# CHAPTER 2 Trigonometry

In this section, an in-depth study of trigonometric identities will be done. It is important, however, to take a look at the building blocks of trigonometry; these building blocks are angles (which can be represented with the symbol θ) sin θ, cos θ and tan θ. An angle is formed where two **straight** lines meet. In other words, an angle is a measure of how wide apart these two straight lines are moving away from each other. Figure 2.1 shows an angle, θ, between two lines meeting each other at point *O*.

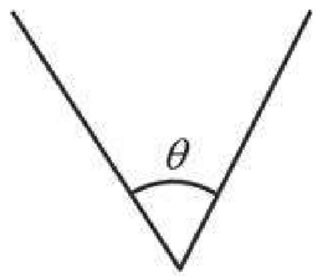


**Fig. 2.1**

Having understood what an angle is, let us examine sin θ, cos θ and tan θ. These three parameters are used to define the relationships between an angle in a right-angled triangle and the three sides of the triangle.

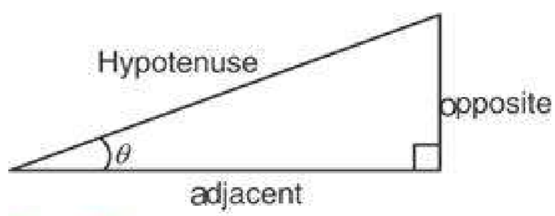


**Fig. 2.2**

From Figure 2.2, sin θ is defined as the ratio of the length of the side opposite angle θ, to the length of the hypotenuse. Thus, $\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$.

Moreover, cos θ is the ratio of the length of the side adjacent to angle θ to the length of the hypothenus.

So, $\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$.

Also, tan θ is the ratio of the length of the opposite side to that of the adjacent side. Therefore, $\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$.

The all time mnemonic for mastering these fundamentals of trigonometry is the So/h Ca/h To/a. Or, simply put, **SohCahToa**. Specific examples are provided below.

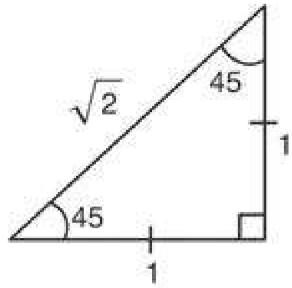


Fig. 2.3

The right-angled triangle in Figure 2.3 has its two other angles equal, therefore, the two adjacent sides to the two angles will also be equal in length. Hence, if one of the sides is 1 unit (as shown in Figure 2.3), the adjacent side will also be 1 unit. Then the hypotenuse can be calculated using Pythagoras theorem as follows:

$(Hypotenuse)^2 = (Opposite)^2 + (Adjacent)^2$;

$(Hypotenuse)^2 = (1)^2 + (1)^2 = 1 + 1 = 2$;

$(Hypotenuse)^2 = 2$; $Hypotenuse = \sqrt{2}$

With this knowledge of how the lengths of the sides came about, further progress can be made to evaluate sine, cosine and tangent of angle 45°.

From the triangle in Figure 2.3, $\sin 45° = \frac{\text{Opp}}{\text{Hyp}} = \frac{1}{\sqrt{2}}$; $\cos 45° = \frac{\text{Adj}}{\text{Hyp}} = \frac{1}{\sqrt{2}}$ $\tan 45° = \frac{\text{Opp}}{\text{Adj}} = \frac{1}{1} = 1$

Another example of a right-angled triangle with angles 30° and 60° is presented below.

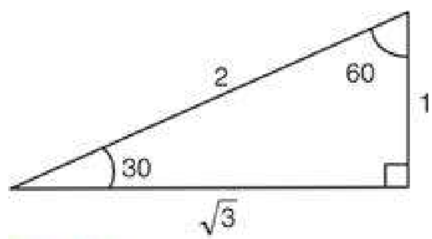


Fig. 2.4

Thus, $\sin 30° = \frac{1}{2}$; $\cos 30° = \frac{\sqrt{3}}{2}$; $\tan 30° = \frac{1}{\sqrt{3}}$ and

$\sin 60° = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{1}{2}$; $\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$.

*Note that these answers can be confirmed by checking the four-figure table, or by using a scientific calculator.*

Finally, a presentation of the basic trigonometric identities is done before resumption at the workshop.

