

CHAPTER 9: Algebraic Fractions

OBJECTIVES

At the end of the chapter, students should be able to:

1. Simplify algebraic fractions to their lowest terms.
2. Simplify algebraic fractions involving addition, subtraction, multiplication and division.
3. Simplify and solve simple equations involving fractions.

I. Simplification of Algebraic Fractions

To simplify an algebraic fraction means to reduce it to its lowest term. This is done by factoring out the common factors in the numerator and the denominator.

Worked Example 1

Simplify .

SOLUTION

(factorising both the numerator and denominator)

(dividing the numerator and denominator by $(x + 2)$)

Worked Example 2

Simplify

SOLUTION

(factorising both the numerator and denominator)

(dividing out the common binomial factor of (y^2))

(dividing out the common monomial factor of $2y$)

Worked Example 3

Simplify

SOLUTION

$$= \frac{(-x+2)(x+4)}{3x(x-2)(x+5)}$$

(factorising both the numerator and the denominator)

$$= \frac{(-1)(x-2)(x+4)}{3x(x-2)(x+5)}$$

(factorising out (-1) from the numerator)

$$= \frac{(-1)(x+4)}{3x(x+5)} \quad \text{(dividing out the common binomial factor of } (x^2))$$

$$= \frac{-x-4}{3x(x+5)}$$

Exercise 1

Simplify the following algebraic fractions:

1. $\frac{abc}{bcd}$

2. $\frac{4a^2b}{6abc}$

3. $\frac{324 - 4x^2}{2x - 18}$

4. $\frac{x^2 - xy}{y^2 - xy}$

5. $\frac{5d^2nv^3}{20d^3n^2v^4}$

6. $\frac{a^2 - ab}{ab}$

7. $\frac{xy - y^2}{x^2 - y^2}$

8. $\frac{4a^2 - 49b^2}{2a^2 + 5ab - 7b^2}$

9. $\frac{a^2 - 5ab + 6b^2}{a^2 + ab - 12b^2}$

10. $\frac{b^2 - bm - bn + mn}{b^2 - bm + bn - mn}$

11. $\frac{u^2 + uv - 6v^2}{u^2 - 3uv + 2v^2}$

12. $\frac{(2m - a)^2 - (m - 2a)^2}{5m^2 - 5a^2}$

II. Operations in Algebraic Fractions

(i) Addition and subtraction of algebraic fractions

Worked Example 4

Simplify $\frac{2}{3x+3} + \frac{1}{2x+4}$

SOLUTION

$$\begin{aligned}
 & \frac{2}{3x+3} + \frac{1}{2x+4} \\
 &= \frac{2(2x+4) + (3x+3)}{(3x+3)(2x+4)} \quad (\text{using the LCM of } (3x+3)(2x+4)) \\
 &= \frac{4x+8+3x+3}{(3x+3)(2x+4)} \\
 &= \frac{7x+11}{(3x+3)(2x+4)}
 \end{aligned}$$

Worked Example 5

Simplify $\frac{y+4}{y^2-3y} - \frac{y-1}{9-y^2}$

SOLUTION

$$\begin{aligned}
 & \frac{y+4}{y^2-3y} - \frac{y-1}{9-y^2} \\
 &= \frac{y+4}{y(y-3)} - \frac{y-1}{(3-y)(3+y)} \\
 &= \frac{y+4}{y(y-3)} + \frac{y-1}{(y-3)(3+y)} \\
 &= \frac{(y+4)(3+y) + y(y-1)}{y(y-3)(3+y)} \\
 &= \frac{3y + y^2 + 12 + 4y + y^2 - y}{y(y-3)(y+3)} \\
 &= \frac{2y^2 + 6y + 12}{y(y-3)(y+3)} \\
 &= \frac{2(y^2 + 3y + 6)}{y(y-3)(y+3)}
 \end{aligned}$$

(ii) Multiplication and division of algebraic fractions

When dealing with multiplication or division of algebraic fractions, first factorise and then divide the denominator by any common factor that they have.

