Chapter 4: Quadratic Equation 2

OBJECTIVES
At the end of the chapter, students should be able to:
1. Express quadratic equations as perfect squares.
2. Solve quadratic equations using completing the square method.
3. Deduce and apply the quadratic formula in solving quadratic equations.
4. Form quadratic equations using the sum and product of the given roots.
5. Form quadratic equations from word problems.

I. Revision of Quadratic Equations

(i) Revision of factorisation of perfect squares

We learnt (in Functional Mathematics Book 1) how to solve quadratic equations using factorisation. The standard or general form of a quadratic equation is ax^2 + bx + c = 0 where a â‰ 0, a, b, c are real numbers and the x value is not fixed.

Examples are as follows:

(a) x^2 + x + 1 = 0

(b) $2x^2$ + x âˆ’ 4 = 0

(c) $3a^2$ âˆ’ 2a = 1

(d) $9a^2$ âˆ’ 25 = 0

(e) $25p^2$ + 40p + 16 = 0

To solve quadratic equations using factorisation method, first factorise the quadratic expression in the given equation. The quadratic expression will then be reduced to a product of two linear expressions in x.

Worked Example 1

Solve the quadratic equation
p^2 + 8p + 16 = 0.

SOLUTION

The quadratic equation is p^2 + 8p + 16. If this is reduced to a perfect square, you obtain:
p^2 + 8p + 16 = $(p + 4)^2$ = (p + 4) (p + 4)
The factors of the constant term 16 whose sum gives the coefficient of p are + 4 and + 4.
âˆ´ p^2 + 8p + 16 = p^2 + 4p + 4p +16
Factorising, we have p(p + 4) + 4 (p + 4) =
(p + 4) (p + 4). Hence, the quadratic equation
p^2 + 8p + 16 = 0 becomes (p + 4) (p + 4) = 0
By solving the equation, we obtain either
p + 4 = 0 or p + 4 = 0.
âˆ´p = âˆ’4 twice.

Worked Example 2a

Solve the following quadratic equation by factorisation: $\mathbf{x}^2$ âˆ’ 18**x** + 81 = 0.

SOLUTION

The factors of the constant term +81 whose sum gives the coefficient **âˆ’**18 of x are **âˆ’**9 and **âˆ’**9.
âˆ´ x^2 **âˆ’** 18x + 81 = x^2 **âˆ’** 9x **âˆ’** 9x + 81
Factorising, we have x(x **âˆ’** 9) **âˆ’**9 (x **âˆ’** 9) = (x **âˆ’** 9) (x **âˆ’** 9) = 0.
Solving, we obtain either x **âˆ’** 9 = 0 or x **âˆ’** 9 = 0.
âˆ´x = 9 twice.

Worked Example 2b

Solve this quadratic equation $3x^2 + 5x + 2 = 0$.

SOLUTION

$3x^2 + 3x + 2x + 2 = 0$

$3x (x + 1) +2 (x + 1) = 0$

$(3x + 2) (x + 1) = 0$

$3x + 2 = 0$ or $x + 1 = 0$

x = **âˆ’** $^2/_3$ or x = **âˆ’** 1

Exercise 1

Solve the following quadratic equations by factorisation:

1. $x^2 + 6x + 9 = 0$
2. x^2 **âˆ’** 12x + 36 = 0
3. $x^2 + 4x + 4 = 0$
4. $x^2 + 10x$ = **âˆ’**25
5. $16x^2$ = **âˆ’**8x **âˆ’** 1
6. $9 + 4x^2 + 12x = 0$
7. $4y^2 + 28y + 49 = 0$
8. $2t^2$ **âˆ’** t **âˆ’** 1 = 0
9. $8r^2$ **âˆ’** 9 = 21r
10. **y**^{2} âˆ’ 8**y** + 16 = 0
11. **s**^{2} âˆ’ 6**s** + 9 = 0
12. **b**^{2} âˆ’ 3**b** + 2 = 0
13. **k**^{2} + 5**k** + 4 = 0
14. **c**^{2} âˆ’ **c** âˆ’ 2 = 0
15. **x**^{2} = 5**x** âˆ’ 4
16. 6**x**^{2} + 13**x** + 6 = 0
17. **b**^{2} +3**b** + 2 = 0
18. **h**^{2} âˆ’ 2**h** âˆ’ 8 = 0
19. 3**x**^{2} âˆ’ 13**x** âˆ’ 10 = 0
20. **b**^{2} âˆ’ 8**b** âˆ’ 9 = 0
21. 81 âˆ’ 18**x** + **x**^{2} = 0
22. 9**x**^{2} = 6**x** âˆ’ 1
23. 9**r**^{2} âˆ’ 24**r** + 16 = 0
24. **b**^{2} âˆ’ 8**b** + 16 = 0
25. **y**^{2} = âˆ’25 âˆ’ 10**y**
26. âˆ’40**x** + 25**x**^{2} = âˆ’16
27. 8**x** + 1 = âˆ’16**x**^{2}
28. 12**x** + 9 = âˆ’4**x**^{2}
29. 4 + 25**y**^{2} = âˆ’20**y**
30. 1 + 36**k**^{2} = 12**k**

(ii) Revision of factorization of non-perfect squares

Unlike the previous Worked Examples 1 and 2, where the constants (**c**) are perfect squares, some times the constants (**c**) might not be perfect squares as can be seen in the following examples. When a quadratic equation is in the form of a non-perfect square, the solution of the quadratic equation is sought out by following the steps below as seen in the Worked Examples.

Worked Example 3

Solve the quadratic equation
4**x**^{2} âˆ’ 3**x** âˆ’ 1 = 0.
SOLUTION
Consider the quadratic expression of the above equation 4**x**$_2$ âˆ’ 3**x** âˆ’ 1 = 0.
Factorise the expression as follows:

Step 1: Multiply the coefficient of the first term (4**x**$_2$) by the constant term (âˆ’1) to obtain âˆ’4**x**^{2}.
Step 2: Find the two factors of this product whose sum equal the middle term (âˆ’3**x**). These are +**x** and âˆ’4**x**.

Step 3: Rewrite the given equation as follows: 4**x**^{2} + **x** âˆ’ 4**x** âˆ’ 1.

Step 4: Factorise as follows: 4**x**^{2} + **x** âˆ’ 4**x** âˆ’ 1.
That is, **x** (4**x** + 1) âˆ’1 (4**x** + 1).
The equation then becomes (**x** âˆ’ 1)(4**x** + 1) = 0.
Solving the equation, we obtain either **x** âˆ’ 1 = 0 or 4**x** + 1 = 0.
That is, either **x** = 1 or **x** = âˆ’ $^1/_4$**.**

Worked Example 4
Solve by factorisation, the equation
2**p**^{2} âˆ’ **p** âˆ’ 1 = 0.
SOLUTION
Following steps 1 to 4 in worked example 3 above, we obtain:
2**p**^{2} âˆ’ 2**p** + **p** âˆ’1 = 0
2**p** (**p** âˆ’ 1) +1 (**p** âˆ’ 1) = 0
(2**p** + 1) (**p** âˆ’ 1) = 0
2**p** + 1 = 0 or **p** âˆ’ 1 = 0
2**p** = âˆ’1 or **p** = 1
p = âˆ’$^1/_2$ or **p** = 1
âˆ´ **p** = âˆ’$^1/_2$ or 1.

Worked Example 5
Solve the quadratic equation x^2 âˆ’ 5x + 6 = 0.
SOLUTION
Consider the quadratic expression from the given quadratic equation: x$_2$ âˆ’ 5x + 6.
Factorise as follows:
Find two factors of the constant term 6 such that their sum equals âˆ’5, the coefficient of x. The factors of 6 are âˆ’2 and âˆ’3 since âˆ’2 âˆ’3 = âˆ’5.

Rewrite x^2 âˆ’ 5x + 6 as follows:
x^2 âˆ’ 2x âˆ’ 3x + 6
Factorising, we obtain x(x âˆ’ 2) âˆ’ 3 (x âˆ’ 2).
Hence, x$_2$ âˆ’ 5x + 6 = (x âˆ’ 3) (x âˆ’ 2). The required equation x$_2$ âˆ’ 5x + 6 = 0 then becomes (x âˆ’ 2) (x âˆ’ 3) = 0.
Solving the equation, we obtain either
x âˆ’ 2 = 0 or x âˆ’ 3 = 0.
That is, either x = 2 or x = 3.
Hence, x = 2 or 3.

Worked Example 6
Solve the quadratic equation x^2 âˆ’ 6x âˆ’ 16 = 0.
SOLUTION
Look for two numbers such that their product is âˆ’16 and their sum is âˆ’6. Using trial and error, these numbers are âˆ’8 and +2.

Hence, we obtain the factors as x âˆ’ 8 and x + 2 i.e. x^2 âˆ’ 6x âˆ’ 16 = (x âˆ’ 8) (x âˆ’ 2) = 0
Solving the equation, we obtain x âˆ’ 8 = 0 or x + 2 = 0
x = 8 or x = âˆ’2
Hence, x = 8 or âˆ’2.

