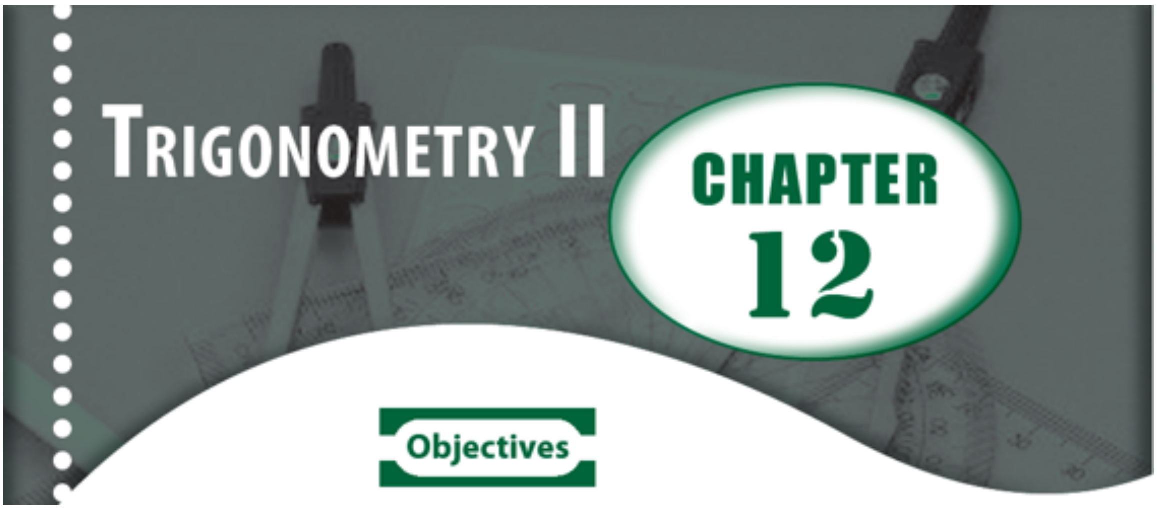
CHAPTER 12



At the end of the chapter, students should be able to draw the graphs of sine, cosine and tangent of angles.

The four quadrants

The *x* and *y* axes divide the circle into four quadrants as shown in Fig. 12.1.

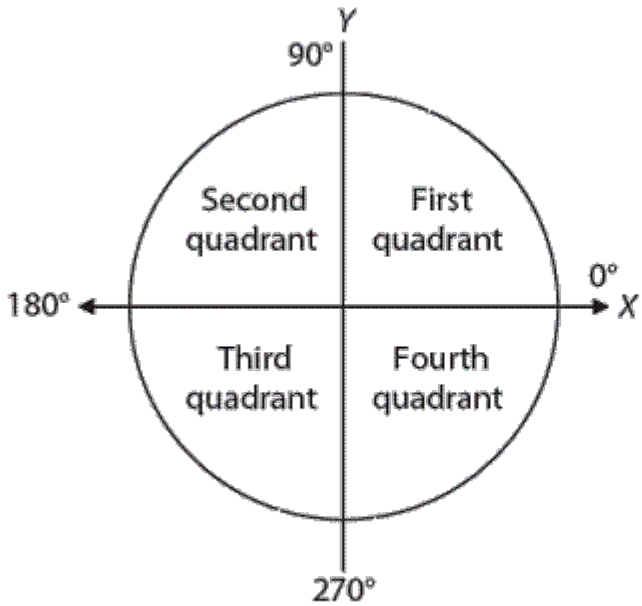


Fig. 12.1

Using the diagram in Fig. 12.1, you will observe the following value ranges of 0°:

First quadrant: $0° < \theta < 90°$

Second quadrant: $90° < \theta < 180°$

Third quadrant: $180° < \theta < 270°$

Fourth quadrant: $270° < \theta < 360°$

An angle may be measured clockwise or anticlockwise from its origin. It is simply done by adding or subtracting from a complete rotation of 360°. See Fig. 12.2 for a detailed illustration.

