CHAPTER 3 Indices, Logarithms and Surds

Indices

The study of indices presents a less burdensome means of doing the basic mathematical operations on large numbers, particularly, large numbers, that can be presented as a repeated multiples of a number. For instance, multiplying 100 000 by 1 000 000 will take a while to evaluate using the conventional multiplication method, but with a good knowledge of indices, one can solve this in just about three lines of a work sheet.

The Concept of Exponents

An exponent is the power of a number. The exponent is commonly referred to as the 'power to which a number is raised.' 10 × 10 × 10 × 10 ×10 × 10 = 1 000 000, but 10 × 10 × 10 × 10 × 10 × 10 can be written as 10^6 (since 10 is multiplying itself in 6 places), and even from an example sited in the previous section, if the two numbers had been expressed in index form, the evaluation will be easier using the laws of indices. Thus, the rules of indices will be expressed shortly.

(i) $x^a \times x^b = x^{a+b}$; for example, $2^2 \times 2^4 = 4 \times 16 = 64$. But, instead of multiplying 4 × 16 all the way, one can simply evaluate $2^2 \times 2^4$ as $2^{2+4} = 2^6 = 64$.

(ii) $x^a \div x^b = x^{a-b}$; $5^5 \div 5^3 = 3125 \div 125 = 25$. Using indices method, $5^5 \div 5^3 = 5^{5-3} = 5^2 = 25$.

(iii) $(x^a)^c = x^{a \times c} = x^{ac}$; For instance, $(2^3)^4 = 2^3 \times 2^3 \times 2^3 \times 2^3 = 2^{3+3+3+3} = 2^{12} = 2^{3 \times 4} = (2^3)^4$. Moreover $\left(x^{\frac{a}{b}}\right)^c = x^{\frac{a}{b} \times c} = x^{\frac{ac}{b}}$.

iv) Anything (be it number, alphabet, symbol, anything at all) raised to the power of zero is equal to 1. $x^\circ = 1$; $1000^\circ = 1$; $(-4)^\circ = 1$;

$\left(\frac{1}{2}\right)^\circ = 1$; $(Ali)^\circ = 1$.

(v) A number raised to the power of a negative index is equal to the reciprocal of the same number raised to the positive index.

That is $x^{-b} = \frac{}{x^b}$. Hence, $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

Logarithm

The logarithm of 100 to the base of 10 (written as $log_{10}100$) is the power to which 10 will be raised so as to get 100; and practically, 10 will be raised to the power of 2 to get 100 (i.e. $10^2 = 100$. Therefore, the logarithm of 100 to the base of 10 is 2. Should one check the four-figure table, or use a scientific calculator, one will notice that $log_{10}11 = 1.0414$. This result simply shows that if 10 is raised to the power of 1.0414, the result will be 11. That is $10^{1.0414} = 11$.

In case one is dealing, for instance in, base 3, such that $log381$ is to be evaluated; $log_3 81$ **CANNOT** be evaluated using the four-figure table because the four figure table gives the logarithm of numbers to base 10, but we are considering 81 to base 3 here. However, recall that, $81 = 3^4$ which means 81 can be gotten by raising 3 to the power of 4, therefore, the logarithm of 81 to the base of 3 is 4 (that is $log_3 81 = 4$). So in general terms, the logarithm of a number, m, to the base of b (written as $log_b m$) is the power to which base, b will be raised to get m.

The basic rules of logarithm are stated below:

(i) $log_a X + log_a Y = log_a XY$.

(ii) $log_a X - log_a Y = log_a \left(\frac{X}{Y}\right)$.

(iii) $log_a X^n = nlog_a X$.

(iv) $log_a a = 1$.

(v) $log_a 1 = 0$.

(vi) $log_a{}^b = \frac{log_c{}^b}{log_c{}^a}$.

Surds

When the word 'surd' is mentioned, it naturally brings to mind 'square root (√),' but it is **not** all mathematical expressions that bear a square root that are surds. For instance, (√4) is **not** a surd because its value is 2, which is a rational number; surds are always irrational numbers.

For this reason, it is important to explain what irrational numbers are. Irrational numbers are numbers that **CANNOT** be expressed as a fraction (either proper or improper fraction). For example, punch √3 into the calculator; a nonrepeating number shows up with a long decimal place. The calculator has only approximated thevalue of √3 to the number of characters that it can display on the screen. Hence, a number like this that contains unending **non-repeated** decimal places (meaning it **cannot** be expressed as a fraction) is called an irrational number. Numbers like this can **only be accurately represented** as a square root, cube root, fourth root, and so on. In the light of this, examples of surds are $\sqrt{3}$, $\sqrt{5}$, $\sqrt{11}$, $^3\sqrt{10}$, $^4\sqrt{80}$. However, the following are **not** surds: $\sqrt{4}$, $\sqrt{9}$, $\sqrt{25}$, $\sqrt{225}$, $\sqrt{10\ 000}$.

