

## Chapter 2

### Chapter 2

# Matrices and Determinants

### OBJECTIVES

At the end of this chapter, students should be able to:

1. define matrix.
2. state the order and notation of a matrix.
3. mention and define the types of matrices.
4. perform the operations of addition and subtraction of matrices.
5. multiply matrix by a scalar quantity and multiply two matrices  $A$  and  $B$ .
6. find the transpose of a matrix.
7. calculate the determinant of  $2 \times 2$  matrix.
8. apply matrices and determinants in solving simultaneous equations.

## I. Definition, Order and Notation of a Matrix

### (i) Definition of matrix

A **matrix** is a rectangular array of real numbers or elements arranged in rows and columns enclosed within brackets.

For example  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$  or  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ .

If there are  $m$  rows and  $n$  columns, the matrix is called an ' $m$  by  $n$  matrix'. In an  $m$  by  $n$  matrix, there are  $mn$  entries altogether. Examples of a matrix include:

$$(1 \ 2 \ 3), \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 5 & 6 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

The numerals or variables (letters) contained in each of the matrices that are shown in the above example are called elements (entries) of the matrix. Matrices are often described by the number of rows and columns they have.

For example, a matrix  $A$  where  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

has two rows and two columns. The first row has  $a$  and  $b$  as its elements while the second row has elements  $c$  and  $d$ . The first column has elements  $a$  and  $c$ , while the second column has elements  $b$  and  $d$ .

### Note

1. The number of rows is always listed first.
2. All the horizontal components form the rows, while the vertical components form the columns.

## (ii) Position of the elements in a matrix

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Each element in a matrix has its own location. This location is described by the position of row or column the element occupies. For easy description, a system of double subscripts is being used. For example, consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{ij}), i=1, 2, 3; j=1, 2, 3$$

where  $a_{11}, a_{12}, a_{13}, \dots, a_{33}$  are the elements in  $A$ . Element  $a_{23}$  in  $A$  is the element in the second row and third column. Similarly,  $a_{34}$  stands for the element in the third row and fourth column. The first number in the subscript stands for row position and the second number for column position.

Generally, if  $a_{ij}$  is an element of a matrix, location wise,  $a_{ij}$  is the element in the  $i$ th row and  $j$ th column.

## Worked Example 1

Describe the position of the elements 7 and 8 in matrix  $A$ , given that  $A$

$$= \begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

### SOLUTION

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$a_{21} = 7$$

$$a_{22} = 8$$

7 is in 2nd row and 1st column while 8 is in 2nd row and 2nd column.

## Worked Example 2

Find the number of rows and columns in each of the following matrices.

$$(a) A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \\ 5 & 7 \end{pmatrix}$$

$$(b) Q = \begin{pmatrix} 2 & 4 & 6 \\ 10 & 11 & 9 \end{pmatrix}$$

$$(c) P = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$$

$$(d) R = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$(e) V = (1 \ 2 \ 3 \ 4)$$

### SOLUTION

- (a) Matrix  $A$  has 3 rows and 2 columns.
- (b) Matrix  $Q$  has 2 rows and 3 columns.
- (c) Matrix  $P$  has 3 rows and 4 columns.

- (d) Matrix  $R$  has 3 rows and 1 column.
- (e) Matrix  $V$  has 1 row and 4 columns.

### **(iii) Order of a matrix**

A matrix is described by its order. The order of a matrix is the number of rows and columns it contains.

For example, matrix  $Q = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}$  has

2 rows and 2 columns. It is said to be 2 by 2 matrix or  $(2 \times 2)$  matrix.  
Similarly,

matrix  $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$  has 3 rows and 2 columns.

This is 3 by 2 or  $(3 \times 2)$  matrix.

In general, a matrix with  $m$  rows and  $n$  columns is said to be a  $m \times n$  matrix. The order is not, however, commutative; that is,  $m \times n$  matrix is not the same as  $n \times m$  matrix. An  $n \times m$  matrix is a matrix with  $n$  rows and  $m$  columns.

### **Worked Example 3**

Give the order of the following matrices:

$$(a) A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \quad (b) B = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 6 & 7 \\ 8 & 9 & 1 \end{bmatrix}$$

### **SOLUTION**

- (a)  $A$  has 2 rows and 3 columns. Therefore,  $A$  is  $(2 \times 3)$  matrix or 2 by 3 matrix. (b)  $B$  has 3 rows and 3 columns. Therefore,  $B$  is 3 by 3 or  $(3 \times 3)$  matrix.

### **Worked Example 4**

What is the order of matrix  $A = (1 \ 0 \ 4)$ ?

### **SOLUTION**

Matrix  $A$  has 1 row and 3 columns. Therefore, matrix  $A$  is of order 1 by 3 or  $(1 \times 3)$ .

## Worked Example 5

Find (i)  $m$  (ii)  $n$  (iii) order (iv)  $m + n$  in the following matrices:

$$(a) P = \begin{pmatrix} 5 & 6 & 7 & 16 \\ 9 & 1 & 0 & -8 \\ -4 & 12 & -3 & -7 \end{pmatrix} \quad (b) Q = \begin{pmatrix} 3 & 5 & 8 \\ 7 & 9 & 4 \\ 1 & 2 & 6 \end{pmatrix}$$

### SOLUTION

- (a) (i)  $m = 3$   
(ii)  $n = 5$   
(iii)  $m$  by  $n = (3 \times 5)$  matrix  
(iv)  $m + n = 3 + 5 = 8$
- (b) (i)  $m = 3$   
(ii)  $n = 3$   
(iii)  $m$  by  $n = (3 \times 3)$  matrix  
(iv)  $m + n = 3 + 3 = 6$

## Exercise 1

Describe the position of each of the elements in terms of  $a_{ij}$  where  $i$  = row and  $j$  = column in the following matrices.

$$1. A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 7 \end{bmatrix}$$

$$2. P = \begin{bmatrix} a & e & i & u & n \\ j & k & l & m & p \\ q & r & b & c & d \end{bmatrix}$$

$$3. B = \begin{bmatrix} 4 & 8 & 9 & 10 \\ 2 & 13 & -5 & 11 \\ 1 & 3 & 7 & 14 \\ 18 & 20 & -7 & -2 \end{bmatrix}$$

$$4. Z = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$$

$$5. V = (a \ e \ i \ o \ u) \quad 6. K = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$7. L = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \quad 8. W = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \\ 21 & 24 & 27 \end{pmatrix}$$

$$9. C = \begin{pmatrix} -4 & -8 & -10 \\ 1 & 2 & 4 \\ 3 & 6 & 12 \end{pmatrix}$$

$$10. Y = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

11. Find the number of rows and columns in the matrices given in Questions 1–10.

12. Find the order of the matrices given in Questions 1–10.

13. Given matrix  $A$  and  $B$  as

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \\ 7 & 8 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$$

- (a) Find the order of matrices  $A$  and  $B$ .  
 (b) What can you say about their  $m$  and  $n$ ?

