CHAPTER 10: Chord Property

OBJECTIVES

At the end of the chapter, students should be able to:
1. Identify angles subtended by chords in a circle.
2. Identify angles subtended by equal chords and derive the rider.
3. Identify perpendicular bisectors of chords and derive the rider.
4. Identify angles in alternate segments and derive the rider.
5. Solve problems on angles subtended by chords in a circle.
6. Solve problems on angles subtended by two equal chords at the centre.
7. Solve problems on perpendicular bisectors of chords.
8. Solve problems on angles in alternate segments.

I. Introduction

A chord of a circle is a line segment whose two endpoints lie on the circle. The chord that passes through the centre of the circle is called diameter and is the largest chord in a circle. A tangent to a circle is a straight line that touches the circle at a point. An arc of a circle is any connected part of the circleâ€™s circumference. A segment is a region bounded by a chord and an arc lying between the chordâ€™s endpoints. A chord which is not a diameter divides the circumference into two arcs of different sizes, a `major arc` and a `minor arc` as shown in Figure 10.1.

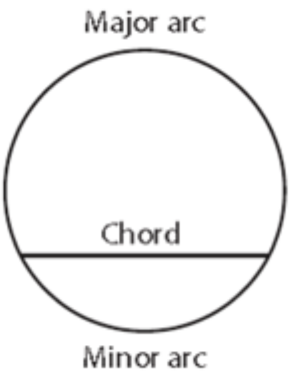

Figure 10.1

The chord also divides the circle into two segments of different sizes, a `major segment` and a `minor segment` as shown in Figure 10.2

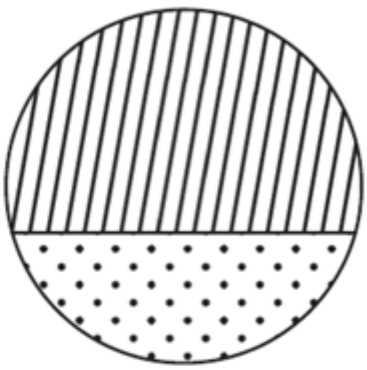

Figure 10.2

II. Angles Subtended by Chords at the Centre and at

the Circumference of a Circle

Theorem 1

A perpendicular line drawn from the centre of a circle to bisect a chord, which is not a diameter, is at right angles to the chord.

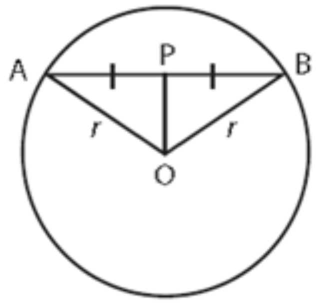

Figure 10.3

Given: A circle with centre O and chord

such that =

To prove: APË† O = BPË† O = 90Â°.

Construction: Join and

Proof:

From the lettering in Figure 10.3,

|OA| = |OB| (radii)

|AP| = |PB| (given)

|OP| = |OP|

®ˆ× ℘„ APO = Î” BPO (SSS)

®ˆ× APË† O = BPË† O

But APË† O + BPË† O = 180Â° (angles on a straight line)

®ˆ× APË† O = BPË† O = = 90Â°

Note:

1. A straight line drawn from the centre of a circle perpendicular to a chord bisects the chord.
2. The perpendicular bisector of a chord of a circle passes through the centre of the circle.

Worked Example 1

A chord of a circle is 8 cm long. The radius of the circle is 9 cm. Calculate the distance of the mid-point of the chord from the centre of the circle.

SOLUTION

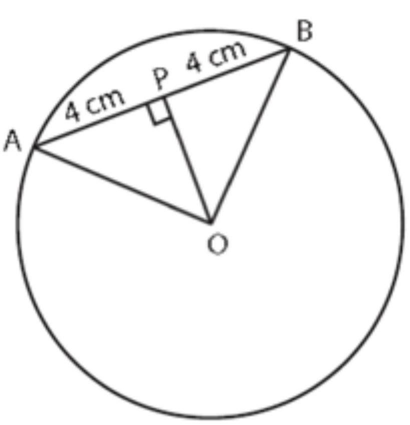

Figure 10.4

In Figure 10.4, O is the centre of the circle and |AB| = 8 cm

|OA| = |OB| = 9 cm

P is the mid-point of

®ˆ× |AP| = |PB| = 4 cm

In Î" APO,

= - (Pythagorasâ€™ theorem)

= âˆ'

= 81 âˆ' 16 = 65

= = 8.06 cm

The mid-point of the chord is 8.06 cm from the centre of the circle.

