Chapter 1

Chapter 1

Surds

OBJECTIVES

At the end of this chapter, students should be able to:

1. differentiate between rational and irrational numbers.
2. state the rules of addition and subtraction and apply them in the simplification of surds.
3. state the rules of multiplication and apply them in the simplification of surds.
4. evaluate conjugate binomial surds using the difference of two squares.
5. apply the concept of surds to problems involving trigonometric ratios of angles $30°$, $60°$ and $45°$.

I. Rational and Irrational Numbers

The ordinary numbers of arithmetic whole numbers, fractions, mixed numbers and decimals, together with their negative images are called **rational numbers**.

Examples include $4, \frac{2}{3}, 0.5, 2\frac{3}{4}, \sqrt{4}, -4$ and $-\sqrt{4}$.

Rational numbers can be expressed mathematically as a fraction $\frac{a}{b}$, where *a* and *b* are integers ($b \neq 0$). An integer itself can be written as a fraction $\frac{a}{b}$ (where $b = 1$); and from arithmetic, we know that a decimal can be written as a fraction. When *a* and *b* are positive, that is, when they are natural numbers, we can always name their ratio. Hence the term, *rational* numbers. Numbers which cannot be expressed as a fraction are called **irrational** or **nonrational numbers**. Examples of irrational numbers are π, $\sqrt{2}, \sqrt{5}, \sqrt{6}$, etc. Irrational numbers of the kind $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{28}$, etc., are called **surds**.

Surds are mathematical expressions containing square roots. However, it must be emphasised that the square roots are 'irrational'. This means, they do not result in a whole number, a terminating decimal or a recurring decimal. Two or more surds are said to be like surds, if the numbers under the square root sign are the same. For example, $\sqrt{2}$, $5\sqrt{2}$, $\frac{\sqrt{2}}{3}$ are like-surds.

II. Simplification of Surds

If the rational number under the square root sign contains a factor which is a square of a number, the surd can be reduced to a simpler or basic form.

Worked Example 1

Reduce each of the following to a simpler form:

(a) $\sqrt{27}$ (b) $\sqrt{98}$ (c) $\sqrt{48}$

SOLUTION

(a) $\sqrt{27} = \sqrt{9 \times 3}$

$= \sqrt{9} \times \sqrt{3}$

$= 3\sqrt{3}$

(b) $\sqrt{98} = \sqrt{49 \times 2}$

$= \sqrt{49} \times \sqrt{2}$

$= 7\sqrt{2}$

(c) $\sqrt{48} = \sqrt{16 \times 3}$

$= \sqrt{16} \times \sqrt{3}$

$= 4\sqrt{3}$

The process of reducing a surd to its simpler or basic for is reversible.

Worked Example 2

Write each of the following in their single surd form:

(a) $3\sqrt{2}$ (b) $5\sqrt{3}$ (c) $6\sqrt{2}$

SOLUTION

(a) $3\sqrt{2} = 3 \times \sqrt{2}$
$= \sqrt{9} \times \sqrt{2}$
$= \sqrt{9 \times 2}$
$= \sqrt{18}$

(b) $5\sqrt{3} = 5 \times \sqrt{3}$
$= \sqrt{25} \times \sqrt{3}$
$= \sqrt{25 \times 3}$
$= \sqrt{75}$

(c) $6\sqrt{2} = 6 \times \sqrt{2}$
$= \sqrt{36} \times \sqrt{2}$
$= \sqrt{36 \times 2}$
$= \sqrt{72}$

Exercise 1

Simplify the following:

1. $\sqrt{18}$
2. $\sqrt{147}$
3. $\sqrt{200}$
4. $\sqrt{72}$
5. $\sqrt{75}$
6. $\sqrt{99}$
7. $\sqrt{54}$
8. $\sqrt{63}$
9. $\sqrt{150}$
10. $\sqrt{300}$

Express each of the following as a single surd:

11. $4\sqrt{3}$
12. $2\sqrt{7}$
13. $10\sqrt{3}$
14. $7\sqrt{2}$
15. $3\sqrt{3}$
16. $2\sqrt{11}$
17. $2\sqrt{2}$
18. $7\sqrt{3}$
19. $2\sqrt{5}$
20. $5\sqrt{2}$

(i) Addition and Subtraction of Surds

Like-surds can be added or subtracted from one another. Before adding or subtracting, the surds should first be reduced to their simplest form, if possible.