$\sin (A + B) = \sin A \cos B + \cos A \sin B$

$\sin (A - B) = \sin A \cos B - \cos A \sin B$

$\cos (A + B) = \cos A \cos B - \sin A \sin B$

$\cos (A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta};\quad \cot\theta = \frac{1}{\tan\theta} = 1 \div \frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\sin\theta};$$

$$\sec\theta = \frac{1}{\cos\theta};\quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

## Trigonometric Identities

1. (a) Find, in surd form, the value of cos 15°.

**(WAEC)**

**Workshop**

(a) cos 15° = cos (45° − 30°). Recall that

cos ($A$ − $B$) = cos $A$ cos $B$ + sin $A$ sin $B$,

thus, cos (45° − 30°) = cos 45° cos 30° + sin 45° sin 30°;

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}};$$

$\frac{\sqrt{3}+1}{2\sqrt{2}}$ is an irrational number, because its denominator is in surd form; it contains $\sqrt{2}$ However, we can make it a rational number by removing $\sqrt{2}$ from its denominator and this could be done by rationalizing it.

$\frac{a}{\sqrt{b}}$ can be rationalised by multiplying $\frac{a}{\sqrt{b}}$ by $\frac{\sqrt{b}}{\sqrt{b}}$

*Note that : (i) 'a' could be a surd, and*

*(ii) multiplying* $\frac{a}{\sqrt{b}}$ *by* $\frac{\sqrt{b}}{\sqrt{b}}$ is as good as

*multiplying* $\frac{a}{\sqrt{b}}$ *by* 1 *since* $\frac{\sqrt{b}}{\sqrt{b}} = 1$.

Thus, if we rationalise $\frac{a}{\sqrt{b}}$, we will get

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b} \times \sqrt{b}}$$

$$= \frac{a\sqrt{b}}{\sqrt{b \times b}} = \frac{a\sqrt{b}}{\sqrt{b_2}} = \frac{a\sqrt{b}}{b}.$$

Rationalising $\frac{\sqrt{3}+1}{2\sqrt{2}}$ gives

$$\frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}\sqrt{3}+\sqrt{2}}{2\sqrt{2}\sqrt{2}} = \frac{\sqrt{2\times 3}+\sqrt{2}}{2\sqrt{2\times 2}}$$

$$= \frac{\sqrt{6}\times\sqrt{2}}{4}.\ \text{Therefore, } \cos 15° = \frac{\sqrt{6}\times\sqrt{2}}{4}.$$

*Dear student, please be advised that when attempting to prove an identity like this, you should take a few seconds to consider which side will be easier to prove from, to arrive at the other side.*

2. Given that $\tan(x-y) = \frac{-2}{3}$ and $\tan x = \frac{-1}{2}$

calculate the value of:

(a) $\tan y$;

(b) $(x+y)$ for $0° < x+y < 360°$, correct to the nearest degree. **(WAEC)**

**Workshop**

(a) Recall that, $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$;

$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{-2}{3}$ ... (*).

From the question, $\tan x = -\frac{1}{2}$;

put $\tan x = -\frac{1}{2}$ in equation (*) to get

$$\frac{\frac{-1}{2} - \tan y}{1 + \left(-\frac{1}{2}\right)\tan y} = \frac{-2}{3}; \quad \frac{\frac{-1}{2} - \tan y}{1 - \frac{\tan y}{2}} = \frac{-2}{3};$$

$$3\left(\frac{-1}{2} - \tan y\right) = -2\left(1 - \frac{\tan y}{2}\right);$$

$$\frac{-3}{2} - 3\tan y = -2 + \frac{2\tan y}{2};$$

$$-\frac{3}{2} + 2 = \tan y + 3\tan y; \; 4\tan y = \frac{1}{2};$$

$\tan y = \frac{\frac{1}{2}}{4} = \frac{1}{2} \div 4 = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$. *(recall that $x + 3x = 4x$, then* $\tan y + 3\tan y = 4\tan y$). Therefore, $\tan y$ is $\frac{1}{8}$.