Exercise 2

Solve the following equations by factorisation:

1. **y**^{2} âˆ’ 10**y** âˆ’ 24 = 0
2. **c**^{2} âˆ’ 5**c** + 6 = 0
3. 15 âˆ’ 2**t** âˆ’ **t**^{2} = 0
4. **x**^{2} âˆ’ 11**x** + 24 = 0
5. **h**^{2} + 25**h** + 150 = 0
6. **S**^{2} + 25**s** âˆ’ 150 = 0
7. **k**^{2} + 4**k** âˆ’ 21 = 0
8. **x**^{2} = 2**x** + 15
9. 2**h**^{2} = 5**h** âˆ’ 3
10. 2**x**^{2} âˆ’ **x** âˆ’ 1 = 0
11. 3**b**^{2} + 2**b** âˆ’ 1 = 0
12. 18**x** âˆ’ 5**x**^{2} = âˆ’35
13. 10**x**^{2} âˆ’ 41**x** âˆ’ 45 = 0
14. 6**x**^{2} âˆ’ **x** âˆ’ 2 = 0
15. **x**^{2} + 4**x** + 3 = 0
16. **x**^{2} âˆ’ **x** âˆ’ 6 = 0
17. **y**^{2} âˆ’ 21**y** + 110 = 0
18. (**x** + 5) (**x** âˆ’ 6) = 0
19. (5 âˆ’ **x**) (12 âˆ’ 2**x**) = 0
20. (2**x** + 7) ^{2} = 0

(iii) Revision of factorization of difference of two squares

Recall the factorisation of a quadratic equation of a the form **ax**^{2} + **bx** + **c** = 0 whose middle term (**bx** = 0) is factorised using the principle of the difference of two squares:
x^2 âˆ’ **y**^{2} = (**x** âˆ’ **y**) (**x** + **y**)
e.g. 4**x**^{2} âˆ’ 36 = (2**x**) ^{2} âˆ’ 6^2
= (2**x** âˆ’ 6) (2**x** + 6)

Worked Example 7

Solve by factorisation the equation 25**k**^{2} âˆ’ 9 = 0.
SOLUTION
Using the principle of the difference of two squares,
x^2 âˆ’ **y**^{2} = (**x** âˆ’ **y**) (**x** + **y**)
25**k**^{2} âˆ’ 9 = (5**k**)^2 âˆ’ 3^2
= (5**k** âˆ’ 3) (5**k** + 3)
Solving the equation 25**k**^{2} âˆ’ 9 = 0 gives 25**k**^{2} âˆ’ 9 = (5**k** âˆ’ 3) (5**k** + 3)
So either 5**k** âˆ’ 3 = 0, i.e. **k** = $^3/_5$, or 5**k** + 3 = 0 i.e. k = $^{âˆ’3}/_5$
The solutions are **k** = $^3/_5$ and **k** = $^{âˆ’3}/_5$.

Worked Example 8

Solve by factorisation the equation

$^1/_4 x^2 = {}^1/_{25}$.

SOLUTION

Using the principle of the difference of two squares,

a^2 âˆ' b^2 = (a + b) (a âˆ' b)

$$\text{e.g. } \frac{1}{4}x^2 - \frac{1}{25} = \left(\frac{1}{2}x\right)^2 - \left(\frac{1}{5}\right)^2 = 0$$

$$\Rightarrow \left(\frac{1}{2}x - \frac{1}{5}\right)\left(\frac{1}{2}x + \frac{1}{5}\right) = 0$$

$$\therefore \frac{1}{4}x^2 - \frac{1}{25} = \left(\frac{1}{2}x - \frac{1}{5}\right)\left(\frac{1}{2}x + \frac{1}{5}\right) = 0$$

So either $\frac{1}{2}x - \frac{1}{5} = 0$ i.e. $x = \frac{1}{5} \div \frac{1}{2}$

$$\Rightarrow x = \frac{2}{5}$$

or $\frac{1}{2}x + \frac{1}{5} = 0$ i.e. $x = \frac{-1}{5} \div \frac{1}{2}$

$$\Rightarrow x = \frac{-2}{5}$$

The solutions are $x = \frac{2}{5}$ and $x = \frac{-2}{5}$.

Exercise 3

Solve the following equations by factorisation:

1. $9t^2$ âˆ' 25 = 0
2. $4b^2$ = 100
3. y^2 âˆ' 9 = 0
4. $4x^2$ = 25
5. $81t^2$ âˆ' 25 = 0
6. $9b^2$ âˆ' 1 = 0
7. $25k^2$ âˆ' 4 = 0
8. $64z^2$ âˆ' 1 = 0
9. c^2 âˆ' 4 = 0
10. y^2 âˆ' 1 = 0
11. k^2 âˆ' 16 = 0
12. h^2 âˆ' 100 = 0
13. b^2 âˆ' 81 = 0
14. b^2 âˆ' 25 = 0
15. $64x^2$ âˆ' 25 = 0

(iv) Revision of solving quadratic equations using graphs

In solving quadratic equations graphically, the nature curve that a quadratic graph produces is either U-shaped or âˆ©-shaped.

The following steps should be taken when solving quadratic equations using graphs:

Step 1: Construction of a suitable table of values.
Step 2: Drawing and labelling of the **x** and **y** axes using the given scales.
Step 3: Plotting the points.
Step 4: Joining the points into a smooth curve using free-hand.
Step 5: Reading and interpreting the curve.

Worked Example 9

Draw the graph of y = x^2 âˆ’ 5x + 4 for values of x from âˆ’2 to +5. Hence, solve x^2 âˆ’ 5x + 4 = 0.
The table of values is shown below:

Table 4.1

x	−2	−1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
$-5x$	10	5	0	−5	−10	−15	−20	−25
+4	+4	+4	+4	+4	+4	+4	+4	+4
y	18	10	4	0	−2	−2	0	4

Plotting the points, we obtain the graph in Figure 4.1.

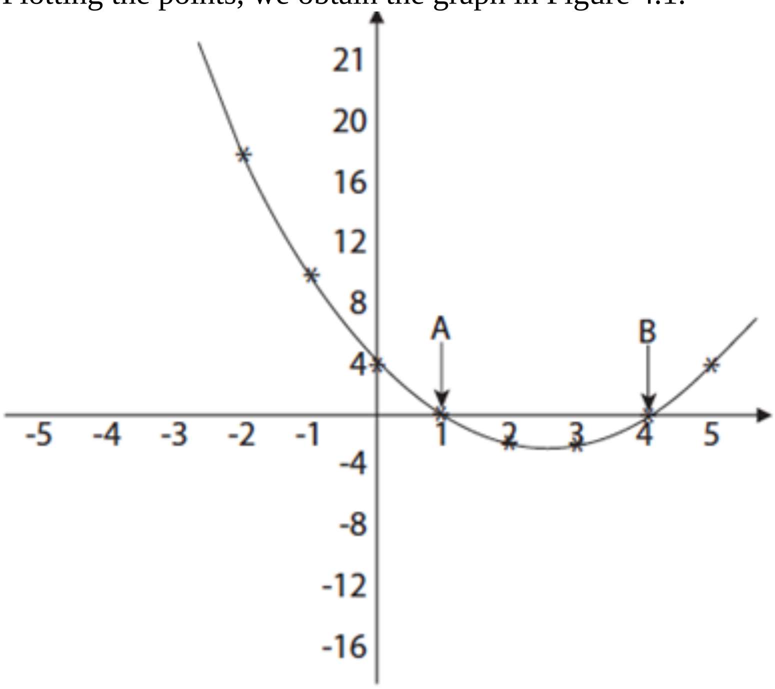


Figure 4.1

From the graph in Figure 4.1 the roots of the equation $\mathbf{x}^2$ âˆ’ 5**x** + 4 = 0 are **x** = 1 and **x** = 4.
Note: The roots of the equation are obtained at the points A and B (the points where the curve meets the **x**-axis).

Worked Example 10

Draw the graph of y = x_2 âˆ’ 3x âˆ’ 10 for âˆ’ 4 â‰¤ x â‰¤ 7. From your graph, find the roots of x_2 âˆ’ 3x âˆ’ 10 = 0.
SOLUTION
The table of values is shown below:

Table 4.2

x	x^2	$-3x$	-10	y
–4	16	12	–10	18
–3	9	9	–10	8
–2	4	6	–10	0
–1	1	3	–10	–6
0	0	0	–10	–10
1	1	–3	–10	–12
2	4	–6	–10	–12
3	9	–9	–10	–10
4	16	–12	–10	–6
5	25	–15	–10	0
6	36	–18	–10	8
7	49	–21	–10	18

The graph of y = x^2 âˆ’ 3x âˆ’ 10 is drawn in Figure 4.2 using the values in Table 4.2.

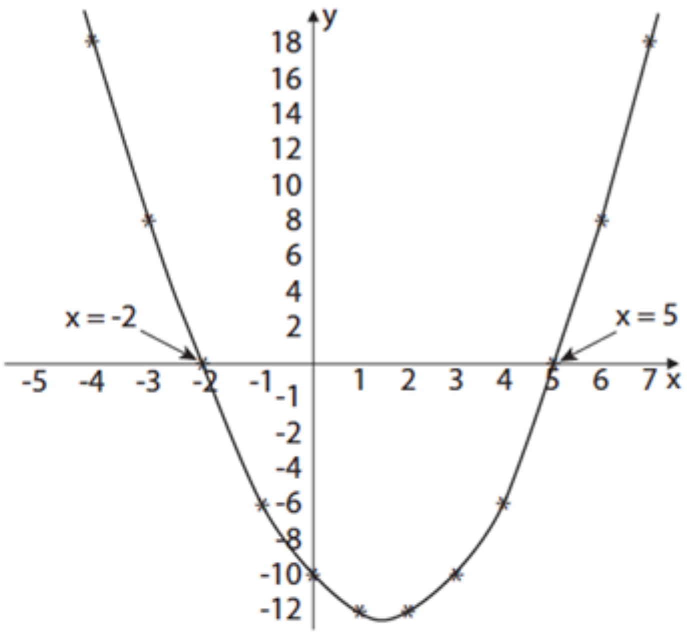


Figure 4.2

From the graph in Figure 4.2, the roots of the equation **x**^{2} âˆ’ 3**x** âˆ’ 10 = 0 are **x** = âˆ’2 and **x** = 5.

Worked Example 11

Draw the graph of **y** = **x**^{2} + 4**x** + 4 for values of **x** from âˆ’2 to +4. From the graph, find the roots of the equation **x**^{2} + 4**x** + 4 = 0.

SOLUTION

The table of values for the equation **y** = **x**^{2} + 4**x** + 4 is shown in Table 4.3.

Table 4.3

x	–2	–1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$4x$	–8	–4	0	4	8	12	16
+4	+4	+4	+4	+4	+4	+4	+4
y	0	1	4	9	16	25	36

The graph of y = x_2 + 4x + 4 is drawn in Figure 4.3 using the values in Table 4.3.