14. Describe the position of the elements 4, 6 and 12 in matrix

$$G = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

15. Define the following, and give a hypothetical example for each case:

- (a)  $(4 \times 3)$  matrix
- (b)  $(1 \times 4)$  matrix
- (c)  $(3 \times 3)$  matrix
- (d)  $(4 \times 5)$  matrix
- (e)  $(2 \times 2)$  matrix

## II. Types of Matrices

The different types of matrices are described in the following:

(a) **Line matrix:** This is a matrix with only one row and it is also called a row matrix, for example  $(1 \ 4 \ 6)$ ,  $[1 \ 0 \ 1]$ ,  $(10 \ 12 \ 14)$ , etc. The order of this matrix is  $(1 \times n)$ .

(b) **Column matrix:** A column matrix is a matrix with only one column, for

example,  $\begin{pmatrix} 5 \\ 4 \\ 6 \end{pmatrix}$ . Its order is  $(m \times 1)$ .

- (c) **Square matrix:** This is a matrix with the same number of rows and columns. For example,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$A$  is a  $2 \times 2$  square matrix and  $B$  is a  $3 \times 3$  square matrix. The order of a square matrix can either be  $(m \times m)$ ,  $(n \times n)$  or  $(m \times n)$ .

- (d) **A zero or null matrix:** This is a matrix with all elements being zero and the order notwithstanding, for example,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A$  is a  $2 \times 2$  null matrix and  $B$  is a  $3 \times 3$  null matrix.

- (e) **Diagonal matrix:** This is a square matrix whose elements are zero except the elements in the leading diagonal. Examples include

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \text{ and } \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$$

The main diagonal (or leading diagonal) of a sequence matrix consists

$$\left[ \begin{array}{cccccc} a_{11} & a_{21} & a_{31} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & \dots & a_{2n} \\ \vdots & & & & & & \text{of those} \\ \vdots & & & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & \dots & \dots & a_{nn} \end{array} \right]$$

entries with the same row and column subscript, that is  $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ .

- (f) **Unit or identity matrix:** This is a square matrix with all the elements in the leading diagonal equal to 1 and every other element being zero.

Examples include  $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where  $I_2$  is  $2 \times 2$  unit matrix and  $I_3$  is  $3 \times 3$  unit matrix.

(g) **Equal matrix:** Two matrices are equal if and only if they are of the same order and the corresponding elements are equal.

$$(i) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

For example,

if and only if  $a = r, b = s, c = t$  and  $d = u$

$$(ii) A = \begin{bmatrix} 4 & 1 \\ \sqrt{9} & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$$

Matrices  $A$  and  $B$  are equal since  
 $4 = 4, 1 = 1, \sqrt{9} = 3$  and  $6 = 6$ .

## Exercise 2

1. Consider matrices  $P$  and  $Q$ :

$$P = \begin{pmatrix} 2 & 3 \\ 9 & 14 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 2 & 3 \\ 9 & 7 \end{pmatrix}$$

Why are they not equal?

2. What is the difference between matrix  $A$  and matrix  $B$ ?

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Give an example of  $I_4$  matrix.
4. Give an example of  $I_3$  matrix.
5. Determine the order of the following matrices:
- (a)  $I_2$
  - (b)  $I_3$
  - (c)  $I_4$
  - (d)  $I_5$
6. Give an example of the following matrices:
- (a)  $2 \times 3$  null matrix
  - (b)  $3 \times 3$  null matrix
  - (c)  $4 \times 4$  null matrix
  - (d)  $1 \times 1$  null matrix
7. Define a square matrix with an example.
8. What is a matrix whose order is  $(n \times n)$ ?
9. Name the matrix for each of the following orders:
- (a)  $(n \times 1)$
  - (c)  $(1 \times n)$
  - (b)  $(1 \times m)$
  - (d)  $(m \times 1)$
10. Write out the elements in the leading diagonal in the following matrices:

$$A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{pmatrix} \quad B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -7 \end{pmatrix}$$

### III. Addition and Subtraction of Matrices

#### (i) Addition of matrices

Given two matrices  $A$  and  $B$  of the same order, then the sum of  $A$  and  $B$ , that is  $A + B$ , is obtained by adding the corresponding elements in each matrix.

For example

$$(a) \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$(b) \begin{pmatrix} s & t & u \\ v & w & x \\ y & z & r \end{pmatrix} + \begin{pmatrix} p & q & l \\ j & k & m \\ n & d & f \end{pmatrix}$$

$$= \begin{pmatrix} s+p & t+q & u+l \\ v+j & w+k & x+m \\ y+n & z+d & r+f \end{pmatrix}$$

In other words, when two matrices are of the same order, they are said to be **conformable** for addition.

The sum of two  $(m \times n)$  matrices  $A = |a_{ij}|_{(m,n)}$  and  $B = |b_{ij}|_{(m,n)}$ , where  $a_{ij}$  is the entry in the  $i$ th row and  $j$ th column of  $A$  and  $b_{ij}$  is the entry in the  $i$ th row and  $j$ th column of  $B$  (which are conformable for addition), is defined as  $A + B = |a_{ij} + b_{ij}|_{(m,n)}$ .

#### Worked Example 6

Given that  $A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{pmatrix}$

Find  $A + B$ .

SOLUTION

$$A + B = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2+3 & 5+3 & 4+6 \\ 5+7 & 6+8 & 7+9 \\ 8+1 & 9+2 & 1+3 \end{pmatrix} = \begin{pmatrix} 5 & 8 & 10 \\ 12 & 14 & 16 \\ 9 & 11 & 4 \end{pmatrix}$$

### Worked Example 7

- (a) Find  $B + A$  in Worked Example 6.  
 (b) What can you say about  $A + B$  and  $B + A$ ?

#### SOLUTION

$$(a) B + A = \begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+2 & 5+3 & 6+4 \\ 7+5 & 8+6 & 9+7 \\ 1+8 & 2+9 & 3+1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 8 & 10 \\ 12 & 14 & 16 \\ 9 & 11 & 4 \end{pmatrix}$$

- (b)  $A + B$  and  $B + A$  are equal.