Worked Example 6

Simplify $\frac{xy}{3x-6y} \times \frac{4x-8y}{x^2y}$

SOLUTION

$$\frac{xy}{3x-6y} \times \frac{4x-8y}{x^2y}$$

$$= \frac{xy}{3(x-2y)} \times \frac{4(x-2y)}{x(xy)} = \frac{4}{3x}$$

Worked Example 7

Simplify $\frac{12ab^3}{15ca^3} \div \frac{9c^3b}{10c^2a^2}$

SOLUTION

$$\frac{12ab^3}{15ca^3} \div \frac{9c^3b}{10c^2a^2}$$

Note: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

$$\therefore \frac{12ab^3}{15ca^3} \div \frac{9c^3b}{10c^2a^2} = \frac{12ab^3}{15ca^3} \times \frac{10c^2a^2}{9c^3b}$$

$$= \frac{3b(4ab^2)}{5ca^2(3a)} \times \frac{5ca^2(2c)}{3b(3c^3)}$$

$$= \frac{4ab^2}{3a} \times \frac{2c}{3c^3}$$

$$= \frac{4b^2}{3} \times \frac{2}{3c^2} = \frac{8b^2}{9c^2}$$

Worked Example 8

Simplify

$$\frac{x^2 + xy}{x^2 - 2xy + y^2} \div \frac{x+3y}{x+2y} \times \frac{xy - x^2}{x^2 + 3xy + 2y^2}$$

SOLUTION

$$\frac{x^2 + xy}{x^2 - 2xy + y^2} \div \frac{x+3y}{x+2y} \times \frac{xy - x^2}{x^2 + 3xy + 2y^2}$$

$$= \frac{x^2 + xy}{x^2 - 2xy + y^2} \times \frac{x+2y}{x+3y} \times \frac{xy - x^2}{x^2 + 3xy + 2y^2}$$

$$= \frac{x(x+y)}{(x-y)(x-y)} \times \frac{x+2y}{x+3y} \times \frac{x(y-x)}{(x+y)(x+2y)}$$

$$= \frac{-x^2(x-y)}{(x-y)(x-y)(x+3y)}$$

$$= -\frac{x^2}{(x-y)(x+3y)}$$

Exercise 2

Simplify the following:

$$1. \frac{x^2 + 2x - 3}{x^2 - 16} \times \frac{x+4}{x^2 + 8x + 15}$$

$$2. \frac{12ab}{10bc} \times \frac{15cd}{24de}$$

$$3. \frac{a+b}{b} \times \frac{ab}{3a+3b}$$

$$4. \frac{u^2 - 4}{u^2 - 3u + 2} \div \frac{u}{u-1}$$

$$5. \frac{2x - 2y + 2z}{8yz} \times \frac{10xyz}{5x - 5y + 5z}$$

$$6. \frac{a^2 - b^2}{a^2 - 2ab + b^2} \div \frac{a^2 + ab}{a^2 - ab}$$

$$7. \frac{u^2 - 5u + 6}{u^2 + 2u - 3} \div \frac{3u - 9}{2u^2 + 6u}$$

$$8. \frac{m^2 - mn - 6n^2}{m^2 + mn - 6n^2} \times \frac{m^2 - mn - 2n^2}{m^2 - 2mn - 3n^2}$$

$$9. \text{ Simplify } \left(\frac{2}{x} - \frac{5}{y} \right) \div \frac{4}{xy} \quad (\text{WAEC})$$

$$10. \frac{m^2 + mn - 2n^2}{m^2 - 2mn - 3n^2} \times \frac{m^2 n^2}{mn + 2n^2}$$
$$\div \frac{m^2 - 2mn + n^2}{m^2 - 3mn}$$

$$11. \frac{18a^2c}{16b^3d^2} \div \frac{24a}{15bc^3} \times \frac{8b^2d^3}{30a^3c}$$

$$12. \frac{x^2 + 3x - 10}{3x^2 + 12x} \div \frac{x^2 - 25}{x^2 - x - 20}$$

III. Equations Involving Algebraic Fractions

In solving equations involving algebraic fractions, we follow the following procedure:

Step 1: Determine the LCM.