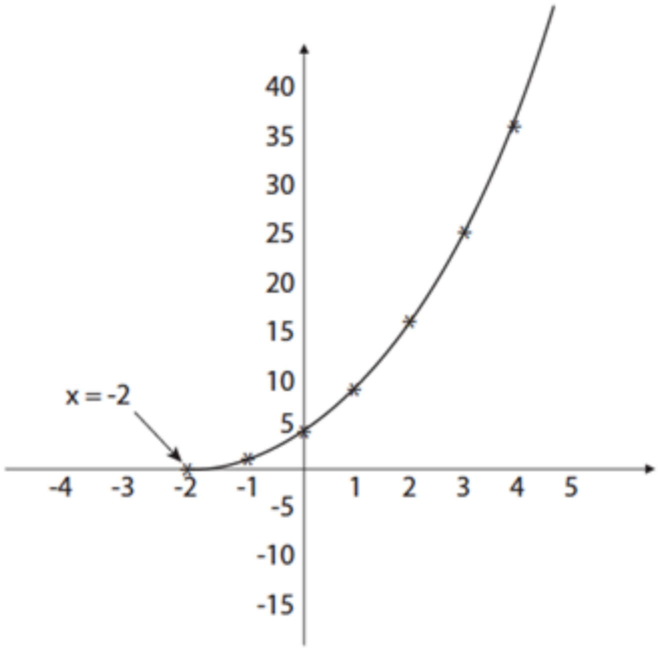


Figure 4.3

Note: Curve y = x_2 + 4x + 4 touches the x-axis at only one point. Thus, the equation has roots x = âˆ’2 twice.

Exercise 4

1. Draw the graph of y = $^1/_2$x (10 âˆ’ x) for âˆ’1 â‰¤ x â‰¤ 1. Find the roots of $^1/_2$x (10 âˆ’ x) = 0 from the graph. (WAEC)
2. Draw the graph of y = x_2 âˆ’ 3x âˆ’ 10 for âˆ’4 â‰¤ x â‰¤ 7. From your graph, find the roots of x^2 âˆ’ 3x âˆ’ 10 = 0. (WAEC)
3. (a) Draw the graph of y = $2x^2$ âˆ’ 3 for values of x from âˆ’4 to 4.
(b) Using your graph, find the roots of $2x_2$ âˆ’ 3 = 0. (WAEC)
4. (a) Draw the graph of y = 12 + 5x âˆ’ $3x^2$ for values âˆ’3 â‰¤ x â‰¤ + 4.
(b) Find the values for which 12 + 5x âˆ’ $3x^2$ = 0. (WAEC)
5. (a) Draw the graph of y = 3 âˆ’ 4x âˆ’ $2x^2$ for âˆ’ 4 â‰¤ x â‰¤ 2.
(b) From your graph, find and correct to one decimal place, the values of x for which 3 âˆ’ 4x âˆ’ $2x^2$ = 0.
(WAEC)
6. (a) Draw the graph of y = x_2 âˆ’ 25 for values of x from âˆ’5 to 5.
(b) From your graph, find the roots of the equation x^2 âˆ’ 25 = 0.
7. Draw the graph of y = $4x_2$ + 4x + 1 for values of x from âˆ’3 to +3. From the graph, find the roots of the equation $4x^2$ + 4x + 1 = 0.
8. Draw the graph of y = x^2 âˆ’ x âˆ’ 2 for values of x from âˆ’2 to +3. Hence, solve **x^2** âˆ’ **x** âˆ’ 2 =

0.

9. The graph in Figure 4.4 is that of a quadratic function y = 2x_2 âˆ’ x âˆ’ 6. Use the information provided by the graph to find:

(a) the roots of the equation.

(b) the minimum value of the equation.

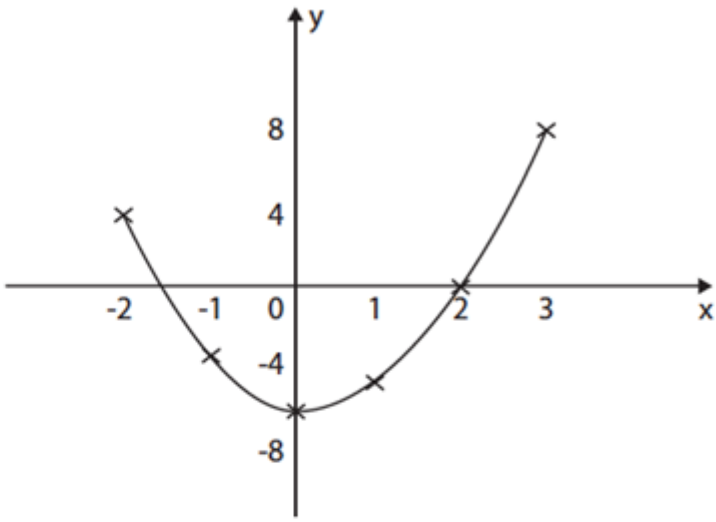


Figure 4.4

(v) Revision of forming quadratic equations with given roots

If the given roots of a quadratic equation are a and b, it is possible to form a quadratic equation from the two given values:

Let the two roots be x = âˆž and x = ÃŸ.

Then x âˆ’ âˆž = 0 or x âˆ’ ÃŸ = 0.

By expansion, we obtain:

(x âˆ’ âˆž) (x âˆ’ ÃŸ) = 0

x (x âˆ’ ÃŸ) âˆ’ âˆž (x âˆ’ ÃŸ) = 0

x_2 âˆ’ xÃŸ âˆ’ âˆžx + âˆžÃŸ = 0

x_2 âˆ’ (âˆž + ÃŸ) x + âˆžÃŸ = 0

Hence, the required equation.

Worked Example 12

Find the quadratic equation whose roots are **âˆ’**3 and 2.

SOLUTION

Let x = **âˆ’**3 and x = 2

This implies that x + 3 = 0 or x **âˆ’** 2 = 0.

Thus, (x + 3) (x **âˆ’** 2) = 0

x (x **âˆ’** 2) + 3 (x **âˆ’** 2) = 0

x^2 **âˆ’** 2x + 3x **âˆ’** 6 = 0

x^2 + x **âˆ’** 6 = 0 is the required quadratic equation.

Worked Example 13

Find the quadratic equation whose roots are $^1/_5$ and **âˆ’** $^1/_4$.

SOLUTION

Let x = $^1/_5$ and x = **âˆ’** $^1/_4$

âˆ´ x **âˆ’** $^1/_5$ = 0 or x + $^1/_4$ = 0

5x **âˆ’** 1 = 0 or 4x + 1 = 0

(5x **âˆ’** 1) (4x + 1) = 0

5x (4x + 1) **âˆ’**1 (4x + 1) = 0

20x^2 + 5x **âˆ’** 4x **âˆ’** 1 = 0

20x^2 + x **â^’** 1 = 0 is the required quadratic equation.
Note: If â^ž and ÃŸ are the roots of a quadratic equation, then the equation will be in the form:
x^2 **â^’** (â^ž + ÃŸ) x + â^žÃŸ = 0.
i.e. **x**2 â^’ (sum)**x** + product = 0.
â^ž + ÃŸ = sum of the roots while â^žÃŸ = product of the roots respectively.

(vi) Construction of quadratic equations from sum and product of roots

From the quadratic equation
ax^2 + bx + c = 0,

$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$....... (i)
If â^ž and ÃŸ are the roots of this equation,
then x = â^ž and x = ÃŸ.
i.e. x â^’ â^ž = 0 or x â^’ ÃŸ = 0
â‡’ (x â^’ â^ž) (x â^’ ÃŸ) = 0
i.e. x2 â^’ â^žx â^’ ÃŸx + â^žÃŸ = 0
â^´ x2 â^’ (â^ž + ÃŸ) x + â^žÃŸ = 0 .... (ii)

Comparing equation (i) with equation (ii), we obtain â^’ (â^ž + ÃŸ) = $\frac{b}{a}$.

i.e. â^ž + ÃŸ = â^’ $\frac{b}{a}$..................... (iii)

â^žÃŸ = $\frac{c}{a}$........................ (iv)
It follows from equation (iii) and (iv) that if â^ž and ÃŸ are the roots of equation (i), then the sum

of the roots (â^ž + ÃŸ) = â^’ $\frac{b}{a}$ and the product of the roots (â^žÃŸ) = $\frac{c}{a}$.
The above deduction is useful in finding the sum and product of roots of a quadratic equation. It can also assist us to form a quadratic equation whose roots are known. We can see that any quadratic equation can be written in the form:

$$\boxed{x^2 - (\text{sum of the roots } (\alpha + \beta))\, x + (\text{product of the roots } (\alpha\beta)) = 0}$$

Worked Example 14

Find the quadratic equation whose roots are 2 and **â^’**3.
SOLUTION
Method 1
Let α = 2 and β = **â^’**3
â^´ α + β = 2 **â^’** 3 = **â^’**1
αβ = 2 (**â^’**3) = **â^’**6

Recall that x^2 **â^’** (α + β) x + αβ = 0
â^´ x^2 **â^’** (**â^’**1)x + (**â^’**6) = 0
â^´ x^2 + x **â^’** 6 = 0
which is the required equation.

Method 2

Since the roots are 2 and â^’3,
â^´ x = 2 or x = â^’3 **â‡’** x â^’ 2 = 0 or x + 3 = 0
Hence,(x **â^’** 2) (x + 3) = 0.

That is, x^2 **â^’** 2x + 3x **â^’** 6 = 0.

x^2 + x **â^’** 6 = 0 is the same as above.

Worked Example 15

Find the quadratic equation whose roots are $\frac{3}{4}$ and **â^’** $\frac{1}{4}$
SOLUTION

Let α = $\frac{3}{4}$ and β = **â^’** $\frac{1}{4}$

â^´ α + β = $\frac{3}{4}$ + (**â^’** $\frac{1}{4}$) = $\frac{1}{2}$

αβ = $\frac{3}{4}$ **x** (**â^’** $\frac{1}{4}$) = â^’ $\frac{3}{16}$
Hence, the required equation gives

x^2 â^’ $\frac{1}{2}$x + (â^’ $\frac{3}{16}$) = 0.
That is, $16x^2$ â^’ 8x â^’ 3 = 0.

