numbers x which are elements of R, except the value of x for which $g(x)$ is undefined. For example,

$g(x) = \frac{2x-1}{5x+3}$, when $x = 1$, $g(x) = g(1)$

$= \frac{2(1)-1}{5(1)+3} = \frac{1}{8}$; when $x = -1$, $g(-1) = \frac{3}{2}$,

when $x = 2$, $g(2) = \frac{3}{13}$ and so on.

Therefore, while $x = 1, -1, 2$ are elements in the domain, $g(x) = \frac{1}{8}, \frac{3}{2}, \frac{3}{13}$ are elements in the co-domain. $g(x)$ is undefined, when the denominator of $g(x)$ is equal to zero i.e. $5x + 3 = 0$. Then, $5x + 3 = 0$.

$5x = -3$; $\therefore x = -\frac{3}{5}$. This means,

$$g\left(-\frac{3}{5}\right) = \frac{2\left(-\frac{3}{5}\right)-1}{5\left(-\frac{3}{5}\right)+3} = \frac{-\frac{6}{5}-1}{-\frac{15}{5}+3} = \frac{-\frac{11}{5}}{-3+3}$$

$$= \frac{-\frac{11}{5}}{0} = \frac{-11}{5} \div 0 = \frac{-11}{5} \times \frac{1}{0}$$

$$= -\frac{11}{0} = -\infty \text{ (negative infinity)},$$

which is not a unique element in the domain of g because it is not a real number (*recall from the question that x is an element of real numbers*).

Therefore, the domain of g contains all elements, x, of real numbers (R), except $-\frac{3}{5}$

(ii) Let $y = g(x) = \frac{2x-1}{5x+3}$; $y = \frac{2x-1}{5x+3}$

Know that if $f(x) = y$, then $x = f^{-1}(y)$. Therefore for this problem, $y = g(x)$, therefore, $g^{-1}(y) = x$.

Hence, make x the subject of the formula, to get $y(5x + 3) = 2x-1$;

$5xy + 3y = 2x - 1$; $3y + 1 = 2x - 5xy$;

$3y + 1 = x(2 - 5y)$; $x = \frac{3y+1}{2-5y}$.

Thus, $g^{-1}(y) = x = \frac{3y+1}{2-5y}$ Recall that if

$f(x) = y = x^2 + 3x$, then,

$f(2) = 2^3 + 3(2)$; also $f(a) = a^2 + 3(a)$.

Therefore, if $g^{-1}(y) = \frac{3y+1}{2-5y}$,

then $g^{-1}(x) = \frac{3x+1}{2-5x}$

(iii) For g^{-1} **not** to exist, the value of the denominator of $g(x)$, will be equal to zero, i.e $g^{-1} = \frac{3x+1}{0} = \infty$.

Hence, $2 - 5x = 0$, $5x = 2$; $x = \frac{2}{5}$

(b) For a function to be onto, its range must be equal to its co-domain; that is, all the elements in the co-domain ($g(x)$) must be the image of at least one element of the domain (x).

Recall that $g(x) = \frac{2x-1}{5x+3}$,

so $g(1) = \frac{2(1)-1}{5(1)+3} = \frac{1}{8}$,

$g(-1) = \frac{3}{2}$, $g(10) = \frac{19}{53}$

$g(2) = \frac{3}{13}$ and so on.

Remember that $g\left(-\frac{3}{5}\right)$ does not exist so, $x = -\frac{3}{5}$ is not an element in the domain.

Applying the above calculations to the elements in the domain, it can be inferred that each element in the domain has a distinct image in the co-domain. This means that g is a one-to-one mapping. But all the elements in the co-domain are images of an element in the domain. This means that the number of elements in the codomain is equal to its range. The range of a function is the number of elements in the codomain that is an image of **at least one** element in the domain; therefore, g is onto mapping.

*Note that, for a mapping to be onto, its range **must** be equal to the number of elements in the co-domain.*

(c) $2x^2 + x - 6 < 0$; $2x^2 + 4x - 3x - 6 < 0$;

$2x(x + 2) - 3(x + 2) < 0$; $(x + 2)(2x - 3) < 0$.

Make a rough sketch of the graph of $y = 2x^2 + x - 6$. The zeros of the equation are $\frac{3}{2}$ and -2, and the graph will have a minimum turning point, since the coefficient of $x2$ is positive. Having known these parameters, a sketch of $y = 2x2 + x - 6$ is shown in figure 5.3.

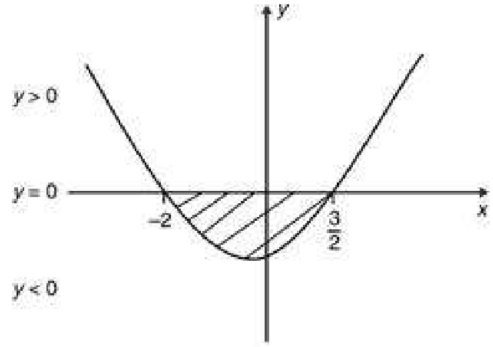

Fig. 5.3

From the graph in Figure 5.3, the range of values of x that makes $y = 2x^2 + x - 6$ **less** than zero is the shaded part of the graph, since that is the region of the graph where y is less than zero; that is the region lying below point $y = 0$.

Therefore, the range of x, for which $y = 2x^2 + x - 6 < 0$, is $-2 < x < \frac{3}{2}$.

Note that the range of x for which $y = 2x_2 + x - 6 > 0$, is the region of the graph above line $y = 0$, as shaded in figure 5.4 below:

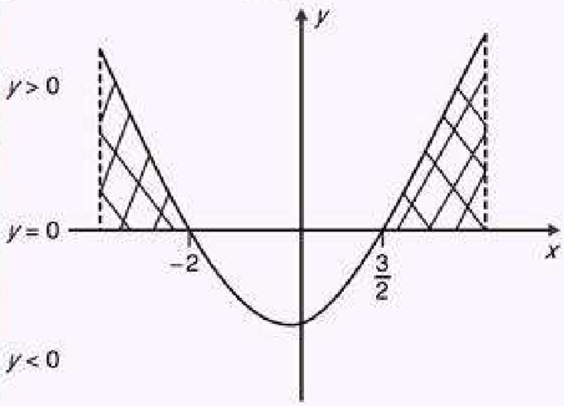

Fig. 5.4

Hence, the range of x for which $y = 2x_2 + x - 6 > 0$, is $x < -2$ and $x > \frac{3}{2}$.