Step 2: Multiply both sides of the equation by the LCM (this step eliminates the fractions).

Step 3: Solve the resulting equation.

Step 4: Check the solution.

Worked Example 9

$$\text{Solve the equation } \frac{t}{4} + \frac{2t}{3} = \frac{55}{12}.$$

SOLUTION

$$\frac{t}{4} + \frac{2t}{3} = \frac{55}{12}$$

Multiply the equation throughout by 12 (LCM of denominators)

$$12\left(\frac{t}{4} + \frac{2t}{3}\right) = 12\left(\frac{55}{12}\right)$$

$$12\left(\frac{t}{4}\right) + 12\left(\frac{2t}{3}\right) = 12\left(\frac{55}{12}\right)$$

$$3t + 8t = 55$$

$$11t = 55$$

$$t = \frac{55}{11}$$

$$t = 5$$

Worked Example 10

Solve the equation $\frac{1}{y} + \frac{1}{2} = \frac{5}{6y} + \frac{1}{3}$.

SOLUTION

$$\frac{1}{y} + \frac{1}{2} = \frac{5}{6y} + \frac{1}{3}$$

Multiply throughout by 6y (LCM of denominators)

$$6y\left(\frac{1}{y}\right) + 6y\left(\frac{1}{2}\right) = 6y\left(\frac{5}{6y}\right) + 6y\left(\frac{1}{3}\right)$$

$$6 + 3y = 5 + 2y$$

$$3y \hat{=} 2y = 5 \hat{=} 6$$

$$y = \hat{=} 1$$

Since $y = \hat{=} 1$ will not make any of the denominators zero, it is the solution.

Worked Example 11

Solve the equation $\frac{1}{3t-1} = \frac{2}{t+1} - \frac{3}{8}$.

SOLUTION

$$\frac{1}{3t-1} = \frac{2}{t+1} - \frac{3}{8}$$

The LCM of the denominators is $8(3t - 1)(t + 1)$.

$$\frac{1}{3t-1} \times 8(3t-1)(t+1) = \frac{2}{t+1} \times$$

$$8(3t-1)(t+1) - \frac{3}{8} \times 8(3t-1)(t+1)$$

$$8(t+1) = 16(3t-1) - 3(3t-1)(t+1)$$

$$8t + 8 = 48t - 16 - (9t - 3)(t + 1)$$

$$8t + 8 = 48t - 16 - 9t^2 - 6t - 3 = 0$$

$$8t + 8 - 48t + 16 + 9t^2 + 6t - 3 = 0$$

$$9t^2 - 34t + 21 = 0$$

$$(t-3)(9t-7) = 0$$

$$t=3 \text{ or } 9t=7$$

$$t=3 \text{ or } t=\frac{7}{9}$$

Since $t = 3$ or $t = \frac{7}{9}$ will not make any of the denominators zero, both of these values are the solutions.

Exercise 3

Solve the following:

$$1. \frac{1}{a} + \frac{1}{2} = \frac{5}{8a} + \frac{1}{4}$$

$$2. \frac{t}{4} + \frac{t}{3} = \frac{21}{12}$$

$$3. \frac{3}{x} = x - 2$$

$$4. \frac{7}{3} + \frac{2}{a} = a$$

$$5. t = \frac{8}{3t+2}$$

$$6. \frac{m-2}{m+4} = m$$

$$7. y+3 = \frac{6}{y+4}$$

$$7. y+3 = \frac{6}{y+4}$$

$$8. \frac{1}{2(a-1)} - \frac{1}{2(a+1)} = \frac{1}{3}$$

$$9. \frac{2}{2x-3} + \frac{3}{2x+3} = \frac{2}{4x^2-9} \quad (\text{WAEC})$$

$$10. \frac{4b-3}{6b+1} = \frac{2b-1}{3b+4}$$

IV. Substitution in Fractions

Worked Example 12

If $\frac{a}{b} = \frac{3}{4}$, evaluate $\frac{2a-b}{2a+b}$.