Worked Example 16

If α and β are the roots of the quadratic equation $3x^2$ â^’ x â^’ 5 = 0, find the equation whose roots are 1/α and 1/β
SOLUTION

From $3x^2$ â^’ x â^’ 5 = 0,

x^2 â^’ x/3 â^’ 5/3 = 0.

Since α and β are the roots of the given equation:

α + β = $\frac{1}{3}$, αβ = â^’ $\frac{5}{3}$
For the required equation, the roots are 1/α and 1/β
Sum of roots

= ^{1}/α + ^{1}/β = β+α/αβ = α + β/αβ = $\frac{1}{3}$/â^’ $\frac{5}{3}$

Product of roots = /α x ^{1}/β = 1/αβ = 1/â^’ $\frac{5}{3}$ = - $\frac{3}{5}$

Hence, the required equation is x^2 + $\frac{1}{5}$x + (- $\frac{3}{5}$) = 0

That is, $\mathbf{x}^2$ + $\frac{1}{5}$**x** â^’ $\frac{3}{5}$ = 0.
â^´ 5$\mathbf{x}^2$ + **x** â^’ 3 = 0 is the required equation

Exercise 5

Find the following quadratic equations whose roots are given below:

1. $\frac{3}{4}$ and - $\frac{1}{4}$
2. 2 and **âˆ’**3
3. $1\frac{1}{4}$ and $\frac{1}{4}$
4. $3\frac{2}{3}$ and $\frac{3}{4}$
5. **âˆ’**7 and 5
6. **âˆ’**0.25 and 0.75
7. $\frac{1}{4}$ and **âˆ’** $\frac{1}{5}$
8. $1/\alpha$ and $1/\beta$
9. $\sqrt{2+1}$ and $\sqrt{}$ 2-1
10. $\frac{4}{3}$ and **âˆ’** $\frac{1}{3}$
11. **âˆ’**3 and 1
12. $\frac{1}{3}$ and 2
13. **âˆ’**5 and **âˆ’**6
14. âˆš3 **âˆ’** âˆš2 and $\sqrt{3+2}$
15. $\frac{1}{\sqrt{1+2}}$ **and** $\frac{1}{\sqrt{1-2}}$
16. $\frac{\sqrt{1+5}}{2}$ **and** $\frac{\sqrt{1+5}}{2}$
17. **âˆ’** $\frac{1}{2}$ and **âˆ’** $\frac{1}{5}$
18. âˆ’ $\frac{1}{7}$ and âˆ’ 6
19. âˆ’4 and $\frac{1}{4}$
20. 0.5 and âˆ’0.75

If α and β are the roots of the following quadratic equations. Find the equations whose roots are $1/\alpha$ and $^1/\beta$.

21. 12$\mathbf{x}^2$ âˆ’ 8**x** + 1 = 0
22. $\mathbf{x}^2$ + 5**x** + 6 = 0
23. 2$\mathbf{y}^2$ + 5**y** âˆ’ 3 = 0
24. 2$\mathbf{x}^2$ + 3**x** âˆ’ 4 = 0
25. $\mathbf{m}^2$ + 2**m** âˆ’ 2 = 0
26. $\mathbf{y}^2$ âˆ’ 2**y** + 1 = 0
27. $\mathbf{m}^2$ âˆ’ 5 = 0
28. $\mathbf{r}^2$ âˆ’ 8**r** âˆ’ 5 = 0
29. 2$\mathbf{b}^2$ âˆ’ **b** âˆ’ 1 = 0
30. 4$\mathbf{y}^2$ âˆ’ 6**y** + 4 = 0

If a and b are the roots of 3**x**^{2} + 5**x** âˆ’ 1 = 0, construct equations whose roots are:
(Note: $\alpha^2 + \beta^2 = (\alpha+\beta)^2$ âˆ’ $2\alpha\beta$)

31. 5α**,** 5β
32. α^2, β^2
33. $1/\alpha$ and $1/\beta$
34. $\alpha+1/\beta$, $\beta+1/\alpha$
35. If the roots of 3**x**^{2} âˆ’ 4**x** âˆ’ 1 = 0 are α and β**,** find $\alpha + \beta$ and $\alpha\beta$.

II. Making Quadratic Expressions Perfect Squares

From our previous examples, it could be seen that all perfect square quadratic expressions are factorisable. It can therefore be deduced that some quadratic expressions are perfect squares while others are non-perfect squares.

However, certain quadratic expressions that are not perfect squares can be made perfect squares by adding a quantity, which is regarded as a constant (say **k**).

Examples of quadratic expressions that are perfect squares include:

(a) **x**^{2} âˆ’ 8**x** + 16

(b) **x**^{2} + 10**x** + 25

(c) 25**x**^{2} âˆ’ 40**x** + 16

(d) 4**x**^{2} âˆ’ 4**x** + 1

From the statement above, it is possible to convert a non-perfect square quadratic expression into a perfect square expression.

Worked Example 17

What should be added to **x**$_2$ + 5**x** to make it a perfect square?

SOLUTION

Method 1

We express **x**^{2} + 5**x** in the form (**x** + **a**)^2, (a âˆˆ Z).

Add a constant **k** to the given expression to obtain **x**^{2} + 5**x** + **k.** Then equate the two expressions as follows:

x^2 + 5**x** + **k** = (**x** + **a**)^2

Note: a and k are real numbers.

Expand $(x + a)^2$ to obtain $(x + a)^2 = x^2 + 2ax + a^2$

âˆ´ $x^2 + 5x + k = x^2 + 2ax + a^2$

By comparing the terms,

$5x = 2ax$ to give $2a = 5$ and $k = a^2$.

âˆ´ $a = \frac{5}{2}$

Hence, $k = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

âˆ´ $x^2 + 5x + k = x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$ which is a perfect square.

The term that should be added to $x^2 + 5x$ is $\frac{25}{4}$

Method 2

Firstly, find half of the coefficient of x that is $\frac{5}{2}$

Secondly, square the above result i.e. $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.

Lastly, add this square $\left(\frac{25}{4}\right)$ to the given quadratic expression to obtain a perfect square expression.

Thus, $x^2 + 5x + \frac{25}{4}$

Worked Example 18

What should be added to $4\mathbf{x}^2 + 3\mathbf{x}$ to make it a perfect square?

Method 1

Add **k** to the given quadratic expression so that the expression becomes $4\mathbf{x}^2 + 3\mathbf{x} + \mathbf{k}$ and equate it to an expression of the form $(\mathbf{ax} + \mathbf{b})^2$ where **a** and **b** are real numbers.

That is, $4\mathbf{x}^2 + 3\mathbf{x} + \mathbf{k} = (\mathbf{ax} + \mathbf{b})^2$.

By comparing the terms on both sides, we get:

$\mathbf{a}^2 = 4$ .... (i)

$2\mathbf{ab} = 3$ .... (ii)

$\mathbf{b}^2 = \mathbf{k}$ .... (iii)

Square both sides of equation (ii) to obtain $4\mathbf{a}^2\mathbf{b}^2 = 9$ and then substitute the value of $\mathbf{a}^2$ and $\mathbf{b}^2$ from (i) and (iii) into (ii) to obtain.

$4 \times 4 \times \mathbf{k} = 9$

$16\mathbf{k} = 9$

Hence, $\mathbf{k} = \frac{9}{16}$ which is the required term.

The quadratic expression then becomes $4\mathbf{x}^2 + 3\mathbf{x} + \frac{9}{16}$

Method 2

Divide the given quadratic expression through by 4 and obtain:

$\frac{4x^2}{4} + \frac{3x}{4} = x^2 + \frac{3x}{4}$, then make $x^2 + \frac{3x}{4}$ a perfect square.

Find half of the coefficient of **x** which is $\left[\frac{1}{2}\left(\frac{3}{4}\right) - \frac{3}{8}\right]$ and then square the result to obtain =

$\left(\frac{3}{8}\right)^2 - \frac{9}{64}$

Add $\frac{9}{64}$ to $x^2 + \frac{3x}{4}$ _ to obtain $x^2 + \frac{3x}{4} + \frac{9}{64}$.

The given expression is $4\mathbf{x}^2 + 3\mathbf{x} = 4\left(x^2 + \frac{3x}{4}\right)$ which becomes $4\left(x^2 + \frac{3x}{4} + \frac{9}{64}\right) = 4\mathbf{x}^2 + 3\mathbf{x} + \frac{9}{16}$

â^´ $\frac{9}{16}$ must be added to $4\mathbf{x}^2 + 3\mathbf{x}$ to make it a perfect square.

Exercise 6

Find the values of d which makes the following perfect squares:

1. $3x^2$ âˆ’ dx + 3
2. $2y^2$ âˆ’ 8xy + d
3. For what value of k will the equation $3x^2$ âˆ’ 2kx + 5 = 0 become a perfect square?
4. For what value of k will the equation x^2 âˆ’ 2x + k = 0 become a perfect square?
5. Find the value of p which makes the quadratic expression $3x_2$ âˆ’ 5x + p a perfect square.
6. Find the value of t which makes the quadratic expression $2y_2$ âˆ’ 2ty + 2 a perfect square.
7. What value of r will make the expression $3x^2$ âˆ’ 6xy + r a perfect square?
8. Given that $8x^2$ + 5x âˆ’ 13 â‰¡ m $(x + p)^2$ + r, find the value of m, p and r.
9. What value of k makes the given expression a perfect square? m^2 âˆ’ 8m + k. (WAEC)

For each of the following expressions, find the term which when added to it will make it a perfect square.

