CHAPTER 8

At the end of the chapter, students should be able to:
1. State the meaning of simple statements with examples.
2. Identify true or false statements.
3. State the negation of a simple statement.
4. Distinguish between simple statements and compound statements.
5. Give examples of compound statements involving conjunction, disjunction, implication and bi-implication.

I. Introduction

The main asset that makes human beings '*superior*' to other creatures is our ability to reason. How well this ability is used depends on each person's power of reasoning. Logical reasoning is the method of applying mathematical concepts, either explicitly or implicitly, to the solution of problems. In this chapter, we discuss basic ideas of *Logical Reasoning* in the context of Mathematics.

II. Statement

A statement is defined as a declaration of facts in speech, writing or an abstract of an account. According to the above definition, the following sentences are statements: Nigeria got its independence in the year 1960, Oh! my goodness, Where are you going, Don't run, The house is ugly. In Mathematics, however, the notion of a statement is more precise.

III. Meaning of Simple Statement

A mathematical statement is a declarative sentence that is true or false, but not both. So, of the five sentences above, only the first one is a statement in the mathematical sense. Its truth value is true. 'Oh, my goodness!' is an exclamation, 'Don't run' is a command, 'Where are you going?' is a question and so neither of these sentences have truth value. In the last sentence, what is ugly may be a matter of opinion, so whether it is true or false is ambiguous, and moreover which house is in question is not specified. Sentences involving variable times such as 'today', 'tomorrow' or 'yesterday' are not statements. This is because it is not known what time is referred to here. The same argument holds for sentences with pronouns unless a particular person is referred to and for variable places such as ‘here’ and ‘there’.

Worked Example 1

Check whether each of the following sentences are statements. State also whether each is true or false.
(a) 8 is less than 6.
(b) Every set is a finite set.
(c) The sun is a star.
(d) Mathematics is fun.
(e) There is no rain without clouds.
(f) How far is Abuja from here?
(g) Tomorrow is Friday.
(h) He is a Mathematics graduate.

Solution

Answers are given in Table 8.1, where Y stands for Yes and N stands for No. In Table 8.1, some statements are either true or false. We call these closed statements. *Closed statements* are about well-defined situations. Other statements, such as (d), (g) and (h), do not involve well-defined situations. We do not know if they are true or false; hence, they are called *Open statements*.

Table 8.1

Expression	A statement?	True or False?
(a)	Y	False
(b)	Y	False
(c)	Y	True
(d)	N	Impossible to say
(e)	Y	True
(f)	N	Not applicable (a question)
(g)	N	Impossible to say
(h)	N	Impossible to say

Mathematical statements are often represented using capital letters. For example, we denote the statement ‘*Fire is always hot*’ by P . This is also written as **P:** Fire is always hot.

Worked Example 2

For each of the following, state whether it is a closed or an open statement or neither. Also check whether it is true or false.
A: How old are you?
B: *y* is an odd number.
C: Physics is a science subject.
D: 2 is an even number.
E: Chemistry is a difficult subject.
F: Chair and table.
G: Women are more intelligent than men.
H: Lagos is the capital of Nigeria.

Table 8.2

Expression	A statement?			True or False
	Closed?	Open?	Neither?	
A			√	Not applicable
B		√		Impossible to say
C	√			True
D	√			True
E		√		Impossible to say
F			√	Not applicable
G		√		Impossible to say
H	√			False

IV. Negation of a Simple Statement

The opposite of a statement is called the negation of the statement. Let us consider the statement:
P: Kano is a city. The negation of this statement is: It is not the case that Kano is a city. This can also be written as: It is false that Kano is a city. This can simply be expressed as: Kano is not a city. If P is a statement, then the negation of P is also a statement denoted by ~P and read as 'not P'.
Note: While forming the negation of a statement, phrases like 'it is not the case' or 'it is false that' are also used.