Worked Example 2

Calculate the length of a chord of a circle of radius 20 cm, if the chord is 8 cm from the centre of the circle.

SOLUTION

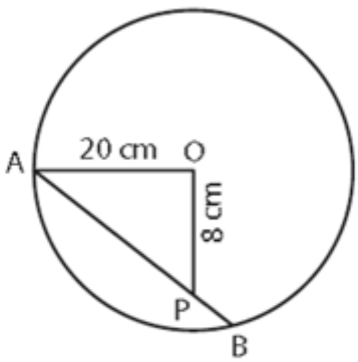

Figure 10.5

In Figure 10.5,
|OA| = 20 cm
|OP| = 8 cm

In Î" APO
|AP| (Pythagorasâ€™ theorem)
= âˆ'
= 400 âˆ' 64
= 336

|AP| = = 18.3 cm

®ˆ× Length of chord AB = 2 Ã— 18.3 = 36.6 cm

Exercise 1

1. A chord of a circle is 10 cm long. The radius of the circle is 11 cm. Calculate the distance of the midpoint of the chord from the centre of the circle.
2. Calculate the length of a chord of a circle of radius 16 cm, if the chord is 9 cm from the centre of the circle.
3. A circle is drawn to pass through the vertices of an isosceles triangle of sides 10 cm, 6 cm and 6 cm. Find the radius of the circle.
4. Prove that a straight line drawn from the centre of a circle and perpendicular to a chord bisects the chord.
5. An isosceles triangle XYZ has its vertices on a circle. If |XY| = 10 cm, |YZ| = 10 cm and |XZ| = 8 cm, calculate
   a. The radius of the circle to the nearest whole centimetre (cm).
   b. The height of the triangle.
6. In a circle of radius 5 cm, calculate the length of a chord which is 4 cm from the centre of the circle.
7.

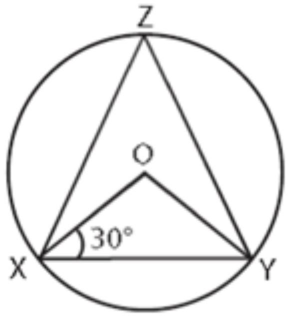

Figure 10.6

In Figure 10.6, O is the centre of the circle. If ®ˆ€OXY = 30Â°, find ®ˆ€XOY.

III. Perpendicular Bisectors of Chords

Theorem 2

The perpendicular bisector of a circle passes through the centre of the circle.

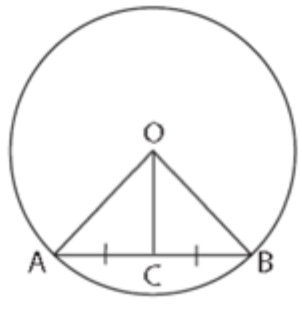

Figure 10.7

Given:

is a chord of a circle with centre â€˜Oâ€™. O is joined to the mid-point â€˜Câ€™ on

To prove:

Construction: Join

and .

Proof: In Î”OAC and Î”OCB,

|OA| = |OB| (radii of the circle)

|AC| = |CB| (given)

is common

Therefore, Î”OAC = Î”OCB (SSS axiom)

Therefore, ®ˆ€OCA = ®ˆ€OCB = 90Â°.

Therefore, is the perpendicular bisector of and is passing through â€˜Oâ€™, the centre of the circle.

Theorem 3

The straight line drawn from the centre of a circle and perpendicular to a chord bisects the chord.

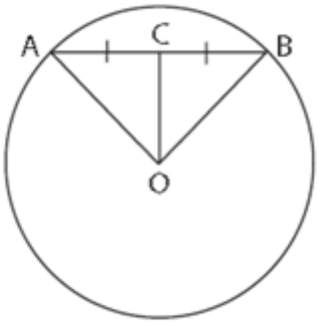

Figure 10.8

Given: A circle with centre O, with perpendicular to .

To prove: |AC| = |CB|.

Construction: Join and .

Proof:

In Î”OAC and Î”OCB,

|OA| = |OB| (radii of the circle)

is common

®ˆ€OCA = ®ˆ€OCB = 90Â° (given)

Therefore, Î”OCA = Î”OCB (RHS axiom)

Therefore, |AC| = |CB|.