Hence,  $A + B$  and  $B + A$  are commutative, that is,  $A + B = B + A$ .

**Note:** Addition is commutative under matrix operations.

### (ii) Subtraction of matrices

Given two matrices  $A$  and  $B$  of the same order, then the difference between  $A$  and

$B$ , that is  $A - B$ , is obtained by subtracting corresponding elements in each matrix. In other words, the difference  $A - B$  of two matrices  $A = [a_{ij}]_{(m,n)}$  and  $B = [b_{ij}]_{(m,n)}$  is obtained by subtracting and writing the results in the

corresponding positions.

### Worked Example 8

Given that  $X = \begin{pmatrix} 5 & 6 & 7 \\ 1 & 3 & 4 \\ 8 & 2 & 9 \end{pmatrix}$  and  $Y = \begin{pmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 8 & 1 & 2 \end{pmatrix}$ ,

find  $X - Y$ .

#### SOLUTION

$$\begin{aligned} X - Y &= \begin{pmatrix} 5 & 6 & 7 \\ 1 & 3 & 4 \\ 8 & 2 & 9 \end{pmatrix} - \begin{pmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 8 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 5-3 & 6-4 & 7-5 \\ 1-6 & 3-7 & 4-8 \\ 8-8 & 2-1 & 9-2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -5 & -4 & -4 \\ 0 & 1 & 7 \end{pmatrix} \end{aligned}$$

### Worked Example 9

- (a) From the given matrices in Worked Example 8, find  $Y - X$ .  
(b) Are the two matrices commutative?

#### SOLUTION

$$(a) Y - X = \begin{pmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 8 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 6 & 7 \\ 1 & 3 & 4 \\ 8 & 2 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 3-5 & 4-6 & 5-7 \\ 6-1 & 7-3 & 8-4 \\ 8-8 & 1-2 & 2-9 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -2 & -2 \\ 5 & 4 & 4 \\ 0 & -1 & -7 \end{pmatrix}$$

$$(b) \text{ Since } \begin{pmatrix} 2 & 2 & 2 \\ -5 & -4 & -4 \\ 0 & 1 & 7 \end{pmatrix} \neq \begin{pmatrix} -2 & -2 & -2 \\ 5 & 4 & 4 \\ 0 & -1 & -7 \end{pmatrix}$$

The two matrices are not commutative, that is  $X - Y \neq Y - X$

### Worked Example 10

$$(a) \text{ Add } A = \begin{bmatrix} 4 & 5 \\ \frac{1}{4} & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 4 \\ \frac{2}{3} & -11 \end{bmatrix}.$$

$$(b) \text{ Subtract } B = \begin{bmatrix} -6 & 1 \\ 5 & -7 \end{bmatrix} \text{ from }$$

$$A = \begin{bmatrix} 7 & 8 \\ 10 & -2 \end{bmatrix}.$$

### SOLUTION

$$(a) A + B = \begin{bmatrix} 4 & 5 \\ \frac{1}{4} & -6 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ \frac{2}{3} & -11 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0 & 5+4 \\ \frac{1}{4}+\frac{2}{3} & -6+(-11) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 \\ \frac{11}{12} & -17 \end{bmatrix}$$

$$(b) \quad A - B = \begin{bmatrix} 7 - (-6) & 8 - 1 \\ 10 - 5 & -2 - (-7) \end{bmatrix}$$

$$= \begin{bmatrix} 7 + 6 & 8 - 1 \\ 10 - 5 & -2 + 7 \end{bmatrix} = \begin{bmatrix} 13 & 7 \\ 5 & 5 \end{bmatrix}$$

### Exercise 3

Given that  $A = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 7 & 6 \\ 8 & 9 & -10 \end{pmatrix}$  and

$$B = \begin{pmatrix} 0 & 1 & 7 \\ 7 & 0 & 9 \\ 4 & -4 & -8 \end{pmatrix}$$

Evaluate the following:

1.  $A + B$
2.  $B + A$
3.  $A - B$
4.  $B - A$

5. Is  $A + B$  equal to  $B + A$ ?

6. Is  $A - B$  equal to  $B - A$ ?

Given that  $P = \begin{pmatrix} 4 & x-y & 6 \\ \sqrt{4} & \frac{x}{y} & 9\frac{1}{2} \\ 4 & 6 & \frac{2}{3} \end{pmatrix}$  and

$$Q = \begin{pmatrix} \sqrt[3]{27} & -6 & 4 \\ x \div y & 8 & 10 \\ 2^4 & \frac{1}{4} & x-y \end{pmatrix}$$

If  $x = 0$  and  $y = -4$ , compute the following:

7.  $P + Q$       8.  $Q + P$

9.  $P - Q$       10.  $Q - P$

11.  $P + P$       12.  $Q + Q$

13.  $P - P$       14.  $Q - Q$

15. What can you say about the results of Questions 11–14?

## IV. Scalar Multiplication and Multiplication of Matrices

### (i) Scalar multiplication of matrices

If  $A [a_{ij}]_{(m,n)}$  is a matrix and  $k$  is a real number, then  $kA = [ka_{ij}]_{(m,n)}$  and  $Ak = [a_{ij}k]_{(m,n)}$ . In other words, to multiply a matrix by a scalar (real number)  $k$ , we multiply each element of a matrix  $A$  by  $k$ . That is, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ r & s & t \end{pmatrix}, \text{ then } kA = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \\ kr & ks & kt \end{pmatrix}$$

### Worked Example 11

Given that  $D = \begin{pmatrix} 3 & 5 & 6 \\ 1 & 2 & 7 \\ 0 & 3 & 4 \end{pmatrix}$ ,

find  $4D$ .