Multiplication and Division of Surds

(i) $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$; for instance,

$\sqrt{100} = \sqrt{25 \times 4} = \sqrt{25} \times \sqrt{4} = 5 \times 2 = 10.$

(ii) $(a + b\sqrt{c})\,(d + e\sqrt{f}) = ad + a(e\sqrt{f}) + b\sqrt{c})d + b\sqrt{c})\,(e\sqrt{f})$; For instance,

$$(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{5}) = \sqrt{2}\sqrt{2} - \sqrt{2}\sqrt{5} + \sqrt{3}\sqrt{2} - \sqrt{3}\sqrt{5}$$

$$= \sqrt{2 \times 2} - \sqrt{2 \times 5} + \sqrt{3 \times 2} - \sqrt{3 \times 5}$$

$$= \sqrt{4} - \sqrt{10} + \sqrt{6} - \sqrt{15} = 2 - \sqrt{10} + \sqrt{6} - \sqrt{15}.$$

(iii) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$; therefore, $\sqrt{\frac{81}{25}} = \frac{\sqrt{81}}{\sqrt{25}} = \frac{9}{5}$.

Addition and Subtraction of Surds

Surds like $5\sqrt{a} - 10\sqrt{a} + 3\sqrt{a}$, which have a common surd term (that is $\sqrt{a}$) can be evaluated as follows: $5\sqrt{a} - 10\sqrt{a} + 3\sqrt{a} = 5\sqrt{a} + 3\sqrt{a} - 10\sqrt{a} = 8\sqrt{a} - 10\sqrt{a} = -2\sqrt{a}$. The expression can be viewed as $5x - 10x + 3x = -2x$, where $x = \sqrt{a}$ .Besides, $7\sqrt{a} - 10\sqrt{b}$ can be viewed as $7x - 10y$,and as this **cannot** be further evaluated, then, $7\sqrt{a} - 10\sqrt{b}$ also **cannot** be further evaluated since $\sqrt{a}$ and $\sqrt{b}$ are **not** the same.

The Concept of Conjugates and Rationalising a Surd

The two surds $\frac{x}{\sqrt{m}}$ and $\frac{y}{1-\sqrt{n}}$ (where $\sqrt{m}$ and $\sqrt{n}$ are surds) do not look tidy as both have irrational numbers (surds) as denominators. It is proper in mathematics to make the denominators of the expressions rational by removing the a surds in the denominator. Hence, this segment aims to shed light on how an expression having a surd as its denominator can be rationalised.

$\frac{x}{\sqrt{m}}$ can be rationalised by multiplying $\frac{x}{\sqrt{m}}$ by $\frac{\sqrt{m}}{\sqrt{m}}$, where x might or might **not** be a surd or **not**. Multiplying $\frac{x}{\sqrt{m}}$ by $\frac{\sqrt{m}}{\sqrt{m}}$ is as good as multiplying it with 1, since $\frac{\sqrt{m}}{\sqrt{m}} = 1$. For instance, $\frac{\sqrt{2}-1}{\sqrt{3}}$ can be rationalised as $\frac{\sqrt{2}-1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{(\sqrt{2}-1)\sqrt{3}}{\sqrt{3}\sqrt{3}} =$

$$\frac{\sqrt{2}\sqrt{3}-\sqrt{3}}{\sqrt{3\times 3}} = \frac{\sqrt{2\times 3}-\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{6}-\sqrt{3}}{3}.$$

Imagine that any one of these surds happens to be at the denominator of a surd: $2\sqrt{3} - 5$, $\sqrt{7} + \sqrt{2}$, $3 + 2\sqrt{2}$. Surds having any of these a its denominator can **only** be rationalised by using the concept of conjugates. Recall that the formula for evaluating the difference of two squares states thus:

$a^2 - b^2 = (a + b)(a - b)$. Therefore, $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2}) = (\sqrt{5})^2 - (\sqrt{2})^2 = 5^{\frac{1}{2}\times 2} - 2^{\frac{1}{2}\times 2} = 5^1 - 2^1 = 5 - 2 = 3$. So, if for example, $\sqrt{5} + \sqrt{2}$

is at the denominator of a surd, the denominator of such surd can be made rational by multiplying the denominator (which is $\sqrt{5} + \sqrt{2}$), by $\sqrt{5} - \sqrt{2}$ to have a rational number (non-surd) at the denominator. Carefully note that, as explained in the case of $\frac{x}{\sqrt{m}}$ above, both the numerator and the denominator has to be multiplied with $\sqrt{m}$ as $\frac{\sqrt{m}}{\sqrt{m}}$ in the same way, the conjugate of $\sqrt{5} + \sqrt{2}$ (which is $\sqrt{5} - \sqrt{2}$) will also be used to multiply the numerator and the denominator of the surd in question

as $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$, and this is also as good as multiplying with 1 as $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = 1$.

Hence, the surd $\sqrt{5} - \sqrt{2}$ is referred to as the conjugate of $\sqrt{5} + \sqrt{2}$. It can also be said the other way around that that surd $\sqrt{5} + \sqrt{2}$ is the conjugate of $\sqrt{5} - \sqrt{2}$; again the two surds can be called conjugates of each other. Furthermore, the conjugate of $2\sqrt{3} - 5$ is $2\sqrt{3} + 5$; the conjugate of $3 + 2\sqrt{2}$ is $3 - 2\sqrt{2}$. In general, the conjugate of a surd $a\sqrt{m} + b\sqrt{n}$ is $a\sqrt{m} - b\sqrt{n}$, or, one can say the two are conjugates of each other.