Worked Example 3

Chord XY on a circle of radius 40 cm is 50 cm long. Calculate the angle which XY subtends at the centre of the circle to the nearest degree?

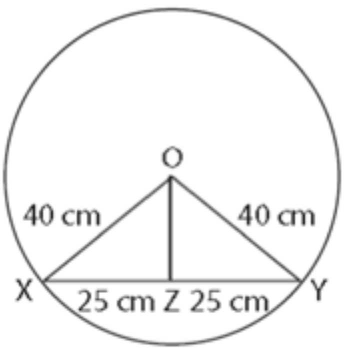

Figure 10.9

SOLUTION

In Figure 10.9, O is the centre of the circle and Z is the mid-point of XY.

®ˆ€XOY = 2®ˆ€ZOX

But sin âˆ ZOX = = 0.6250

®ˆ×®ˆ€ZOX =

= 38.7Â° 39Â°

®ˆ× XY subtends an angle approximate to 78Â° at the centre of the circle.

Worked Example 4

Find the radius of the circle, if a chord 9 cm long is 6 cm from the centre of the circle.

SOLUTION

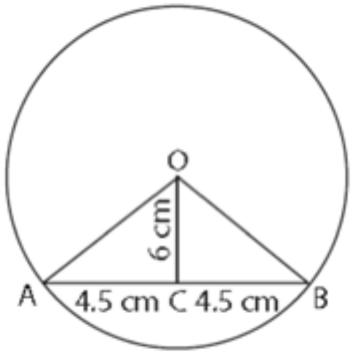

Figure 10.10

Since OC bisects AB

®ˆ× |AC| = |CB| = 4.5 cm

By Pythagorasâ€™ theorem

= +

= +

= (36 + 20.25) cm

= 56.25 cm

®ˆ× |AO| = 7.5 cm

Exercise 2

i. What angle does chord XY subtend at the centre of the circle, to the nearest degree, if chord XY is 60 cm long in a circle of radius 30 cm?

ii. Calculate the radius of a circle, if a chord 10 cm long is 5 cm from the centre of the circle.

iii. O is the centre of a circle and AB is any chord of the circle. From O, a perpendicular line is drawn to meet AB at C. By considering a pair of congruent triangles, prove that |AC| = |CB|. (WAEC)

iv. A tangent TAS touches a circle, with centre O, at A and a chord AB is drawn such that ®ˆ€TAB is acute. is the diameter parallel to , and the points P and B lie on opposite sides of AQ. Prove that

(a) ®ˆ€PAS = ®ˆ€QAB

(b) ®ˆ€PAS = 45ℜ° ®ˆ’ ®ˆ€BAT

(c) If AB = BQ, ®ˆ€PAS = 30Â°. (WAEC)

v. What can you say about the lengths of two chords which are drawn in a circle of radius 10 cm and each chord is 7 cm from the centre of the circle?

vi. How far from the centre of a circle of radius 4 cm is a chord 3 cm long?

vii. In a circle of radius 10 cm in which XY is a chord of length 12 cm and XZ is a diameter. Calculate

(a) |YZ| (b) ®ˆ€YXZ

viii. A chord of a circle is 16 cm long. The perpendicular distance of this chord from the centre of the circle is 5 cm.

Calculate

(a) The radius of the circle.

(b) The angle subtended by the chord at the centre of the circle, to the nearest degree.

ix. A chord 15 cm long is 8 cm from the centre of a circle. Find the radius of the circle.

x. Chord is 50 cm long in a circle of radius 25 cm. Calculate the angle that chord subtends at the centre of the circle.

IV. Angles in Alternate Segments

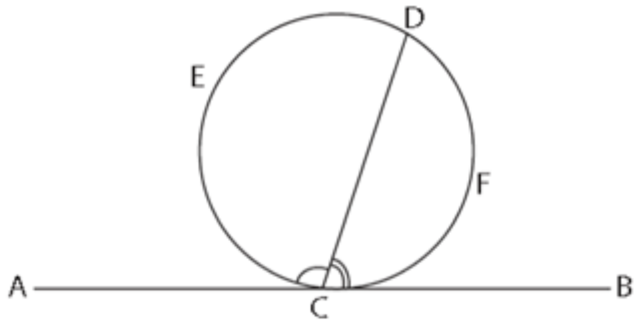

Figure 10.11

In Figure 10.11,

ACB is a tangent to the circle at C. is a chord dividing the circle into two segments CED and CFD. Chord at the point of contact with the tangent makes two angles ®ˆ€DCA and ®ˆ€DCB. The segment CED is the alternate segment to ®ˆ€DCB, i.e. it is on the other side of from ®ˆ€DCB. Similarly, segment CFD is the alternate segment to ®ˆ€DCA.