SOLUTION

To evaluate

$$\frac{2a-b}{2a+b}, \text{ if } \frac{a}{b} = \frac{3}{4}$$

Divide the numerator and denominator of

$$\frac{2a-b}{2a+b} \text{ by } b$$

$$\frac{2a-b}{2a+b} = \frac{2\left(\frac{a}{b}\right) - 1}{2\left(\frac{a}{b}\right) + 1}$$

Substitute in the expression.

$$= \frac{2\left(\frac{3}{4}\right) - 1}{2\left(\frac{3}{4}\right) + 1} = \frac{\frac{3}{2} - 1}{\frac{3}{2} + 1} = \frac{\frac{3-2}{2}}{\frac{3+2}{2}}$$

$$= \frac{3-2}{2} \times \frac{2}{3+2} = \frac{1}{5}$$

Worked Example 13

If , express in terms of a.

SOLUTION

To express in terms of a, if .

Substitute for x in the given expression.

$$\frac{2x+1}{3x+1} = \frac{2\left(\frac{a+3}{2a-1}\right) + 1}{3\left(\frac{a+3}{2a-1}\right) + 1}$$

Multiply the numerator and denominator by $(2a - 1)$.

$$\frac{2x+1}{3x+1} = \frac{2(a+3) + (2a-1)}{3(a+3) + (2a-1)}$$

$$= \frac{2a+6+2a-1}{3a+9+2a-1}$$

$$= \frac{4a+5}{5a+8}$$

Worked Example 14

If $a:b = 5:3$, evaluate

$$\frac{6a+b}{a-\frac{1}{3}b}$$

SOLUTION

To evaluate $\frac{6a+b}{a-\frac{1}{3}b}$, if $a:b = 5:3$,

If $a:b = 5:3$, then $\frac{a}{b} = \frac{5}{3}$.

Divide the numerator and denominator of

$\frac{6a+b}{a-\frac{1}{3}b}$ by b .

$$\frac{6a+b}{a-\frac{1}{3}b} = \frac{6\left(\frac{a}{b}\right) + 1}{\frac{a}{b} - \frac{1}{3}}$$

Substitute $\frac{a}{b}$ for b in the expression.

$$\frac{6a+b}{a-\frac{1}{3}b} = \frac{6\left(\frac{a}{b}\right) + 1}{\frac{a}{b} - \frac{1}{3}}$$

$$= \frac{6\left(\frac{5}{3}\right) + 1}{\frac{5}{3} - \frac{1}{3}}$$

$$= \frac{\frac{10+1}{4}}{\frac{3}{3}} = \frac{11}{4}$$

$$= 11 \div \frac{4}{3} = 11 \times \frac{3}{4}$$

$$= \frac{33}{4} = 8 \frac{1}{4}$$

Exercise 4

1. Given that $a:b = 9:4$, evaluate

$$\frac{8a-3b}{a-\frac{3}{4}b}.$$

2. If $x = \frac{2a+3}{3a-2}$, express $\frac{x-1}{2x+1}$ in terms of a .
(WAEC)

3. Given $\frac{x}{y} = \frac{2}{7}$, evaluate $\frac{7x+y}{x-\frac{1}{7}y}$.

(WAEC)

4. If $x = \frac{2b+3}{3b-2}$, express b in terms of x .

5. Given $m:n = 9:5$, evaluate $\frac{15m - 2n}{5m + 16n}$.

6. If $x = \frac{3b - 5}{3b + 5}$, express $\frac{x-1}{x+1}$ in terms of b.

7. If $a:b = 5:2$, evaluate $\frac{6a+b}{a-\frac{1}{2}b}$.

8. If $a = \frac{b+1}{b-1}$, express $\frac{a+1}{a-1}$ in terms of b.

9. If $x = \frac{3a-1}{a+2}$, express $\frac{2x-3}{3x-1}$ in terms of a.

10. If $x = \frac{2w+1}{2w-1}$, express $\frac{2x+1}{2x-1}$ in terms of w.

V. Simultaneous Linear Equations Involving Fractions

To solve simultaneous equations involving fractions, first find the LCM of each equation and multiply each LCM by its corresponding equation to remove the fractions. Use any suitable method of solving simultaneous equations to find the solution.