And so, the surd $\frac{2-\sqrt{3}}{\sqrt{2}-2\sqrt{3}}$ can be rationalised thus: the denominator here is $\sqrt{2}-2\sqrt{3}$, and its conjugate will be $\sqrt{2}+2\sqrt{3}$. So, multiplying the numerator and the denominator with $\sqrt{2}+2\sqrt{3}$

$$=\frac{(2-\sqrt{3})(\sqrt{2}+2\sqrt{3})}{(\sqrt{2}-2\sqrt{3})(\sqrt{2}+2\sqrt{3})}$$

$$=\frac{2\sqrt{2}+2(2\sqrt{3})-\sqrt{3}\sqrt{2}-\sqrt{3}(2\sqrt{3})}{(\sqrt{2}\sqrt{2}+\sqrt{2}(2\sqrt{3})-2\sqrt{3}\sqrt{2}-4\sqrt{3}\sqrt{3}}$$

$$=\frac{2\sqrt{2}+4\sqrt{3}-\sqrt{3}\sqrt{2}-2\sqrt{3}\sqrt{3}}{\sqrt{2\times 2}+2\sqrt{2}\sqrt{3}-2\sqrt{2}\sqrt{3}-4\sqrt{3\times 3}}$$

$$=\frac{2\sqrt{2}+4\sqrt{3}-\sqrt{3}\sqrt{2}-2\sqrt{3\times 3}}{\sqrt{4}+2\sqrt{2\times 3}-2\sqrt{2\times 3}-4\sqrt{9}}$$

$$=\frac{2\sqrt{2}+4\sqrt{3}-\sqrt{3}\sqrt{2}-2\sqrt{9}}{2+2\sqrt{6}-2\sqrt{6}-4\times 3}$$

$$=\frac{2\sqrt{2}+4\sqrt{3}-\sqrt{3}\sqrt{2}-2\times 3}{2+2\sqrt{6}-2\sqrt{6}-12}$$

$$=\frac{2\sqrt{2}+4\sqrt{3}-\sqrt{3}\sqrt{2}-6}{2+0-12}$$

$$=\frac{2\sqrt{2}+4\sqrt{3}-\sqrt{3}\sqrt{2}-6}{-10}.$$

Indices and Logarithms

1. Solve $x^{\frac{2}{3}}+5\,x^{\frac{1}{3}}+6=0$.. **(WAEC)**

Workshop

$x^{\frac{2}{3}}-5x^{\frac{1}{3}}+6=0;\ x^{\left(\frac{1}{3}\times 2\right)}-5x^{\frac{1}{3}}+6=0;\ x^{\left(\frac{1}{3}\right)^2}-5x^{\frac{1}{3}}+6=0$

Let $x^{\frac{1}{3}}=m$; *so that the equation can be rewritten as;* $m_2-5m+6=0$;

$m_2-2m-3m+6=0$; $m(m-2)-3(m-2)=0$; $(m-2)(m-3)=0$; $m-2=0$ *or* $m-3=0$; $m=2$ *or* $m=3$.

Recall that $x^{\frac{1}{3}}=m$; *so, when* $m=3$; $x^{\frac{1}{3}}=3$; $x^{\frac{1}{3}}=3^{1}$.

To know *x*, we have to make the power of *x* to be 1, and how to achieve this is to multiply the power of *x* by 3. However, we will have to multiply the power of the other side of the equation by 3 to keep the equation balanced.

Multiply the powers on both sides by 3 so that; $x^{\frac{1}{3}\times 3}=3^{1\times 3}$

> *Recall that to keep an equation balanced, whatever is done to one side of the equation must be done to the other side of the equation as we did.*

$x^{\frac{1}{3}\times 3} = 3^{1\times 3}$; $x^{\frac{3}{3}} = 3^3$; $x^1 = 3^3$; $x = 3^3 = 27$. When $m = 2$; $x^{\frac{1}{3}} = 2$, $x^{\frac{1}{3}} = 2^1$; multiply the powers on both sides by 3 to get $x^{\frac{1}{3}\times 3} = 2^{1\times 3}$; $x^{\frac{3}{3}} = 2^3$; $x^1 = 2^3$; $x = 2^3 = 8$. Therefore, the values of x satisfying the equation in question are 27 and 8.

2. If $2^{2x + 3y} = 32$ and $\log_y x = 2$, find the values of x and y. **(WAEC)**

Workshop

$2^{2x - 3y} = 32 = 2^5$; $2^{2x - 3y} = 2^5$.

Since the base are equal the powers are also equal, thus,

$2x - 3y = 5$ ....(i)

$\log_y x = 2$; recall that if $\log_m n = p$, then, $n = m^p$. T

herefore, if $\log_y x = 2$, then $x = y^2$ ....(ii).

Put $x = y^2$ into equation (i) to get

$2(y^2) - 3y = 5$; $2y^2 - 3y - 5 = 0$; $2y^2 - 5y + 2y - 5 = 0$; $y(2y - 5) + 1(2y - 5) = 0$;

$(2y - 5)(y + 1) = 0$; $2y - 5 = 0$ or $y + 1 = 0$;
$2y = 5$ or $y = -1$

$y = \frac{5}{2}$ or $y = -1$; when $y = \frac{5}{2}$, $x = y^2 = \left(\frac{5}{2}\right)^2 = \frac{5^2}{2^2} = \frac{25}{4}$;
when $y = -1$, $x = y^2 = (-1)^2 = -1 \times -1 = 1$.