10. a^2 âˆ’ 4ab
11. $3x^2$ âˆ’ 5x
12. b^2 + 4b
13. a^2 âˆ’ 3a
14. $5b^2$ + 7bc
15. $3x^2$ âˆ’ 5xy
16. $3r^2$ âˆ’ $\frac{1}{2}$rs
17. b^2 + $\frac{1}{7}$b
18. s^2 âˆ’ $\frac{5}{7}$s
19. k^2 âˆ’ 10k
20. w^2 + $\frac{4}{5}$wy
21. x^2 + 0.5xy
22. $0.75y^2$ âˆ’ 0.5y
23. a^2 âˆ’ 6ad
24. $\mathbf{x}^2$ âˆ’ 3**xy**
25. $\mathbf{k}^2$ âˆ’ 0.75**k**
26. $\mathbf{c}^2$ + $\frac{3}{2}$ **c**
27. $\mathbf{p}^2$ + $\frac{2}{3}$**p**
28. $\mathbf{q}^2$ âˆ’ 4$\frac{2}{5}$**q**
29. $\mathbf{y}^2$ + $\frac{3}{5}$ **y**
30. $\mathbf{Z}^2$ + $\frac{2}{5}$**z**
31. $3\mathbf{y}^2$ + $\frac{1}{3}$**yx**

32. 5**k**^{2} + 7**kr**

33. âˆ' $\frac{3}{4}$**x** + **x**^{2}

34. **x**^{2} + 2**xy**

35. $\frac{y^2}{3} + \frac{1}{3}xy$

III. Solutions of Quadratic Equations by Completing the Square Method

We have considered before this section, other methods of solving quadratic equations. In this section, we shall solve quadratic equations by considering the method which is referred to as **completing the square**. This method is very useful whenever the quadratic expression is not factorisable.

Worked Example 19

Solve the equation **x**^{2} âˆ' 4**x** âˆ' 2 = 0.

SOLUTION

The quadratic expression from the given quadratic equation is not factorisable i.e. **x**^{2} âˆ' 4**x** âˆ' 2.
Rewrite the given quadratic equation to make it a perfect square.

That is: **x**^{2} âˆ' 4**x** âˆ' 2 = 0

Add 2 to both sides:

$$x^2 - 4x - 2 + 2 = 2$$
$$x^2 - 4x = 2$$

Add the term that will make **x**^{2} âˆ' 4**x** a perfect square to both sides i.e. **x**^{2} âˆ' 4**x** +

$$\left(\frac{4}{2}\right)^2 - 2 + \left(\frac{4}{2}\right)^2$$

i.e. **x**^{2} âˆ' 4**x** + 4 = 2 + 4

i.e. (**x** âˆ' 2)^2 = 6

Taking the square of both sides,

$$x - 2 = \pm\sqrt{6}$$
$$x = 2 \pm \sqrt{6}$$

i.e. either $x = 2 + \sqrt{6}$ or $2 - \sqrt{6}$.

Worked Example 20

Solve the equation **a**^{2} + 9**a** + 19 = 0.

SOLUTION

$$a^2 + 9a + 19 = 0$$
$$a^2 + 9a = -19$$

Add $\left(\frac{+9}{2}\right)^2$ to both sides

$$a^2 + 9a + \left(\frac{9}{2}\right)^2 = -19 + \frac{81}{4}$$
$$= -\frac{19}{1} + \frac{81}{4}$$
$$= \frac{-76 + 81}{4}$$
$$= \frac{5}{4}$$
$$\Rightarrow \left(a + \frac{9}{2}\right)^2 = \frac{5}{4}$$
$$\Rightarrow \left(a + \frac{9}{2}\right) = \pm\sqrt{\frac{5}{4}}$$
$$= \pm\frac{\sqrt{5}}{2}$$
$$\Rightarrow a = -\frac{9}{2} \pm \frac{\sqrt{5}}{2}$$
$$a = \frac{-9 \pm 5}{2}$$
$$\text{either } a = \frac{-9 \pm \sqrt{5}}{2} \text{ or } \frac{-9 - \sqrt{5}}{2}$$

Worked Example 21

Solve the equation 3**y**^{2} + 4**y** âˆ’ 6 = 0.

SOLUTION

The quadratic expression 3**y**^{2} + 4**y** âˆ’ 6 is not factorisable. Next, rewrite or rearrange the quadratic equation as follows:

3**y**^{2} + 4**y** = 6. Then, divide the left hand side and right hand side by the coefficient of **y**$_2$ i.e.

$$y^2 + \frac{4y}{3} - \frac{6}{3} - y^2 + \frac{4y}{3} - 2$$

Make $y^2 + \frac{4y}{3}$ a perfect square and add the required terms to the R.H.S.

i.e. $y^2 + \frac{4y}{3} + \left(\frac{4}{6}\right)^2 - 2 + \left(\frac{4}{6}\right)^2$

i.e $\left(y + \frac{4}{6}\right)^2 - 2 + \frac{16}{36}$

Taking square root of both sides

$$y+\frac{4}{6}=\pm\sqrt{\frac{22}{9}}=\pm\frac{\sqrt{22}}{3}$$

$$\therefore y=-\frac{2}{3}\pm\frac{\sqrt{22}}{3}=-2\pm\frac{\sqrt{22}}{3}$$

$$\therefore y=\frac{-2+\sqrt{22}}{3}\text{ or }y=\frac{-2-\sqrt{22}}{3}$$

Exercise 7

Solve the following equations by completing the squares:

1. **d**^{2} + 5**d** + 6 = 0
2. **e**^{2} + 7**e** = 10
3. **d**^{2} + 11**d** = 18
4. **b**^{2} + 6**b** + 8 = 0
5. **r**$_2$ âˆ’ 2**r** + 1 = 0
6. 4**x**$_2$ âˆ’ 6**x** = 0
7. 2**p**$_2$ + 3**p** âˆ’ 4 = 0
8. **s**$_2$ âˆ’ 2 = 0
9. **v**$_2$ âˆ’ 6**v** âˆ’ 7 = 0
10. 2**w**$_2$ + 2**w** = 5
11. **b**$_2$ âˆ’ **b** âˆ’ 12 = 0
12. **t**$_2$ âˆ’ 6**t** + 9 = 0
13. **x**$_2$ âˆ’ **x** âˆ’ 3 = 0
14. **y**$_2$ âˆ’ 7**y** âˆ’ 11 = 0
15. **n**$_2$ âˆ’ 10**n** + 25 = 0
16. 2**x**$_2$ âˆ’ 5**x** + 2 = 0
17. **Z**$_2$ âˆ’ 14**z** âˆ’ 3 = 0
18. **K**$_2$ + 5**k** + 4 = 0
19. **V**$_2$ âˆ’ 8**v** âˆ’ 1 = 0
20. **c**$_2$ + 4**c** âˆ’ 21 = 0
21. **y**$_2$ âˆ’ 9 = 0
22. **j**$_2$ âˆ’ 2**j** + 1 = 0
23. **q**$_2$ + 5**q** = 2
24. 4**k**$_2$ âˆ’ 6**k** = 0
25. 2**y**$_2$ + 2**y** âˆ’ 4 = 0
26. 2**x**$_2$ âˆ’ 3**x** âˆ’ 4 = 0
27. **x**$_2$ + 3**x** âˆ’ 3 = 0
28. 9 + **k**$_2$ âˆ’ 12**k** = 0
29. **b**$_2$ âˆ’ 8**b** = 5
30. 25**x**$_2$ âˆ’ 16 = 0

IV. Solutions of Quadratic Equations of the Form (ax + b)$_2$ = C

Given a quadratic equation of the form (**ax** + **b**)$_2$ = **c** where **a, b** and **c** are real numbers and **x** is not fixed. The solution of such quadratic equation is sought for.
Thus, if **y**$_2$ = 25, then **y** = $\underline{+}$ âˆš25 = $\underline{+}$ 5.

Worked Example 22

Solve the equation (**x** âˆ’ 1)^2 = 4.

SOLUTION

Take the square root of both sides
x âˆ’ 1 = $\underline{+}$ âˆš4
â‡’ **x** âˆ’ 1 = $\underline{+}$ 2
â‡’ **x** âˆ’ 1 = +2 or **x** âˆ’ 1 = âˆ’2

â‡’ **x** = +2 + 1 or **x** = âˆ’2 + 1
â‡’**x** = 3 or **x** = âˆ’1

Worked Example 23
Solve the equation (2**x** + 5)^2 = 9.
SOLUTION
Take the square root of both sides

$$2x + 5 = \pm\sqrt{9}$$
$$\text{i.e. } 2x = -5 \pm 3$$
$$x = \frac{-5 \pm 3}{2}$$
$$\Rightarrow x = \frac{+3-5}{2} \text{ or } x = \frac{-3-5}{2}$$
$$x = \frac{-2}{2} \text{ or } x = \frac{-8}{2}$$
$$x = -1 \text{ or } x = -4$$

Worked Example 24
Solve the equation (6r âˆ’ 1)^2 = 3.
SOLUTION
Take the square root of both sides
6r âˆ’ 1 = ±âˆš3

$$\text{i.e. } 6r = 1 \pm \sqrt{3}$$
$$r = \frac{1}{6} \pm \frac{\sqrt{3}}{6}$$
$$\text{i.e either } r = \frac{1}{6} + \frac{\sqrt{3}}{6} \text{ or } r = \frac{1}{6} - \frac{\sqrt{3}}{6}$$
$$r = \frac{1+\sqrt{3}}{6} \text{ or } r = \frac{1-\sqrt{3}}{6}$$

Exercise 8
Solve the following quadratic equations.

1. (y + 7)^2 = 4
2. (y + 2)^2 = 25
3. (x + 4)^2 = 3
4. (x + 2)^2 = 2
5. (y âˆ’ 6)^2 = 36
6. (y âˆ’ 1)^2 = 7
7. (7 + x)^2 = 5
8. (b + 9)^2 = 3
9. (x + 10)^2 = 8
10. (k âˆ’ 6)^2 = 3
11. (k âˆ’ 8)^2 = 3
12. (x + 3)^2 = 49
13. (x âˆ’ 4)^2 = 10
14. (h âˆ’ 3)^2 = 5
15. (h âˆ’ 1)^2 = 2
16. (z + 3)^2 = 4
17. (z âˆ’ 2)^2 = 9
18. $(x+5)^2 = \frac{1}{9}$

19. $\left(y-1\right)^2 = \frac{9}{25}$
20. (y âˆ’ 2.5)$_2$ = 6.25
21. (3j + 2)$_2$ = 1
22. (y + 1)$_2$ = 2.25
23. (x + 3)$_2$ = 49

24. $\left(4+\frac{1}{2}x\right)^2 = 25$

25. $\left(\frac{3x}{5}-\frac{1}{9}\right)^2 = 64$

V. Nature of Roots of Quadratic Equations

Given a quadratic equation ax$_2$ + bx + c = 0, the two roots are either real and unequal or may be real and equal or may also be imaginary.

