Chapter 13

Chapter 13

Integration

OBJECTIVES

At the end of this chapter, students should be able to:

1. recognise integration as the reverse of differentiation.
2. recognise some standard integrals of polynomials and algebraic fractions.
3. apply some techniques of integration such as
   (a) Integration by substitution
   (b) Integration by parts
   (c) Integration by partial fractions
4. apply integration in real-life situations.

I. Integration as the Reverse of Differentiation

In Chapter 12, we learnt that if $y = f(x)$, then $\frac{dy}{dx} = f'(x)$.
Hence, $dy = f'(x)\,dx$ and dy is called the diff erential of y (the variable dx used in the definition is the differential of the particular function $y = f(x)$). Differentials have limited application, but notation is very useful when applying methods of integration. The integral of a given function $f(x)$ is defined as the function whose differential is $f(x)\,dx$ and is denoted by $\int f(x)dx$.
Thus, the symbol $\int$ means 'the function whose differential is'. In other words, $\int f(x)dx$ is a function, which when differentiated gives $f(x)$, that is

$\int f(x)dx = f(x)$ implies that

$f'(x) = f(x)$

The process of forming the function
$\int f(x)dx$ from $f(x)$ is called ***integration***. For example, $\int 2x\,dx = x2$.
Since,

$\frac{d}{dx}(x^2) = 2x$ or in terms of differentials where c is an arbitrary constant, since $d(x2 + c) = 2x\,dx$.
c is called the constant of integration and $x2 + c$ is referred to as the indefinite integral of $2x$. $d(x2) = 2x\,dx$
Note that
$\int 2x\,dx = x2 + c$
where c is an arbitrary constant, since $d(x2 + c) = 2x\,dx$. c is called the constant of integration and $x2 + c$ is referred to as the indefinite integral of $2x$.

Note

1. $d\left(\int 2x\,dx\right) = 2x\,dx$
2. $\int d\,(x^2) = x^2 + c$

In general, for any suitable function $f(x)$

(a) $d\left\{\int f(x)dx\right\} = f(x)dx$ and that

(b) $\int d\{f(x)\} = f(x) + c$

From (a) and (b) we can see that, apart from the constant c, the differential operator d is the inverse of the integral operator $\int$ and vice versa.
The rules for differentiating xn and trigonometric functions derived in Chapter 11 can now be reversed to establish standard forms for integration as follows:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1},\ n \neq -1$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

Note: For any constant k, we have

1. $\int k f(x)\, dx = k\int f(x)dx + c$

2. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx + c$

Worked Example 1

Determine the following:

(a) $\int (x^2 + 3x + 5)dx$

(b) $\int \left(x^{\frac{1}{2}} + x^{\frac{1}{3}} + x\right)dx$

(c) $\int \left(\sqrt{x} + \frac{3}{\sqrt{x}}\right)dx$

(d) $\int \cos 2x\, dx$

SOLUTION

(a) $\int (x^2 + 3x + 5)dx = \int x^2\, dx + \int 3x\, dx + \int 5\, dx$

$= \int x^2\, dx + 3\int x\, dx + 5\int x^0\, dx$

$= \frac{x^3}{3} + \frac{3x^2}{2} + 5x + C$

(b) $\int \left(x^{\frac{1}{2}} + x^{\frac{1}{3}} + x\right)dx = \int x^{\frac{1}{2}}dx + \int x^{\frac{1}{3}}dx + \int x\, dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{x^2}{2} + C$

$= \frac{2}{3}x^{\frac{2}{3}} + \frac{3}{4}x^{\frac{4}{3}} + \frac{x^2}{2} + C$

(c) $\int \left(\sqrt{x} + \frac{3}{\sqrt{x}}\right)dx = \int \left(x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}\right)dx$

$= \int x^{\frac{1}{2}}dx + 3\int x^{-\frac{1}{2}}dx$

Worked Example 2

Obtain the following functions:

(a) $\int \frac{z^3 + z^2 + z}{z^2} dz$

(b) $\int (x^2 + 2)^2 dx$

(c) $\int \frac{(P-4)^2}{P^4} dp$

SOLUTION

(a) $\int \frac{z^3 + z^2 + z}{z^2} dz = \int (z + 1 + 2z^{-2}) dz$

$$= \frac{z^2}{2} + z - \frac{2}{z} + C$$

(b) $\int (x^2 + 2)^2 dx = \int (x^4 + 2x^2 + 4) dx$

$$= \frac{x^5}{5} + \frac{2}{3}x^3 + 4x + C$$

(c) $\int \frac{(P-4)^2}{P^4} dp = \int \frac{P^2 - 8P + 16}{P^4} dP$

$$= \int (P^{-2} - 8P^{-3} + 16P^{-4}) dp$$

$$= -\frac{1}{P} + \frac{8}{2} P^{-2} + \frac{16}{-3} P^{-3} + C$$

$$= \frac{4}{P^2} - \frac{1}{P} - \frac{16}{3P^3} + C$$

Note: If *y* is a function of *x*, $\int y\, dx$ means the integral of *y* with respect to *x*. The integral sign ' $\int$' cannot be divorced from *dx* if the integral is with respect to *x*.