(b) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$$= \frac{\frac{-1}{2} + \frac{1}{8}}{1 - \left(\frac{-1}{2}\right)\left(\frac{1}{8}\right)} = \frac{-\frac{3}{8}}{1 - \left(-\frac{1}{16}\right)}$$

$$= \frac{-\frac{3}{8}}{1 + \frac{1}{16}};$$

$$\tan(x+y) = \frac{-\frac{3}{8}}{\frac{17}{16}} = \frac{-3}{8} \div \frac{17}{16} = \frac{-3}{8} \times \frac{16}{17}$$

$$= \frac{-6}{17} = -0.3529;$$

$\tan(x+y) = -0.3529; x+y = \tan^{-1}(-0.3529).$

$\tan(x + y) = -0.3529$; $x + y = \tan^{-1}(-0.3529)$.

Let $x + y$ be equal to $t$, so that $\tan(x + y) = \tan t = -0.3529$. To know $t$, we will first find an angle $\alpha$, such that $\tan \alpha = 0.3529$ from the four figure table. $\tan \alpha = 0.3529$; $\alpha = \tan-1\ 0.3529$.

From the four figure table, $\tan-1\ 0.3529 = 19.44°$, then, $\tan 19.44° = 0.3529$.

Now, let us take a look at the trigonometric quadrants. It is already known that $\tan t = -0.3529$.

Hence, the two possible values of angle $t$ will be an angle in the **second** quadrant, **or** an angle in the **fourth** quadrant, where $\tan t$ is always negative.

For the second quadrant, recall that we made $x + y = t$ thus, $\tan(x + y) = \tan t$; $\tan t = -0.3529$; recall that $\tan 19.44° = 0.3529$, so, $-\tan 19.44° = -0.3529$;

$\tan t = -0.3529 = -\tan 19.44°$. Also, recall that in the second quadrant,

$\tan \theta = -\tan(180° - \theta)$, so that

$\tan t = -\tan(180° - t)$

$= -\tan 19.44°$. So,

$-\tan(180° - t) = -\tan 19.44°$;

$\tan(180° - t) = \tan 19.44°$;

$180° - t = \tan 19.44°$;

$t = 180° - 19.44° = 160.56°$,

$t = x + y = 160.56°$.

For the fourth quadrant, $\tan t = -0.3529$ because $\tan 19.44° = 0.3529$, then, $-\tan 19.44° = -0.3529$; therefore, $\tan t = -0.3529 = -\tan 19.44°$. Also, recall that, in the fourth quadrant, $\tan \theta = -\tan(360° - \theta)$, so that $\tan t = -\tan(360° - t) = -\tan 19.44°$; $\tan(360° - t) = \tan 19.44°$; $360° - t = 19.44°$;

$t = 360° - 19.44° = 340.56°$; but $t = x + y = 340.56°$, thus, $x + y = 340.56°$. Therefore, the possible values of $\tan-1(-0.3529)$ are $160.56°$ and $340.56°$. Remember you are to find $(x + y)$ within the range $0° < x + y < 360°$. Therefore, the values of $x + y$, in the range $0° < x + y < 360°$, correct to the nearest degree are $161°$ and $341°$.

3. Evaluate $\sin 70° \cos 10° - \sin 20° \cos 80°$, leaving your answer in surd form. **(WAEC)**

**Workshop**

sin 70° cos 10° – sin 20° cos 80°. Angles 70° and 20° are complementary angles.

Recall that if $A$ and $B$ are angles in degrees such that $A° + B° = 90°$, then angles $A$ and $B$ are complementary angles. Thus, 70° + 20° = 90°, and so sin 70°= cos 20° and sin 20°=cos 70°. Also, angles 80° and 10° are complementary angles so that cos 80° = sin 10° and sin 80° = cos 10°. Since sin 70°= cos 20° and cos 80° = sin 10°, then sin 70° cos 10° – sin 20° cos 80° = cos 20° cos 10° – sin 20° sin 10°.