Worked Example 3

Simplify the following:

(a) $4\sqrt{7} - 2\sqrt{7}$

(b) $5\sqrt{2} + 8\sqrt{2}$

(c) $\sqrt{8} + \sqrt{18}$

(d) $\sqrt{48} - \sqrt{27}$

(e) $\sqrt{18} + \sqrt{72} - \sqrt{98}$

SOLUTION

(a) $4\sqrt{7} - 2\sqrt{7} = (4-2)\sqrt{7} = 2\sqrt{7}$

(b) $5\sqrt{2} + 8\sqrt{2} = (5+8)\sqrt{2} = 13\sqrt{2}$

(c) $\sqrt{8} + \sqrt{18} = \sqrt{4 \times 2} + \sqrt{9 \times 2}$

$= \sqrt{4} \times \sqrt{2} + \sqrt{9} \times \sqrt{2}$

$= 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$

(d) $\sqrt{48} - \sqrt{27} = \sqrt{16 \times 3} - \sqrt{9 \times 3}$

$= \sqrt{16} \times \sqrt{3} - \sqrt{9} \times \sqrt{3}$

$= 4\sqrt{3} - 3\sqrt{3} = \sqrt{3}$

(e) $\sqrt{18} + \sqrt{72} - \sqrt{98}$

$= \sqrt{9 \times 2} + \sqrt{36 \times 2} - \sqrt{49 \times 2}$

$= \sqrt{9} \times \sqrt{2} + \sqrt{36} \times \sqrt{2} - \sqrt{49} \times \sqrt{2}$

$= 3\sqrt{2} + 6\sqrt{2} - 7\sqrt{2}$

$= 9\sqrt{2} - 7\sqrt{2}$

$= 2\sqrt{2}$

Note: $5\sqrt{2} + 3\sqrt{3}$ cannot be added because the surds are diff erent (i.e., they are not like-surds).

(ii) Multiplication of Surds

When multiplying two or more surds, they must first be simplified. Then, surds must be multiplied with surds, and whole numbers with whole numbers.

Note

1. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
2. $m\sqrt{a} \times n\sqrt{b} = mn\sqrt{ab}$
3. $\sqrt{a} \times \sqrt{a} = \left(\sqrt{a^2}\right) = a$

Worked Example 4

Simplify the following:

(a) $\sqrt{3} \times \sqrt{2}$

(b) $\sqrt{5} \times \sqrt{15}$

(c) $2\sqrt{50} \times 3\sqrt{32} \times 4\sqrt{18}$

(d) $(4\sqrt{3})^2$

SOLUTION

(a) $\sqrt{3} \times \sqrt{2} = \sqrt{3 \times 2} = \sqrt{6}$

(b) $\sqrt{5} \times \sqrt{15} = \sqrt{5 \times 15} = \sqrt{5 \times 5 \times 3}$

$= \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3}$

$= 5\sqrt{3}$

(c) $2\sqrt{50} \times 3\sqrt{32} \times 4\sqrt{18}$

$= 2\sqrt{25 \times 2} \times 3\sqrt{16 \times 2} \times 4\sqrt{9 \times 2}$

$= 2 \times \sqrt{25} \times \sqrt{2} \times 3 \times \sqrt{16} \times \sqrt{2} \times 4 \times \sqrt{9} \times \sqrt{2}$

$= 2 \times 5\sqrt{2} \times 3 \times 4\sqrt{2} \times 4 \times 3\sqrt{2}$

$= 10\sqrt{2} \times 12\sqrt{2} \times 12\sqrt{2}$

$= 1\,440\sqrt{2} \times \sqrt{2} \times \sqrt{2} = 1440 \times 2 \times \sqrt{2}$

$= 2\,880\sqrt{2}$

(d) $(4\sqrt{3})^2 = 4\sqrt{3} \times 4\sqrt{3}$

$= 16 \times 3 = 48$

Alternatively, $(4\sqrt{3})^2 = 4^2 \times (\sqrt{3})^2$

$= 16 \times 3 = 48$

When multiplying surds involving brackets, the brackets are expanded as usual.

Note

1. $(\sqrt{a} + \sqrt{b})(\sqrt{c} + \sqrt{d}) = \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd}$
2. $(a + \sqrt{b})(c + \sqrt{d}) = ac + a\sqrt{d} + c\sqrt{b} + \sqrt{bd}$

Worked Example 5

Simplify the following:

(a) $(1 + \sqrt{3})(2 - \sqrt{8})$

(b) $(3\sqrt{3} + 2)(\sqrt{3} + 3)$

(c) $(3\sqrt{5} - 2\sqrt{2})(\sqrt{2} - 2\sqrt{5})$

(d) $(\sqrt{3} + \sqrt{7})^2$

SOLUTION

(a) $(1 + \sqrt{3})(2 - \sqrt{8})$

$= 2 - \sqrt{8} + 2\sqrt{3} - (\sqrt{3} \times \sqrt{8})$

$= 2 - \sqrt{8} + 2\sqrt{3} - \sqrt{3 \times 8}$

$= 2 - \sqrt{4 \times 2} + 2\sqrt{3} - \sqrt{24}$

$= 2 - 2\sqrt{2} + 2\sqrt{3} - \sqrt{4 \times 6}$

$= 2 - 2\sqrt{2} + 2\sqrt{3} - 2\sqrt{6}$

(b) $(3\sqrt{3}+2)(\sqrt{3}+3)$

$= 3\sqrt{3}(\sqrt{3}) + (3\sqrt{3})3 + 2\sqrt{3} + 2(3)$

$= 9 + 9\sqrt{3} + 2\sqrt{3} + 6$

$= 15 + 11\sqrt{3}$

(c) $(3\sqrt{5}-2\sqrt{2})(\sqrt{2}-2\sqrt{5})$

$= 3\sqrt{5}(\sqrt{2}) - 3\sqrt{5}(2\sqrt{5}) - 2\sqrt{2}(\sqrt{2})$
$+ 2\sqrt{2}(2\sqrt{5})$

$= 3\sqrt{10} - 6\times5 - 2\times2 + 4\sqrt{10}$

$= 7\sqrt{10} - 34$

(d) $(\sqrt{3}+\sqrt{7})^2 = (\sqrt{3}+\sqrt{7})(\sqrt{3}+\sqrt{7})$

$= (\sqrt{3}\times\sqrt{3}) + (\sqrt{3}\times\sqrt{7})$
$+ (\sqrt{7}\times\sqrt{3}) + (\sqrt{7}\times\sqrt{7})$

$= 3 + 2\sqrt{3\times7} + 7$

$= 10 + 2\sqrt{21}$

(iii) Division of Surds

It is untidy to have a surd as the denominator.
When we divide a number by a surd, this can be 'tided up' by multiplying the numerator and the denominator by the surd. This is known as **rationalization** of the denominator, you are changing the denominator from an irrational to a rational number. Since the surds are irrational numbers.

Note

1. $\frac{a}{\sqrt{b}} = \frac{a\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}} = \frac{a\sqrt{b}}{b}$

2. $\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}} = \frac{\sqrt{ab}}{b}$

Worked Example 6

Simplify the following:

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{9}{\sqrt{3}}$

(c) $\frac{7}{\sqrt{50}}$ (d) $\frac{4}{\sqrt{18}}$

SOLUTION

(a) $\frac{1}{\sqrt{2}}=\frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{2}}{2}=\frac{1}{2}\sqrt{2}$

(b) $\frac{9}{\sqrt{3}}=\frac{9\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{9\sqrt{3}}{3}=3\sqrt{3}$

(c) $\frac{7}{\sqrt{50}}=\frac{7}{5\sqrt{2}}=\frac{7\times\sqrt{2}}{5\sqrt{2}\times\sqrt{2}}=\frac{7\sqrt{2}}{5\times 2}=\frac{7\sqrt{2}}{10}$

(d) $\frac{4}{\sqrt{18}}=\frac{4\times\sqrt{2}}{3\sqrt{2}\times\sqrt{2}}=\frac{4\sqrt{2}}{3\times 2}=\frac{4\sqrt{2}}{6}=\frac{2\sqrt{2}}{3}$

Note: Surds should be simplified first, if possible, before rationalising.

Worked Example 7

Simplify the following:

(a) $\frac{\sqrt{32}}{\sqrt{2}}$ (b) $\frac{\sqrt{7}}{\sqrt{3}}$

(c) $\sqrt{\frac{16}{5}}$ (d) $\frac{5\sqrt{5}\times 2\sqrt{3}}{\sqrt{45}\times\sqrt{12}}$

SOLUTION

(a) $\frac{\sqrt{32}}{\sqrt{2}}=\sqrt{\frac{32}{2}}=\sqrt{16}=4$

(b) $\frac{\sqrt{7}}{\sqrt{3}}=\frac{\sqrt{7}\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{\sqrt{7\times 3}}{3}=\frac{1}{3}\sqrt{21}$

(c) $\sqrt{\frac{16}{5}}=\frac{\sqrt{16}}{\sqrt{5}}=\frac{4}{\sqrt{5}}=\frac{4\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{4\sqrt{5}}{5}$

(d) $\frac{5\sqrt{5}\times 2\sqrt{3}}{\sqrt{45}\times\sqrt{12}}=\frac{5\sqrt{5}\times 2\sqrt{3}}{3\sqrt{5}\times 2\sqrt{3}}=\frac{5}{3}$

Exercise 2

Simplify the following:

1. $5\sqrt{3} + 7\sqrt{3}$
2. $8\sqrt{5} - 3\sqrt{5}$
3. $\sqrt{50} + \sqrt{72}$
4. $\sqrt{98} - \sqrt{18}$
5. $4\sqrt{3} + 6\sqrt{27}$
6. $45\sqrt{6} - 3\sqrt{216}$
7. $\sqrt{32} + \sqrt{50} + \sqrt{128}$
8. $3\sqrt{50} + 4\sqrt{18}$
9. $\sqrt{108} - \sqrt{75}$
10. $3\sqrt{50} - 4\sqrt{8} + 7\sqrt{18}$
11. Show that $8\sqrt{45} - 8\sqrt{20} + \sqrt{245} - 3\sqrt{125} = 0$.