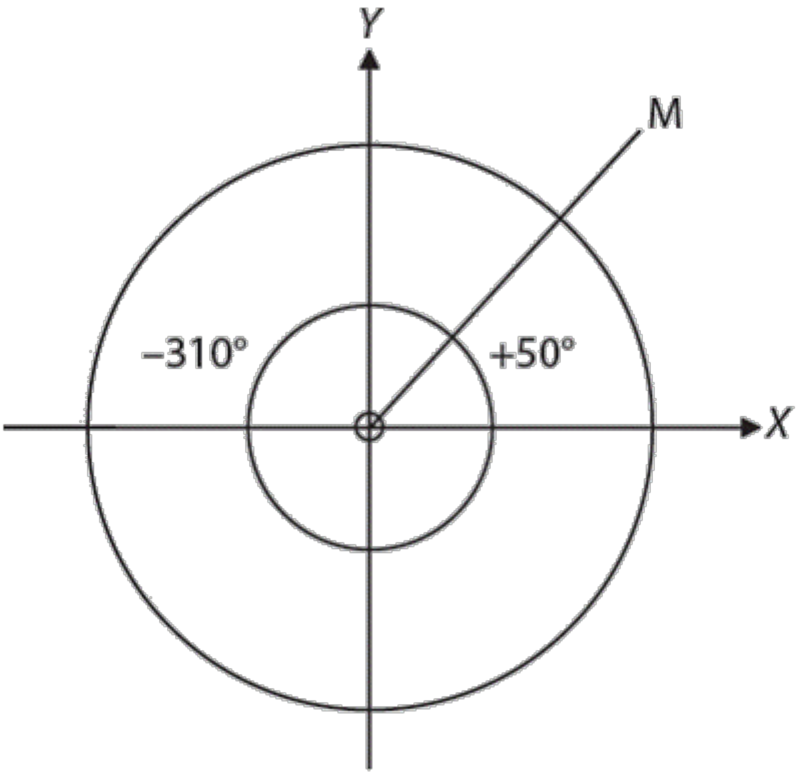


Fig. 12.2

In Fig. 12.2, +50° is measured from OX to OM in the anticlockwise direction, while –310° is measured from OX to OM in the clockwise direction. This then implies that +50° = –310°.

+130° = (310 − 360)° = –230°
+215° = (215 − 360)° = –145°
+345° = (345 – 360)° = –15°
−40° = (−40 + 360)° = +320°
−65° = (−65 + 360)° = +295°

Similarly,

+120° = (120 – 360)° = –240°
+480° = (480 – 360)° = 120°

Also,

920° = (920 – 360 – 360)°
= 200°

and −160° = (–160 + 360)°
= 200°

Exercise 1

1. Identify the quadrant where the following angles could be found.

(a) $+135°$	(b) $+156°$
(c) $+34°$	(d) $+219°$
(e) $-15°$	(f) $-189°$
(g) $-197°$	(h) $-137°$
(i) $+239°$	(j) $-310°$
(k) $+119°$	(l) $-114°$
(m) $+137°$	(n) $-241°$
(o) $-331°$	

2. Find the equivalent positive angles of the following negative angles between 0° and 360°.

(a) $-60°$	(b) $-136°$
(c) $-14°$	(d) $-143°$
(e) $-123°$	(f) $-39°$
(g) $-141°$	(h) $-310°$

3. What is the equivalent negative angle of the following positive angles between 0° and 360°?

(a) $+39°$	(b) $+210°$
(c) $+113°$	(d) $+136°$
(e) $+191°$	(f) $+156°$
(g) $+157°$	(h) $+189°$

II. Trigonometrical Ratios

We can recall the already learnt trigonometrical ratios of acute angles using the right-angled triangle that states that given a right-angled Δ,

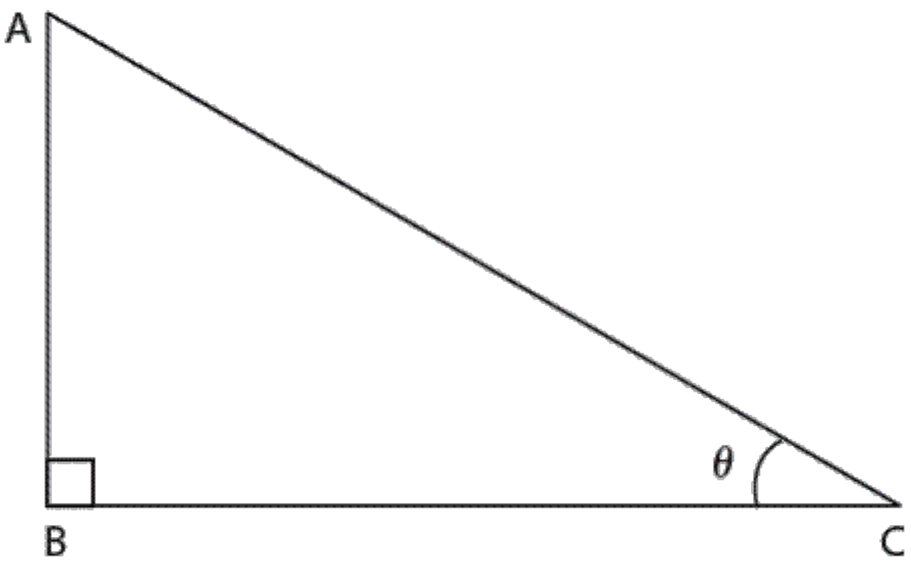


Fig. 12.3

$$\sin\theta = \frac{Opp}{Hyp} = \frac{AB}{AC}$$

$$\cos\theta = \frac{Adj}{Hyp} = \frac{BC}{AC}$$

$$\tan\theta = \frac{Opp}{Adj} = \frac{AB}{BC}$$

We have now reached a stage where we need to also find the trigonometrical ratios of obtuse and reflex angles which may not necessarily require the use of a right-angled triangle.

Note: In the unit circle, the circle is divided into four quadrants. We shall be using the concept of unit circle to determine the trigonometrical ratios of any given angle (general angles).