**SOLUTION**

$$4D = 4 \begin{pmatrix} 3 & 5 & 6 \\ 1 & 2 & 7 \\ 0 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 \times 3 & 4 \times 5 & 4 \times 6 \\ 4 \times 1 & 4 \times 2 & 4 \times 7 \\ 4 \times 0 & 4 \times 3 & 4 \times 4 \end{pmatrix}$$
$$= \begin{pmatrix} 12 & 20 & 24 \\ 4 & 8 & 28 \\ 0 & 12 & 16 \end{pmatrix}$$

**Worked Example 12**

Given that  $A = \begin{pmatrix} -4 & 2^5 & \frac{1}{4} \\ \sqrt{16} & \frac{3}{4} & 0.5 \\ 4 & -7 & 0 \end{pmatrix}$

Find the following:

- (a)  $\frac{1}{4}A$       (b)  $-7A$

**SOLUTION**

$$A = \begin{pmatrix} -4 & 2^5 & \frac{1}{4} \\ \sqrt{16} & \frac{3}{4} & 0.5 \\ 4 & -7 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 32 & \frac{1}{4} \\ 4 & \frac{3}{4} & \frac{1}{2} \\ 4 & -7 & 0 \end{pmatrix}$$

$$(a) \frac{1}{4}A = \frac{A}{4} = \begin{vmatrix} \frac{1}{4} \times -4 & \frac{1}{4} \times 32 & \frac{1}{4} \times \frac{1}{4} \\ \frac{1}{4} \times 4 & \frac{1}{4} \times \frac{3}{4} & \frac{1}{4} \times \frac{1}{2} \\ \frac{1}{4} \times 4 & \frac{1}{4} \times -7 & \frac{1}{4} \times 0 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & 32 & \frac{1}{16} \\ 4 & \frac{3}{16} & \frac{1}{8} \\ 4 & -\frac{7}{4} & 0 \end{vmatrix} = \begin{vmatrix} -1 & 8 & \frac{1}{16} \\ 1 & \frac{3}{16} & \frac{1}{8} \\ 1 & -\frac{7}{4} & 0 \end{vmatrix}$$

$$(b) A = \begin{vmatrix} -4 & 32 & \frac{1}{4} \\ 4 & \frac{3}{4} & \frac{1}{2} \\ 4 & -7 & 0 \end{vmatrix} \text{ then}$$

$$-7A = \begin{vmatrix} -7 \times -4 & -7 \times 32 & -7 \times \frac{1}{4} \\ -7 \times 4 & -7 \times \frac{3}{4} & -7 \times \frac{1}{2} \\ -7 \times 4 & -7 \times -7 & -7 \times 0 \end{vmatrix}$$

$$= \begin{vmatrix} 28 & -224 & -\frac{7}{4} \\ -28 & -\frac{21}{4} & -\frac{7}{2} \\ -28 & 49 & 0 \end{vmatrix}$$

## (ii) Multiplication of matrices

Given two matrices  $A$  and  $B$ , the product  $AB$  is possible if and only if the number of columns in  $A$  is equal to the number of rows in  $B$ . Similarly, the product  $BA$  is possible if and only if the number of columns in  $B$  is equal to the number of rows in  $A$ .

For example,

if  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$  the product

$AB$  is possible since the number of columns in  $A$  is 2 and the number of rows in  $B$  is also 2.  $BA$  is not possible because the number of columns in  $B$  is 1 and the number of rows in  $A$  is 2. For multiplication of matrices, we shall consider a line matrix  $R = (a \ b \ c)$  and

a column matrix  $Q = \begin{pmatrix} r \\ s \\ t \end{pmatrix}$

Since  $R$  has three columns and  $Q$  has three rows, the product  $RQ$  is possible.

$$RQ = (a \ b \ c) \begin{pmatrix} r \\ s \\ t \end{pmatrix} = (ar + bs + ct).$$

Multiplication of matrices is also called dot product of matrices given that  $C = A \cdot B = AB$ , then  $C_{ij} = (\text{i-th row of } A) \cdot (\text{j-th column of } B)$

**Note:**

1. The number of columns of  $A$  must be equal to the number of rows of  $B$ . This is conformability condition for multiplication.
2. Multiplication of matrix is not commutative, that is,  $AB \neq BA$ .

### Worked Example 13

Given that  $A = \begin{pmatrix} x & y \\ u & t \end{pmatrix}$  and  $D = \begin{pmatrix} u & v \\ r & s \end{pmatrix}$

Find  $AD$ .

**SOLUTION**

$$AD = \begin{pmatrix} x & y \\ w & t \end{pmatrix} \cdot \begin{pmatrix} u & v \\ r & s \end{pmatrix} = \begin{pmatrix} xu + yr & xv + ys \\ wu + tr & wv + ts \end{pmatrix}$$

### Worked Example 14

If  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} 2 & 0 & 4 \\ 3 & 1 & 5 \end{pmatrix}$ ,

find (a)  $AB$  and (b)  $BA$ .

.....  
**SOLUTION**  
.....

$$\begin{aligned}(a) \quad AB &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 3 & 1 & 5 \end{pmatrix} \\ &= \begin{pmatrix} (1 \times 2) + (2 \times 3) & (1 \times 0) + (2 \times 1) & (1 \times 4) + (2 \times 5) \\ (3 \times 2) + (4 \times 3) & (3 \times 0) + (4 \times 1) & (3 \times 4) + (4 \times 5) \end{pmatrix} \\ &= \begin{pmatrix} (2+6) & (0+2) & (4+10) \\ (6+12) & (0+4) & (12+20) \end{pmatrix} \\ &= \begin{pmatrix} 8 & 2 & 14 \\ 18 & 4 & 32 \end{pmatrix}\end{aligned}$$

$$(b) \quad BA = \begin{pmatrix} 2 & 0 & 4 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The number of columns in  $B$  is 3 while the number of rows in  $A$  is 2. Since the number of columns in  $B$  is not the same as the number of rows in  $A$ , it is, therefore, not possible to find the product  $BA$ . Hence,  $BA$  is not possible.

**Worked Example 15**

Compute  $PQ$  if  $P = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$  and  $Q = \begin{pmatrix} 3 & 5 \\ 4 & 0 \end{pmatrix}$

.....  
**SOLUTION**  
.....

$$\begin{aligned}(a) \quad PQ &= \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 4 & 0 \end{pmatrix} \\ &= \begin{bmatrix} (3 \times 3) + (4 \times 4) & (3 \times 5) + (4 \times 0) \\ (1 \times 3) + (2 \times 4) & (1 \times 5) + (2 \times 0) \end{bmatrix}\end{aligned}$$

$$= \begin{pmatrix} 9+16 & 15+0 \\ 3+8 & 5+0 \end{pmatrix} = \begin{pmatrix} 25 & 15 \\ 11 & 5 \end{pmatrix}$$

(b)  $QP = \begin{pmatrix} 3 & 5 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

Since the number of columns in  $A$  is equal to the number of rows in  $B$ ,  $QP$  is possible.