Recall that cos $(A + B)$ = cos $A$ cos $B$ – sin $A$ sin $B$, so cos 20° cos 10° – sin 20° sin 10° = cos (20° + 10°) = cos 30° $\frac{\sqrt{3}}{2}$

Therefore, the value of sin 70° cos 10° – sin 20° cos 80° leaving answer in surd form is $\frac{\sqrt{3}}{2}$

4. Given that $A = 45°$, $B = 30°$; sin $(A + B)$ = sin $A$ cos $B$ – sin $B$ cos $A$ and cos $(A + B)$ = cos $A$ cos $B$ – sin $A$ sin $B$

(a) show that: sin 15° = $\frac{\sqrt{6}-\sqrt{2}}{4}$ and cos 15° = $\frac{\sqrt{6}+\sqrt{2}}{4}$

(b) hence find tan 15°. **(WAEC)**

**Workshop**

(a) $A = 45°$; $B = 30°$; sin 15° = sin (45° – 30°).

But from the question, we were told to show that sin 15° = $\frac{\sqrt{6}-\sqrt{2}}{4}$ given that sin $(A + B)$ = sin $A$ cos $B$ + sin $B$ cos $A$.

Meaning we need to express sin 15° in terms of angles 45° and 30° in the form sin $(A + B)$ and **not** sin $(A – B)$. So, we need to rewrite sin 15° = sin (45° – 30°) as follows:

sin 15° = sin (45° – 30°) = sin (45° + (– 30°)).

From the question,

sin $(A + B)$ = sin $A$ cos $B$ + sin $B$ cos $A$;

sin (45° + (– 30°)) = sin 45° cos (– 30°) + sin (– 30°) cos 45°.

Recall that for any positive angle, θ,

sin (−θ) = −sin θ

cos (−θ) = cos θ and

tan (−θ) = −tan θ

$LHS$ = sin 15° = sin (45° + (−30°))

= sin 45° (cos 30°) + (−sin 30°) cos 45°.

*LHS means left hand side, while RHS is right hand side.*

$$\sin(45° + (-30°)) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

To simplify this, we need to rationalize $\frac{\sqrt{3}-\sqrt{1}}{2\sqrt{2}}$ by multiplying the numerator "$\sqrt{3}-\sqrt{1}$" and the denominator "$2\sqrt{2}$" by the surd that is at the denominator i.e. " $\sqrt{2}$ "

to get $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{3}\ \sqrt{2}-\sqrt{2}}{2\sqrt{2}\ \sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{2\times\sqrt{2}\times 2} = \frac{\sqrt{6}-\sqrt{2}}{2\times\sqrt{4}} = \frac{\sqrt{6}-\sqrt{2}}{2\times 2}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}.$

Therefore, sin 15° is truly equal to $\frac{\sqrt{6}-\sqrt{2}}{4}$.

Also, cos 15° = cos (45° − 30°), but in the question, we are to find cos 15° given that cos $(A + B)$ = cos $A$ cos $B$ − sin $A$ sin $B$, so we need to rewrite cos 15° as follows: cos 15° = cos (45° − 30°) = cos (45° + (− 30°)).

From the question, cos $(A + B)$ = cos $A$ cos $B$ − sin $A$ sin $B$ thus,

cos(45° + (−30°)) = cos 45° cos (−30°) − sin 45° sin (−30°).

Recall that cos (−θ) = cos θ and sin (−θ), where θ is a positive angle; therefore,

cos (45° + (−30°)) = cos 45° cos 30° − sin 45°(−sin 30°)

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} - \left(-\frac{1}{2\sqrt{2}}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$

$\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}+\sqrt{2}}{2\sqrt{2}\ \sqrt{2}}$

$= \frac{\sqrt{3\times 2}+\sqrt{2}}{2\sqrt{2\times 2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{2\sqrt{4}} = \frac{\sqrt{6}+\sqrt{2}}{2\times 2} = \frac{\sqrt{6}+\sqrt{2}}{4} = RHS.$

Therefore, cos 15° is truly equal to $\frac{\sqrt{6}+\sqrt{2}}{4}$.

*Note that when you are given specific formulas or instructions for a problem, you are to use these formulas and instructions and* ***not*** *another method you prefer. For example, in this question, we know cos 15° = cos (45° − 30°). Then, the formula for cos (A − B) would have been the easiest formula to use, but the question told us to use the formula of cos (A + B) to solve the problem, then we must find a way to use it to solve the problem as we just did.*

(b) Recall that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, so,

$$\tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4} \div \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4} \times \frac{4}{\sqrt{6}+\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}.$$

We need to simplify this answer as it is **not** proper for its denominator to be in irrational (surd) form, so it should be rationalised. The denominator is $\sqrt{6} + \sqrt{2}$ and its conjugate will be $\sqrt{6} - \sqrt{2}$.