Therefore, the values of x are 1 and $\frac{25}{4}$ and the values of y are -1 and $\frac{5}{2}$.

3. Solve the simultaneous equations: $8^y = 4^{(2x + 5)}$, $\log_3 y + \log_3 x + 2$. **(WAEC)**

Workshop

$8^y = 4^{(2x + 5)}$ ....(1)

$\log_3 y = \log^3 x + 2$ ....(2)

Considering (1) $8^y = 4^{(2x+5)}$; $(4 \times 2)^y = 4^{(2x+5)}$;

$\left(4_1 \times 4^{\frac{1}{2}}\right)^y = 4^{(2x+5)}$; $\left(4^{\left(1+\frac{1}{2}\right)}\right)^y = 4^{(2x+5)}$; $\left(4^{\frac{3}{2}}\right)^y = 4^{(2x+5)}\ 4^{\frac{3}{2}xy}$

$= 4_{(2x+5)}; 4^{\left(\frac{3y}{2}\right)}; = 4_{(2x+5)}$.

Because the bases are equal, then the powers are also equal so that $\frac{3y}{2} = 2x + 5$; $3y = 2(2x + 5)$;

$3y = 4x + 10$; $3y - 4x = 10$..............................(a)

From equation (2) above, $\log_3 y = (\log_3 x) + 2 = (\log_3 x) + 2(1)$

$= \log_3 x + 2(\log_3 3)$; recall that $\log_a a = 1$, therefore, $\log_3 3 = 1$; thus, $\log_3 y = \log_3 x + 2\log_3 3 = \log_3 x + \log_3 3_2$; $\log_3 y = \log_3(x \times 3_2)$;

$\log_3 y = \log_3 9x$. As the log bases on both sides are equal, then $y = 9x$ ....(b)

Put $y = 9x$ into equation (a), to get $3(9x) - 4x = 10$;

$27x - 4x = 10$;

$23x = 10$; $x = \frac{10}{23}$; $y = 9x = 9\left(\frac{10}{23}\right) = \frac{90}{23}$.

Therefore, $x = \frac{10}{23}$ and $y = \frac{90}{23}$.

4. Solve the equation: $\log_{10}(4p^2 + 1) - 2\log_{10} P - \log_{10} 2 = 1$. **(WAEC)**

Workshop

Recall that $n \log_a x = \log_a x_n$; thus, $\log_{10}(4P + 1) - 2\log_{10} P - \log_{10} 2 = \log_{10}(4P_2 + 1) - \log_{10} P_2 - \log_{10} 2 = 1$. Moreover, recall that $\log_m a - \log_m b = \log_m\left(\frac{a}{b}\right)$, and so, $\log_{10}(4P_2 + 1) - \log_{10} P_2 - \log_{10} 2$

$= \log_{10}\left[\frac{(4P_2 + 1)}{(P_2)(2)}\right] = 1$; (*recall that if* $\log_a b = c$, *then* $b = a_c$)

Hence, $\log_{10}\left[\frac{(4P_2 + 1)}{(P_2)(2)}\right] = 1$; $\frac{4P_2 + 1}{(P_2)(2)} = 10_1$;

$4P_2 + 1 = 2P_2(10)$;

$4P_2 + 1 = 20P_2$; $16P_2 = 1$; $P_2 = \frac{1}{16}$; $P = \pm\sqrt{\frac{1}{16}}$;

$P = \pm\frac{\sqrt{1}}{\sqrt{16}} = \pm\sqrt{\frac{1}{16}} = \pm\frac{1}{4}$. Therefore, the values of P are $+\frac{1}{4}$ and $-\frac{1}{4}$.

5. Find the values of x for which:

(a) $\sqrt{x} + \sqrt{x} + 5 = 5$,

(b) $\log_2 2x + \log_2(x + 4) = 6$. **(WAEC)**

Workshop

(a) $\sqrt{x} + \sqrt{x} + 5 = 5$; $\sqrt{x} + 5 = 5 - \sqrt{x}$; Square both sides of the equation to get $(\sqrt{x} + 5)^2$

$= (5 - \sqrt{x})^2$; $\left((x + 5)^{\frac{1}{2}}\right)^2 = 25 - 10\sqrt{x} +$
$\sqrt{x}\sqrt{x}$;

$(x + 5)^{\frac{1}{2}\times 2} = 25 + (\sqrt{x})^2 - 10\sqrt{x}$; $x + 5 = 25$
$+ x - 10\sqrt{x}$;

$x - x + 5 - 25 = -10\sqrt{x}$; $-20 = -10\sqrt{x}$;
$\sqrt{x} = \frac{-20}{-10} = 2,$
$(\sqrt{x})^2 = 2_2$; $\left(x^{\frac{1}{2}}\right)^2 = 2_2$; $x^{\frac{1}{2}\times 2} = 2_2$; $x = 2_2 = 4$.