The nature of the roots of a quadratic equation is best determined by the value of the discriminant, D = b$_2$ âˆ’ 4ac. If (i) D = 0, the roots are real and equal, (ii) if D > 0, the roots are real and unequal and

(iii) if D < 0, the roots are imaginary.

Worked Example 25

Determine the nature of the roots of the following quadratic equations:

(a) x$_2$ + 6x âˆ’ 9 = 0

(b) 2x$_2$ âˆ’ x + 3 = 0

(c) x$_2$ + 4x + 4 = 0

SOLUTION

(a) From x$_2$ + 6x âˆ’ 9 = 0

a = 1, b = 6, c = âˆ’9

âˆ´ D = b âˆ’ 4ac

= 6$_2$ âˆ’ 4 (1) (âˆ’9)

= 36 + 36 = 72 > 0

âˆ´ D > 0.

Hence, the roots are real and unequal.

(b) For 2x$_2$ âˆ’ x + 3 = 0

a = 2, b = âˆ’1, c = 3

âˆ´ D = b^2 âˆ’ 4ac = (âˆ’1)^2 âˆ’ 4(2) (3)

= 1 âˆ’ 24 = âˆ’23 < 0

i.e. D < 0.

Hence, the roots are imaginary.

(c) For x$_2$ + 4x + 4 = 0

a = 1, b = 4, c = 4

âˆ´D = b$_2$ âˆ’ 4ac = (4)$_2$ âˆ’ 4(1) (4)

= 16 âˆ’ 16 = 0

âˆ´ D = 0

Hence, the roots are real and equal.

Worked Example 26

Find the possible values of the constant k for which the quadratic equation 9x$_2$ âˆ’ kx âˆ’ 5x + 4 = 0 has real and equal roots. i.e. 9x$_2$ âˆ’ (k + 5) x + 4 = 0

Hence, a = 9, b = âˆ’(k + 5) and c = 4

Rearrange the equation in the form ax$_2$ + bx + c = 0 and obtain 9x$_2$ âˆ’ kx âˆ’ 5x + 4 = 0

i.e. 9x$_2$ âˆ’ (k + 5) x + 4 = 0

Hence, $a = 9$, $b = -(k + 5)$ and $c = 4$

For real and equal roots, $b^2 - 4ac = 0$.

i.e. $[(-k + 5)]^2 - 4(9)(4) = 0$

i.e. $k^2 + 10k + 25 - 144 = 0$

$k^2 + 10k - 119 = 0$

i.e. $(k + 17)(k - 7) = 0$

$\therefore$ $k = -17$ or $k = 7$

The given quadratic equation will have equal roots when $k = -17$ or 7.

Note: Square roots of negative numbers e.g. $\sqrt{-7}$, $\sqrt{-3}$ *, are imaginary.*

Exercise 9

Determine the nature of the roots of the following quadratic equations:

1. $\mathbf{x}^2 + \mathbf{x} - 3 = 0$
2. $3\mathbf{x}^2 - \mathbf{x} + 3 = 0$
3. $4\mathbf{x}^2 - 4\mathbf{x} + 1 = 0$
4. $\mathbf{d}^2 + 5\mathbf{d} + 6 = 0$
5. $16\mathbf{x}^2 - 25 = 0$
6. $\mathbf{r}^2 - 8\mathbf{r} - 1 = 0$
7. $\mathbf{k}^2 - 12\mathbf{k} + 1 = 0$
8. $\mathbf{y}^2 - 7\mathbf{y} - 11 = 0$
9. $\mathbf{b}^2 - 2\mathbf{b} + 1 = 0$
10. $(6\mathbf{r} - 1)^2 = 3$
11. $2\mathbf{x}^2 - \mathbf{x} - 1 = 0$
12. $3\mathbf{x}^2 - 2\mathbf{x} + 4 = 0$
13. $\mathbf{x}^2 = \mathbf{x} - 5$
14. $\mathbf{x}^2 + 5\mathbf{x} - 1 = 0$
15. $4\mathbf{x}^2 - 28\mathbf{x} + 49 = 0$

Find the possible values of the constant **k** for which the following quadratic equations have real and unequal, and real and equal roots:

16. $\mathbf{x}^2 - \mathbf{kx} - 12 = \mathbf{x} - 12$
17. $3\mathbf{x}^2 - 5\mathbf{x} + \mathbf{k} = 3\mathbf{x} + 5$
18. $\mathbf{kx}^2 + 6\mathbf{x} - 8 = 2\mathbf{x} - 5$
19. $8\mathbf{x}^2 - \mathbf{kx} - 1 = -\mathbf{x} - 1$
20. $\mathbf{x}^2 - 2\mathbf{x} - \mathbf{k} = -5\mathbf{x} + 3$

VI. Deducing the Quadratic Formula from Completing the Square

The standard form of a quadratic equation is $\mathbf{ax}^2 + \mathbf{bx} + \mathbf{c} = 0$. The roots of the general quadratic equation are deduced by completing the square. This formula is another method of solving quadratic equations. The formula is very useful, as it can be used to solve all forms of quadratic equations-like method of completing the square. Unlike factorisation method that is limited to solving some problems, quadratic formula and method of completing the square are not limited to particular problems.

The quadratic formula is deduced as follows:

$$ax^2 + bx + c = 0$$

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$ (Divide through by the coefficient of x^2)

$$\Rightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$$ (Collect the like terms)

$$\Rightarrow x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

(Add to both sides, the square of half the coefficient of x)

$$= -\frac{c}{a} + \frac{b^2}{4a^2}$$ (Simplify RHS)

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$ (Take square of both sides)

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x - \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(Make x the subject of the formula)

There are two values of x. They are

$x = -b + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x = -b - \frac{\sqrt{b^2 - 4ac}}{2a}$

Note: the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is called the quadratic formula.

Worked Example 27

Find the roots of the quadratic equation correct to 2 decimal places 4**x**^{2} + 7**x** âˆ’ 2 = 0.

SOLUTION

Comparing the given quadratic equation with the general form of quadratic equations, we get:

4**x**$_2$ + 7**x** âˆ’ 2 = 0 with **ax**$_2$ + **bx** + **c** = 0

a = 4, **b** = 7, **c** = 2

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 4 \times 2}}{2 \times 4}$$

$$x = \frac{-7 \pm \sqrt{49 - 32}}{8}$$

$$x = \frac{-7 \pm \sqrt{17}}{8}$$

$$x = \frac{-7 \pm 4.12}{8}$$

$$x = \frac{-7 + 4.12}{8} \quad \text{or} \quad x = \frac{-7 - 4.12}{8}$$

$$x = \frac{-2.88}{8} \quad \text{or} \quad x = \frac{-11.12}{8}$$

$$x = -0.36 \quad \text{or} \quad x = -1.39$$

Worked Example 28

Solve the equation **y**2 + 3**y** â̂’ 10 = 0 using the quadratic formula.

SOLUTION

Comparing **y**2 + 3**y** â̂’ 10 = 0 with **ax**2 + **bx** + **c** = 0,

a = 1, **b** = 3, **c** = â̂’10

Using the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - (4 \times 1 \times -10)}}{2 \times 1}$$

$$x = \frac{-3 \pm \sqrt{9 + 40}}{2}$$

$$x = \frac{-3 \pm \sqrt{49}}{2}$$

$$x = \frac{-3 \pm 7}{2}$$

$$x = \frac{-3 + 7}{2} \quad \text{or} \quad x = \frac{-3 - 7}{2}$$

$$x = \frac{4}{2} \quad \text{or} \quad x = \frac{-10}{2}$$

$$x = 2 \quad \text{or} \quad x = -5$$

Note: Worked Example 28 can also be solved by the method of factorisation.

Thus,

$$y^2 + 3y - 10 = 0$$

$$y^2 - 2y + 5y - 10 = 0$$

$$y(y - 2) + 5\,(y - 2) = 0$$

$$(y + 5)\,(y - 2) = 0$$

$$y + 5 = 0 \text{ or } y - 2 = 0$$

$$y = -5 \text{ or } y = 2$$

Exercise 10

Solve the following quadratic equations using the quadratic formula:

1. y^2 âˆ’ $5y$ + 6 = 0
2. x^2 âˆ’ $4x$ âˆ’ 5 = 0
3. y^2 âˆ’ $5x$ + 4 = 0
4. y^2 + $2y$ + 1 = 0
5. y^2 âˆ’ $2y$ âˆ’ 4 = 0
6. $3x^2$ âˆ’ $5x$ âˆ’ 7 = 0
7. $2x^2$ âˆ’ $5x$ âˆ’ 3 = 0
8. x^2 + $2x$ âˆ’ 3 = 0
9. x^2 + $2x$ +1 = 0
10. $2x^2$ âˆ’ $2x$ âˆ’ 5 = 0
11. $3x^2$ âˆ’ $5x$ âˆ’ 3 = 0
12. $2x^2$ âˆ’ x âˆ’ 1 = 0
13. $4y^2$ âˆ’ y = 0
14. $10y^2$ âˆ’ y âˆ’ 1 = 0
15. $3x^2$ âˆ’ $13x$ + 10 = 0
16. $6x^2$ âˆ’ $7x$ âˆ’ 5 = 0
17. $6x^2$ âˆ’ x âˆ’ 2 = 0
18. $4x^2$ + $7x$ âˆ’2 = 0
19. $3x^2$ âˆ’ $13x$ âˆ’ 10 = 0
20. $2x^2$ âˆ’ $5x$ + 2 = 0
21. $3x^2$ + $7x$ + 3 = 0
22. $5x^2$ + $3x$ âˆ’ 3 =0
23. x^2 + px + 18 = 0
24. $2y^2$ âˆ’ $2ty$ + 2 = 0
25. $3x^2$ âˆ’ $6xy$ + 8 = 0

VII. Maximum and Minimum Values of Quadratic Equations

The graph of $y = ax^2 + bx + c$ has the following shapes: When $a > 0$, the minimum value A is obtained as shown in Figure 4.5. When $a < 0$, the maximum value of B is obtained as shown in Figure 4.6.