Theorem 4 (The Alternate Segment Theorem)

An angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

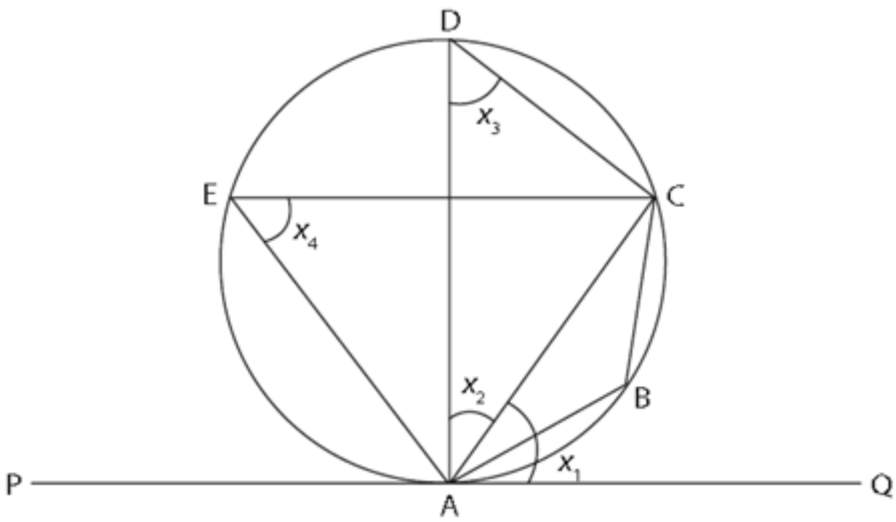

Figure 10.12

Given: A circle with tangent at A and Chord dividing the circle into two segments AEC and ABC. Segment AEC is alternate to ®ˆ€QAC.

To prove: ®ˆ€QAC = ®ˆ€AEC and ®ˆ€PAC = ®ˆ€ABC.
Construction: Draw the diameter .
Join .

Proof: From the lettering in Figure 10.12,
 + = 90Â° ....(1)   (   )
Also, ®ˆ€ACD = 90Â° (angle in a semicircle)
In Î”ACD,
 + +®ˆ€ACD = 180Â° (sum of angles in a triangle)
 + + 90Â° = 180Â°
®ˆ× + = 90ℜ° .... (2)

Subtracting x_2 from equations (1) and (2)

®ˆ× = =

®ˆ× ®ˆ€QAC = ®ˆ€AEC

Also, B is a point in the minor segment.

®ˆ€PAC+®ˆ€CAQ = 180Â° (angles on a straight line)

®ˆ€PAC + = 180Â°

®ˆ€PAC = 180Â° âˆ'

= 180Â° âˆ' (proved =)

= ®ˆ€ABC (opposite angles of a cyclic quadrilateral)

Worked Example 5

In Figure 10.13, ABC is a tangent to circle BDE. Calculate ®ˆ€DBC.

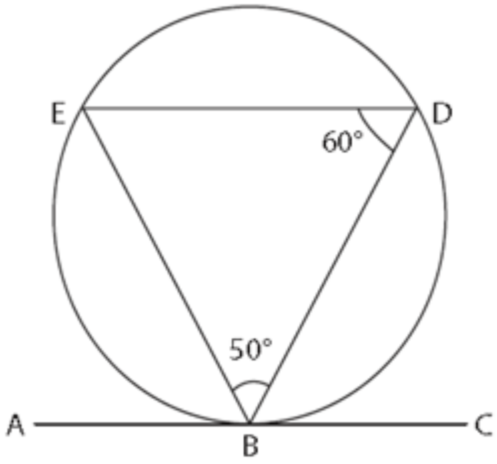

Figure 10.13

SOLUTION

In Î"BDE,

Î"BED = 180Â° âˆ' (60Â° + 50Â°) (sum of angles in a triangle) = 70Â°

®ˆ× ®ˆ€DBC = 70Â° (angles in alternate segments)