Therefore, the value of x for which

$\sqrt{x} + \sqrt{(x + 5)} = 5$ is 4.

b) $\log_2 2x + \log_2(x + 4) = 6$; $\log_2(2x)(x + 4) = 6$; recall that if $\log_a b = x$,

than, $b = a^x$; therefore, $2x(x + 4) = 2^6$; $2x(x + 4) = 64$;

$2x^2 + 8x = 64$; divide the equation through

by 2 to get $x^2 + 4x = 32$;

$x^2 + 4x - 32 = 0$; $x^2 + 8x - 4x - 32 = 0$;

$x(x + 8) - 4(x + 8) = 0$;

$(x + 8)(x - 4) = 0$; $(x + 8) = 0$; or $(x - 4) = 0$;

$x = -8$ or $x = 4$.

Therefore, the values of x for which

$\log_2 2x + \log_2(x + 4) = 6$ are 4 and -8

6. Simplify $\frac{2\sqrt{15} + \sqrt{3}}{\sqrt{15} + \sqrt{3}}$. **(WAEC)**

Workshop

To simplify this surd, we must first of all make sure the denominator is not in surd form (the denominator is the set of numbers below the division sign). This can be done by multiplying the numerator and the denominator by the conjugate of the denominator. For example, the conjugate of $\sqrt{a}+\sqrt{b}$ is $\sqrt{a}-\sqrt{b}$, while the conjugate of $\sqrt{m}-\sqrt{n}$ is $\sqrt{m}+\sqrt{n}$. The denominator is $\sqrt{15}+\sqrt{3}$, its conjugate will be $\sqrt{15}-\sqrt{3}$.

So, $\dfrac{2\sqrt{15}+\sqrt{3}}{\sqrt{15}+\sqrt{3}} = \dfrac{2\sqrt{15}+\sqrt{3}}{\sqrt{15}+\sqrt{3}} \times \dfrac{\sqrt{15}-\sqrt{3}}{\sqrt{15}-\sqrt{3}}$

$$= \frac{2\sqrt{15}\sqrt{15} - 2\sqrt{15}\ \sqrt{3} + \sqrt{3}\sqrt{15} - \sqrt{3}\sqrt{3}}{\sqrt{15}\sqrt{15} - \sqrt{15}\sqrt{3} + \sqrt{15}\sqrt{3} - \sqrt{3}\sqrt{3}}$$

$$= \frac{2(\sqrt{15})^2 - 2\sqrt{15\times 3} + \sqrt{3\times 15} - (\sqrt{3})^2}{(\sqrt{15})^2 - \sqrt{15\times 3} + \sqrt{15\times 3} - (\sqrt{3})^2}$$

$$= \frac{2(15) - 2\sqrt{45} + \sqrt{45} - 3}{15 - \sqrt{45} + \sqrt{45} - 3}$$

(Recall that $-2x + x = -x$, *then* $-2\sqrt{45} + \sqrt{45} = -\sqrt{45}$*)*

Therefore, $\dfrac{2(15) - 2\sqrt{45} + \sqrt{45} - 3}{15 - \sqrt{45} + \sqrt{45} - 3}$

$$= \frac{30 - 3 - \sqrt{45}}{15 - 3}$$

$$= \frac{27 - \sqrt{3\times 3\times 5}}{12} = \frac{27 - 3\sqrt{5}}{12} = \frac{27}{12} - \frac{3\sqrt{5}}{12}$$

$$= \frac{9}{4} - \frac{\sqrt{5}}{4}.$$

7. Solve the equation $\sqrt{4x-3} - \sqrt{2x-5} = 2$.

(WAEC)

Workshop

$\sqrt{4x-3} - \sqrt{2x-5} = 2;\ \sqrt{4x-3} = 2 + \sqrt{2x-5};$

Square both sides of the equation to get

$(\sqrt{4x-3})^2 = (2 + \sqrt{2x-5})^2;\ \left((4x-3)^{\frac{1}{2}}\right)^2$

$= (2 + \sqrt{2x-5})(2 + \sqrt{2x-5});$

$(4x-3)^{\frac{1}{2}\times 2} = 4 + 2\sqrt{2x-5} + 2\sqrt{2x-5} + (\sqrt{2x-5})^2.$

If $2x + 2x = 4x$, then $2\sqrt{2x-5} + 2\sqrt{2x-5}$ will be equal to $4\sqrt{2x-5}$;

therefore, $(4x-3)^1 = 4 + 4\sqrt{2x-5} + \left((2x-5)^{\frac{1}{2}}\right)^2;$

$4x - 3 = 4 + 4\sqrt{2x-5} + (2x-5)^{\frac{1}{2}\times 2};\ 4x - 3 = 4 + 4\sqrt{2x-5} + (2x-5)^1;$

$4x - 3 = 4 + 4\sqrt{2x-5} + 2x - 5;\ 4x - 3 = 4\sqrt{2x-5} + 2x - 1;$

$4x - 3 + 1 - 2x = 4\sqrt{2x-5};\ 2x - 2 = 4\sqrt{2x-5}.$

Square both sides of the equation to get $(2x-2)^2 = (4\sqrt{2x-5})^2$; $4x^2 - 8x + 4 = 4^2(\sqrt{2x-5})^2$;

$4x^2 - 8x + 4 = 16(2x-5)$; $4x^2 - 8x + 4 = 32x - 80$;

$4x^2 - 8x + 4 - 32x + 80 = 0$; $4x^2 - 40x + 84 = 0$;

divide through this equation by 4 to get

$x^2 - 10x + 21 = 0$; $x^2 - 7x - 3x + 21 = 0$;

$x(x-7) - 3(x-7) = 0$;

$(x-7)(x-3) = 0$; $x-7 = 0$ *or* $x-3 = 0$; $x = 7$ *or* $x = 3$.