Worked Example 3

Write the negation of the following statements:

(a) **A:** Both the diagonals of a rectangle have the same length.

(b) **B:** $\sqrt{7}$ is a rational number.

(c) **C:** Everyone in Nigeria speaks English Language.

(d) **D:** David is a lecturer.

(e) **E:** $x = 10$.

Solution

The negations of these statements are:

(a) **~A:** It is false that both the diagonals of a rectangle have the same length.

(b) **~B:** It is not the case that $\sqrt{7}$ is a rational number or $\sqrt{7}$ is not a rational number.

(c) **~C:** Not everyone in Nigeria speaks English Language.

(d) **~D:** David is not a lecturer.

(e) **~E:** $x \neq 10$

Worked Example 4

Write the negation of the following statements and check whether the resulting statements are true or false.

(a) Africa is a continent.

(b) There does not exist a quadrilateral which has all its sides equal.

(c) Every natural number is greater than 0.

(d) The sum of 3 and 4 is 9.

Solution

(a) It is false that Africa is a continent or Africa is not a continent. We know that this statement is false.

(b) It is not the case that there does not exist a quadrilateral which has all its sides equal or there exists a quadrilateral which has all its sides equal. This statement is true because we know that square is a quadrilateral such that its four sides are equal.

(c) It is false that every natural number is greater than 0 or there exists a

natural number which is not greater than 0. This is a false statement.
(d) It is false that the sum of 3 and 4 is 9 or the sum of 3 and 4 is not equal to 9. This statement is true.

Note: It is often possible to form negations without using the word ‘not’. In such cases, we look at the sense of the given statement and use appropriate language to form the negation. See Worked Example 5.

Worked Example 5

Write the negation of the following statements without using the word ‘not’.
(a) **A:** This word ends with a vowel.
(b) **B:** She ran at variable speed.

Solution

(a) ~**A:** This word ends with a consonant.
(b) ~**B:** She ran at constant speed.
Note: Care should be taken when using the opposite to form a negation of a statement. This method does not always give the correct negation, especially if
the language is vague.

Worked Example 6

Worked Example 6
Consider whether B is the negation of A in the following, if not, give reasons.
(a) **A:** Nura scored the highest mark in the competition.
B: Nura scored the lowest mark in the competition.
(b) **A:** The teacher was inside the class.
B: The teacher was outside the class.

Solution

(a) B is not a negation of A , because Nura could have scored any mark lower than the highest.
(b) B is not a negation of A , because the teacher might have been standing at the door.
Note: If ‘not’ had been used in the above statements, the negation would be correct.

Exercise 1

1. Check whether the following sentences are statements. Also, say whether they are true or false.

   Provide your answers in a tabular form as in Table 8.1.

   (a) **A:** Two plus two equals four.

   (b) **B:** The sum of two positive numbers is positive.

   (c) **C:** All prime numbers are odd numbers.

   (d) **D:** The sum of x and y is greater than 0.

   (e) **E:** How beautiful!

   (f) **F:** Open the door.

   (g) **G:** $(2 + 4)^2 = 2^2 + 4^2$.

   (h) **H:** $3x + 2 = 10$.

2. Which of the following sentences are statements? If it is a statement, say whether it is a closed or an open statement or neither; also say whether it is true or false. Provide your answers in a tabular form as in Table 8.2.

   (a) **A:** There are 35 days in a month.

   (b) **B:** The sum of 5 and 7 is greater than 10.

(c) **C:** The square of a number is an even number.

(d) **D:** The sides of a quadrilateral have equal length.

(e) **E:** Answer this question.

(f) **F:** The product of (-1) and 8 is 8.

(g) **G:** The sum of all interior angles of a triangle is $180°$.

(h) **H:** Today is a windy day.

(i) **I:** All real numbers are complex numbers.

3. Give five examples of sentences which are not statements and five which are statements. Give reasons for the answers.

4. Write the negation of the following statements.

   (a) She is a beautiful lady.