Worked Example 6

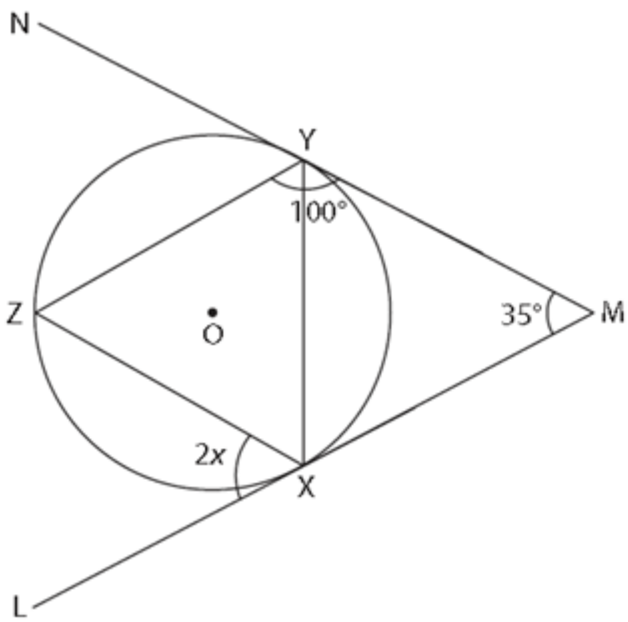

Figure 10.14

In Figure 10.14,
and are tangents to the circle with centre O. Find x.

SOLUTION

®ˆ€XYZ = 2xÂ° (angles in alternate segments)
®ˆ€ZYN + 100Â° = 180Â° (angles on a straight line)
®ˆ× ®ˆ€ZYN = 80Â°
®ˆ€ZXY = 80Â° (angles in alternate segments)

®ˆ€XZY = 100 âˆ’ 2xÂ° (angles in alternate segments)
®ˆ€MXY = ®ˆ€XZY = 100 âˆ’ 2xÂ° (angles in alternate segments)
®ˆ×®ˆ€MXY = ®ˆ€MYX = 100 âˆ’ 2xÂ°
Î”MXY is an isosceles triangle.
®ˆ× 35ℜ° + 2(100 ®ˆ’ 2xÂ°) = 180Â° (sum of angles in a triangle)
2(100 âˆ’ 2xÂ°) = 145Â°
100 âˆ’ 2xÂ° = 72.5Â°
2xÂ° = 100Â° âˆ’ 72.5Â°
2xÂ° = 27.5Â°
®ˆ× x = 13.75Â°

Exercise 3

1. In Figure 10.15, is a tangent to circle QPS. Calculate ®ˆ€SQX.

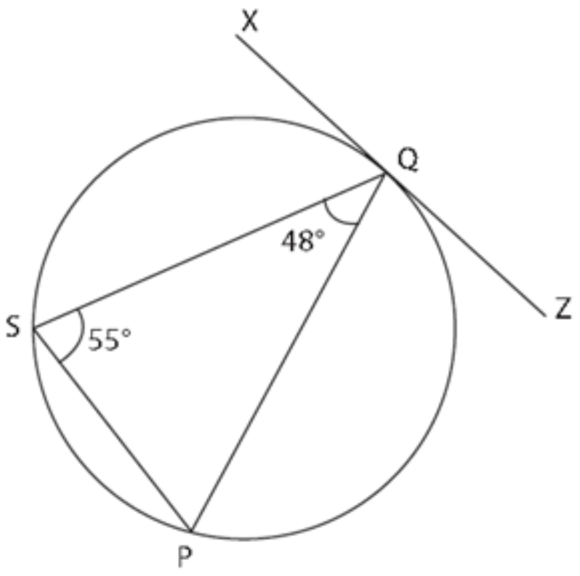

Figure 10.15

(WAEC)

2. In Figure 10.16, PQR is a circle with centre O. is a tangent to the circle at Q and ®ˆ€QPR = 36ℜ°. Find ®ˆ€PQT. (WAEC)

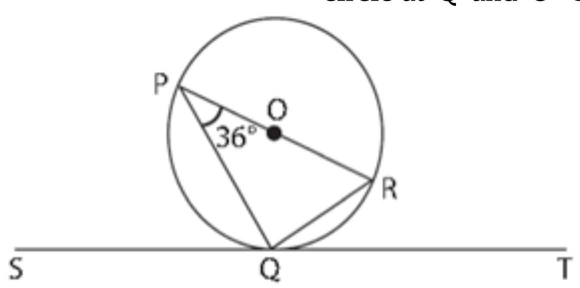

Figure 10.16

3. In Figure 10.17, XYZ is a tangent to the circle at Y. Name an angle equal to ®ˆ€YQP.

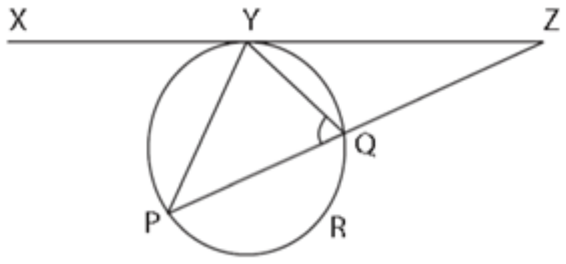

Figure 10.17

(WAEC)

4.