Therefore, the values of x for which $\sqrt{4x-3} - \sqrt{2x-5} = 2$ are 7 and 3.

8. Simplify $\frac{\sqrt{75}-3}{\sqrt{3}+1}$ leaving your answer in the form $a + b\sqrt{c}$ where a, b and c are rational numbers. **(WAEC)**

Workshop

To simplify $\frac{\sqrt{75}-3}{\sqrt{3}+1}$ we need to multiply the numerator and the denominator by the conjugate of the denominator. The denominator is $\sqrt{3}+1$ and its conjugate will be $\sqrt{3}-1$.

Hence, $\frac{\sqrt{75}-3}{\sqrt{3}+1}$

$$= \frac{\sqrt{75}-3}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{\sqrt{75}\sqrt{3} - \sqrt{75} - 3\sqrt{3} + 3}{\sqrt{3}\sqrt{3} - \sqrt{3} + \sqrt{3} - 1};$$

$$= \frac{\sqrt{25 \times 3}\sqrt{3} - \sqrt{25 \times 3} - 3\sqrt{3} + 3}{\sqrt{3 \times 3} - 1}$$

$$= \frac{(\sqrt{25}\sqrt{3}\sqrt{3}) - (\sqrt{25}\sqrt{3}) - 3\sqrt{3} + 3}{\sqrt{9} - 1};$$

$$= \frac{\sqrt{25}\sqrt{9} - \sqrt{25}\sqrt{3} - 3\sqrt{3} + 3}{3 - 1}$$

$$= \frac{(5 \times 3) - 5\sqrt{3} - 3\sqrt{3} + 3}{2};$$

$$= \frac{15 - 8\sqrt{3} + 3}{2} = \frac{18 - 8\sqrt{3}}{2} = \frac{18}{2} - \frac{8\sqrt{3}}{2} = 9 - 4\sqrt{3}$$

9. Simplify the expression $\frac{3\sqrt{50} + 3\sqrt{18} - 8}{1 - \sqrt{18}}$. **(WAEC)**

Workshop

$$\frac{3\sqrt{50} + 3\sqrt{18} - 8}{1 - \sqrt{18}}$$

$$= \frac{3\sqrt{5 \times 5 \times 2} + 3\sqrt{3 \times 3 \times 2} - 8}{1 - \sqrt{18}}$$

$$= \frac{3 \times \sqrt{5 \times 5}\sqrt{2} + 3\sqrt{3 \times 3}\sqrt{2} - 8}{1 - \sqrt{18}}$$

$$= \frac{3 \times 5\sqrt{2} + 3 \times 3\sqrt{2} - 8}{1 - \sqrt{18}}$$

$$= \frac{15\sqrt{2} + 9\sqrt{2} - 8}{1 - \sqrt{18}} = \frac{24\sqrt{2} - 8}{1 - \sqrt{18}}.$$

Note that $15x + 9x = 24x$, so $15\sqrt{2} + 9\sqrt{2} = 24\sqrt{2}$, no magic! Right? Ok.

Now, to further simplify this surd, we need to rationalize it. This can be achieved by multiplying the numerator and denominator by the conjugate of the denominator. The conjugate of $1 - \sqrt{18}$ is $1 + \sqrt{18}$, therefore,

$$\frac{24\sqrt{2} - 8}{1 - \sqrt{18}} = \frac{24\sqrt{2} - 8}{1 - \sqrt{18}} \times \frac{1 + \sqrt{18}}{1 + \sqrt{18}}$$

$$= \frac{24\sqrt{2} + 24\sqrt{2}\sqrt{18} - 8 - 8\sqrt{18}}{2};$$

recall that, $a_2 - b_2 = (a + b)(a - b)$, hence, $(1 - \sqrt{18})(1 + \sqrt{18}) = 1^2 - \sqrt{18}^2$.

$$\frac{24\sqrt{2} + (24\sqrt{2 \times 18}) - 8 - (8\sqrt{3 \times 3 \times 2})}{1 - 18}$$

$$= \frac{24\sqrt{2} + (24\sqrt{36}) - 8 - (8\sqrt{3 \times 3}\sqrt{2}}{-17}$$

$$= \frac{24\sqrt{2} + (24 \times 6) - 8 - (8 \times 3\sqrt{2}}{-17}$$

$$= \frac{24\sqrt{2} + (144) - 8 - 24\sqrt{2}}{-17}$$

$$= \frac{24\sqrt{2} - 24\sqrt{2} + 144 - 8}{-17} = \frac{136}{-17} = -8.$$

Sequence and Series

1. If $(x + 1)$, $\frac{1}{2}x$ and $(x - 5)$ are the first three terms of a linear sequence, (A.P.), find their common difference. **(WAEC)**

Workshop

$(x + 1), \frac{1}{2}x, (x - 5)$, ... Since these are the first three terms of an AP, then,

$T_1 = (x + 1), T_2 = \frac{1}{2}x, T_3 = (x - 5)$. For any arithmetic progression,

$T_2 - T_1 = T_3 - T_2 =$ common difference (d).