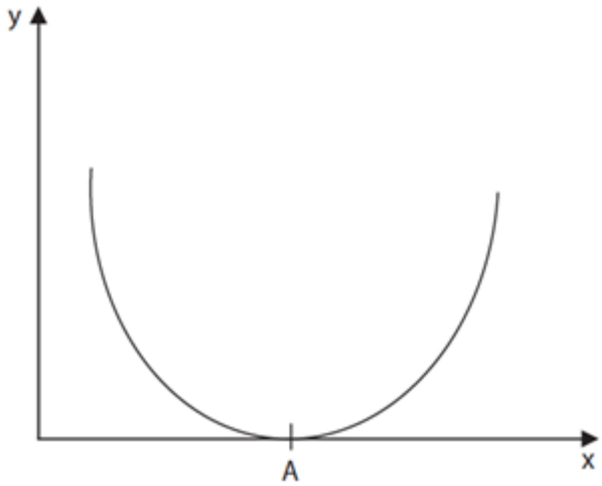


Figure 4.5

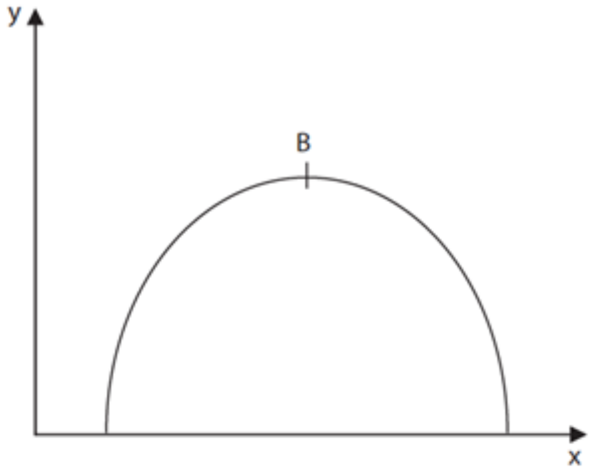


Figure 4.6

We are interested in fi nding at what value of x is the minimum or maximum value attained in each case.

(i) If a > 0, the minimum value of y is given by $\frac{4ac - b^2}{4a}$ and the corresponding value of $x = \frac{-b}{2a}$.

(ii) If a < 0, the maximum value of y is given by $\frac{4ac - b^2}{4a}$ and the corresponding value of $x = \frac{-b}{2a}$. In both cases, the line of symmetry is $x = \frac{-b}{2a}$.

Note: The minimum and maximum values of y and the corresponding values of x look the same formula-wise. The conditions attached make the difference. The formula is applicable to quadratic equations only.

Worked Example 29

Find the maximum value of y = 2x âˆ' 3x$_2$ + 4 and obtain the value of x for which the maximum value is attained.

SOLUTION

y = 2x âˆ' 3x$_2$ + 4

â‡' y = âˆ' 3x$_2$ + 2x + 4

Here, a = âˆ' 3 < 0, b = 2, c = 4

a < 0 â‡'y has maximum value.

âˆ´Maximum value y_{max} = $\frac{4ac - b^2}{4a}$

$$= \frac{4\,(-3)\,(4) - 2^2}{4(-3)}$$

$$= \frac{-52}{-12} = \frac{13}{3} = 4\frac{1}{3}$$

$$\therefore y_{max} = 4\frac{1}{3}$$

The value of x for which y_{max} is attained is

$$x = \frac{-b}{2a} = \frac{-2}{2(-3)} = \frac{-2}{-6} = \frac{1}{3}$$

Worked Example 30

Find the maximum or minimum value of y = 2x$_2$ âˆ' x + 3 and its line of symmetry.

SOLUTION

Compare equation y = 2x^2 âˆ’ x + 3 with y = ax^2 + bx + c.

a = 2 > 0, b = âˆ’1 and c = 3.

Since a > 0, y has a minimum value.

$$\text{But } y_{\min} = \frac{4ac - b^2}{4a} = \frac{4(2)\,(3) - (-1)^2}{4(2)}$$

$$= \frac{23}{8} = 2\frac{7}{8}$$

$$\therefore y_{\min} = 2\frac{7}{8}$$

The line of symmetry is given by

$$x = \frac{-b}{2a} = \frac{-1\,(-1)}{2\,(2)} = \frac{1}{4}$$

Hence, the line of symmetry is x = $\frac{1}{4}$

Exercise 11

1. What is the minimum value of 3$\mathbf{x}^2$ âˆ’ 2**x** + 1 and for what value of **x** does it occur?
2. Find the maximum value of the function 5 âˆ’ **x** âˆ’ 2$\mathbf{x}^2$.
3. Find the maximum or minimum value of **y** = 3$\mathbf{x}^2$ + 2**x** âˆ’ 3. Find the maximum or minimum values of the following functions and the values of **x** at which they occur?
4. 2$\mathbf{x}^2$ âˆ’ **x** âˆ’ 1
5. 3$\mathbf{x}^2$ âˆ’ **x** âˆ’ 6
6. 2 âˆ’ 4**x** âˆ’ $\mathbf{x}^2$
7. 1 âˆ’ **x** âˆ’ $\mathbf{x}^2$
8. $\mathbf{x}^2$ âˆ’ 20**x**
9. (**x** + 3) (**x** âˆ’ 1)
10. $\mathbf{x}^2$ + b**x** + **c**
11. (1 âˆ’ **x**) (2 + **x**)
12. (x âˆ’ 3) (x + 2)
13. x^2 âˆ’ 4x + 6
14. b^2 âˆ’ 10b + 15
15. 4x^2 + 8x + 10

VIII. Deduction of the Quadratic Equation of a Curve from its Graph

Given any quadratic equation graph, its equation can be deduced or written out. Hence, quadratic equations can be constructed from any given curve of the quadratic equation.

Worked Example 31

Use Figure 4.7 to find the equations of each of the curves given.

(a)

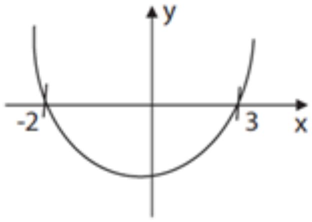


Figure 4.7 (a)

(b)

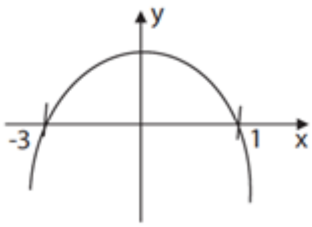


Figure 4.7 (b)

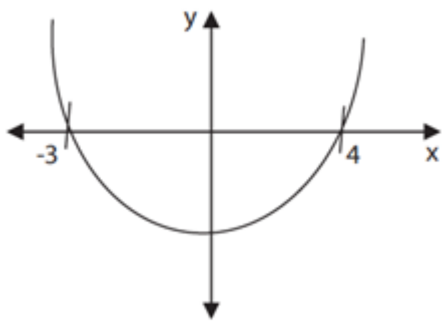
SOLUTION

In (a), the curve passes through the points
y = 0, x = â^'2 and x = 3.
â‡' x + 2 = 0 or x â^' 3 = 0
i.e. (x + 2) (x â^' 3) = 0
â^´x2 â^' x â^' 6 = 0
But y = 0, hence the required equation is
y = x2 â^' x â^' 6.
In (b), the curve passes through y = 0,
x = â^'3 and x = 1.
â‡'x + 3 = 0 or x â^' 1 = 0
â^´(x + 3) (x â^' 1) = 0
i.e. x2 + 2x â^' 3 = 0
â^´ y = x2 + 2x â^' 3

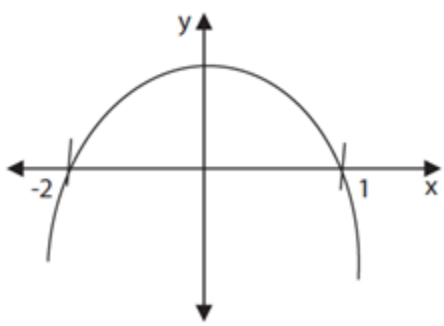
Exercise 12

Find the equations of the curves given in Figure 4.8 (1)–(10).

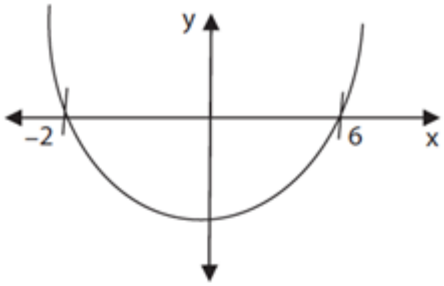
1.



2.



3.