Let P be a point (x, y) on the circle such that /OP/ = r and ∠POM = θ in any of the four quadrants.

First quadrant: $0° < \theta < 90°$

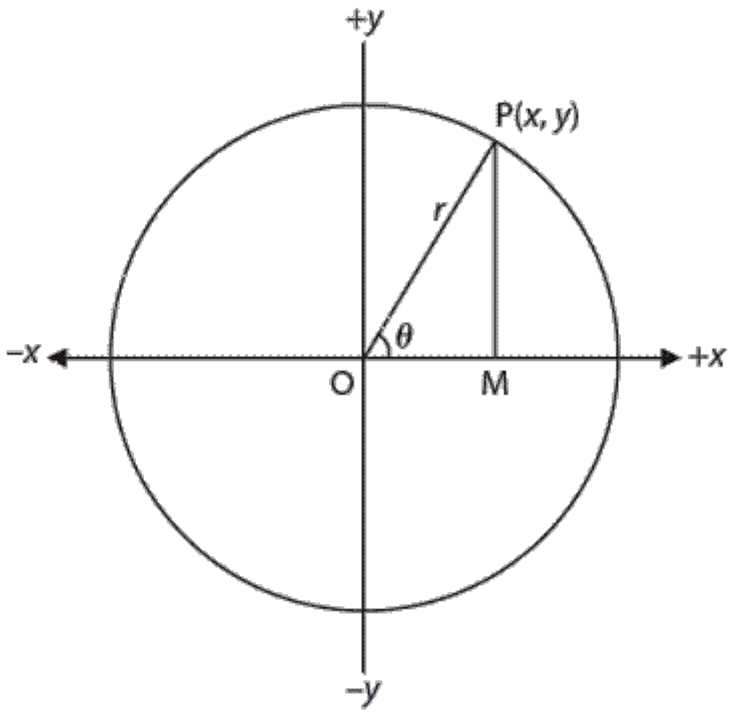


Fig. 12.4

In Fig. 12.4,

$$\sin\theta = \frac{PM}{OP} = \frac{y}{r}$$

$$\cos\theta = \frac{OM}{OP} = \frac{x}{r}$$

$$\tan\theta = \frac{PM}{OM} = \frac{y}{x}$$

Observe that, while *r* (hypotenuse) is always positive, *x* and *y* are both positive. Therefore, in the first quadrant:

sin θ = +

cos θ = +

tan θ = +

Second quadrant: 90° < θ < 180°

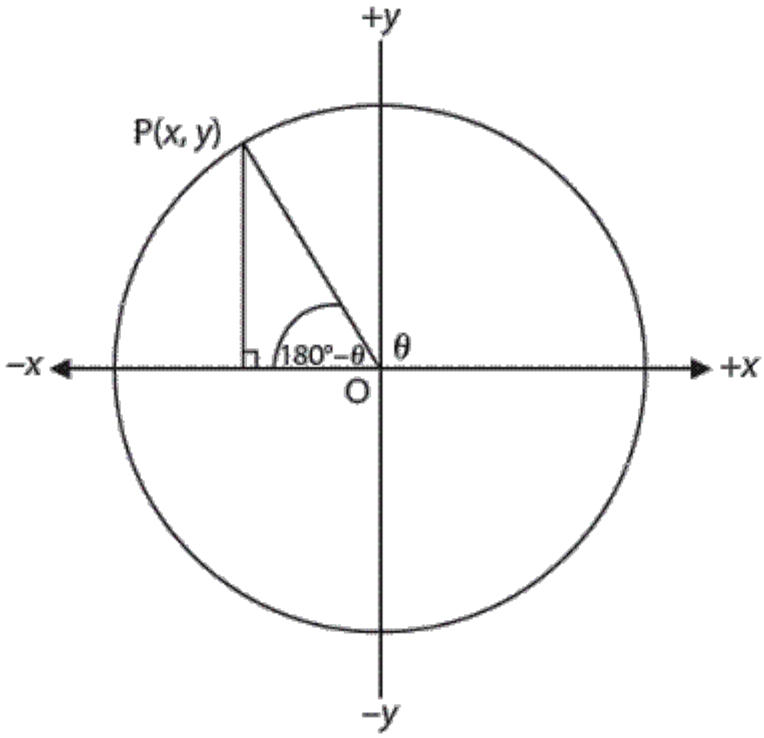


Fig. 12.5

In Fig. 12.5, x is negative and y is positive.