Worked Example 15

Solve the following simultaneous equations:

$$(a) \quad \frac{x}{2} + \frac{y}{4} = 1, \quad \frac{x}{4} - \frac{y}{4} + 1 = 0$$

$$(b) \frac{1+2y}{x} = 5, \frac{3+4y}{2x} = \frac{7}{2}$$

SOLUTION

$$\frac{x}{4} - \frac{y}{4} + 1 = 0 \quad \dots \dots \dots \quad (2)$$

Multiply equation (1) by its LCM = 4

$$4\left(\frac{x}{2} + \frac{y}{4} = 1\right)$$

$$4\left(\frac{x}{2}\right) + 4\left(\frac{y}{4}\right) = 4$$

Multiply equation (2) by its LCM = 4

$$4\left(\frac{x}{4} - \frac{y}{4} + 1 = 0\right)$$

$$4\left(\frac{x}{4}\right) - 4\left(\frac{y}{4}\right) + 4(1) = 4(0)$$

From equation (3), $2x + y = 4$

$$v \equiv 4 \hat{A}' 2x \dots \quad (5)$$

Substitute equation (5) into equation (4).

Substitute equation (5)

$$x \hat{a}''(4 \hat{a}' - 2x) + 4 = 0$$

$$x \hat{=} 4 + 2x + 4 = 0$$

$$x \neq 4$$

$$4. \frac{1}{3}(a - 3b) = 2, \frac{a+b}{4} = \frac{1}{2}$$

$$5. x + 18y = xy, \frac{1}{x} + \frac{1}{y} + \frac{8}{9} = 0$$

$$6. \frac{3}{x} - \frac{4}{y} = \frac{1}{3}, \frac{2}{x} - \frac{5}{y} = 1$$

$$7. \frac{x-1}{5} + \frac{y-1}{3} + \frac{1}{15} = 0,$$

$$7y + 2x + 10 = 0 \quad (\text{WAEC})$$

$$8. \frac{1}{x} - \frac{5}{y} = 7, \frac{2}{x} + \frac{1}{y} = 3 \quad (\text{WAEC})$$

$$9. \frac{2}{x} - \frac{5}{y} = 15, \frac{5}{x} + \frac{7}{y} = 18$$

$$10. y = 3x - 6, \frac{x}{3} + \frac{y}{2} = 1$$

VI. Undefined Value of a Fraction

An algebraic fraction, whose numerator and denominator have no common factor, is said to be undefined when the denominator is equal to zero. In general, we cannot divide any number by zero. If both the numerator and the denominator are zero, respectively, then the algebraic expression is said to be indeterminate.

Worked Example 16

Find the value of x for which the fraction is undefined.

SOLUTION

is undefined when $x^2 = 0$.

If $x^2 = 0$, then $x = 0$.

The fraction is undefined when $x = 0$.

Worked Example 17

$$\frac{x(x-1)}{(x^2-1)}$$

Find the value of x for which the fraction is undefined.

SOLUTION

Here, we should simplify the algebraic expression first.

$$\frac{x(x-1)}{(x^2-1)} = \frac{x(x-1)}{(x-1)(x+1)} = \frac{x}{x+1}$$

Equating the denominator of the simplified algebraic fraction to zero,

$$x+1=0$$

$$x=\hat{=}1$$

The fraction is undefined when $x = \hat{=}1$.

Worked Example 18

For what value of x is $\frac{a}{x-2} + \frac{b}{x^2-2x} - \frac{c}{x+3}$ undefined?

SOLUTION

$$\frac{a}{x-2} + \frac{b}{x^2-2x} - \frac{c}{x+3} = \frac{a}{x-2} + \frac{b}{x(x-2)}$$

The expression is undefined, if any of the fractions has a denominator equal to zero e.g. is undefined when $x \neq 2$ or $x = 2$

is undefined when $x(x - 2) = 0$.

If $x(x - 2) = 0$, then either $x = 0$ or $x - 2 = 0$ i.e. either $x = 0$ or $x = 2$
is undefined when $x + 3 = 0$ or $x = -3$

The expression is undefined when $x = 0, 2, -3$.