(*Recall that the conjugate of* $\sqrt{x} + \sqrt{y}$ *is* $\sqrt{x} - \sqrt{y}$, *the conjugate of* $x + \sqrt{y}$ *is* $x - \sqrt{y}$ *and the conjugate of* $\sqrt{x} - \sqrt{y}$ *is* $\sqrt{x} + \sqrt{y}$, *etc*).

To simplify the surd, we will multiply both numerator and denominator by the conjugate of the denominator. The denominator is $\sqrt{6} + \sqrt{2}$ , so, its conjugate will be $\sqrt{6} - \sqrt{2}$ . Therefore,

$$\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} \times \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}$$

$$= \frac{\sqrt{6}\,\sqrt{6} - \sqrt{6}\,\sqrt{2} - \sqrt{2}\,\sqrt{6} + \sqrt{2}\,\sqrt{2}}{\sqrt{6}\,\sqrt{6} - \sqrt{6}\,\sqrt{2} + \sqrt{2}\,\sqrt{6} - \sqrt{2}\,\sqrt{2}};$$

$$= \frac{\sqrt{6\times 6} - \sqrt{6\times 2} - \sqrt{2\times 6} + \sqrt{2\times 2}}{\sqrt{6\times 6} - \sqrt{6\times 2} + \sqrt{2\times 6} - \sqrt{2\times 2}}$$

$$= \frac{\sqrt{36} - \sqrt{12} - \sqrt{12} + \sqrt{4}}{\sqrt{36} - \sqrt{12} + \sqrt{12} - \sqrt{4}};$$

$$= \frac{6 - \sqrt{12} - \sqrt{12} + 2}{6 + \sqrt{12} - \sqrt{12} - 2} = \frac{8 - 2\sqrt{12}}{6-2}$$

$$= \frac{8 - 2\sqrt{4\times 3}}{4} = \frac{8 - 2(\sqrt{4}\,\sqrt{3})}{4} = \frac{8 - 2\,(2\sqrt{3})}{4}$$

$$= \frac{8 - 4\sqrt{3}}{4} = \frac{8}{4} - \frac{4\sqrt{3}}{4} = 2 - \sqrt{3}. \text{ Therefore,}$$

$\tan 15^\circ = 2 - \sqrt{3}$.

> *Note that the phrase, 'hence, find,* $\tan 15^\circ$' *means using your answers from the previous solution (that is 4(i)), find* $\tan 15^\circ$, *so you* DO NOT *need to use* $\tan 15^\circ = \tan(45^\circ - 30^\circ) = ....$

5. Show that $\tan\alpha - \cot\alpha = 2\cos 2\alpha \operatorname{cosec} 2\alpha$. **(WAEC)**

**Workshop**

It is easier to prove from the Left Hand Side (*L.H.S*) to the Right Hand Side (*R.H.S*) as we will do shortly: *L.H.S* = $\tan\alpha - \cot\alpha$.

Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, while $\cot\theta = \frac{1}{\tan\theta}$

$$= \frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}.$$

Hence, $L.H.S = \tan\alpha - \cot\alpha = \frac{\sin\alpha}{\cos\alpha} - \frac{\cos\alpha}{\sin\alpha}$

$$= \frac{\sin\alpha\sin\alpha - \cos\alpha\cos\alpha}{\cos\alpha\sin\alpha}$$

$$= \frac{\sin^2\alpha - \cos^2\alpha}{\cos\alpha\sin\alpha}$$

$$= \frac{-(\cos^2\alpha - \sin^2\alpha)}{\cos\alpha\sin\alpha}$$

$$= \frac{-(\cos^2\alpha - \sin^2\alpha)}{\sin\alpha\cos\alpha}$$

Furthermore, recall that $\cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$, and $\sin 2\theta = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = \sin\theta\cos\theta + \sin\theta\cos\theta = 2\sin\theta\cos\theta$. If $\sin 2\theta = 2\sin\theta\cos\theta$, then $\frac{\sin 2\theta}{2} = \sin\theta\cos\theta$.

Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$, and $\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$, therefore,

$$\frac{-(\cos^2\alpha - \sin^2\alpha)}{\sin\alpha\cos\alpha} = \frac{-(\cos 2\alpha)}{\frac{\sin 2\alpha}{2}} = \frac{-2(\cos 2\alpha)}{\sin 2\alpha}$$

$= -2\cos 2\alpha \cdot \frac{1}{\sin 2\alpha}$. Again, recall that $\frac{1}{\sin\theta} = \operatorname{cosec}\theta$. Thus,

$$-2\cos 2\alpha \cdot \frac{1}{\sin 2\alpha} = -2\cos 2\alpha \operatorname{cosec} 2\alpha = R.H.S.$$

However, if you wish, you can prove from the *R.H.S.* to the *L.H.S.* as we will do now; but it is a longer path:

*R.H.S.* = $-2\cos 2\alpha \operatorname{cosec} 2\alpha = \tan\alpha - \cot\alpha$.

Recall that $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, so,

$\operatorname{cosec} 2\alpha = \frac{1}{\sin 2\alpha}$, therefore,

$$R.H.S. = -2\cos 2\alpha \operatorname{cosec} 2\alpha$$

$$= -2\left(\cos 2\alpha \times \frac{1}{\sin 2\alpha}\right) = -2\left(\frac{\cos 2\alpha}{\sin 2\alpha}\right);$$

Also, recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ *then*, $\frac{1}{\tan\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}}$

$$= 1 \times \frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\sin\theta},$$

so that $-2\left(\frac{\cos 2\alpha}{\sin 2\alpha}\right) = -2\left(\frac{1}{\tan 2\alpha}\right)$

$= -2(1 \div (\tan 2\alpha))$. Again, recall that,

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, therefore,

$\tan 2\alpha = \tan(\alpha + \alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}.$

Thus, $-2\,(1 \div (\tan 2\alpha))$

$$= -2\left(1 \div \left[\frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}\right]\right);$$

$$-2\left(1 \times \left[\frac{1 - \tan\alpha\tan\alpha}{\tan\alpha + \tan\alpha}\right]\right) = -2\left(\frac{1 - \tan^2\alpha}{2\tan\alpha}\right)$$

$$= \frac{-2(1 - \tan^2\alpha)}{2\tan\alpha}$$

$$\left(\frac{-(1 - \tan^2\alpha)}{\tan\alpha}\right) = \frac{\tan^2\alpha - 1}{\tan\alpha} = \frac{\tan^2\alpha}{\tan\alpha} - \frac{1}{\tan\alpha}.$$

Also, recall that $\cot\alpha = \frac{1}{\tan\alpha}.$

Therefore, $\frac{\tan^2\alpha}{\tan\alpha} - \frac{1}{\tan\alpha} = \tan\alpha - \frac{1}{\tan\alpha}$

$= \tan\alpha - \cot\alpha = L.H.S.$

## Circular Measure and Radians

## (Mensuration)

1. The perimeter of a sector *OPQ*, of a circle centre *O*, is 15 cm. If the radius of the circle is 5 cm, calculate the area of:

   (a) the sector

   (b) $\Delta OPQ$. (WAEC)

**Workshop**

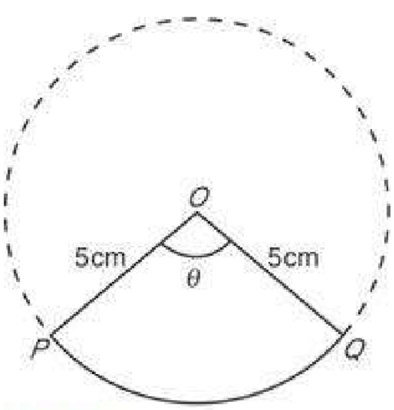


Fig. 2.5

(a) Area of sector $OPQ = \frac{\theta}{360}° \times \pi r^2$. But, to be able to calculate the area of sector $OPQ$, θ should be known first.