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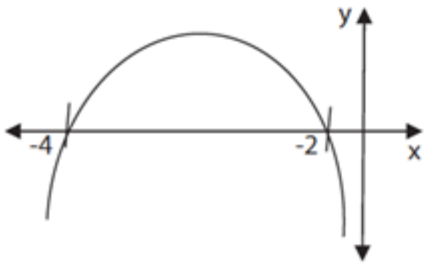

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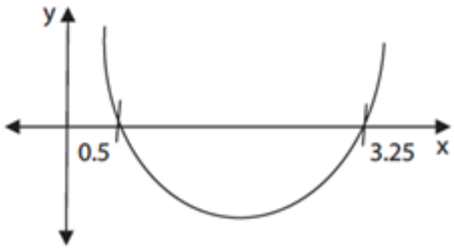

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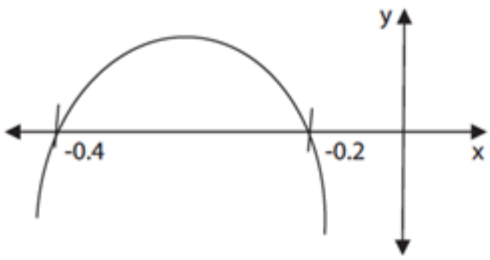

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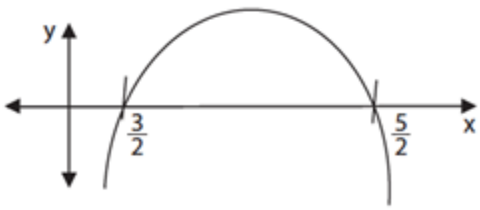

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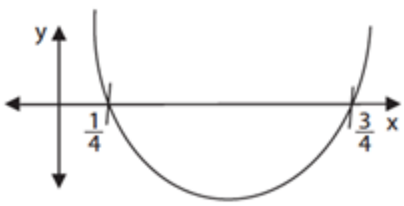

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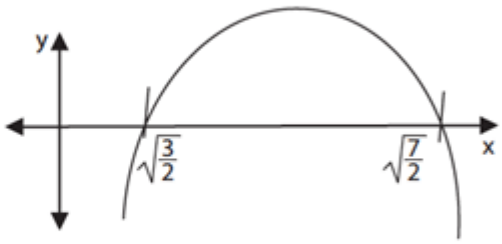

10.

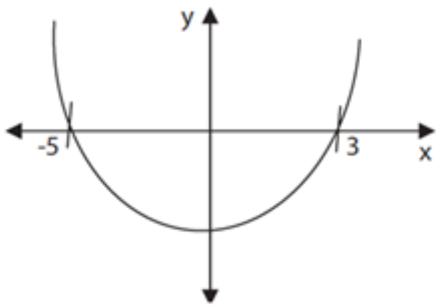

Figure 4.8

IX. Word Problems Involving Quadratic Equations

At times, word problems can lead to quadratic equations which can be solved using any of the methods discussed in this chapter.

Worked Example 32

Find two numbers whose difference is 4 and whose product is 45.

SOLUTION

Let x and y be the two numbers.

x â^’ y = 4 .... (i)

xy = 45 .... (ii)

Make x the subject of the formula in equation (i) i.e. x = 4 + y.

Substitute the value of x in equation (ii) to obtain (4 + y) y = 45.

Open the bracket (4y + y_2 = 45) and equate it to zero.

4y + y_2 â^’ 45 = 0

Rewrite in quadratic form:

y_2 + 4y â^’ 45 = 0.

y_2 â^’ 5y + 9y â^’ 45 = 0

y(y â^’ 5) + 9 (y â^’ 5) = 0

(y â^’ 5) (y + 9) = 0

y â^’ 5 = 0 or y + 9 = 0

y = 5 or y = â^’9

Since x = 4 + y â‡’ when y = 5, x = 4 + 5 = 9

and when y = â^’9, x = 4 â^’ 9 = â^’5.

i.e. x = â^’5 when y = 9 and x = 9 when y = 5.

Worked Example 33

Two numbers have a difference of 4. The difference of their squares is 64. Find the numbers.

SOLUTION

Let a and b be the two numbers.

â^´ a â^’ b = 4 .... (i)

a_2 â^’ b_2 = 64 .... (ii)

(i) â‡’ a = 4 + b

Substituting for a in (ii) gives $(4 + b)^2$ â^’ b^2 = 64

16 + 8b = 64

8b = 64 â^’ 16 = 48

b = _48_8_ = 6

a = 4 + b = 4 + 6 = 10

Hence, the numbers are 6 and 10.

Worked Example 34

A woman is 3 times as old as her daughter. Eight years ago the product of their ages was 112. Find their present ages.

SOLUTION

Let the daughterâ€™s age be x years. Then the motherâ€™s age is 3x years. Eight years ago, the daughterâ€™s age was (x âˆ’ 8) years and the motherâ€™s age was (3x âˆ’ 8) years.

The product of their ages was (x âˆ’ 8) (3x âˆ’ 8).

Hence, (x âˆ’ 8) (3x âˆ’ 8) = 112

$3x^2$ âˆ’ 32x + 64 = 112

$3x^2$ âˆ’ 32x âˆ’ 48 = 0

$3x^2$ âˆ’ 36x + 4x âˆ’ 48 = 0

3x (x âˆ’ 12) + 4 (x âˆ’ 12) = 0

(3x + 4) (x âˆ’ 12) = 0

3x + 4 = 0 or x âˆ’ 12 = 0

3x = âˆ’ 4 or x = 12

x = âˆ’ _43_ or x = 12

Discard 3x + 4 = 0 since the value is not reasonable, and conclude that x = 12.

Therefore, the daughter is 12 years old and the mother is 30 years old.

Exercise 13

1. Twice a certain whole number subtracted from 4 times the square of the number gives 6. Find the numbers.
2. Find two numbers which differ by 6 and whose product is 72.
3. A certain whole number subtracted from twice the square of the number gives 3. Find the number.
4. The square of a certain number is 22 less than 13 times the original number. Find the number.
5. Find two consecutive even numbers whose product is 224.
6. Find two consecutive odd numbers whose product is 195.
7. The length of a football field is 4 m less than the width. If its area is 45 m^2, fi nd the dimensions of the football field.
8. The area of a rectangular box is 60 cm^2. The length is 11 cm more than the width. Find the width.
9. The product of two numbers is 40 and their sum is 13. Find the number.
10. The difference of the squares of two numbers is 57 and their difference is 3. Find the numbers.
11. The breadth of a rectangle is 3 cm less than the length. If its area is 88 cm^2, find its breadth.
12. If 5 times a certain integer is subtracted from twice the square of the integer, the result is 63. Find the integer. (WAEC)
13. The ages of Eunice and Lizzy are 8 and 11 years respectively. In how many years time will the product of their ages be 208.
14. Four times the square of a number is added to 7 times the number. If the result is 11, find the number.
15. The area of a rectangle is 45 cm^2 and the perimeter is 28 cm. Find the length and breadth.
16. The sum of two numbers is 56 and their product is 300. What are the numbers?
17. The difference of two numbers is 5 and their product is 24. Find the numbers.
18. A number is added to 20 and 22. If the product of the sums obtained is 88, find the number.
19. The length of a rectangle is 6 cm longer than its breadth. If the area is 72 cm^2, find the length of the rectangle.
20. Find two consecutive numbers whose product is 156.
21. A man is 37 years old and his childâ€™s age is 8. How many years ago was the product of their ages 96?
22. Find two numbers whose difference is 5 and whose product is 150.

23. A mother is 4 times older than her child. Eight years ago, the product of their ages was 128. Find their present ages.
24. A certain number is subtracted from 18 and 13. The product of the two numbers obtained is 66. Find the first number.
25. A boy is 6 years younger than his brother. The product of their ages is 135. Find their ages.

SUMMARY

In this chapter, we have learnt the following:

â– How to solve quadratic equations using:
(a) factorisation method
(b) completing the square method
(c) formula method
(d) Graph method.
â– If a and b are the roots of a quadratic equation, then the equation whose roots are given is **x**$_2$ âˆ’ (a + b)**x** + ab = 0.
â– The root of a quadratic equation may be
(a) real and unequal here **b**$_2$ âˆ’ 4**ac** > 0.
(b) real and equal here **b**$_2$ âˆ’ 4**ac** = 0.
(c) imaginary here **b**$_2$ âˆ’ 4**ac** < 0.
â– The maximum value of a quadratic equation is obtained when **a** < 0 and the maximum value is

given as $y = \frac{4ac-b^2}{4a}$

GRADUATED EXERCISES

1. If a and b are the roots of the equation 3x$_2$ âˆ’ 5x + 2 = 0, for a > b, fi nd the value of (a) ab$_2$ âˆ’ a$_2$b and (b) a$_2$ âˆ’ ab. (WAEC)
2 If a and b are the roots of the quadratic equation 3x$_2$ âˆ’ 2x âˆ’ 5 = 0, fi nd the equation

whose roots are $\frac{1}{2\alpha}$ and $\frac{1}{2\beta}$. (WAEC)
3. Solve the equation 2x$_2$ âˆ’ 5x + 2 = 0. (WAEC)
4. If y = 2x$_2$ + 9x âˆ’ 35, fi nd the range of values for which y < 0. (JAMB)
5. The roots of the equation 4x$_2$ âˆ’ 12x âˆ’ 7 = 0 are α and β, with α greater than β. Find the value of (a) α âˆ’ β and (b) α^2 âˆ’ β^2. (WAEC)

6. Solve the equation $\frac{x+1}{4} - \frac{5}{2x-1} = 2$ giving your answer corrected to two decimal places. (WAEC)
7. Solve the equation 3x$_2$ + 8x + 1 = 0, giving your answer corrected to two decimal places. Hence, write down the values of y which satisfy 3(y + 1)^2 + 8(y + 1) + 1 = 0. (WAEC)
8. Find the sum and product of the roots of the equation (2x + 5) (x âˆ’ 1) = 0.

9. Find the sum and product of the roots of the equation $\frac{1}{7}$x^2 âˆ’ 3x + 5 = 0.
10. Find the quadratic equation whose roots are âˆ’3/4 and âˆ’ ^{1}/4.
11. One of the roots of the quadratic equation dx$_2$ + cx + p = 0 is twice the other. Find the relationship between d, c and p.
12. If a, b are the roots of the indicial equation 2^{2}x + 16 = 10 (2x), find (a) α + β and (b) 2αβ
13. Find the sum and product of the roots of the following quadratic equations:
(a) x^2 + _ 5x_ âˆ’ _ 12_ = 0
(b) 3x^2 âˆ’ 2x âˆ’ 3 = 0

14. For what value of g does the function $y = 3x^2 + gx + 4$ have a minimum value of 1?
15. For what value of k does the function $y = 2x_2$ âˆ’ $4x + k$ have a minimum value of 2?