Simplify the following:

12. $\sqrt{27} \times \sqrt{50}$
13. $\sqrt{45} \times 3\sqrt{60} \times \sqrt{12}$
14. $(2\sqrt{5})^2$
15. $\sqrt{3} \times \sqrt{6}$
16. $\sqrt{2} \times \sqrt{3} \times \sqrt{5} \times \sqrt{12} \times \sqrt{45} \times \sqrt{50}$
17. $\sqrt{5} \times \sqrt{24} \times \sqrt{30}$
18. $\sqrt{8} \times \sqrt{32}$
19. $2\sqrt{3} \times 3\sqrt{2}$
20. $2\sqrt{6} \times \sqrt{5} \times \sqrt{30}$
21. $(2\sqrt{2})^3 + (\sqrt{3})^3$

22. $(1+\sqrt{2})(1-\sqrt{2})$

23. $(\sqrt{2}-\sqrt{5})(\sqrt{2}+\sqrt{5})$

24. $(2\sqrt{3}+2)(2\sqrt{3}-2)$

25. $(4\sqrt{3}+3\sqrt{2})(4\sqrt{3}-3\sqrt{2})$

26. $(7-\sqrt{10})(7+\sqrt{10})$

27. $(2\sqrt{6}+1)(2\sqrt{6}-1)$

Simplify the following:

28. $\frac{6}{\sqrt{3}}$

29. $\frac{7}{\sqrt{18}}$

30. $\frac{\sqrt{5}}{\sqrt{2}}$

31. $\frac{5}{\sqrt{5}}$

32. $\frac{5}{\sqrt{15}}$

33. $\sqrt{\frac{16}{7}}$

34. $\sqrt{\frac{18}{2}}$

35. $\sqrt{\frac{5}{2}}$

36. $\frac{\sqrt{18}\times\sqrt{20}\times\sqrt{24}}{\sqrt{8}\times\sqrt{30}}$

37. $\frac{\sqrt{3}\times\sqrt{18}\times\sqrt{39}}{\sqrt{24}\times\sqrt{26}}$

III. Conjugate of a Binomial Surd

Expression of the form $\sqrt{a}+\sqrt{b}$ or $a+\sqrt{b}$ is called **binomial surds**. Examples include $2+\sqrt{2}$, $\sqrt{3}+2$ and $3\sqrt{2}+3$. The difference of two squares is used to rationalize a binomial surd, for example, to rationalise $\sqrt{x}+y$ we multiply it by $\sqrt{x}-y$ to get $(\sqrt{x})^2-y^2$, which is a rational number.
Thus, $\sqrt{x}+y$ and $\sqrt{x}-y$ are said to be conjugates of one another. Similarly, $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$ are also the conjugates of each other.

Worked Example 8

Write the conjugate of the following:

(a) $1+\sqrt{2}$ (b) $\sqrt{2}-\sqrt{3}$ (c) $3+\sqrt{7}$

SOLUTION

(a) $1-\sqrt{2}$ (b) $\sqrt{2}+\sqrt{3}$ (c) $3-\sqrt{7}$

Worked Example 9

Evaluate the following:

(a) $(2\sqrt{2}-4)(2\sqrt{2}+4)$

(b) $(6+3\sqrt{3})(6-3\sqrt{3})$

SOLUTION

(a) $(2\sqrt{2}-4)(2\sqrt{2}+4)$

$=(2\sqrt{2})^2-4^2$ (difference of two squares)

$=4\times 2-16$

$=8-16$

$=-8$

(b) $(6+3\sqrt{3})(6-3\sqrt{3})$

$=6^2-(3\sqrt{3})^2$ (difference of two squares)

$=36-9\times 3$

$=36-27$

$=9$

Note: When the denominator of a fraction is a binomial surd, use the conjugate of the denominator to rationalize it.