$$\sin\theta = \frac{y}{r} = \sin(180-\theta)^\circ$$

$$\cos\theta = \frac{-x}{r} = -\cos(180-\theta)^\circ$$

$$\tan\theta = y = -\tan(180-\theta)^\circ$$

Therefore, in the second quadrant

$\sin\theta = +$

$\cos\theta = -$

$\tan\theta = -$

Third quadrant: $180° < \theta < 270°$

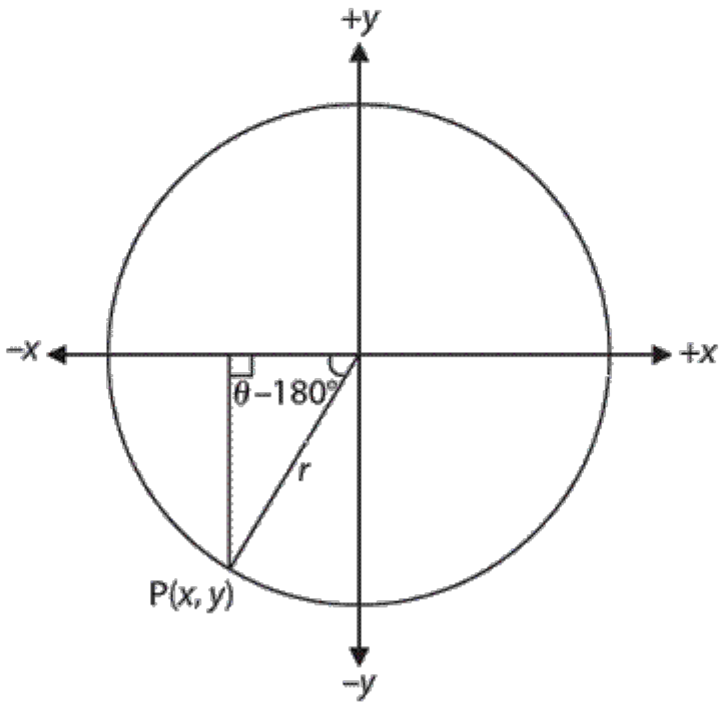


Fig. 12.6

In Fig. 12.6, both *x* and *y* are negative.
Therefore,

$$\sin\theta = \frac{-y}{r} = -\sin(\theta-180°)$$

$$\cos\theta = \frac{-x}{r} = -\cos(\theta-180°)$$

$$\tan\theta = \frac{-y}{-x} = \frac{y}{x} = \tan(\theta-180°)$$

Therefore,

$$\sin\theta = -$$

$$\cos\theta = -$$

$$\tan\theta = +$$

Fourth quadrant: $270° < \theta < 360°$

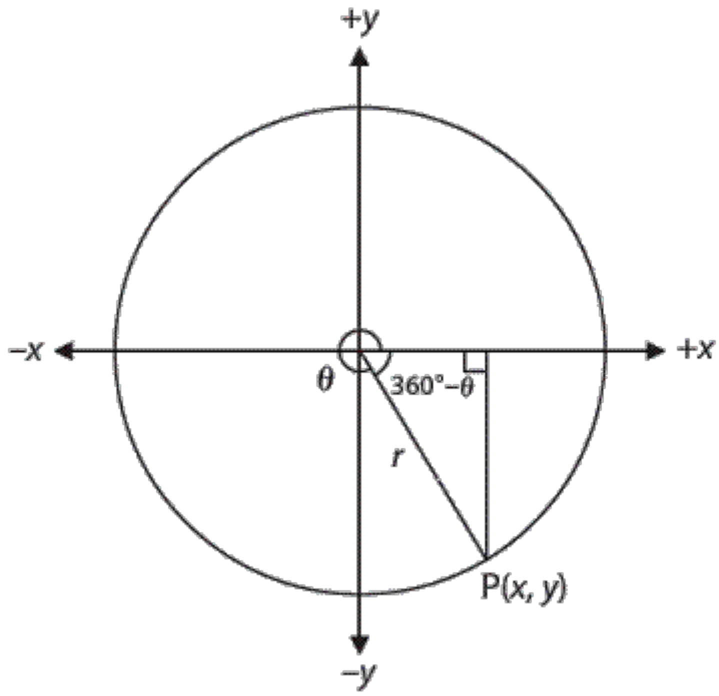


Fig. 12.7

In Fig. 12.7, x is positive and y is negative.
Therefore,

$$\sin\theta = \frac{-y}{r} = -\sin(360-\theta)^\circ$$

$$\cos\theta = \frac{x}{r} = \cos(360-\theta)^\circ$$

$$\tan\theta = \frac{-y}{x} = -\tan(360-\theta)^\circ$$

Therefore, in the fourth quadrant:

$\sin\theta = -$
$\cos\theta = +$
$\tan\theta = -$

All the above facts can be summarised as shown in Fig. 12.8.

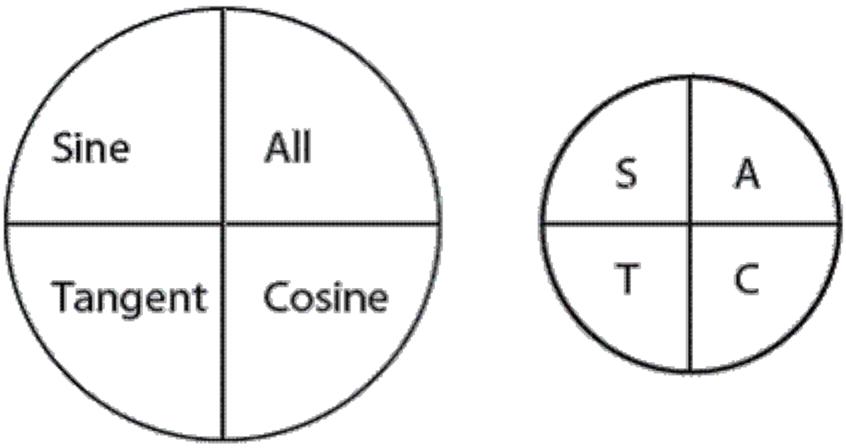


Fig. 12.8

All trigonometric ratios in the first quadrant are positive.
Only sine is positive in the second quadrant.
Only tangent is positive in the third quadrant.