Note: If part of an expression is not defined, then the whole expression is not defined (i.e. undefined).

Exercise 6

Find the values of x for which the following fractions are undefined:

1. $\frac{1}{x}$

2. $\frac{1}{x^2}$

3. $\frac{10}{x^2 - 9}$

4. $\frac{5+x}{x^2 + 4}$

5. $\frac{3}{5+x}$

6. $\frac{2}{x+2}$

7. $\frac{10}{x^2 + 25}$

8. $\frac{x^2 - 2}{x^2 + 2x + 1}$

9. $\frac{x+b-1}{x-b+1}$

10. $\frac{3x-1}{x^2 - 8x - 20}$

11. $\frac{x+1}{(x+4)(x+1)}$

12. $\frac{10}{(x+6)(x-6)}$

13. $\frac{x-3}{x(x+4)(x-9)}$

14. $\frac{2m^2 + m - 15}{m^2 - 9}$

15. $\frac{x^2 + 5}{x^2 + 4x - 5} - \frac{2x - 1}{x^2 + 8x + 15}$

SUMMARY

In this chapter, we have learnt the following:

- v To simplify an algebraic fraction means to reduce it to its lowest term. We do this by first factorising its denominators and numerators and then dividing out the common factors in the numerators and denominators as shown in Worked Examples 1–3.
- v To add or subtract the algebraic fractions, find the LCM of the denominators of the fractions. Then, express each fraction with the LCM as the denominator. Add or subtract and then simplify as shown in Worked Examples 4–5.
- v To multiply or divide the algebraic fractions, first factorise fully and then divide the numerator and denominator by any factors they have in common as shown in Worked Examples 6–8.
- v The steps to follow in solving an equation involving fractions are listed under ‘Equations Involving Algebraic Fractions™’.
- v To solve simultaneous equations involving fractions, first find the LCM of each equation and then multiply each LCM by its corresponding equation to remove the fraction. Use any suitable method of solving simultaneous equations to find the solution as shown in Worked Example 15.
- v An algebraic fraction is undefined, if its denominator is zero as shown in Worked Examples 16–18.

GRADUATED EXERCISES

1. Simplify $\frac{x^2 - y^2}{x^2 + xy}$. (WAEC)

2. Simplify $\frac{1 - x^2}{x - x^2}$. (WAEC)

3. Simplify $\frac{y^2 + 2y^2 + z^2}{y^2 - z^2}$. (WAEC)

4. Simplify $\frac{3x+2}{3} + \frac{x-1}{4} - \frac{5}{12}$. (WAEC)

5. Reduce $\frac{3}{x} - \frac{x}{2} + 5$ to a single fraction. (WAEC)

6. Simplify the expression $\frac{2m^2 + m - 15}{m^2 - 9}$ and state the value of m for which the simplified expression is not defined.

Simplify the following algebraic fractions:

7. $\frac{x^2 - y^2}{xy + x^2} \times \frac{2x^3}{xy - x^2}$

8. $\frac{m^2 - n^2}{m^2 + mn} \div \frac{2m - 2n}{mn}$

9. $\frac{a^2 - 9}{a^2 - a - 6} \times \frac{a^2 + 2a}{a^2}$

10. $\frac{m^2 - 3m - 4}{m^2 - 4m} \div \frac{m^2 - 4m + 4}{m^2 - 4}$

11. $\frac{7x}{2x+3y} - 3$

12. $\frac{3ab}{2a^2 + 2b^2} + \frac{5ab}{3a^2 + 3b^2}$

13. Given that $\frac{5y-x}{8y+3x} = \frac{1}{5}$, find the value of $\frac{x}{y}$ to two decimal places. (WAEC)

14. If $p = \frac{2u}{1-u}$ and $q = \frac{1+u}{1-u}$, express (a) $p+q$ (b) $p-q$ and (c) $\frac{p+q}{p-q}$ in terms of u in its simplest form in each case. (WAEC)

Solve the following equation:

15. $\frac{1}{5(a+1)} + \frac{1}{7(a+1)} = \frac{3}{2}$