Worked Example 10

Simplify the following:

(a) $\frac{5}{3\sqrt{7}+4}$

(b) $\frac{8}{2\sqrt{5}-1}$

(c) $\frac{2\sqrt{2}+2}{2\sqrt{2}-2}$

SOLUTION

(a) $\dfrac{5}{3\sqrt{7}+4} = \dfrac{5(3\sqrt{7}-4)}{(3\sqrt{7}+4)(3\sqrt{7}-4)}$

$$= \frac{15\sqrt{7}-20}{(3\sqrt{7})^2-4^2}$$

$$= \frac{15\sqrt{7}-20}{9\times 7-16}$$

$$= \frac{15\sqrt{7}-20}{63-16}$$

$$= \frac{15\sqrt{7}-20}{47}$$

(b) $\dfrac{8}{2\sqrt{5}-1} = \dfrac{8(2\sqrt{5}+1)}{(2\sqrt{5}-1)(2\sqrt{5}+1)}$

$$= \frac{16\sqrt{5}+8}{(2\sqrt{5})^2-1^2}$$

$$= \frac{16\sqrt{5}+8}{4\times 5-1}$$

$$= \frac{16\sqrt{5}+8}{20-1}$$

$$= \frac{16\sqrt{5}+8}{19}$$

(c) $\dfrac{2\sqrt{2}+2}{2\sqrt{2}-2} = \dfrac{(2\sqrt{2}+2)(2\sqrt{2}+2)}{(2\sqrt{2}-2)(2\sqrt{2}+2)}$

$$= \frac{8+4\sqrt{2}+4\sqrt{2}+4}{(2\sqrt{2})^2-2^2}$$

$$= \frac{12+8\sqrt{2}}{8-4}$$

$$= \frac{12+8\sqrt{2}}{4}$$

$$= \frac{4(3+2\sqrt{2})}{4}$$

$$= 3+2\sqrt{2}$$

Exercise 3

Evaluate the following:

1. $(2+\sqrt{2})(2-\sqrt{2})$
2. $(\sqrt{3}+1)(\sqrt{3}-1)$
3. $(1+\sqrt{3})(1-\sqrt{3})$
4. $(2\sqrt{7}+1)(2\sqrt{7}-1)$
5. $(3+\sqrt{5})(3-\sqrt{5})$
6. $(4+2\sqrt{5})(4-2\sqrt{5})$
7. $(\sqrt{7}+10)(\sqrt{7}-10)$
8. $(15+2\sqrt{6})(15-2\sqrt{6})$

Simplify the following:

9. $\dfrac{1}{3\sqrt{5}+2}$
10. $\dfrac{\sqrt{5}}{7\sqrt{5}-1}$
11. $\dfrac{2}{1+2\sqrt{2}}$
12. $\dfrac{8}{3\sqrt{3}-2}$
13. $\dfrac{11\sqrt{7}}{\sqrt{7}+5}$
14. $\dfrac{3\sqrt{3}+3}{3\sqrt{3}-3}$
15. $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$
16. $\dfrac{2\sqrt{2}-1}{2\sqrt{2}+1}$
17. $\dfrac{7+\sqrt{7}}{7-\sqrt{7}}$
18. $\dfrac{5\sqrt{3}+2}{5\sqrt{3}-2}$
19. $\dfrac{\sqrt{6}+\sqrt{3}}{2\sqrt{3}-1}$
20. $\dfrac{2\sqrt{6}-\sqrt{3}}{2\sqrt{3}-1}$
21. $\dfrac{12}{\sqrt{24}-\sqrt{6}}$
22. $\dfrac{4}{\sqrt{18}-\sqrt{2}}$
23. $\dfrac{1}{\sqrt{5}+2}$
24. $\dfrac{2}{3-\sqrt{5}}$
25. $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$
26. $\dfrac{4+3\sqrt{5}}{4-3\sqrt{5}}$

IV. Application of Surds in Solving Triangles Involving Trigonometric Ratios of Special Angles 30°, 60° and 45°

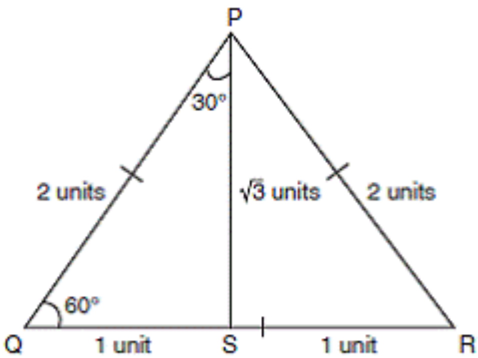


Figure 1.1

Consider an equilateral triangle PQR of side 2 units as in Figure 1.1. The altitude PS is $\sqrt{3}$ units long (using Pythagoras' theorem). Thus,

$\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$ and $\tan 30° = \frac{1}{\sqrt{3}}$,

and

$\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$ and $\tan 60° = \sqrt{3}$

Consider an isosceles right-angled triangle ABC with two sides of 1 unit as in Figure 1.2.

Its hypotenuse $\overline{AC}$ is $\sqrt{2}$ units long (using Pythagoras' theorem).