Only cosine is positive in the fourth quadrant.
All these may be written in a funny statement from the first quadrant as "ALL STUDENTS TAKE COCACOLA".

Worked Example 1

Find the sine, cosine and tangent of
(a) $30°$ (b) $143°$
(c) $265°$ (d) $350°$

Solution

(a) $30°$ is in the first quadrant
$\sin 30° = +\sin 30°$ (sine is positive in the first quadrant).
Consulting the tables
$\sin 30° = 0.5000$
$\cos 30° = +\cos 30°$ (cosine is positive in the first quadrant)
Consulting the tables
$\cos 30° = 0.8660$
$\tan 30° = +\tan 30°$ (tangent is positive in the first quadrant)
Consulting the tables
$\tan 30° = 0.5774$
(b) $143°$ is in the second quadrant
$\sin 143° = +\sin(180 - 143)°$ (sine is positive in the second quadrant)
$= +\sin 37°$
$= 0.6018$
$\cos 143° = -\cos(180 - 143)°$ (cosine is negative in the second quadrant)
$= -\cos 37°$
$= -0.7536$
$\tan 143° = -\tan(180 - 143)°$ (tangent is negative in the second quadrant)
$= -\tan 37°$
$= -0.7536$
(c) $265°$ is in the third quadrant
$\sin 265° = -\sin(265 - 180)°$ (sine is negative in the third quadrant)
$= -\sin 85°$
$= -0.9962$
$\cos 265° = -\cos(265 - 180)°$ (cos is negative in the third quadrant)
$= -\cos 85°$
$= -0.0872$
$\tan 265° = +\tan(265 - 180)°$ (tan is positive in the third quadrant)
$= +\tan 85°$
$= 11.4301$
(d) $350°$ is in the fourth quadrant
$\sin 350° = -\sin(360 - 350)°$ (sine is negative in the fourth quadrant)
$= -\sin 10°$
$= -0.1736$
$\cos 350° = \cos(360 - 350)°$ (cosine is positive in the fourth quadrant)
$= \cos 10°$
$= 0.9848$
$\tan 350° = -\tan(360 - 350)°$ (tangent is negative in the fourth quadrant).

= –tan 10°
= 0.1763

With the aid of the worked examples in 1, we can now tabulate the trigonometric ratios of angles from 0° to 360° as shown in Table 12.1.

Table 12.1

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
tan θ	0	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{3}}{1}$	$+\alpha$	$-\sqrt{3}$	$\frac{1}{3}$	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\alpha$	$-\sqrt{3}$	$\frac{1}{-\sqrt{3}}$	0

Exercise 2

1. Find the sine of the following angles without using tables. Leave the solutions in square root form if it appears as such.

   (a) 0° (b) 60° (c) 120°

   (d) 180° (e) 240°

2. What is the cosine of the following angles without using trigonometric tables?

   (a) 30° (b) 90° (c) 150°

   (d) 210° (e) 270°

3. Find the tangent of the following angles.

(a) 0° (b) 120°

(c) 240° (d) 330°

(e) 360°

4. Find the sine of the following angles using tables.

(a) 35° (b) 69°

(c) 135° (d) 169°

(e) 275° (f) 296°

(g) 335° (h) 350°

5. What is the cosine of the following angles using tables?

(a) 40° (b) 70° (c) 140°

(d) 170° (e) 285°

6. Using tables, identify the tangent of the following angles.

(a) 25° (b) 55° (c) 123°

(d) 347° (e) 265°

III. The Trigonometric Graphs

We shall now learn how to plot the sine, cosine and tangent graphs from 0° to 360°.

Worked Example 2

(a) Copy and complete Table 12.2a below. Express all answers to 1 decimal place.

(b) Using Table 12.2a, plot the following graphs at an interval of 30°.