Worked Example 3

If $\frac{dy}{dx} = 4$ and $y = 3$ when $x = -1$, find *y* in terms of *x*.

SOLUTION

$$\frac{dy}{dx} = 4$$

$$dy = 4dx$$

$$\int dy = \int 4dx + C$$

$$y = 4x + C$$

But $y = 3$ when $x = -1$, it follows that

$$3 = -4 + C$$

$$\therefore \quad C = 7, \text{ hence}$$

$$y = 4x + 7$$

Worked Example 4

The gradient of a line which passes through the point (3, −1) is 2. Find the equation of the line.

SOLUTION

The gradient of the line $\frac{dy}{dx} = 2$ Since the line passes through the point $(3, -1)$, it follows that

$$\frac{y+1}{x-3} = 2.$$

$$\therefore \quad y + 1 = 2(x - 3)$$

$$y + 1 = 2x - 6$$

$$\therefore y - 2x + 7 = 0$$

is the required equation.

Worked Example 5

The gradient of a curve at the origin is zero if at any point of the curve $\frac{dy}{dx} = 2x^2 - x,$ find the equation of the curve.

SOLUTION

$$\frac{dy}{dx} = 2x^2 - x$$

$$\therefore \quad dy = (2x^2 - x)dx$$

$$y = \int (2x^2 - x)dx + C$$

$$y = \frac{2x^3}{3} - \frac{x^2}{2} + C$$

is the required equation of the line.

Exercise 1

Determine the following integrals:

1. $\int (y^2 + 3y + 2)dy$
2. $\int \left(x^{\frac{1}{2}} + \frac{2}{\sqrt{x}}\right)dx$
3. $\int \frac{(y^2 - 1)}{y^2} dy$
4. $\int (6x^3 - 3x^2 + 2x + 4)dx$
5. $\int \sin 8x \, dx$
6. $\int \sec^2 (1 - x)dx$

Integrate the following with respect to *x:*

7. x^2
8. $x^2 - 2x$
9. $x^3 + 6$
10. $(x + 2)(x - 4)$
11. $(x + 1)(x + 2)$
12. $\frac{1}{\sqrt[3]{x}}$
13. $\frac{1 + x^2}{x^4}$
14. $\frac{1 + x}{\sqrt{x}}$
15. Determine $\int (y^3 - 3y^2 + 2)dy$.

II. Techniques of Integration

(i) Simple substitutions

Many integrals can be reduced to one of the standard forms by a change in variable. For example, consider $\int(x + 1)7\ dx$, substitute $u = (x + 1)$ therefore $du = dx.$

$$\text{Thus} \int(x+1)^7\,dx = \int u^7\,du = \frac{1}{8}u^8 + C$$

$$= \frac{1}{8}(x+1)^8 + C$$

Similarly, consider $\int \sin 4x\ dx$
If we substitute $u = 4x$, we have $du = 4dx$ and dx ¼ du

$$\text{Hence,} \int \sin u \cdot \frac{1}{4}du = \frac{1}{4}\int \sin u\,du$$

$$= -\frac{1}{4}\cos u + C$$

$$\therefore \quad \int \sin 4x\,dx = -\frac{1}{4}\cos 4x + C$$

Worked Example 6

Use substitution to determine the following integrals:

(a) $\int(2y + 1)^3 dy$ (b) $\int(x^2 + 1)5x\ dx$

SOLUTION

(a) $\int(2y+1)^3 dy$

Substitute

$$u = (2y+1) \therefore du = 2dy, dy = \frac{du}{2}$$

Hence,

$$= \frac{1}{2}\int u^3\,du = \frac{1}{8}u^4 + C$$

$$= \frac{1}{8}(2y+1)^4 + C \qquad \int(2y+1)^3 dy = \int u^3 \cdot \frac{du}{2}$$

(b) $\int(x^2+1)^5 x\,dx$

Substitute
$u = (x^2+1) \therefore du = 2xdx$

$$xdx = \frac{1}{2}du, \text{ hence } \int (x^2+1)^5 xdx = \int u^5 \cdot \frac{1}{2}du$$

$$= \frac{1}{2}\int u^5\, du$$

$$-\frac{1}{12}u^6 + C$$

$$= \frac{1}{12}(x^2+1)^6 + C$$

Note: When making a change in the variable, we replace a differential $f(x)dx$ by another differential $g(u)du$. Thus, when making a substitution, we must remember to replace both $f(x)$ and the dx parts with the expressions involving the new variable. This technique is known as **integration by substitution**.