Perimeter of sector $OPQ = |OP| + |PQ| + |QO| = 15$ cm, where $|PQ|$ is the length of the minor arc, $PQ$.

Recall that radius, $r$, of the circle $= |OP| = |OQ| = 5$ cm so that

$5 + |PQ| + 5 = 15$; $|PQ| = 15 - 5 - 5 = 5$ cm.

Also, from the diagram, $PQ$ is an arc, and the length of an arc is calculated

as $\frac{\theta}{360°} \times 2\pi r$.

So, length of arc $PQ = \frac{\theta}{360°} \times 2\pi(5)$.

Recall that length of arc $PQ = 5$ cm,

therefore, $\frac{\theta}{360} \times 2\pi(5) = 5$; $\frac{\theta \times 10\pi}{360°} = 5$;

$\theta \times 10\pi = 360° \times 5$;

$\theta = \frac{360° \times 5}{10\pi} = \frac{180°}{\pi}$. So, $\theta \times \frac{180°}{\pi}$;

area of sector $OPQ$ will be

$\frac{\theta}{360°} \times \pi r_2 = \frac{180°/\pi}{360°} \times \pi \times 5_2 =$

$\frac{180°}{360°\ \pi} \times \pi \times 25 = \frac{180°}{360°} \times 25 = \frac{1}{2} \times 25$

$= 12.5$ cm2. Therefore, the area of sector $OPQ$ is 12.5 cm2.

(b) The area of any triangle, with two of its sides having lengths $a$ and $b$, with lengths $a$ and $b$ meeting at an angle, θ, is given by

$\frac{1}{2} ab \sin\theta$. Hence, area of triangle $OPQ$

$= \frac{1}{2}(5)(5)\sin\theta$. Recall that $\theta = \frac{180°}{\pi}$

$= 180° \div \pi$

$= 180° \div \frac{22}{7} = 180° \times \frac{7}{22} = 57.3°$. Then, the area of triangle will be expressed as

$OPQ = \frac{1}{2} \times 5 \times 5 \times \sin 57.3°$

$= \frac{1}{2} \times 25 \times 0.8415 = 10.5$ cm2.

Therefore, the area of triangle $OPQ$ is 10.5 cm2.

2. The lengths in cm, of the sides of a triangle are, $x$, $(x + 1)$ and $(x + 2)$. If the cosine of the smallest angle is $\frac{5}{7}$, calculate:

(a) the value of $x$,

(b) the largest angle of the triangle. **(WAEC)**

**Workshop**

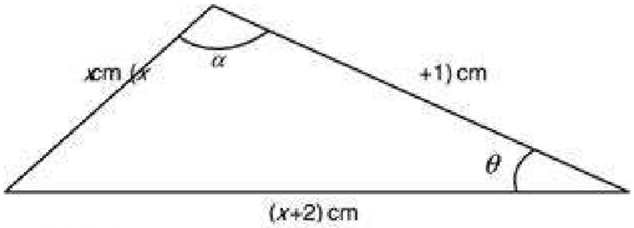


**Fig. 2.6**

(a) From Figure 2.6, the smallest angle θ is the angle opposite the side with the smallest length, which is the side of length *x*.

> *Please note that before you can tell which of the angles is the smallest, you must draw the lines with lengths x, (x + 1) and (x + 2)cm in a way that side (x + 2)cm will be longer than (x + 1) cm and side (x + 1)cm will be longer than xcm.*

By the cosine rule, which states that: $a2 = b2 + c2 - 2bc \cos A$, where $a$, $b$ and $c$ are the lengths of the sides of a triangle, and $A$ is the angle opposite the side of length

$a$; $x^2 = (x + 1)^2 + (x + 2)^2 - 2(x + 1)(x + 2) \cos \theta$.