$$\begin{aligned} QP &= \begin{bmatrix} (3 \times 3) + (5 \times 1) & (3 \times 4) + (5 \times 2) \\ (4 \times 3) + (0 \times 1) & (4 \times 4) + (0 \times 2) \end{bmatrix} \\ &= \begin{bmatrix} a+5 & 12+10 \\ 12+0 & 16+0 \end{bmatrix} \\ &= \begin{pmatrix} 14 & 32 \\ 12 & 16 \end{pmatrix} \end{aligned}$$

Hence  $PQ \neq QP$

#### Exercise 4

1. Compute  $AB$  if  $A = \begin{pmatrix} 6 & 8 \\ 1 & 4 \end{pmatrix}$ ,  $B = \begin{pmatrix} 3 & 5 \\ 4 & 0 \end{pmatrix}$
2. From Question 1 compute  $BA$ .
3. From Questions 1 and 2, compare  $AB$  and  $BA$ .
4. Let  $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 2 \\ 4 & \frac{1}{2} \end{bmatrix}$

- (a) Calculate  $AB$ .
- (b) Calculate  $BA$ .
- (c) What do you notice?

5. Let  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Show that  $AI = IA = A$ .

6. Given that  $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

Calculate  $A^2$ .

7. Given that  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Calculate  $A^2$ .

8. Let  $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  and  $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Compute  $CD$ .

9. If  $X = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$  and  $Y = \begin{bmatrix} 0 & 1 \\ 5 & 7 \end{bmatrix}$

- (a) Compute  $XY$ .
- (b) Compute  $YX$ .
- (c) What do you notice?

10. Let  $A = \begin{pmatrix} 7 & 2 \\ 9 & 14 \end{pmatrix}$ . Find  $5A$ .

11. If  $B = \begin{pmatrix} 4 & 6 & 8 \\ 2 & 12 & 9 \\ -3 & 5 & 7 \end{pmatrix}$ ,

compute the following:

(a)  $-6B$       (b)  $\frac{-5}{7}B$       (c)  $\frac{B}{4}$

12. If  $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ , compute the following:

(a)  $AB$       (b)  $BA$

13. If  $A = \begin{pmatrix} 3 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 & 4 \\ 5 & 2 \\ 6 & 1 \end{pmatrix}$ , find  $AB$  and  $BA$ .

## V. Transpose and Trace of a Matrix

### (i) Transpose of a matrix

**Transpose of a matrix**  $A$  is the matrix obtained by interchanging the rows and columns of  $A$ . In this case, the first row of  $A$  becomes the first column of the transpose and so on.

**Note:** The transpose  $A$  is often denoted by  $A_T$  or  $A'$ .

### Worked Example 16

(a) If  $B = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ , find  $B^T$

(b) Given that  $X = \begin{pmatrix} 3 & 4 & 5 \\ 7 & 8 & 9 \\ 1 & 2 & 6 \end{pmatrix}$ , find  $X^T$ .

### SOLUTION

(a)  $B = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$  then  $B^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$

(b)  $X = \begin{pmatrix} 3 & 4 & 5 \\ 7 & 8 & 9 \\ 1 & 2 & 6 \end{pmatrix}$  then  $X^T = \begin{pmatrix} 3 & 7 & 1 \\ 4 & 8 & 2 \\ 5 & 9 & 6 \end{pmatrix}$

### (ii) Trace of a matrix

The **trace of a matrix** is the sum of the elements in its principal diagonals. For example, consider matrix

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

The elements in the principal diagonal are 1, 5 and 9.  
 $\therefore \text{Trace}(X) = \text{Tr}(X) = 1 + 5 + 9 = 15$

### Worked Example 17

Consider the following matrices:

$$(a) Y = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 3 & 5 \\ 5 & 1 & 6 \end{pmatrix} \text{ and } (b) P = \begin{pmatrix} -5 & 8 \\ 7 & -7 \end{pmatrix}$$

Find the trace of matrix  $Y$  and matrix  $P$ .

#### SOLUTION

$$(a) Y = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 3 & 5 \\ 5 & 1 & 6 \end{pmatrix}, \text{ its trace} = 3 + 3 + 6 = 12$$

$$(b) P = \begin{pmatrix} -5 & 8 \\ 7 & -7 \end{pmatrix}, \text{ its trace} = -5 + (-7)$$

$$= -5 - 7 = -12$$

## VI. Symmetric Matrix

A matrix  $A$  is said to be **symmetric** if it is equal in all respects to its transpose, that is  $A = A_T$ . For example, if

$$Y = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}, \text{ then } Y^T = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$$

Hence,  $Y = Y_T$ . Therefore,  $Y$  is a symmetric matrix.

### Worked Example 18

Given that  $B = \begin{pmatrix} 2 & 4 & 3 \\ 4 & 1 & 5 \\ 3 & 5 & 3 \end{pmatrix}$

Is  $B$  a symmetric matrix?

**SOLUTION**

$$B = \begin{pmatrix} 2 & 4 & 3 \\ 4 & 1 & 5 \\ 3 & 5 & 3 \end{pmatrix}, \text{ then } B^T = \begin{pmatrix} 2 & 4 & 3 \\ 4 & 1 & 5 \\ 3 & 5 & 3 \end{pmatrix}$$

Here  $B = B^T$ . Since  $B = B^T$ ,  $B$  is a symmetric matrix.

### Exercise 5

1. Find the transpose of the matrix

$$A = \begin{pmatrix} 3 & 2 & 5 \\ 5 & 2 & 4 \\ 6 & 1 & 3 \end{pmatrix}.$$

2. Is matrix  $Z = \begin{pmatrix} 3 & 6 & 7 \\ 4 & 1 & 2 \\ 5 & 5 & 1 \end{pmatrix}$  symmetric?

3. Find the trace of the matrix

$$(a) \quad T = \begin{pmatrix} 5 & 7 \\ 6 & 3 \end{pmatrix} \quad (b) \quad Z = \begin{pmatrix} g & r & m \\ v & u & t \\ a & b & c \end{pmatrix}$$

4. Given that matrix  $A = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 6 \\ 2 & 6 & 2 \end{pmatrix}$

- (a) Find its transpose.  
 (b) What do you notice?

5. Is matrix  $Y = \begin{pmatrix} 1 & 9 & 6 \\ 9 & 4 & 8 \\ 6 & 8 & 2 \end{pmatrix}$  symmetric?

If yes, state your reason(s).