Worked Example 7

Determine the following:

(a) $\displaystyle\int \frac{x^2}{(2+x^3)^2}dx$ (b) $\displaystyle\int x\sin\left(x^2 - \frac{\pi}{4}\right)dx$

SOLUTION

(a) $\displaystyle\int \frac{x^2}{(2+x^3)^2}dx$

We substitute

$u = 2 + x^3$

$\therefore \quad du = 3x^2\, dx \text{ and } x^2dx = \dfrac{du}{3}$

$$\therefore \quad \int \frac{x^2}{(2+x^3)^2}dx = \int \frac{1}{3}\frac{du}{u^2}$$

$$= \frac{1}{3}\int \frac{du}{u^2} = \frac{1}{3}\int u^{-2}du$$

$$= -\frac{1}{3}u^{-1} + C = \frac{-1}{3}(2x+3)^{-1} + C$$

(b) $\displaystyle\int x\sin\left(x^2 - \frac{\pi}{4}\right)dx$

Substitute

$u = x^2 - \frac{\pi}{4},\ du = 2xdx$

$$xdx = \frac{1}{2}du$$

$$\therefore \quad \int x\sin\left(x^2 - \frac{\pi}{4}\right)dx = \int \frac{1}{2}\sin u\, du$$

$$= \frac{1}{2}\int \sin u\, du$$

$$= -\frac{1}{2}\cos u + C$$

$$= \frac{1}{2}\cos\left(x^2 - \frac{\pi}{4}\right) + C$$

Note

1. In (a) the multiplier x^2 is proportional to the derivative of the function $2 + x^3$, hence, we substitute $u = 2 + x^3$. Similarly in (b), the multiplier x is proportional to the derivative of the function $\left(x^2 - \frac{\pi}{4}\right)$. Thus, we substitute $u = \left(x^2 - \frac{\pi}{4}\right)$.
2. From (a) and (b), it follows that in general, $\int F\{f(x)\}f'(x)dx = \int F(u)\, du$.

Worked Example 8

Determine $\int(1 + \cos 2\, u) \sin u\, du$.

SOLUTION

$$\int (1 + \cos^2 u) \sin u\, du$$

The multiplier sin u is proportional to the derivative of the function cos u. Substituting $z = \cos u$, $dz = -\sin u\, du$, that is, $\sin u\, du = -dz$.

$$\therefore \quad \int (1 + \cos^2 u) \sin u\, du = -\int (1 + z^2)dz$$

$$= -\left(z + \frac{1}{3}z^3\right) + C$$

$$= -\cos u - \frac{1}{3}\cos^3 u + C$$

Note that:

$$\int (f(x))^n\, f'(x)dx = \frac{(f(x))^{n+1}}{n+1} + C,$$

$n \neq -1.$

Exercise 2

Determine the following:

1. $\int (2x+3)^7 dx$
2. $\int (x^3+4)^2 x^2 dx$
3. $\int (x^3+2)^2 dx$
4. $\int \frac{x}{(x+1)^4} dx$
5. $\int \frac{3t^2}{\sqrt{t^3+4}} dt$
6. $\int \frac{\sin x}{\sqrt{\cos^3 x}} dx$
7. $\int \frac{2u+3}{\sqrt{u^2+3u+4}} du$
8. $\int 3\sqrt{27-y}\, dy$
9. $\int \sin\left(\frac{\pi}{2} - 7x\right) dx$
10. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

(ii) Integration by parts

Recall the product rule for differentiation stated in chapter 12.

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

where u and v are functions of x. Integrating both sides of the above equation, we have

$$\int \frac{d}{dx}(uv) = \int \left(v\frac{du}{dx} + u\frac{dv}{dx}\right) dx$$

$$\therefore \quad uv = \int \left(v\frac{du}{dx}\right) dx + \int \left(u\frac{dv}{dx}\right) dx$$

That is,

$$uv = \int v\, du + \int u\, dv$$

Hence,

$$\int u\, dv = uv - \int v\, du$$

where *du* and *dv* are the differentials of *u* and *v*. The equation

$$\int u\,dv = uv - \int v\,du$$

is known as the rule for **integration by parts**. We use this rule when integrating the product of two functions $\int f(x) \cdot g(x)dx$ and also for integrating inverse functions.

Worked Example 9

Determine the function

$$\int xe^x dx.$$

SOLUTION

Here we substitute

$u = x$ and $dv = e^x$

$\therefore \quad du = dx$ and $v = \int dv = \int e^x dx$

$- e^x$

$$\int u\,dv = uv - \int v\,du$$

$$\therefore \quad \int xe^x dx = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

Worked Example 10

Determine the following:

(a) $\int x\cos 2x\, dx$ (b) $\int x2ex\, dx$

SOLUTION

(a) $\int x\cos 2x\,dx$

Let $u = x$ and $dv = \cos 2x$

$\therefore \quad du = dx$ and $v = \int \cos 2x\,dx = \frac{1}{2}\sin 2x$

$$\int x\cos 2x\,dx = \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx + C$$

$$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

(b) $\int x^2 e^x\, dx$

Let $u = x^2$ and $dv = e^x$

$du = 2x\, dx$ and $v = \int dv = \int e^x = e^x$

$$\therefore \quad \int x^2 e^x\, dx = x^2 e^x - \int 2xe^x\, dx$$

$$= x^2 e^x - 2\int xe^x\, dx$$

Here, we apply the rule for integration by parts on $\int xex\ dx$ to have $\int xex\ dx = xex - ex + C.$

Finally, $\int x2ex\ dx = x2ex - 2[xex - ex] + C = x2ex - 2xex + 2ex + C$

Note: When carrying out the integration

$\int f(x) \cdot g(x)\ dx$, it is not necessary to first define u and dv and then determine $du$and v. The choice of u and dv satisfying $udv = f(x) \cdot g(x)dx$ is not unique, but the decision in making this choice normally requires that the integral $\int v\ du$ on the right-hand side of the formula $\int u\ dv = uv - \int v\ du$ is simpler than the given integral.

Exercise 3

Apply the method of integration by parts to the following integrals:

1.	$\int \lambda nx\, dx$	2.	$\int x^2 dx$
3.	$\int x^4 \log x dx$	4.	$\int x^3 e^x dx$
5.	$\int x \cos x dx$	6.	$\int e^{5x} \sin 3x dx$
7.	$\int x^2 \sin x dx$	8.	$\int x^3 e^{2x}$
9.	$\int e^{3x} \sin x dx$	10.	$\int x^2 e^{3x}$

(iii) Integration by partial fractions

A rational function may be integrated by expressing the function in the form of partial fraction. We must consider various types of rational functions and their corresponding partial fractions,

Case 1: If $f(x) = \dfrac{ax + b}{(a_1x + b_1)(a_2x + b_2)}$,

the corresponding partial fraction will be

$\frac{A_1}{ax_1 + b_1} + \frac{A_2}{a_2x + b_2}$, where $A1$ and $A2$ are constants.

Worked Example 11

Determine the function

$$\int \frac{x+2}{x^2+6x+5}\,dx.$$

SOLUTION

We first resolve $\frac{x+2}{x^2+6x+5} = \frac{x+2}{(x+1)(x+5)}$ into partial fractions before integrating

$$\frac{x+2}{(x+1)(x+5)} = \frac{A}{x+1} + \frac{B}{x+5}$$

$$\Rightarrow \quad x+2 = A(x+5) + B(x+1)$$

$$\therefore \quad x+2 = (A+B)x + (5A+B)$$

$$A + B = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$5A + B = 2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(2) − (1) gives $4A = 1$, hence, $A = \frac{1}{4}$ and $B = \frac{3}{4}$.

Finally, $\frac{x+2}{x^2+6x+5} = \frac{1}{4\,(x+1)} + \frac{3}{4\,(x+5)}$

$$\therefore \int \frac{x+2}{x^2+6x+5}\,dx = \int \left[\frac{1}{4(x+1)} + \frac{3}{4(x+5)}\right] dx$$

$$=\frac{1}{4}\int\frac{dx}{x+1}+\frac{3}{4}\int\frac{dx}{x+5}+C$$

$$=\frac{1}{4}\ln(x+1)+\frac{3}{4}\ln(x+5)+C$$

$$=\ln(x+1)^{\frac{1}{4}}+\ln(x+5)^{\frac{3}{4}}+C$$

$$=\ln\left[(x+1)^{\frac{1}{4}}(x+5)^{\frac{3}{4}}\right]+C$$

$$=\ln[(x+1)(x+5)^3]^{\frac{1}{4}}+C$$

$$=\frac{1}{4}\ln(x+1)(x+5)^3+C$$

Case 2: If $f(x)=\dfrac{ax+b}{(a_1x+b_1)^n}$

the corresponding partial fraction will be

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_1x+b_1}+\ldots+\frac{A_n}{(a_1x+b_1)^n},$$

where A_1, A_2, …, A_n, a_1, b_1 are constants.

Worked Example 12

Determine:

$$\int\frac{x+1}{(2x+1)^2}\,dx.$$

SOLUTION

We first resolve $\dfrac{x+1}{(2x+1)^2}$ into partial fraction form as follows:

$$\frac{x+1}{(2x+1)}=\frac{A_1}{2x+1}+\frac{A_2}{(2x+1)^2}$$

$$\Rightarrow\ x+1=A_1(2x+1)+A_2$$

Hence, $2A_1 = 1$ and $(A_1 + A_2) = 1$

$$\therefore \quad A_1 = \frac{1}{2} \text{ and } A_2 = \frac{1}{2}$$

$$\frac{x+1}{(2x+1)^2} = \frac{1}{2(2x+1)} + \frac{1}{2(2x+1)^2}$$

$$\therefore \quad \int \frac{x+1}{(2x+1)^2}\,dx = \frac{1}{2}\int \frac{dx}{2x+1} + \frac{1}{2}\int \frac{dx}{(2x+1)^2}$$