  (i) $y = \sin\theta$

  (ii) $y = \cos\theta$

  (iii) $y = \tan\theta$

(c) Use the plotted graphs to solve the following equations:

  (i) $\sin\theta = 0.8$ (ii) $y = \sin 240°$

  (iii) $\cos\theta = 0.7$ (iv) $y = \cos 95°$

  (v) $\tan\theta = -1.5$ (vi) $y = \tan 35°$

Table 12.2a

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$													
$\cos\theta$													
$\tan\theta$													

Solution

Table 12.2b

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	0.0	0.5	0.9	1.0	0.9	0.5	0.0	−0.5	−0.9	−1.0	−0.9	−0.5	0.0
$\cos\theta$	1.0	0.9	0.5	0.0	−0.5	−0.9	−1.0	−0.9	−0.5	0.0	0.5	0.9	1.0
$\tan\theta$	0.0	0.6	1.7	$+\alpha$	−1.7	−0.6	0.0	0.6	1.7	$-\alpha$	−1.7	−0.6	0.0

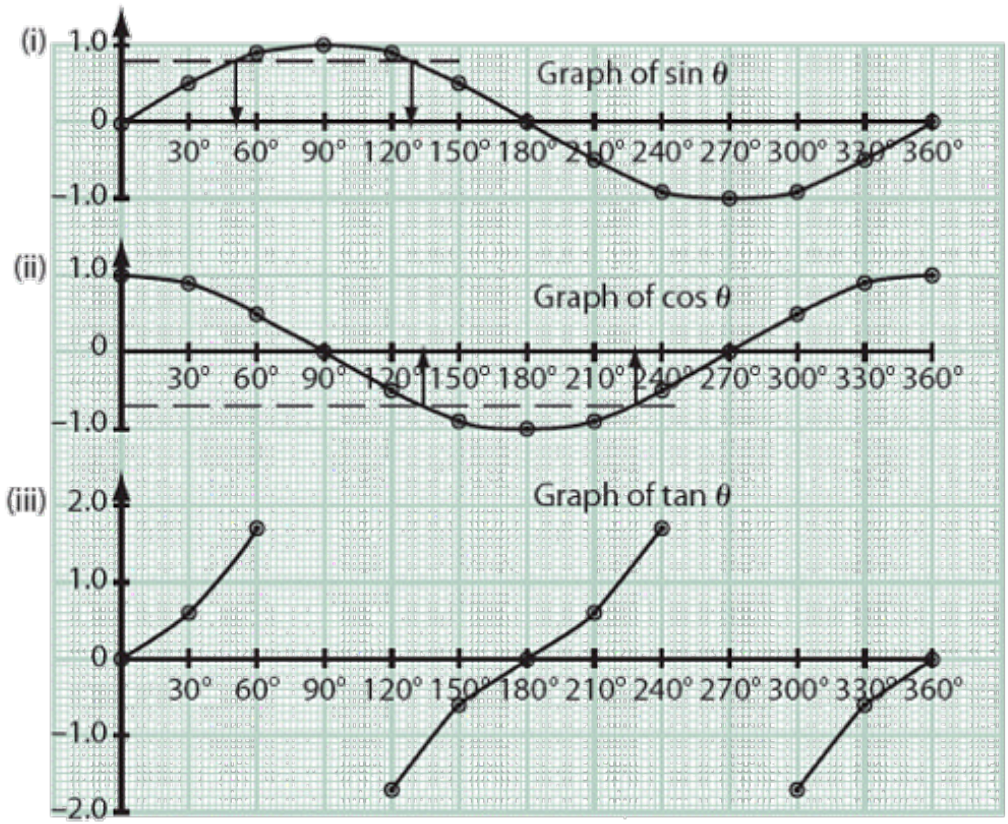


Fig. 12.9

Scale:

2 cm to 30° on the x-axis

2 cm to 1 unit on the y-axis

Note: From the sine and cosine graph, we can establish the fact that sin(θ + 90°) = cos θ. And that the tangents of 90° and 270° do not exist between 0° and 360°
(i) θ = 53° or θ = –127° (ii) y = 0.9
(iii) θ = 134° or θ = 229° (iv) y = 0.1
(v) θ = 123° (vi) *y* = 0.7

Solution

(a)
Table 12.3

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
Sin θ	0	1.5	2.6	3	2.6	1.5	0	−1.5	−2.6	−3	−2.6	−1.5	0
Cos θ	2	1.7	1	0	−1	−1.7	−2	−1.7	−1	0	1	1.7	2
Tan θ	2	3.2	3.6	3	1.6	−0.2	−2	−3.2	−3.6	−3	−1.6	0.2	2