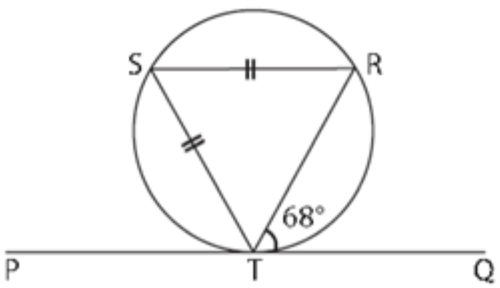

Figure 10.18

In Figure 10.18, is a tangent to circle RST at T. |SR| = |ST| and ®ˆ€RTQ = 68ℜ°. Find ®ˆ€PTS. (WAEC)

5. In Figure 10.19, is a tangent to circle PQRS at S. If |PR| = |PS| and ®ˆ€PQR = 117ℜ°, calculate ®ˆ€RST.

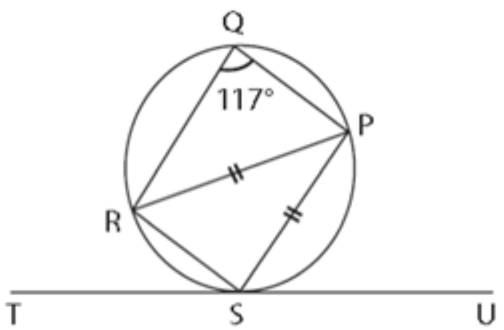

Figure 10.19

(WAEC)

6.

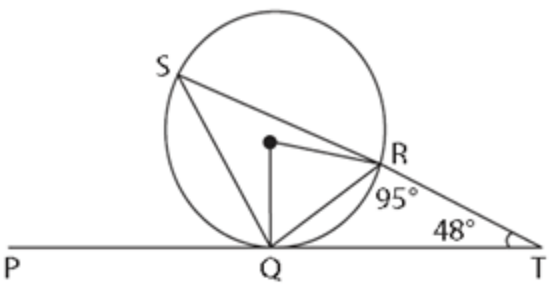

Figure 10.20

In Figure 10.20, is a tangent to circle QRS at Q. ®ˆ€QTR = 48ℜ° and ®ˆ€QRT = 95ℜ°. Find ®ˆ€QST. (WAEC)

7. In Figure 10.21, AB is a tangent to circle ACDE, |DC| = |DA| and ®ˆ€CAB = 90ℜ°. Calculate ®ˆ€DEA.

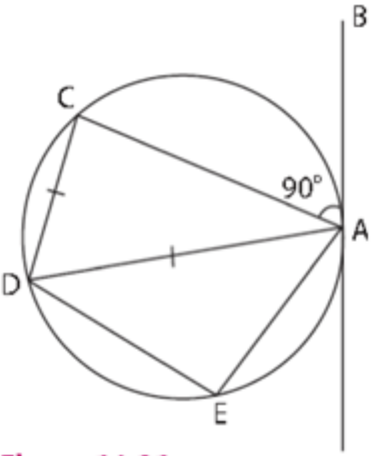

Figure 10.21

8.

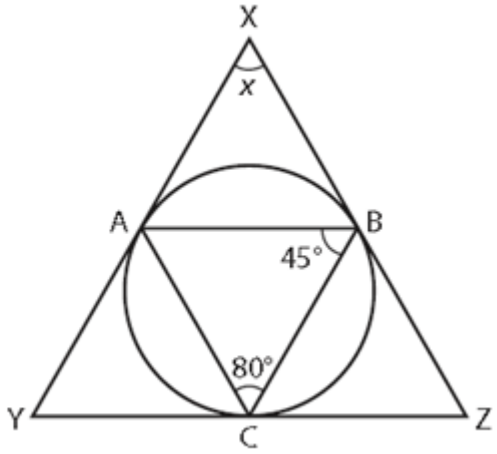

Figure 10.22

8.