$$= \frac{1}{4}\int \frac{2dx}{2x+1} + \frac{1}{2}\int (2x+1)^{-2}\,dx$$

$$= \frac{1}{4}\int \frac{2}{2x+1}\,dx + \frac{1}{4}\int u^{-2}\,du$$

(Substitute $u = 2x + 1$)

$$= \frac{1}{4}\ln(2x+1) - \frac{1}{4u} + C$$

$$= \frac{1}{4}\ln(2x+1) - \frac{1}{4\,(2x+1)} + C$$

Case 3: If $f(x) = \dfrac{ax+b}{(a_1x+b_1)\,(a_2x_2^2+b_2x+c_1)}$, where $a_2 x_2^2 + b_2x + c_1$ cannot be factorised into linear factors. The corresponding partial fraction is

$$\frac{A}{a_1x+b_1} + \frac{Bx+C}{a\,x_2^2+b_2x+c_1}$$

Worked Example 13

Determine $\displaystyle\int \frac{3}{(x-2)\,(x^2+5)}\,dx.$

SOLUTION

We first resolve $\dfrac{3}{(x-2)\,(x^2+5)}$ into partial fraction form as follows:

$$\frac{3}{(x-2)(x^2+5)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+5}$$

$$3 = A(x^2+5) + (Bx+C)(x-2)$$

$$3 = (A+B)x^2 + (C-2B)x + (5A-2C)$$

$$A + B = 0 \text{ .... } (1)$$

$$C - 2B = 0 \text{ .... } (2)$$

$$5A - 2C = 3 \text{ .... } (3)$$

Solving (1), (2) and (3), we have

$A = \frac{1}{3}, B = -\frac{1}{3}$ and $C = \frac{-2}{3}$

$$\therefore \quad \frac{3}{(x-2)(x^2+5)} = \frac{1}{3(x-2)} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+5}$$

$$= \frac{1}{3(x-2)} - \frac{(x+2)}{3(x^2+5)}$$

$$= \frac{1}{3}\left(\frac{1}{x-2}\right) - \frac{1}{3}\left(\frac{x+2}{x^2+5}\right)$$

$$\therefore \quad \int \frac{3}{(x-2)(x^2+5)}\,dx$$

$$= \frac{1}{3}\int \frac{x}{x^2+5}\,dx - \frac{1}{3}\int \frac{x+2}{x^2+5}\,dx$$

$$= \frac{1}{3}\int \frac{dx}{x-2} - \frac{1}{3}\int \frac{x}{x^2+5}\,dx - \frac{1}{3}\int \frac{2}{x^2+5}\,dx$$

$$= \frac{1}{3}\int \frac{dx}{x-2} - \frac{1}{6}\int \frac{2x}{x^2+5}\,dx - \frac{2}{3}\int \frac{dx}{x^2+(\sqrt{5})^2}\,dx + C$$

$$= \frac{1}{3}\log_e(x-2) - \frac{1}{6}\log_e(x^2+5) - \frac{2}{3} \times \frac{1}{\sqrt{5}}$$

$$\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

$$= \ln\left(\frac{(x-2)^{\frac{1}{3}}}{(x^2+5)^{\frac{1}{6}}}\right) - \frac{2\sqrt{5}}{15}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

Case 4: If the given rational function is an improper rational function, we first divide the numerator by the denominator using the long division method.

Worked Example 14

Determine $\int \frac{2x^2 + 9x^2 + 14x + 6}{2x^2 + 3x + 1} dx$.

SOLUTION

Here, the degree of the numerator is greater than the degree of the denominator.
Therefore, a long division must be done, giving the quotient $(x + 3)$ and remainder $(4x + 3)$.
Hence,

$$\frac{2x^3 + 9x^2 + 14x + 6}{2x^2 + 3x + 1} = (x + 3) + \frac{4x + 3}{2x^2 + 3x + 1}$$

$$= (x + 3) + \frac{4x + 3}{(x + 1)(2x + 1)}$$

$$= (x + 3) + \frac{1}{x + 1} + \frac{2}{3x + 1}$$

$$\therefore \quad \int \frac{2x^2 + 9x^2 + 14x + 6}{2x^2 + 3x + 1} dx$$

$$= \int (x + 3) + \int \frac{dx}{x + 1} + \int \frac{2}{2x + 1} dx$$

$$= \frac{x^2}{2} + 3x + \ln(x + 1) + \ln(2x + 1) + C$$

$$= \frac{x^2}{2} + 3x + \ln(x + 1)(2x + 1) + C$$

Exercise 4

Carry out the following integrations:

1. $\int \frac{x^4}{x^3 + 8} dx$
2. $\int \frac{2x + 15}{x^2 + 5x + 6} dx$
3. $\int \frac{4x^2 + 1}{x(2x - 1)^2} dx$
4. $\int \frac{x^2 + 5}{(x + 2)(2x - 1)^2} dx$
5. $\int \frac{x^2 - 2x - 6}{x^2 - x - 2} dx$
6. $\int \frac{x^2 + 4x - 2}{(x + 1)(x^2 + 4)}$

7. $\int \frac{x^2+1}{(x+2)^2} dx$

8. $\int \frac{x\,dx}{(x^2+1)(4x^2+1)}$ Substitute $u = x^2$

9. $\int \frac{dx}{x^2-1}$

10. $\int \frac{x^3}{x-2} dx$

III. Application of Integration in Calculating the Area Under the Curve

(i) Definite integral

The process of integrating $f(x)$ to obtain $F(x)$ and then evaluating $F(b) - F(a)$ is described by the notation

$$\int_a^b f(x)dx$$

This is called **defi nite integral** of $f(x)$ over the closed interval between a and b, a and b are called the **limits of integration**.

Note

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is any indefi nite integral of $f(x)$.

Worked Example 15

Evaluate $\int_2^4 (3x^2+2)\,dx$.

SOLUTION

$$\int_2^4 (3x^2+2)\,dx = [x^3+2x]_2^4$$
$$= (4^3 + 2\,xy) - (2^3 + 2\times 2)$$
$$= (64+8) - (8\times 4)$$
$$= 72 - 12$$
$$= 60$$

Note

1. The variable x in $\int_a^b f(x)dx$ is called dummy variable.

2. The value of the definite integral $\int_a^b f(x)dx$ depends on the function f and the integration limits a and b

.

Properties of definite integral

1. $\int_a^b f(x)dx = -\int_a^b f(x)dx$

2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

3. $\frac{d}{dt}\int_a^t f(x)dx = f(t)$

Worked Example 16

Evaluate the following:

(a) $\int_2^4 \left(x^2 + \frac{1}{x^2}\right) dx$ (b) $\int_0^{2\pi} \cos^2\theta\, d\theta$

SOLUTION

(a) $\int_2^4 \left(x^2 + \frac{1}{x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{x}\right]_2^4$
$$= \left(\frac{64}{3} - \frac{1}{4}\right) - \left(\frac{8}{3} - \frac{1}{2}\right)$$
$$= \frac{227}{12}$$

(b) $\int_0^{2\pi} \cos^2\theta \, d\theta$

Since $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$\therefore \quad \int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} (1 + \cos\theta) d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\left(2\pi + \frac{1}{2}\sin 4\pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= \pi$$

(ii) Area under the curve

The area under the curve $y = f(x)$, $f(x) \geq 0$, in the interval $a \leq x \leq b$ is given by the definite integral

$$\int_a^b f(x) dx$$

Worked Example 17

Calculate the area under the curve $y = 1 + x2$ between $x = 1$ and $x = 3$.

SOLUTION

$$\text{Area} = \int_1^3 (1 + x^2) dx$$

$$= \left[x + \frac{x^3}{3}\right]_1^3$$

$$= \left(3 + \frac{27}{3}\right) - \left(1 + \frac{1}{3}\right)$$

$$= 12 - \frac{4}{3}$$

$$= \frac{32}{3} \text{ square unit}$$

Worked Example 18

Calculate the area enclosed between the curves

$y = x(2-x)$ and $y = \frac{x}{2}$.

SOLUTION

The curves intersect where

$$x(2-x) = \frac{x}{2}$$

that is, $\frac{3x}{2} - x^2 = 0$.

Hence, $x = 0$ or $x = \frac{3}{2}$.

$$\text{Required area} = \int_0^{\frac{3}{2}} \left(\frac{3x}{2} - x^2\right) dx$$

$$= \left[\frac{3x^2}{4} - \frac{x^3}{3}\right]_0^{\frac{3}{2}}$$

$$= \left(\frac{27}{16} - \frac{9}{8}\right) - 0$$

$$= \frac{9}{16} \text{ square units}$$

(iii) Approximate integration

Trapezoidal rule

The area under the curve $y = f(x)$, $a \le x \le b$, divided into n strips of equal width w is given as

$$\text{Area} = \int_a^b f(x)dx \cong \frac{w}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

The formula given above is called **trapezoidal rule**.

Simpson's rule

The area under the curve $y = f(x)$, $a \le x \le b$, divided into n strips of equal width w is given as

$$\text{Area} = \int_a^b f(x)dx \cong \frac{w}{3}[y_0 + 4(y_1 + y_2 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2}) + y_n]$$

The formula given above is called **Simpson's rule**.

Note: In the above given trapezoidal rule and Simpson's rule,

Worked Example 19

Find an approximate value for $\int_0^2 \sqrt{1+x^3}\,dx$ using the following:

(a) The trapezoidal rule with $n = 4$.

(b) Simpson's rule with $n = 4$.

(c) Simpson's rule with $n = 10$.