   (b) Franklin is my friend.

   (c) y is not an even number.

   (d) The bag is black.

   (e) It is hot in Maiduguri.

   (f) Yassir is older than John.

   (g) The river is flowing.

   (h) The figure is a triangle.

   (i) The school is closed.

   (j) She studies chemistry.

5. Determine whether B is the negation of A in the following statements. If not, then write a correct negation of A.

   (a) **A:** Mike is a good boy.
   B: Mike is a bad boy.

(b) **A:** This is an easy problem.
B: This is a difficult problem.

(c) **A:** Q is an odd number.
B: Q is an even number.

(d) **A:** He earns more than 100 naira.
B: He earns less than 100 naira.

(e) **A:** Regina is older than Maryam.
B: Regina is younger than Maryam.

V. Compound Statements

Many mathematical statements are obtained by combining one or more statements using some connecting words like 'and', 'or', etc. as shown below.

(i) Consider the following statement:

P: There is something wrong with the bulb or with the wiring. This statement tells us that there is something wrong with the bulb or there is something wrong with the wiring. It means that the given statement is actually made up of two statements:

Q: There is something wrong with the bulb.

R: There is something wrong with the wiring.

It is therefore a compound statement connected by 'or'.

(ii) Now, suppose two statements are given as below:

Q: 7 is an odd number.
R: 7 is a prime number.

(b) **A:** This is an easy problem.
B: This is a difficult problem.
(c) **A:** Q is an odd number.
B: Q is an even number.
(d) **A:** He earns more than 100 naira.
B: He earns less than 100 naira.
(e) **A:** Regina is older than Maryam.
B: Regina is younger than Maryam.

These two statements can be combined with 'and'. **P:** 7 is both odd and prime number. This is a compound statement. This leads us to the following definition:

A *Compound Statement* is a statement which is made up of two or more statements. Each of the statements in a compound statement is called a *component statement*.

Worked Example 7

Find the component statements of the following compound statements:

(a) The sky is blue and the grass is green.

(b) It is raining and it is cold.
(c) All rational numbers are real and all real numbers are complex.
(d) 0 is a positive number or a negative number

Solution

(a) The component statements are
P: The sky is blue.
Q: The grass is green.
The connecting word is ‘and’.
(b) The component statements are
P: It is raining.
Q: It is cold.
The connecting word is ‘and’.
(c) The component statements are
P: All rational numbers are real.
Q: All real numbers are complex.
The connecting word is ‘and’.
(d) The component statements are
P: 0 is a positive number.
Q: 0 is a negative number.
The connecting word is ‘or’.

(i) Conjunction

Given two statements P and Q , the compound statement P and Q is called the conjunction, and it is denoted by $P \wedge Q$ defined by the following truth table.

Table 8.3

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Note: The conjunction, $P \wedge Q$, is true only when both P and Q are true.

WORKED EXAMPLE

Worked Example 8

Find the component statements of the following and say whether the conjunction is true or false.
(a) A square is a quadrilateral and its four sides are equal.
(b) Abuja is the capital of Nigeria and Ibadan is the capital of Osun.
(c) Olympic is organised by CAF and World cup is organised by FIFA.
(d) Accra is the capital of Ghana and Niger.
(e) $2 > 3$ and $5 < 3$.
(f) 24 is a multiple of 5, 7 and 9.

Solution

(a) The component statements are

P: A square is a quadrilateral.
Q: A square has all its sides equal.
Both these statements are true. The conjunction is true.

(b) The component statements are
P: Abuja is the capital of Nigeria.
Q: Ibadan is the capital of Osun.
The first statement is true but the second is false. The conjunction is false.

(c) The component statements are
P: Olympic is organised by CAF.
Q: World cup is organised by FIFA.
The first statement is false but the second is true. The conjunction is false.