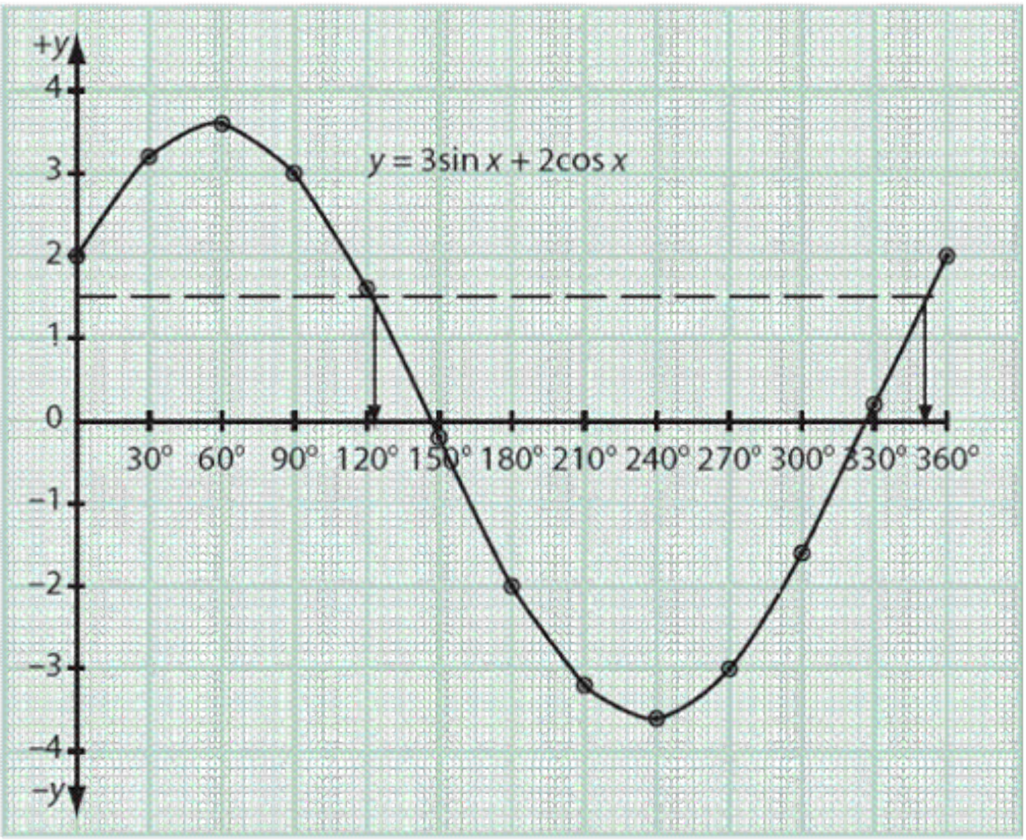


Fig. 12.10

(b) The minimum (least) point on the graph which occurred at 240° is – 3.6. Therefore, y = – 3.6.
(c) Draw a straight line at *y* = 1.5 and determine the values of *x* at the point where the line across (intersects) the graph.
Y = 3 sin *x* + 2 cos *x.*
Therefore, either *x* = 123° or *x* = 345°.