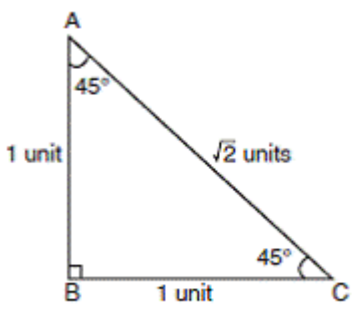


Figure 1.2

Thus,

$\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$ and $\tan 45° = 1$

Worked Example 11

In Figure 1.3, if $|PQ| = 7$ cm, calculate x and y. (Leave the answers in the surd form).

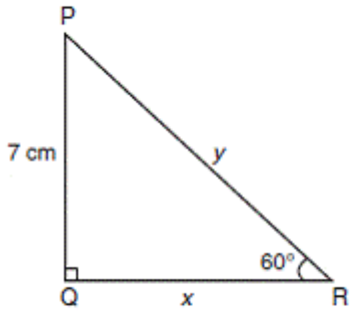


Figure 1.3

SOLUTION

In ΔPQR

$\tan 60° = \frac{7}{x}$

$\Rightarrow \sqrt{3} = \frac{7}{x}$

$\Rightarrow \quad x = \frac{7}{\sqrt{3}}$ cm

$\therefore \quad x = \frac{7\sqrt{3}}{3}$ cm

Also,

$\sin 60° = \frac{7}{y}$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{7}{y}$

$\Rightarrow \quad y = \frac{14}{\sqrt{3}}$ cm

$= \frac{14\sqrt{3}}{3}$ cm

Worked Example 12

From the top of the tower, the angle of depression of a boat is 60. If the tower is 10 m high, how far is the boat from the foot of the tower? Leave your answer in the surd form with a rational denominator.

SOLUTION

First, draw a sketch. In Figure 1.4, $\overline{AB}$ represents the tower; C is the position of the boats.

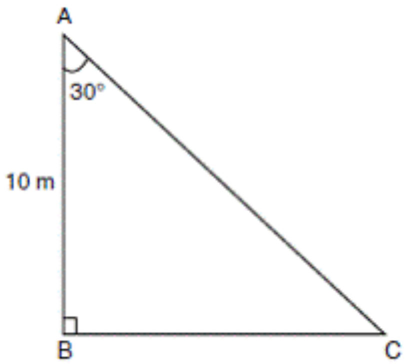


Figure 1.4

In ΔABC,

$$\tan 30^\circ = \frac{|BC|}{10}$$

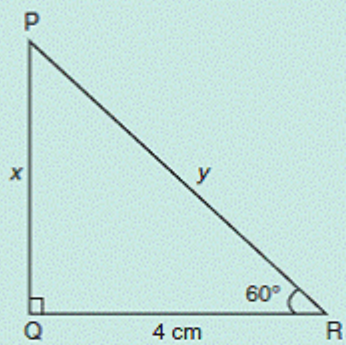
$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{|BC|}{10}$$

$$\Rightarrow |BC| = \frac{10}{\sqrt{3}} \text{ cm}$$

$$= \frac{10\sqrt{3}}{3} \text{ cm}$$

Exercise 4

In each part of Figure 1.5, find the lengths marked x and y. (Leave your answers in the surd form.)

1. 

2.

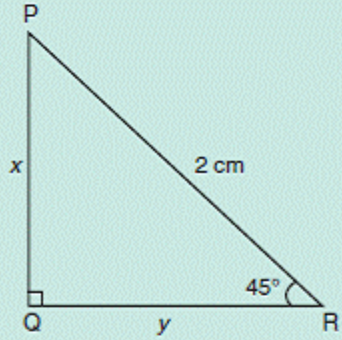
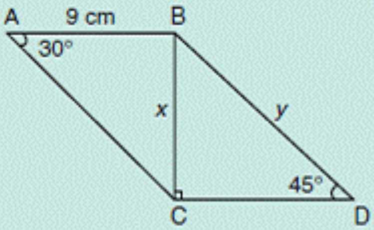
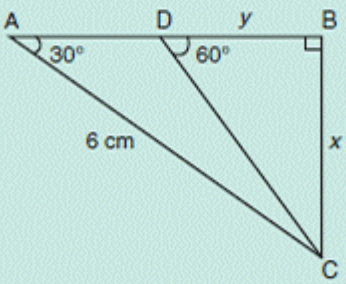


Figure 1.5

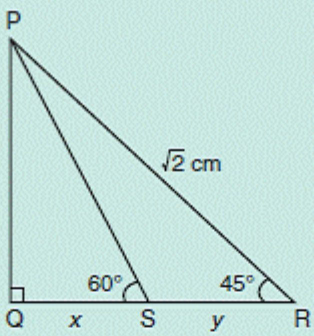
3.