SOLUTION

(a) $\int_0^2 \sqrt{1+x^2}\,dx$

$$n = 4;\ w = \frac{2}{4} = \frac{(b-a)}{n} = 0.5$$

x	0	0.5	1.0	1.5	2.0
$y = \sqrt{1+x^3}$	1	1.0606	1.4142	2.0916	3
	y_0	y_1	y_2	y_3	y_4

Using trapezoidal rule

$$\int_0^2 \sqrt{1+x^3}\,dx \cong \frac{w}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$\cong \frac{0.5}{2}[1 + 3 + 2(1.0606 + 1.4142 + 2.0916)]$$

$$\cong \frac{0.5}{2}[4 + 2 \times 4.5664]$$

$$\cong \frac{0.5}{2} \times 13.1325$$

$$\cong 3.283$$

(b) $n = 4;\ w = \frac{2}{4} = \frac{(b-a)}{n} = 0.5$

x	0	0.5	1.0	1.5	2.0
$y=\sqrt{1+x^3}$	1	1.0606	1.4142	2.0916	3
	y_0	y_1	y_2	y_3	y_4

Using Simpson's rule

$$\int_0^2 \sqrt{1+x^2}\,dx \cong \frac{w}{3}[y_0+4(y_1+y_3)+2(y_2)+y_4]$$

$$\cong \frac{0.5}{3}[1+4(1.0606+2.0916)+2(1.4142)+3]$$

$$\cong \frac{0.5}{3}[1+4\times 3.1522+2\times 1.4142+3]$$

$$\cong \frac{0.5}{3}[4+12.6080+2.8284]$$

$$\cong 3.240$$

(c) $n=10,\ w=\frac{2}{10}=\frac{(b-a)}{n}=0.2$

x	$y=\sqrt{1+x^3}$	
0	1	y_0
0.2	1.0039	y_1
0.4	1.0315	y_2
0.6	1.1027	y_3
0.8	1.2296	y_4
1.0	1.4142	y_5
1.2	1.6517	y_6
1.4	1.9349	y_7
1.6	2.2574	y_8
1.8	2.6138	y_9
2.0	3	y_{10}

$$y_1 + y_3 + y_5 + y_7 + y_9 = 8.0695$$

$$y_2 + y_4 + y_6 + y_8 = 6.1702$$

Using Simpson's rule

$$\int_0^2 \sqrt{1+x^2}\, dx \cong \frac{w}{3}[y_0 + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8) + y_{10}]$$

$$\cong \frac{0.2}{3}(1 + 4 \times 8.095 + 2 \times 6.1702 + 3)$$

$$\cong 3.241$$

Exercise 5

Evaluate the following integrals:

1. $\int_0^1 \frac{2x^3 + 9x^2 + 14x + 16}{2x^2 + 3x + 1} dx$
2. $\int_0^2 (5x^2 + 3x + 2)\, dx$
3. $\int_1^2 \frac{dx}{x}$
4. $\int_0^1 dx$
5. $\int_{-2}^1 (3t - 2)\, dt$
6. $\int_0^1 \frac{x^2 + 1}{x^2} dx$
7. $\int_2^3 \left(10t - \frac{1}{2}\right) dt$
8. $\int_1^4 \frac{dx}{\sqrt{x}}$
9. $\int_1^2 \frac{y^2 - 1}{y^2} dy$
10. $\int_0^2 (t - 1) dt$

IV. Application of Integration in Real-Life Situations

In Chapter 12, we learnt that in a straight line, the distance travelled *s,* the velocity *v* and the acceleration a are related by

$v = \frac{ds}{dt}, a = \frac{d^2s}{dt^2}$ where *t* is time. It follows that, provided *a* is known as a function of *t, v* may be obtained by the process of integration from the differential.

$dv = \left(\frac{dv}{dt}\right)dt = adt$ that is,

$$v = \int adt \text{ .... } (1)$$

Similarly,

$$s = \int vdt \text{ .... } (2)$$

Worked Example 20

A stone is dropped from the highest point of a river bridge and hits the river surface 2.5 seconds later. Neglecting air resistance, and the given acceleration due to gravity being 9.8 m/s2, calculate the height of the bridge.

SOLUTION

For this constant acceleration situation, the equation of motion is

$\frac{dv}{dt} = a = 9.81.$

Therefore,

$$v = \int 9.81dt + C_1$$

$$= 9.81\ t + C_1$$

But the stone starts from rest. It follows that $v = 0$ when, $t = 0$

$\therefore\ C1 = O$

Hence, $v = 9.81\ t$

$$s = \int vdt$$

$$\therefore \quad s = \int 9.81t\,dt + C_2$$

$$= \frac{9.81t^2}{2} + C_2$$

$$= 4.905\,t^2 + C_2$$
$s = 0$, when $t = 0$
$$\Rightarrow \quad C_2 = 0$$
$$s = 4.905\,t^2$$
The distance covered by the stone is the height of the bridge. Hence $s = h = 4.905\ t2$. When $t = 2.5$ sec, $h = 4.905 \times 2.52 = 30.66$ m.