(d) The component statements are
P: Accra is the capital of Ghana.
Q: Accra is the capital of Niger.
The first statement is true but the second is false. The conjunction is false.

(e) The component statements are
P: $2 > 3$
Q: $5 < 3$
Both these statements are false. The conjunction is false.

(f) The component statements are
P: 24 is a multiple of 5.
Q: 24 is a multiple of 7.
R: 24 is a multiple of 9.
All the three statements are false. The conjunction is false.

(ii) Disjunction

The statement P or Q is called the disjunction. It is denoted by $P \vee Q$ and is defined by the truth table below.

Table 8.4

P	Q	P V Q
T	T	T
T	F	T
F	T	T
F	F	F

Note: P or Q is true, if at least one of the statements is true.

Worked Example 9

Find the component statements of the following and say whether the disjunction is true or false.

(a) All prime numbers are either even or odd.

(b) A person who has taken Physics or Geology can go for Geophysics.

(c) $\sqrt{2}$ is a rational number or an irrational number.

(d) 5 is a prime number or 7 is an even number.

Solution

(a) The component statements are

P: All prime numbers are even.
Q: All prime numbers are odd.

Both these statements are false. The disjunction is false.

(b) The component statements are

P: A person who has taken Physics can go for Geophysics.
Q: A person who has taken Geology can go for Geophysics.

Both these statements are true. The disjunction is true.

(c) The component statements are

P: $\sqrt{2}$ is a rational number.
Q: $\sqrt{2}$ is an irrational number.

The first statement is false and the second is true. The disjunction is true.

(d) The component statements are

P: 5 is a prime number.

Q: 7 is an even number.

The first statement is true and the second statement is false. The disjunction is true.

(iii) Implication

The statement, 'If the day is Tuesday, then Mary is in school' uses the connective, if … then, to combine the two statements P : the day is Tuesday, Q : Mary is in school. This type of compound statement is called an implication and is denoted by $P \Rightarrow Q$. The truth table for the implication is not as intuitive as the previous truth tables. However, if we consider this statement as a fact, then the only time the fact is broken, or the implication is false, is *if the day is Tuesday and Mary is not in school*. That is, the only time the statement is false is *if P is true and Q is false*.

Note: If the day is not Tuesday, Mary may or may not be in school and the fact about what happens on Tuesday is not broken. With this reasoning, we make the following definition.

The statement, if P, then Q, called an implication and denoted by $P \Rightarrow Q$, is defined by the truth Table 8.5.

Table 8.5

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: The only time implication is false is when P is true and Q is false.

WORKED EXAMPLE

Worked Example 10

For each of the following compound statements, first identify the corresponding
component statements. Then check whether the statement is true or false.

(a) **S:** If a triangle *ABC* is equilateral, then it is isosceles.

(b) **S:** If *a* and *b* are integers, then *ab* is a rational number.

Solution

(a) The component statements are given by

P: Triangle *ABC* is equilateral.

Q: Triangle *ABC* is isosceles.

Since an equilateral triangle is isosceles, we infer that the given compound statement is true.

(b) The component statements are given by

P: *a* and *b* are integers.

Q: *ab* is a rational number.

Since the product of two integers is an integer and therefore a rational number, the compound statement is true. In the implication S : P ⇒ Q , the substatement P is called the *antecedent*, while the sub-statement Q is called the *consequent* of P ⇒ Q . The arrow shows that Q follows P . P ⇒ Q is not the same as Q ⇒ P. The statement P ⇒ Q is sometimes called a *conditional statement*.

(iv) Contrapositive and converse

Contrapositive and converse are certain other statements which can be formed from a given statement with 'if–then'.

Worked Example 11

Write the contrapositive of the following statements:

(a) If the physical environment changes, then the biological environment changes.

(b) If a number is divisible by 9, then it is divisible by 3.

Solution

Write the contrapositive of the following statements:

(a) If the physical environment changes, then the biological environment changes.