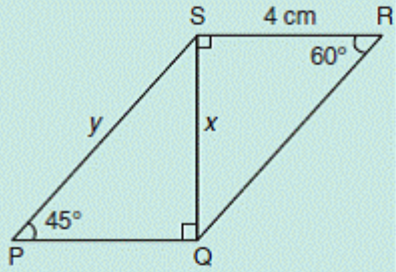
4



5.



6



7. ABC is an isosceles triangle with $|AB| = |AC| = 6$ cm and $\angle BAC = 120°$. Calculate the length of $|BC|$. **(WAEC)**

8. From the top of a tower 60 cm high, two boats are seen in a direction due south. The angles of depression of the boats from the top of the tower are 45° and 30°. Find the distance between the boats.

9. From the top of a tower, the angle of depression of a boat is 30°. If the tower is 20 m high, how far is the boat from the foot of the tower? **(WAEC)**

10. Two huts and a radio mast are on level ground such that one hut is due east of the mast and the other is due west of it. From the top of the mast, the angles of depressions of the huts are 60° and 45°, respectively. If the mast is 150m high, find the distance between the huts. **(WAEC)**

11. The angle of elevation of A from B is 60°. If $|AB| = 20$ m, how high is A above the level of B? **(WAEC)**

12. From the top of a tower, the angle of depression of a boat is 60°. If the tower is 10 m high, how far is the boat from the foot of the tower? (WAEC)

13. The angle of elevation of X from Y is 30°. If $|XY| = 40$ cm, how high is X above the level of Y? (WAEC)

14. The top of a building 24 m high is observed from the top and from the bottom of a vertical tree. The angles of elevation are found to be 45° and 60°. Find the height of the tree.

V. Evaluation of Expressions Involving Surds

While dealing with expressions with surds as denominators, for example $\frac{1}{\sqrt{13}}$, it is best to rationalise the denominator of the expression to a whole number before evaluating.

Worked Example 13

Given that $\sqrt{2} = 1.414$, evaluate $\frac{5}{\sqrt{2}}$ to 3 d.p.

SOLUTION

$$\frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2} = \frac{5 \times 1.414}{2} = 3.535$$

Exercise 5

1. Given $\sqrt{3} = 1.732$, evaluate $\frac{3}{\sqrt{3}}$ to 2 d.p.
2. $\sqrt{1.225} = 1.107$, $\sqrt{12.25} = 3.5$ and $\sqrt{100} = 10$. Evaluate $\sqrt{1225}$. (WAEC)

Evaluate the following:

3. $\frac{8}{\sqrt{50} - \sqrt{2}}$ (WAEC)
4. $(\sqrt{19} + \sqrt{11})(\sqrt{19} - \sqrt{11})$ (WAEC)
5. $(5\sqrt{2.5} - \sqrt{10})\sqrt{0.4}$ (WAEC)
6. $(\sqrt{0.6} - \sqrt{15})^2$ (WAEC)

Given that $\sqrt{5} = 2.236$ and $\sqrt{7} = 2.646$, evaluate the following to 2 d.p.

7. $\frac{2}{\sqrt{5}}$
8. $\frac{6}{\sqrt{7}}$
9. $\frac{1}{\sqrt{7}}$
10. $\frac{2}{2\sqrt{5}}$
11. $\frac{10}{2\sqrt{7}}$
12. $\frac{4}{\sqrt{125}}$
13. $\frac{1}{\sqrt{125}}$
14. $\frac{2}{\sqrt{147}}$
15. $\frac{1}{\sqrt{75}}$

SUMMARY

In this chapter, we have learnt the following:

❖ The ordinary numbers of arithmetic the whole numbers, fractions, mixed numbers and decimals, together with\ their negative images are called rational numbers. Examples include $2, \frac{1}{2}, 0.6, \sqrt{9}$ and $1\frac{1}{3}$.

❖ Numbers which cannot be expressed as a fraction $\frac{a}{b}$ are called irrational or non-rational numbers, for example, $\pi, \sqrt{2}, \sqrt{5}$, etc.

❖ Surds are mathematical expressions containing square roots. However, it must be emphasized that the square roots are irrational, for example, $\sqrt{2}, \sqrt{7}, \sqrt{8}, \sqrt{3}$, etc.

❖ Two or more surds are said to be like surds if the numbers under the square root sign are the same. For example $\sqrt{2}, 5\sqrt{2}, \frac{\sqrt{2}}{3}$, etc. are like surds.

❖ Like surds can be added or subtracted from one another.

❖ While multiplying surds, multiply surds with surds, and whole numbers with whole numbers.

❖ When multiplying surds involving brackets, the brackets are expanded as usual.

❖ To rationalise the denominator means to make the denominator of an expression into a rational number.