Worked Example 21

A particle, having an initial velocity *u* m/sec travels in a straight line with a constant acceleration *a* m/sec2. Derive expressions for its velocity *v* (m/sec) and displacement *s* (m) from its initial position in terms of time *t* (sec).

SOLUTION

Here, the equation of motion is
$$\frac{d^2s}{dt^2} = a$$
$$\frac{ds}{dt} = \int a\,dt + C_1$$
$$= at + C_1$$
$$s = \int (at + C_1)dt + C_2$$
$$= \frac{at^2}{2} + C_1 t + C_2$$
The initial conditions are $s = 0$ when $t = 0$ and $\frac{ds}{dt} - u$ when $t = 0$. It follows that $C1 = u$ and $C2 = 0$. Therefore, the velocity $v = u + at$ and the displacement's $= ut + \frac{1}{2}at^2.$

Exercise 6

1.The velocity of a particle is given : by $\frac{ds}{dt} = 3\sin t.$ Satisfying the condition $s = 1$, when $t = 0$, express *s* in terms of *t.*
2. The acceleration of a moving vehicle is 2 m/sec. Given that the initial values of velocity *v* and position *s* are 0 m/sec and 10 m, respectively, express *s* in terms of *t.*
3. A particle moves in a straight line such that its velocity is (2*t*2 + 5) m/sec after *t* seconds. What is the distance travelled by the particle in the first 4 seconds?
4. A particle moves in a straight line. Its velocity *v* is (*t*3 − 2*t*2 + 2) m/sec after *t* seconds. What is the distance travelled by the particle after *t* seconds given that $s = 20$ when $t = 65$ seconds.

SUMMARY

In this chapter, we have learnt the following:

- $\int \frac{dy}{dx}\, dx = y$
- $\int f'(x)dx = f(x) + c$
- $\int ax^n dx = \frac{a}{n+1} x^{n+1} + C, n \neq 1$
- $\int \frac{dx}{x} = \log_e x + C = \ln x + C$
- $\int \sin x\, dx = -\cos x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int \sec x \tan x\, dx = \sec x + C$
- $\int \text{cosec}\, x = -\cot x + C$
- $\int udv = uv - \int vdu$
- $\int \cos x \cot x = -\text{cosec}\, x + C$
- $\int e^{ax} dx = \frac{1}{2} e^{ax} + C$
- $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$
- Trapezoidal rule:

$$\int_a^b f(x) \cong \frac{W}{2}[y_0 + y_n + 2\,(y_1 + y_2 + \cdots + y_{n-1})]$$

- Simpson's rule:

$$\int_a^b f(x)dx \cong \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-1})$$
$$+2(y_2 + y_4 + \cdots + y_{n-2}) + y_n]$$

where $w = \frac{b-a}{n}$.

GRADUATED EXERCISES

Integrate the following:

1. $\int (2x^2 + x + 1)dx$

2. $\int (x+2)(x-4)dx$

3. $\int \left(\frac{y^2 + y^3}{y}\right)dy$

4. $\int (6x+5)^2$

5. $\int x \sin x \, dx$

6. $\int \left(\frac{z + z^2}{z^2}\right)dz$

7. Find the equation of the curve which passes through the point (1, 2) and whose gradient at any point is 6*x*2 − 5*x*.

8. Find the area bounded by the curve *y* = *x*2(2 − *x*) in the *x*-axis and the ordinates at *x* = 1 and *x* = 2.

9. Find the equation of the curve that passes through the point (2, 3) with gradient 3*x*2 + 4*x* + 3.

10. Evaluate $\int_{-1} f(x)dx$ given that $f(x) =$ $x(x^2 + 3x + 1)$.

11. Evaluate $\int_0^{\pi} (\sec^2 x - \tan^2 x)\, dx$.

12. Integrate $\int \frac{2}{y^2} + \frac{1}{y^3} dy$.

13. Integrate $3\sqrt{2y}$ with respect to y.

Determine the following integrals:

14. $\int \frac{2x^2 - 3x + 1}{(x+1)(x^2+5x+1)} dx$

15. $\int \frac{1-x}{(x^2-3x+2)(x^2+4)} dx$

16. $\int \frac{1-x}{(2x^2-3)(x^2-x-1)} dx$

17. $\int_0^2 t\sqrt{5+t^2}\, dt$

18. The function $F(t)$ is defined by

$$F(t) = \int_1^t (1+u^2)^3\, du$$

Verify that $F'(t) = (1 + t2)2$

19. Calculate the area enclosed between x-axis and the curve $y = x(x-1)(x-2)$ in the interval $0 \le x \le 2$.

20. Evaluate $\int_0^3 \frac{dx}{(9+x^2)^2}$ using trapezoidal rule with $n = 10$.