(b) If a number is divisible by 9, then it is divisible by 3.

Solution

olution

The contrapositive of these statements is (a) If the biological environment does not change, then the physical environment does not change.

(b) If a number is not divisible by 3, it is not divisible by 9.

Note that both these statements convey the same meaning.

Worked Example 12

Worked Example 12

Write the converse of the following statements:

(a) If a number *n* is even, then n^2 is even.

(b) If you do all the exercises in the book, you get an A grade in the class.

(c) If two integers *a* and *b* are such that $a > b$, then $a - b$ is always a positive integer.

Solution

The converse of these statements is

(a) If a number n_2 is even, then *n* is even.

(b) If you get an A grade in the class, then you have done all the exercises of the book.

(c) If two integers *a* and *b* are such that $a - b$ is always a positive integer, then $a > b$.

(v) Bi-Implication

The last connective to consider is the bi-implication statement, P if and only if Q as in the statement: I can get a refund if and only if I have my receipt. The bi-implication, P if and only if Q, denoted by $P \Leftrightarrow Q$ is defined by the truth table 8.6.

Table 8.6

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: The bi-implication, $P \Leftrightarrow Q$, is a true statement only when P and Q have the same truth value and is sometimes called a *bi-conditional statement*.

Worked Example 13

Given below are two pairs of statements. Combine these two statements using ‘if and only if’.

(a) **P:** If a rectangle is a square, then all its four sides are equal.
Q: If all the four sides of a rectangle are equal, then the rectangle is a square.

(b) **P:** If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.
Q: If a number is divisible by 3, then the sum of its digits is divisible by 3.

Solution

(a) A rectangle is a square if and only if all its four sides are equal.

(b) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Exercise 2

1. Find the component statements of the following compound statements and check whether they are true or false.

   (a) Number 3 is prime or it is odd.

   (b) All integers are positive or negative.

(c) A point occupies a position and its location can be determined.

(d) 42 is divisible by 5, 6 and 7.

2. Find the component statements of the following and say whether the conjunction is true or false.

   (a) A line is straight and extends indefinitely in both directions.

   (b) 0 is less than every positive integer and every negative integer.

   (c) All living things have two legs and two eyes.

   (d) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

3. For each of the following compound statements, first identify the corresponding component statements. Then check whether the compound statement is true or false.

   (a) **S:** If a number is a multiple of 9, then it is a multiple of 3.

   (b) **S:** If you are born in some country, then you are a citizen of that country.

4. Write the contrapositive and converse of the following statements:

   (a) If you are born in Nigeria, then you are a citizen of Nigeria.

   (b) If a triangle is equilateral, then it is isosceles.

(c) If x is a prime number, then x is odd.

(d) If the two lines are parallel, then they do not intersect in the same plane.

(e) You cannot comprehend geometry, if you do not know how to reason deductively.

5. Identify the statements in (a) and (b) as contrapositive or converse of each other.

(a) If you live in Delhi, then you have winter clothes.

(i) If you do not have winter clothes, then you do not live in Delhi.

(ii) If you have winter clothes, then you live in Delhi.

(b) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(i) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

(ii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

VI. Algebraic Laws of Logical Statements

Algebra of logical statements has a close relationship with the algebra of sets. However, as operations, logical connectives obey the laws of algebra. Let P , Q and R be the propositions, then the following algebraic laws of logical statements holds.