❖ If one or both of the terms in an expression contains a surd, we call this a binomial surd. One binomial surd is the conjugate of the other, they differ only in the sign connecting the two terms, e.g. $(\sqrt{2}+1)$ and $(\sqrt{2}-1)$. When the denominator of a fraction is a binomial surd, use the conjugate of the denominator to rationalise it.

❖ Trigonometric ratios of special angles in the surd form are:

$$\sin 30° = \frac{1}{2}, \cos 30° = \frac{\sqrt{3}}{2}, \tan 30° = \frac{1}{\sqrt{3}}$$

$$\sin 60° = \frac{\sqrt{3}}{2}, \cos 60° = \frac{1}{2}, \tan 60° = \sqrt{3}$$

$$\sin 45° = \frac{1}{\sqrt{2}}, \cos 45° = \frac{1}{\sqrt{2}}, \tan 45° = 1$$

GRADUATED EXERCISES

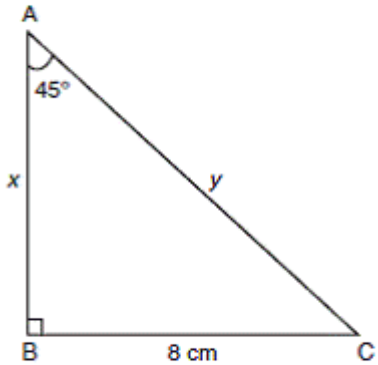
Simplify the following expressions:

1. $2\sqrt{3}+\sqrt{3}$
2. $4\sqrt{2}-2\sqrt{2}$
3. $\sqrt{72}-\sqrt{50}$
4. $5\sqrt{27}+\sqrt{3}$
5. $\sqrt{98}-\sqrt{72}$
6. $\sqrt{40}\times\sqrt{200}$
7. $\sqrt{90}\times\sqrt{600000}$
8. $(1-\sqrt{5})(1+\sqrt{5})$
9. $(\sqrt{7}+3)(\sqrt{7}-3)$
10. $\sqrt{3}\times\sqrt{27}$

Rationalise the following expressions:

11. $\frac{1}{\sqrt{11}}$
12. $\frac{30}{\sqrt{3}}$
13. $\frac{5}{\sqrt{2}}$
14. $\frac{1}{\sqrt{7}}$
15. $\sqrt{\frac{3}{27}}$
16. $\frac{1}{(1-\sqrt{3})}$
17. $\frac{1}{(\sqrt{7}+\sqrt{2})}$
18. $\frac{2}{(\sqrt{5}-\sqrt{7})}$
19. $\frac{1}{\sqrt{5}-\sqrt{3}}$
20. $\frac{1}{1+\sqrt{2}}$

Find the lengths marked x and y in each part of Figure 1.6. Leave your answers in the surd form with rational denominators.

21. 

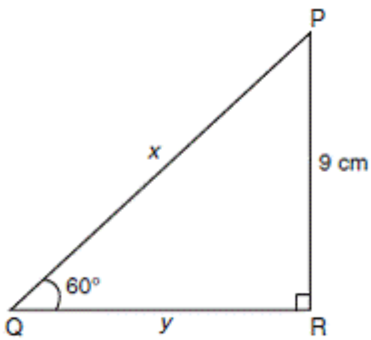
22. 

Figure 1.6

23. If $\tan x = \frac{1}{\sqrt{3}}$, find $\cos x - \sin x$ such that $0° \leq x \leq 90°$. **(WAEC)**

24. Express $\frac{8 - 3\sqrt{2}}{2\sqrt{3} - 3\sqrt{2}}$ in the form $m\sqrt{3} + n\sqrt{2}$, where *m* and *n* are rational numbers. **(WAEC)**

25. Without using table or calculator ,evaluate $\frac{3\sqrt{7} + 5}{3\sqrt{7} - 5} + \frac{3\sqrt{7} - 5}{3\sqrt{7} + 5}$ **(WAEC)**

26. Simplify $\frac{\sqrt{2}}{2\sqrt{2} - \sqrt{3}} + \frac{\sqrt{3}}{2\sqrt{2} + \sqrt{3}}$. **(WAEC)**

27. (a) Simplify $\frac{2}{3 - \sqrt{2}} + \frac{1}{3 + \sqrt{2}}$ leaving your answer in the form $a + b\sqrt{2}$, where *a* and *b* are integers.

(b) Find integers *m* and *n* such that $(m + n\sqrt{2})(1 - \sqrt{2})^2 = 2 - \sqrt{2}$. **(WAEC)**

28. When the angle of elevation of sun is 30°, the shadow of a building is 20 m longer than when the elevation of the sun is 60°. Find the length of the building.

Given that $\sqrt{2} = 1.414$ and $\sqrt{5} = 2.236$.

Evaluate the following:

29. $\sqrt{50}$

30. $\sqrt{10}$