From the question $\cos \theta = \frac{5}{7}$

$$\therefore x^2 = x^2 + 2x + 1 + x^2 + 4x + 4 - 2(x^2 + 3x + 2)\left(\frac{5}{7}\right);$$

$$x^2 - x^2 - 2x - 1 - x^2 - 4x - 4 = -2(x^2 + 3x + 2)\left(\frac{5}{7}\right);$$

$$x^2 - 2x^2 - 6x - 5 = -2 \times \frac{5}{7} \times (x^2 + 3x + 2);$$

$$-x^2 - 6x - 5 = \frac{-10}{7}(x^2 + 3x + 2);$$

$$-x^2 - 6x - 5 = \frac{-10(x^2 + 3x + 2)}{7};$$

$$7(-x^2 - 6x - 5) = -10(x^2 + 3x + 2);$$

$$-7x^2 - 42x - 35 = -10x^2 - 30x - 20;$$

$$3x^2 - 12x - 15 = 0; x^2 - 4x - 5 = 0;$$

$x2 - 5x + x - 5 = 0$; $x(x - 5) + 1(x - 5) = 0$;

$(x - 5)(x + 1) = 0$;

$(x - 5) = 0$ or $(x + 1) = 0$; $x = 5$ or $x = -1$.

The length of a line cannot be negative, hence, $x = 5$ cm.

(b) As $x = 5$ cm, $(x + 1)$ cm $= (5 + 1)$ cm $= 6$ cm, and $(x + 2)$ cm $= (5 + 2)$ cm $= 7$ cm. If you look closely at Figure 2.6, the largest angle in the triangle is the angle α, that is opposite the longest side of the triangle, which is $x + 2 = 7$ cm.

By the cosine rule,

$7^2 = 5^2 + 6^2 - 2(5)(6)(\cos \alpha)$;

$$\cos\alpha = \frac{7^2 - 5^2 - 6^2}{-2(5)(6)} = \frac{49 - 25 - 36}{-60}$$

$$= \frac{-12}{-60} = \frac{12}{60} = 0.2$$

$\alpha = \cos^{-1} 0.2 = 78.46°.$

Therefore, the largest angle of the triangle is 78.46°.

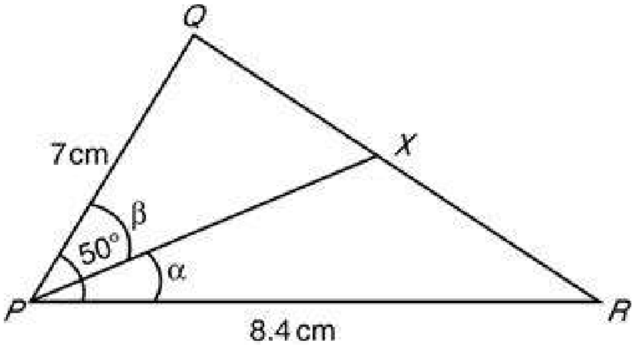


Fig. 2.7

In Figure 2.7, $|PQ| = 7$ cm $|PR| = 8.4$ cm and $<QPR = 50°$. The bisector of $<QPR$ meets $QR$ at $X$. Using the fact that the area of triangle $PQR$ is the sum of areas of $\Delta PQX$ and $\Delta PXR$, calculate the length of $PX$.

**Workshop**

Line $PX$ is a bisector of angle $QPR$, thus, $\beta = \alpha = \frac{5}{20} = 25°$.

The area of a triangle with two of its sides having lengths $a$ and $b$ and these two lines intersecting at an acute angle $\theta$ is expressed as Area,

$A = \frac{1}{2}ab \sin\theta.$

Thus, Area of $\Delta PQR = \frac{1}{2} \times 7 \times 8.4 \times \sin 50°$

$$= \frac{58.8 \times 0.7660}{2}$$

$$= 22.52 \text{ cm}^2.$$

Area of $\Delta PQX = \frac{1}{2} \times 7 \times (PX) \times \sin 25°$

$$= \frac{7 \times 0.4226 \times PX}{2}$$

$$= 1.48\ PX.$$

Area of $\Delta PXR = \frac{1}{2} \times 8.4 \times (PX) \times \sin 25°$

$$= \frac{8.4 \times 0.4226 \times PX}{2}.$$

$$= 1.77\ PX.$$

A careful look at Figure 2.7 shows that

$\Delta PQX$ + *Area of* $\Delta PXR$ = *Area of* $\Delta PQR$, so, 1.48

$PX + 1.77\ PX = 22.52$ cm$^2$; $3.25\ PX = 22.52$;

$PX = \frac{22.52}{3.25} = 6.93$ cm. Therefore, the length

of $PX$ is 6.93 cm.