1. Commutative laws

(a) $P \wedge Q = Q \wedge P$

(b) $P \vee Q = Q \vee P$

2. Associative laws

(a) $P \wedge (Q \vee R) = (P \wedge Q) \wedge R$

(b) $P \vee (Q \vee R) = (P \vee Q) \vee R$

3. Distributive laws

(a) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

(b) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

4. Laws of absorption

(a) $P \wedge (P \vee Q) = P$

(b) $P \vee (P \wedge Q) = P$

5. Idempotent laws

(a) $(P \wedge Q) = P$

(b) $(P \vee P) = P$

6. De-Morgan's laws

(a) $\sim(P \wedge Q) = \sim P \vee \sim Q$

(b) $\sim(P \vee Q) = \sim P \wedge \sim Q$

7. Laws of contrapositivity

$P \Rightarrow Q = \sim Q \Rightarrow \sim P$

8. Laws of complementation

(a_1) $P \wedge \sim Q = F$

(a_2) $P \vee \sim Q = T$

(b_1) $\sim F = T$

(b_1) $\sim T = F$

(c_1) $\sim(\sim P) = P$

Note: All these laws can be verified using truth table technique.

SUMMARY

In this chapter, we have learnt the following:

- A mathematically acceptable statement is a sentence which is either true or false.
- Negation of a statement P: If P denotes a statement, then the negation of P is denoted by ~P.
- Compound statements and their related component statements.

  A statement is a compound statement, if it is made up of two or more smaller statements. The smaller statements are called component statements of the compound statement.

  (a) The compound statement P and Q is called the conjunction and is denoted by $P \wedge Q$.

  (b) The compound statement P or Q is called disjunction and is denoted by $P \vee Q$.

  (c) An implication is a statement in the form, 'if P then Q' and is written as $P \Rightarrow Q$.

  A sentence with if P then Q can be written in the following ways:

  (a) P implies Q (denoted by $P \Rightarrow Q$).

  (b) P is a sufficient condition for Q.

  (c) Q is a necessary condition for P.

  (d) ~Q implies ~P.

- The contrapositive of a statement $P \Rightarrow Q$ is the statement $\sim Q \Rightarrow \sim P$. The converse of a statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.
- A bi-implication is a statement in the form 'P if and only if Q', denoted by $P \Leftrightarrow Q$.

GRADUATED EXERCISES

1. What is logic?
2. What is the difference between simple statements and compound statements?

State which of the following are statements in logical context:

(a) Put off the engine.
(b) Stop writing.
(c) The equator is a great circle.
(d) $B = \{x: 1 < x < 3, x \in z\}$.
(e) Oh, what a wonderful girl?
(f) $y > 8$.
(g) Aso Rock at Abuja is a magnificent building.

3. State the truth value of the following statements.

(a) 9 is greater than 18.
(b) The median of the numbers 2, 6, 8, 5, 4, 9, 10 is 6.

(c) The volume V cm^3 of a circular cone of radius r cm and height h cm is given by $\frac{3}{4}\pi r^3 h$.

(d) The average B of two numbers p and q is given by $B = \frac{1}{2}(p + q)$

(e) There are 12 states in Nigeria.

4. Write the negation of each of the following statements.
   (a) Kanu scored the first goal.
   (b) Maizube Farm is in Niger State.
   (c) The diagonals of a kite bisect each other.
   (d) The train is moving fast.
   (e) Sani is the shortest boy in the class.
5. List the four connectives you know.
6. How are compound statements formed from simple statements.

(a) **P:** Bola is a youth corps member, and **Q:** She has a degree.

(b) **P:** She is smiling, and **Q:** She is happy.

(c) **P:** $-\infty < x < 10$, and **Q:** $100 < x^2 < \infty$.

7. Write down the converse of each of the following statements.
   (a) If it is perpendicular to the radius then a line is a tangent to the circle.
   (b) If the harvest will be good then it rains sufficiently.
   (c) If it is an equilateral triangle then the three sides are equal.
8. Write down the contrapositive of each of the following statements.
   (a) If two sides of a triangle are equal, then it is an isosceles triangle.
   (b) If two circles have the same centre, then they are concentric circles.
9. If P and Q are two logical statements, copy and complete the following truth table.

P	Q	P V Q	~(P V Q)	~P	(P V Q) Λ P	~(P V Q) ⇒ ~P
T	T					
T	F					
F	T					
F	F					

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