6. Given that matrix  $Z^T =$

$$\begin{pmatrix} 1 & 5 & 4 & 7 \\ 9 & 2 & 3 & 10 \\ 8 & 6 & 12 & 14 \\ 14 & 20 & 32 & 22 \end{pmatrix}$$

Find matrix  $Z$ .

7. What is the trace of matrix  $Z$  in Question 6.

8. Given that  $\begin{pmatrix} x & 3 \\ y & z \end{pmatrix} = \begin{pmatrix} 10 & 5e \\ 5 & -2 \end{pmatrix}$ ,

find the values of  $x, y, z$  and  $e$ .

9. If matrix  $X = \begin{pmatrix} 2 & 3 & 7 \\ 5 & 6 & 8 \\ 3 & 4 & -2 \end{pmatrix}$ , find  $X^T$ .

10. Given that  $A = \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & y \end{pmatrix}$

- (a) Find  $A^T$ .
- (b) Is  $A$  a symmetric matrix?
- (c) Find the trace of the matrix  $A$ .

## VII. Determinant of a Matrix

The determinant of a square matrix is denoted as  $\det(A)$  or  $|A|$ .

## (I) Determinant of a $2 \times 2$ matrix

Given that

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{Then } |X| = \det(X) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

### Worked Example 19

Find the determinant of the following matrices:

(a)  $C = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$       (b)  $D = \begin{pmatrix} 3 & 5 \\ -2 & -4 \end{pmatrix}$

#### SOLUTION

(a)  $C = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$ , then  $\det(C) = |C| = \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$   
 $= (3 \times 5) - (2 \times 4)$

$$= 15 - 8$$

$$= 7$$

(b)  $D = \begin{pmatrix} 3 & 5 \\ -2 & -4 \end{pmatrix} = \det(D) = |D| = \begin{vmatrix} 3 & 5 \\ -2 & -4 \end{vmatrix}$   
 $= (3 \times -4) - (-2 \times 5)$   
 $= -12 - (-10)$   
 $= -12 + 10 = -2$

#### Note

1. We use two vertical and parallel straight lines for determinants instead of parenthesis or square bracket used for matrices.
2. We can only find the determinant of a square matrix.

## (II) Minor of a matrix

Consider a  $3 \times 3$  matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The determinant obtained when the row and column containing  $a_{11}$  are removed is the minor of  $a_{11}$ . It is always denoted by

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly, the minor of  $a_{12}$  is denoted

$$\text{by } A_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and the minor of } a_{13} \text{ is}$$

$$\text{denoted by } A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

**Note:** The minor of each of the elements in matrix  $A$  can be similarly obtained.

### (III) Cofactors of a matrix

This is obtained by associating/assigning an appropriate sign to each of the minors obtained.

#### Determination of the sign to be assigned/associated

We determine the sign associated to each of the minors by finding the sum of the subscript attached to each of them.

For example, for minor  $A_{ij}$ , the sum of the subscript is  $i + j$ . If  $i + j$  is even, the sign associated will be positive (+). If  $i + j$  is odd, the sign will be negative (-). For example, for minor  $A_{11}$ , the sum of subscript =  $1 + 1 = 2$  which is even and hence the sign for minor  $A_{11}$  is positive (+).

Hence, we have:

$$+A_{11} = + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

For minor  $A_{12}$ , the sum is  $1 + 2 = 3$  which is odd and hence the sign for minor  $A_{12}$  is (-).

Hence, we have:

$$-A_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}.$$

The signs for other minor are similarly determined. The signed minor is called the cofactor. Thus, if  $C_{ij}$  is the corresponding cofactor of the minor  $A_{ij}$ ,

$$C_{11} = +A_{11}, C_{12} = -A_{12},$$

$$C_{13} = +A_{13}, \dots, C_{33} = +A_{33}$$

In general, the signs associated with the minor of each of the element in a  $3 \times 3$  matrix depending on the location are given below:

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

For a  $4 \times 4$  matrix, we have:

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

## Worked Example 20

If  $A = \begin{pmatrix} 2 & 5 & -3 \\ 4 & 6 & 3 \\ 8 & -2 & 7 \end{pmatrix}$ , find the minor and

cofactor of the following elements in A.



### SOLUTION

(a) The minor of 2 =  $\begin{vmatrix} 6 & 3 \\ -2 & 7 \end{vmatrix}$

$$\equiv (6 \times 7) - (3 \times -2)$$

$$= 42 - (-6)$$

$$= 42 + 6$$

≡ 48

Since this is a  $3 \times 3$  matrix, the corresponding cofactor is  $+(48) = 48$ .

(b) The minor of 3 =  $\begin{vmatrix} 2 & 5 \\ 8 & -2 \end{vmatrix}$

$$= (2 \times -2) - (5 \times 8)$$

$$= -4 - 40 = -44$$

The corresponding cofactor = -(44)

#### (iv) Determinant of a $3 \times 3$ matrix

Given that

$$X = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \det X = |X| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - qf) + c(dh - qe)$$

## Worked Example 21

If  $Q = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{pmatrix}$ , find  $|Q|$ .

.....  
**SOLUTION**  
.....

$$|Q| = \begin{vmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ -3 & -2 \end{vmatrix}$$

$$\begin{aligned} &= 2[2 - (-6)] - 1[1 - (-9)] - 2[-2 - (-6)] \\ &= 2(2 + 6) - 1(1 + 9) - 2(-2 + 6) \\ &= 2(8) - 1(10) - 2(4) \\ &= 16 - 10 - 8 \\ &= 16 - 18 = -2 \end{aligned}$$

**(v) Inverse of  $2 \times 2$  matrix**

Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is a  $2 \times 2$  matrix, the inverse of  $A$  is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where  $ad - bc = |A| = \det(A)$ . If  $|A| = 0$ , the inverse of  $A$  does not exist. Matrix  $A$  is then said to be singular, that is, it has no multiplicative inverse.

**Worked Example 22**

Find the inverse of matrix  $C = \begin{pmatrix} 3 & 2 \\ -5 & 7 \end{pmatrix}$ .

.....  
**SOLUTION**  
.....