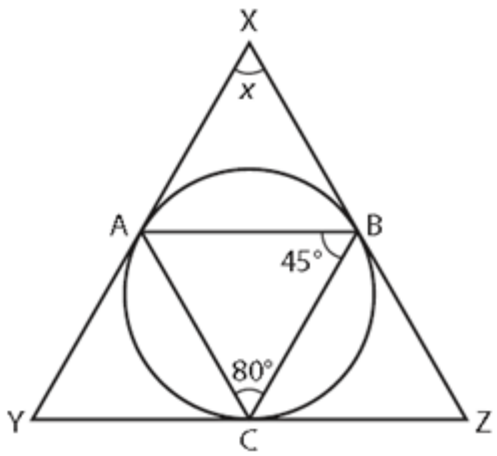

Figure 10.22

In Figure 10.22, , and are tangents to a circle at A, B and C, respectively. If ®ˆ€ACB = 80ℜ°and ®ˆ€CBA = 45Â°, find the value of x.

9.

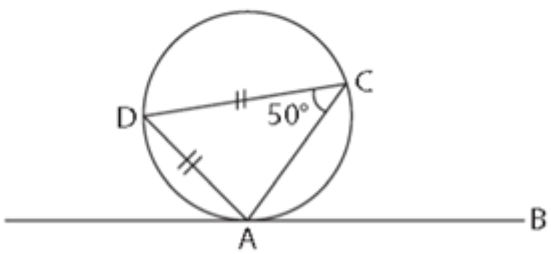

Figure 10.23

In Figure 10.23, is a tangent to circle ACD. If ®ˆ€ACD = 50Â° and |DC| = |DA|, find

®ˆ€DAC ?

10.

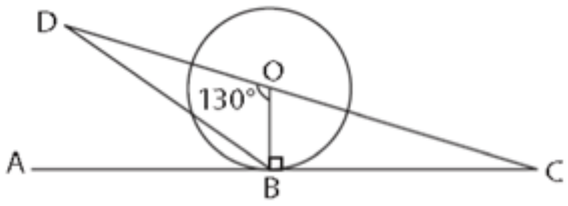

Figure 10.24

In Figure 10.24, is a tangent at point B to the circle with centre O. If ®ˆ€DOB = 130ℜ°, calculate ®ˆ€BOC and ®ˆ€BCO.

SUMMARY

In this chapter, we have learnt the following:

- v A chord of a circle is a line segment whose two endpoints lie on the circle. The diameter, passing through the circleâ€™s centre, is the largest chord in a circle.
- v A perpendicular line drawn from the centre of a circle to bisect a chord, which is not a diameter, is at right angles to the chord.
- v The angle which an arc of a circle subtends at the centre is twice the angle which it subtends at any point on the remaining part of the circumference.
- v Any angle subtended at the circumference by a diameter (or a semi-circle) is a right angle.
- v The perpendicular bisector of a circle passes through the centre of the circle.
- v The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord.
- v An angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

GRADUATED EXERCISES

1. A chord of a circle is 10 cm long. If its distance from the centre of the circle is 5 cm calculate the radius of the circle.
2. Two parallel chords lie on opposite sides of the centre of a circle of radius 12 cm. What is the distance between the chords, if their lengths are 10 cm and 15 cm?
3. Calculate the length of a chord, if the distance of the chord from the centre of a circle of radius 5 cm is 4 cm.
4.

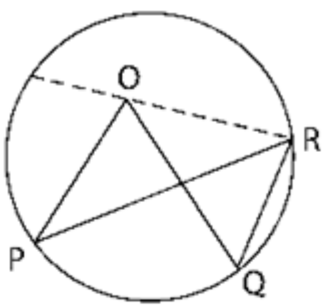

Figure 10.25

In Figure 10.25, O is the centre of the circle. P, Q and R are points on the circumference. Prove that ®ˆ€POQ = 2®ˆ€PRQ. (WAEC)