Then, $\frac{1}{2}x - (x + 1) = (x - 5) - \frac{1}{2}x = d$

$\frac{x}{2} - x - 1 = x - 5 - \frac{x}{2}; \frac{x}{2} + \frac{x}{2} - x - x - 1 + 5 = 0;$

$\frac{2x}{2} - 2x + 4 = 0;$

$x - 2x + 4 = 0; -x + 4 = 0; -x = -4; x = 4$. Recall that,

$d = T_2 - T_1 = \frac{1}{2}x - (x + 1)$ and $x = 4$, so that

$d = \frac{1}{2}(4) - (4 + 1) = 2 - 5 = -3.$

Therefore, the common difference of the linear sequence is −3.

2. Given the sequence 1, x, x^2, x^3 ...

(a) find the sum of its first n terms;

(b) hence, evaluate 1 + 2 + 4 + 8 + ... 256.

(WAEC)

Workshop

To calculate the sum of the first n terms of the sequence 1, x, $x2$, $x3$..., one should, first of all, check if the sequence is an *AP*, *GP* or any other sequence. In this case,

$T_1 = 1, T_2 = x, T_3 = x2, \frac{T_2}{T_1} = \frac{x2}{1} = x; \frac{T_3}{T_2} = \frac{x2}{x} = x.$

Because, $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} =$ and so on, the sequence has a *common ratio*, x, which makes the sequence, a geometric progression (G.P).

(a) The sum of the first n terms of a

G.P. is $\frac{a(r^n - 1)}{r - 1}$.

For the sequence in question, $a = 1$ and common ratio, $r = x$.

Then, the sum of its first n terms

$\frac{1(x^n - 1)}{x - 1} = \frac{x^n - 1}{x - 1}.$

(b) 'Hence evaluate' means that one is to evaluate, using the previous solution. Here, the result from question 2(a) must be used in evaluating question 2(b). Then you **must** use the result from question 2(a) to solve question 2(b).

The series 1 + 2 + 4 + 8 + ... 256 can be written as $1 + 2 + 2^2 + 2^3 + ... 2^8$.

You will observe that this is also a G.P. For this question, our nth term is 256, and recall that the n^{th} term of a G.P is given by

$T_n = ar^{n-1}$, where $r = \frac{T_2}{T_1} = \frac{2}{1} = \frac{T_3}{T_2} = \frac{2^2}{2} = 2.$

$a = 1, r = 2$, so $256 = T_n = ar^{n-1} = 1 \times 2^{n-1};$

$2^{n-1} = 256 = 2^8;\ 2^{n-1} = 2^8$.

Equate the powers, since the bases are the same. Therefore, $n - 1 = 8$, $n = 8 + 1 = 9$. So, 256 is the 9th term; therefore, the last term of the series in question, is the 9th term, which is 256. From 2(a), the sum of the first n terms of the sequence

$1, x, x^2, x^3 \dots$ is $\frac{x^n - 1}{x - 1}$.

We can write the sum of the sequence 1, x, x^2, x^3 ... in the form of a series as $1 + x + x^2 + x^3$ ... Comparing this series with $1 + 2 + 2^2 + 2^3 + \dots$, we see that $x = 2$. Then, the sum of the 9 terms (n = 9), of the series in question 2(b) will be

$$\frac{x^n - 1}{x - 1} = \frac{2^9 - 1}{2 - 1} = \frac{512 - 1}{1} = 511.$$

Therefore, 1 + 2 + 4 + 8 + ... + 256 = 511.

Note that $1, x, x^2, x^3$ ... is a sequence, while $1 + x + x^2 + x^3 + \dots$ is a series.

3. (a) Find the number of terms of the linear sequence (A.P.) $\frac{1}{2}, 9, 11\frac{1}{2} \dots$ required to make a sum of 126.

   (b) If the 1st and 3rd terms of the A.P in (a) are also the 1st and 3rd terms of a geometric progression (G.P.), find the common ratio. **(WAEC)**

Workshop

(a) $a = 4$, as the sequence is an A.P, d =

$T_2 - T_1 = 6\frac{1}{2} - 4 = 2\frac{1}{2} = \frac{5}{2}$. The sum, S_n, of the first n terms of an A.P, is expressed as $S_n = \frac{n}{2}[2a + (n - 1)d]$ $\therefore$ $S_n = \frac{n}{2}[2(4) + (n - 1)\, 2\frac{1}{2}]$. Recall that we want to find the number of terms, n, in the sequence that when added together will add up to 126; therefore, we can equate S_n to 126 and calculate for n as $S_n = \frac{n\left(8 + (n + 1)\frac{5}{2}\right)}{2} = 126;$

$$n\left(8 + \frac{5}{2}n - \frac{5}{2}\right) = 2 \times 126;$$

$$\frac{5}{2}n^2 + 8n - \frac{5}{2}n = 252;\ \frac{5}{2}n^2 + \frac{11}{2}n - 252 = 0.$$

Multiply the equation through by 2 to get $5n^2 + 11n - 504 = 0$. By using the quadratic formula (almighty formula) which states that given the general quadratic equation $ax2 + bx + c = 0$;

$x = \frac{-b \pm \sqrt{b_2 - 4ac}}{2a}$; therefore,

$$n = \frac{-11 \pm \sqrt{11^2 - (4 \times 5 \times -504)}}{2 \times 5}$$

$$= \frac{-11 \pm \sqrt{10201}}{10} = \frac{-11 \pm 101}{10};$$

$$n = \frac{-11 + 101}{10} \text{ or } n = \frac{-11 - 101}{10};$$

$$n = \frac{90}{10} \text{ or } n = \frac{-112}{10} \therefore n = 9 \text{ or } -11.2.$$

It follows, that the number of terms **cannot** be negative, therefore, $n = 9$.