In the given matrix  $C = \begin{pmatrix} 3 & 2 \\ -5 & 7 \end{pmatrix}$

$a = 3, b = 2, c = -5$  and  $d = 7$

$$\text{But } C^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\therefore C^{-1} = \frac{1}{(3 \times 7) - (2 \times -5)} \begin{pmatrix} 7 & -2 \\ -(-5) & 3 \end{pmatrix}$$

$$= \frac{1}{21 + 10} \begin{pmatrix} 7 & -2 \\ 5 & 3 \end{pmatrix}$$

$$= \frac{1}{31} \begin{pmatrix} 7 & -2 \\ 5 & 3 \end{pmatrix}$$

### Exercise 6

1. Find the minor and cofactor of the elements 8 and 12 in matrix

$$Z = \begin{pmatrix} 4 & 6 & 9 \\ 8 & 2 & 3 \\ 10 & 12 & 7 \end{pmatrix}.$$

Find the determinant of the following matrices:

2.  $A = \begin{pmatrix} 6 & 3 \\ 9 & 2 \end{pmatrix}$

3.  $B = \begin{pmatrix} 5 & 3 & 4 \\ 1 & 2 & 6 \\ 2 & 4 & 6 \end{pmatrix}$

4. Find the inverse of matrix  $P = \begin{pmatrix} 8 & 6 \\ 5 & 7 \end{pmatrix}$

5. Show that matrix  $A = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 5 & 1 \\ 4 & -5 & 5 \end{pmatrix}$

is singular (that is, it has no multiplicative inverse).

6. Given that  $M = \begin{pmatrix} 2 & -3 & 4 \\ 2 & 4 & -2 \\ 3 & -1 & 2 \end{pmatrix}$

Determine  $\det M$ .

7. Show that  $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{9} & \frac{2}{9} \end{pmatrix}$  is the inverse of  $\begin{pmatrix} 2 & -3 \\ 1 & 3 \end{pmatrix}$ .

8. Show that  $A = \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}$  and  $B = \begin{pmatrix} 9 & -4 \\ -2 & 1 \end{pmatrix}$  are multiplicative inverses.

### 9. Show that the multiplicative

$$\text{inverse of matrix } A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$\text{is } A^{-1} = \begin{pmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}.$$

10. Let  $A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

## Calculate

- (a)  $AB \cdot B^{-1}$
  - (b)  $A^{-1}$
  - (c)  $B^{-1}A^{-1}$
  - (d) Is  $(AB)^{-1} = A^{-1}B^{-1}$ ?

## VIII. Application of Determinant in Solving Simultaneous Linear Equations

Consider the following simultaneous equations:

These equations can be written in matrix form as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \quad \dots \dots \dots \quad (3)$$

where

- (i) the matrix of the coefficients of the variable (coefficient matrix) represented by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- (ii) the column matrix of the unknown denoted by

$$X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

- (iii) the column matrix of constants denoted by

$$c = \binom{p}{q}.$$

Thus, From (3),

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix}$$

that is,

where  $ad - bc = |A|$ . From (4), the numerical values of  $x$  and  $y$  are obtained.

### Worked Example 23

Solve the simultaneous equations  $x + y = 5$  and  $2x - y = 1$ .

## SOLUTION

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Therefore,  $|A| = [1 \times (-1)] - (1 \times 2)$

$$= -1 - 2 = -3$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$= \frac{-1}{3} \begin{pmatrix} -5 & -1 \\ -10 & 1 \end{pmatrix}$$

$$= \frac{-1}{3} \begin{pmatrix} -6 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Hence,  $x = 2, y = 3$ .

## (I) Solution of Simultaneous Linear Equations Using Cramer's Rule

Consider the following simultaneous linear equations:

$$ax + by = p \quad (1)$$

$$cx + dy = q \quad (2)$$

Solving using Cramer's rule, we proceed as follows:

$$x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{|A|}; y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{|A|}$$

where  $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$  is the determinant of the coefficient matrix A.

### Worked Example 24

Use Cramer's rule to solve the simultaneous linear equations  $x + y = 5$  and  $2x - y = 1$ .

**SOLUTION**

Coefficient matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

where  $a = 1$ ,  $b = 1$ ,  $c = 2$  and  $d = -1$ .

$$|A| = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3; \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

By Cramer's rule;

$$\begin{aligned} x &= \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}} \\ &= \frac{5(-1) - 1(1)}{-3} \\ &= \frac{-6}{-3} = 2 \end{aligned}$$

$$\therefore x = 2$$

$$\begin{aligned} y &= \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}} \\ &= \frac{(1 \times 1) - (2 \times 5)}{-3} \\ &= \frac{-9}{-3} = 3 \end{aligned}$$

$$\therefore y = 3$$

Hence,  $x = 2, y = 3$

### Worked Example 25

Solve the equation

$$\begin{vmatrix} 5 & a & 6 \\ 5 & 2 \end{vmatrix} = 10.$$

\*\*\*\*\*  
**SOLUTION**  
\*\*\*\*\*

$$\begin{vmatrix} 5 & 6 \\ 5 & 2 \end{vmatrix} = 10 \Rightarrow (5a \times 2) - (6 \times 5) = 10$$

$$\Rightarrow 10a - 30 = 10$$

$$10a = 10 + 30 = 40$$

$$\therefore a = 4$$

### Exercise 7

1. Solve for  $x$  and  $y$  in the equation

$$\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \end{pmatrix}.$$

2. Use Cramer's rule to solve the following equations:

$$5x - 2y = 14$$

$$2x + 2y = 14$$

3. Solve the equation

$$\begin{vmatrix} a^2 & a \\ 5 & 2 \end{vmatrix} = 25.$$

4. Solve the equation

$$\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

5. Solve for  $a$ ,  $c$  and  $d$  in the matrix equation

$$\frac{1}{10} \begin{pmatrix} a \\ 5 \\ c \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 \\ d \\ 7 \end{pmatrix}.$$

6. Solve the simultaneous linear equations

$$2x - 3y = 1$$

$$x + 3y = 1$$

7. Show that the following set of equations has no solution:

$$x + 3y + 4z = 6$$

$$2x + 3y + 7z = 9$$

$$x - 3z = 0$$

8. Find the solutions of the system of equations

$$x + 2y - 6z = 4$$

$$-3x - 6y + 9z = 3$$

$$y - 3z = 1$$

9. Solve the system of linear equations

$$2x + 3y + z = -12; x - z = 3; x + y + z = 4.$$

10. If

$$A = \begin{pmatrix} 3 & 3a \\ 2a & a \end{pmatrix},$$

solve the equation  $|A| = 0$ .