5. In Figure 10.26, O is the centre of the circle ACDB. If ®ˆ€CAO = 26ℜ° and ®ˆ€AOB = 130Â°,

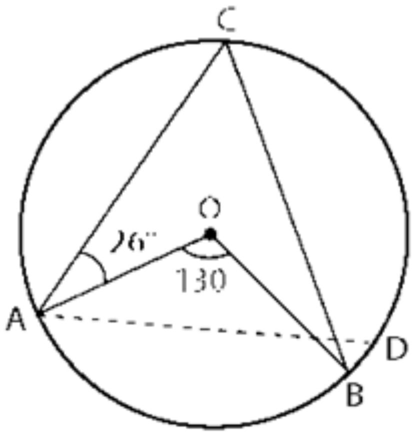

Figure 10.26

Calculate (a) ®ˆ€OBC (b) ®ˆ€COB (WAEC)

6. O is the centre of a circle and AB is any chord in the circle. From O, a perpendicular is drawn to AB at C. By considering a pair of congruent triangles, prove that AC = CB. (WAEC)

7. In Figure 10.27, P, R and S are points on a circle with centre O. If ®ˆ€POR = 150Â°, calculate ®ˆ€PSR. (WAEC)

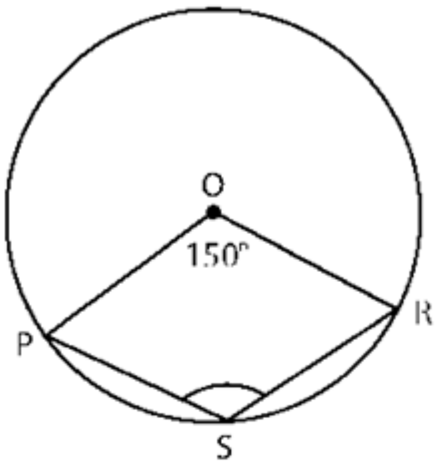

Figure 10.27

8. In Figure 10.28, P, Q and R are points on a circle with centre O. If ®ˆ€OPQ = 36ℜ°, find ®ˆ€PRQ. (WAEC)

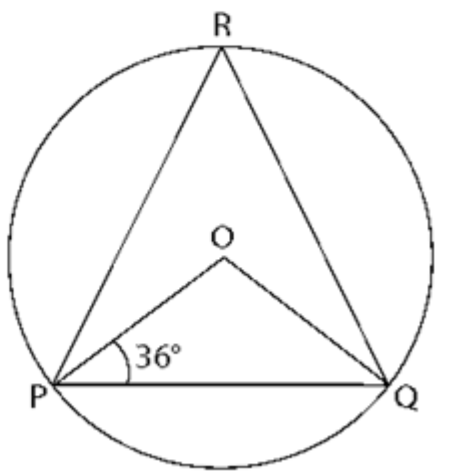

Figure 10.28

9. Find the lettered angles in Figure 10.29. O is the centre of the circle.

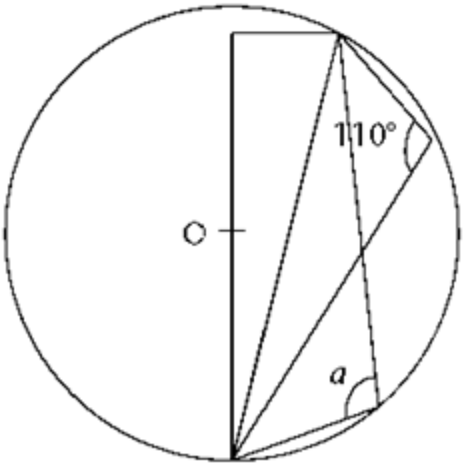

Figure 10.29

10. In Figure 10.30, XYZ is a semicircle while XY and YZ are chords. YD is the perpendicular from Y to the diameter XZ. If |XY| = 6 cm and |DZ| = 5 cm, find |YD|.

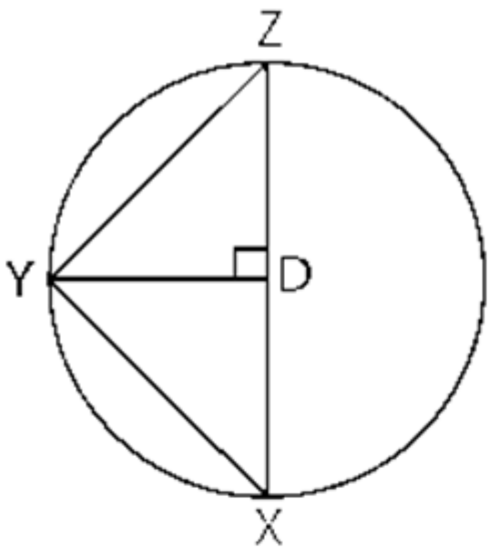

Figure 10.30