Thus, the number of terms in the A.P, required to make a sum of 126, is 9.

(b) The nth term of a G.P. is given by

$Tn = ar^{n-1}$.For the G.P. in question,

$a = T_1 = 4$, $T_3 = 9$. $T_3 = 4r^{3-1} = 9$; $4r^2 = 9$;

$r_2 = \frac{}{4}; r = \frac{}{4}; r = \frac{}{2}$

Therefore, the common ratio of the G.P is $\frac{3}{2}$.

4. The first and the last term of a linear sequence (A.P.) are −12 and 40 respectively. If the sum of the sequence is 196, find:
   (a) the number of terms;
   (b) the common difference;
   (c) the 12th term. **(WAEC)**

Workshop

a) The sum of all the terms in an A.P. having first term, a, and last term, l, is

$$S_n = \frac{n}{2}[-12 + 40] = 196; \frac{n(28)}{2} = 196;$$

$$n = \frac{196 \times 2}{28} = \frac{392}{28} = 14.$$

Therefore, the number of terms in the sequence is 14.

(b) From the answer obtained in (a), the last term of the sequence is the 14th term, and the nth term of an AP is given by $T_n = a + (n - 1)d$.

$S_n = \frac{n}{2}[a + l]$. Where n is the number of terms in the sequence. Hence,

$$S_n = \frac{n}{2}[-12 + 40] = 196; \frac{n(28)}{2} = 196;$$

$$n = \frac{196 \times 2}{28} = \frac{392}{28} = 14.$$

Therefore, the common difference of the sequence is 4.

(c) The 12th term $T_{12} = a + (12 - 1)d = -12 + (12 - 1)4$;

$T_{12} = -12 + (11)4 = -12 + 44 = 32$.

Therefore, the 12th term of the sequence is 32.

5. The sum of the 2nd and 5th terms of an arithmetic progression (A.P.) is 42. If the difference between the 6th and 3rd terms is 12,

find the:

(a) common difference;

(b) first term;

(c) 20th term. **(WAEC)**

Workshop

The nth term of an arithmetic progression (A.P.)is given by $T_n = a + (n - 1)d$; where a is the first term and d is the common difference of the A.P.

$T_2 = a + (2 - 1)d = a + b;$

$T_5 = a + (5 - 1)d = a + 4d$; $T_2 + T_5 = a + d + (a + 4d) = a + d + a + 4d = 42$;

$2a + 5d = 42$......(i).

$T_3 = a + (3 - 1)d = a + 2d$;

$T_6 = a + (6 - 1)d = a + 5d$;

$T_6 - T_3 = a + 5d - (a + 2d) = 12$; $a + 5d - a - 2d = 12$; $a - a + 5d - 2d = 12$;

$3d = 12$...(ii); $d = \frac{12}{3} = 4$. Put $d = 4$ into equation (i) to get $2a + 5(4) = 42$;

$2a + 20 = 42$; $2a = 42 - 20 = 22$; $a = \frac{22}{2} = 11$.

a) Therefore, the common difference d of the A.P is 4 and

(b) The first term of the AP is 11

(c) $T20 = a + (20 - 1)d = 11 + (19)4 = 11 + 76 = 87$

Therefore, the 20th term of the A.P is 87.

6. The second term of a geometric progression is 3. If its sum to infinity is $\frac{25}{2}$, find the values of the common ratio. **(WAEC)**

Workshop

$T_2 = 3$; sum infinity, $S_\infty = \frac{25}{2}$. The nth term of a Geometric Progression is given by $Tn = ar^{n-1}$; where a is the first term and r is the common ratio of the G.P.

So, $T_2 = ar^{2-1} = 3$; $ar = 3$; $a = \frac{3}{r}$ ....(i).

The sum to infinity, S_∞, of a geometric progression is given by $S_\infty = \frac{a}{1-r}$.

Hence, $S_\infty = \frac{a}{1-r} = \frac{25}{2}$; $2a = 25(1-r)$..........*(ii)*.

Put $a = \frac{3}{r}$ from equation (*i*) into equation (*ii*) to get $2\left(\frac{3}{r}\right) = 25(1-r)$; $\frac{6}{r} = 25 - 25r$;

$6 = r(25 - 25r)$; $6 = 25r - 25r^2$; $25r^2 - 25r + 6 = 0$;

$25r^2 - 15r - 10r + 6 = 0$;

$5r(5r-3) - 2(5r-3) = 0$; $(5r-3)(5r-2) = 0$;

$5r - 3 = 0$ *or* $5r - 2 = 0$;

$r = \frac{3}{5}$ *or* $r = \frac{2}{5}$. Therefore, the possible values of the common ratio are $\frac{3}{5}$ *and* $\frac{2}{5}$.