## SUMMARY

### In this chapter, we have learnt the following:

- v A rectangle array of real numbers arranged in rows and columns is called a matrix.
- v If there are  $m$  rows and  $n$  columns, the matrix is called an ' $m$  by  $n$ ' matrix.
- v In an  $m$  by  $n$  matrix, there are altogether  $mn$  entries. It is of order  $m$  by  $n$ , written as  $m \times n$ .
- v The number of rows is always listed first.
- v If the number of rows is the same as the number of columns of a matrix, the matrix is called a square matrix.
- v The main diagonal of a square matrix consists of those entries with the same row and column subscript.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

- v A matrix with a single column is called column matrix.
- v A matrix with a single row is called a row matrix.
- v Addition is possible on two matrices if the matrices are conformable.
- v The difference of two matrices is obtained by subtracting the corresponding elements and writing the answers in the corresponding position.
- v The number of rows of another matrix before the matrices are conformable for multiplication.
- v If the determinant of a matrix is zero, the inverse of the matrix does not exist. Such a matrix is called a singular matrix, otherwise a non-singular matrix.
- v A singular matrix has no multiplicative inverse.
- v Matrices are very useful in solving solutions of simultaneous linear equations.

## GRADUATED EXERCISES

1. Given that  $I$  is a  $(2 \times 2)$  unit matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix},$$

- (a) Determine the constant  $k$  such that  $A^2 = kA + mI$ .
- (b) If  $BA = I$ , where  $B$  is a  $(2 \times 2)$  matrix, deduce that  $B$  may be expressed in the form  $\alpha A + \lambda I$  stating the values of the constants  $\alpha$  and  $\lambda$ .

2. Find the multiplicative inverse of matrix

$$\begin{pmatrix} -2 & -1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \end{pmatrix}.$$

3. Let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 7 \end{pmatrix}, B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$$

Calculate the following:

- (a)  $(AB) C$ .
- (b)  $A(BC)$
- (c)  $A(B + C)$
- (d)  $AB + AC$

4. Given the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- (a) Show that  $A_2 = B$ .
- (b) Show that  $B_2 = A$ .
- (c) Show that  $AB = BA$ .

5. If

$$S = \begin{pmatrix} 3 & 1 \\ 3 & 5 \end{pmatrix} \text{ and } P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that

$$\begin{pmatrix} 3 & 1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ find } P.$$

6. If

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix},$$

find  $x + y$ .

7. Express the simultaneous linear equations in matrix forms:

- (a)  $x_1 + x_2 = 5$   
 $2x_1 + 3x_2 = 16$
- (b)  $x = 4 - y$   
 $x + 2 = y$

8. Find the inverse of the following matrices:

$$(a) \begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix} \quad (b) \begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}$$

9. Solve the equation

$$\begin{vmatrix} -3 & x & 1 \\ x & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

10. Evaluate  $\begin{pmatrix} 3 & 5 & 6 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} / \begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ .

11. If  $A = (a, b)$  and  $B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ , find  $AB$ .

12. Given that  $P = \begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix}$  and  $Q = \begin{pmatrix} 0 & 3 \\ 4 & 2 \end{pmatrix}$

Find (a)  $PQ$  (b)  $|QP|$ .

13. Find the inverse of matrix  $\begin{pmatrix} 3 & 5 \\ 6 & 2 \end{pmatrix}$ .

14. If  $A = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$

such that  $AB = I$  where  $I$  is the identity matrix, find  $B$ .

15. Describe the position of the elements 4 and 13 in matrix

$$B = \begin{pmatrix} 5 & 7 & 8 \\ 4 & 3 & 9 \\ 12 & 13 & -3 \end{pmatrix}.$$

16. Give the order of the following matrices:

(a)  $\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(b)  $\begin{pmatrix} a & b & c \\ d & h & f \\ g & e & i \end{pmatrix}$

17. Given that

$$A = \begin{pmatrix} 4 & 1 & 4 \\ 6 & 7 & 3 \\ 1 & 2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 5 \\ 7 & 0 & 7 \\ 1 & 2 & 3 \end{pmatrix}$$

Find (a)  $A + B$  (b)  $B + A$

18. Given that  $Z = \begin{pmatrix} 5 & 1 & 7 & 6 \\ 1 & 0 & 7 & 3 \\ 4 & 0 & 8 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

Find  $2Z$ .

19. Given that

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 5 & 6 & 7 \\ 12 & 11 & 10 \\ 3 & 2 & 1 \end{pmatrix}$$

Find

- (a)  $X - Y$ .
- (b)  $Y - X$ .
- (c) Is  $X - Y$  equal to  $Y - Z$ ?

20. If  $A = \begin{pmatrix} 4 & 2 \\ 5 & 6 \end{pmatrix}$  and  $B = \begin{pmatrix} 4 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ , find  $AB$  and  $BA$ .

21. If  $A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 & 8 \\ 4 & 1 \\ 2 & 5 \end{pmatrix}$ , find  $AB$

and  $BA$ .

22. Find the transpose of the following matrices:

(a)  $A = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 7 & 0 \\ 5 & 0 & 4 \end{pmatrix}$       (b)  $T = \begin{pmatrix} 5 & 1 & 6 \\ 9 & 1 & 2 \\ 3 & 1 & 2 \end{pmatrix}$

(c)  $Z = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 7 & 1 & 7 & 2 \\ 9 & 2 & 1 & 2 \end{pmatrix}$

(d)  $B = \begin{pmatrix} 7 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 4 \end{pmatrix}$

23. Which of the matrices in Question 22 are symmetric?

24. Find the trace of the following matrices:

(a)  $\begin{pmatrix} 4 & 7 & 6 \\ 2 & 4 & 5 \\ 9 & 3 & 1 \end{pmatrix}$

(b)  $\begin{pmatrix} a & b & c \\ d & e & f \\ k & l & m \end{pmatrix}$

(c)  $\begin{pmatrix} -1 & -4 & 7 \\ 6 & -5 & 3 \\ 4 & 3 & -9 \end{pmatrix}$

(d)  $\begin{pmatrix} 6 & 0 & 4 \\ 7 & 0 & 1 \\ -3 & 4 & -5 \end{pmatrix}$

25. Find the determinant of the following matrices:

(a)  $B = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$

(b)  $P = \begin{pmatrix} 5 & 1 & 4 \\ 2 & 1 & 5 \\ 5 & 3 & 6 \end{pmatrix}$

(c)  $T = \begin{pmatrix} 2 & 4 \\ 7 & 6 \end{pmatrix}$

(d)  $C = \begin{pmatrix} -6 & -5 \\ 7 & -2 \end{pmatrix}$