Exercise 3

1. The graph in Fig. 12.11 represents the function:

   (a) $y = \cos\theta$ (b) $y = 3\sin\theta$

   (c) $y = 3\cos\theta$ (d) $y = \sin\theta$

   (e) $y = \cos\theta + \sin\theta$ **(WAEC)**

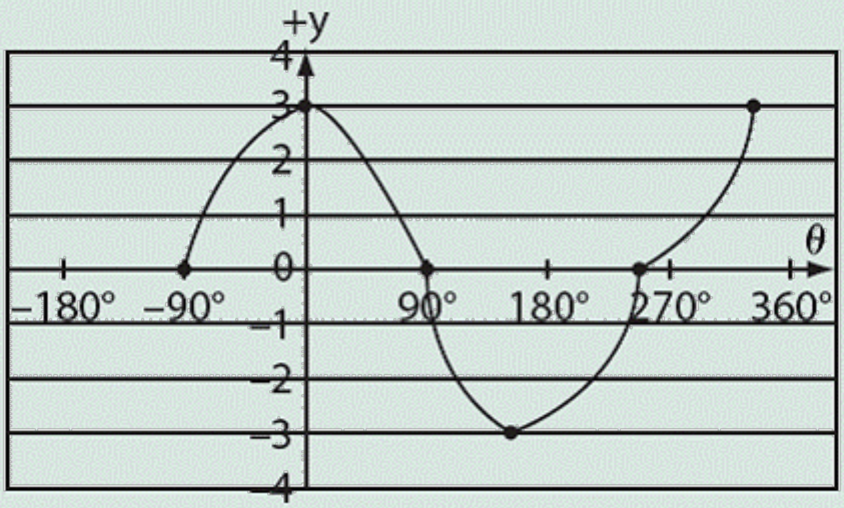


Fig. 12.11

2. Which of the following represents the graph in Fig. 12.12?

   (a) $y = \sin 1\frac{x}{4}$

   (b) $y = \cos 4x$

   (c) $y = 4\sin x$

   (d) $y = 4\cos x$

   (e) $y = \cos 1\frac{x}{4}$ **(NECO)**

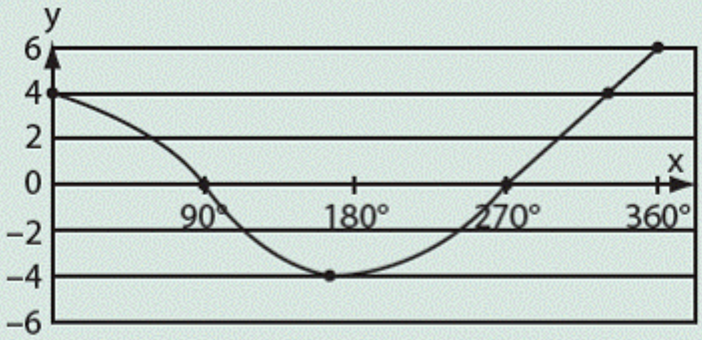
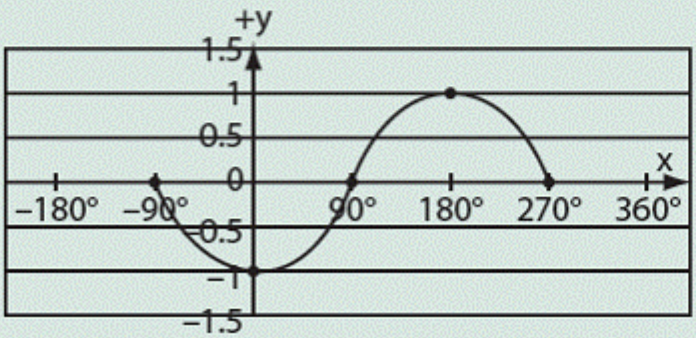


Fig. 12.12

3. Which of the following in Fig. 12.13 is the graph of sin θ for $-90° \leq \theta \geq 270°$?

(a)



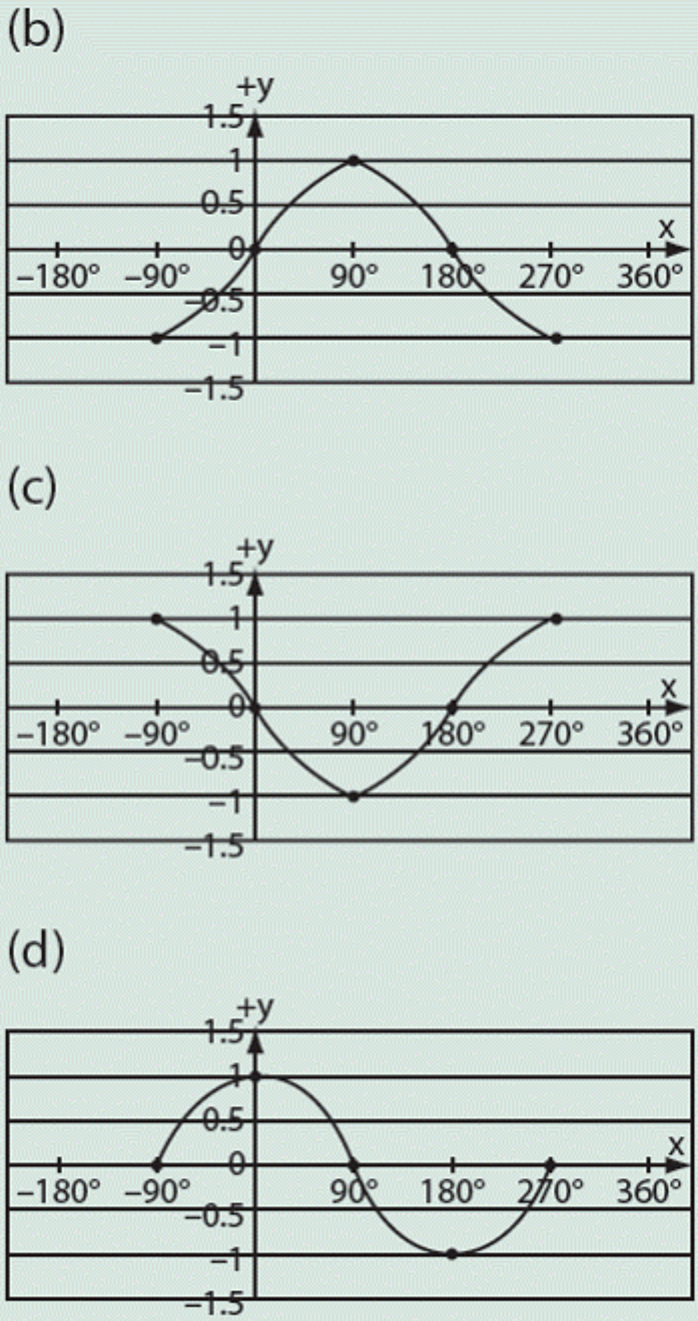

(b)
+y
1.5
1
0.5
0
x
−180°
−90°
90°
180°
270°
360°
−0.5
−1
−1.5
(c)
+y
1.5
1
0.5
0
x
−180°
−90°
90°
180°
270°
360°
−0.5
−1
−1.5
(d)
+y
1.5
1
0.5
0
x
−180°
−90°
90°
180°
270°
360°
−0.5
−1
−1.5

Fig. 12.13

4\. (a) Copy and complete the table of values for $y = \sin x + 2\cos x$, correct to 1 decimal place.

Table 12.4

x	0°	30°	60°	90°	120°	150°	180°	210°	240°
y		2.2				−1.2	−2.0		−1.9

(b) Using a scale of 2 cm to 30° on the x-axis and 2 cm to 1 unit on the y-axis, draw the graph of $y = \sin x + 2\cos x$ for $0° \leq x \geq 240°$.

(c) Use your graph to solve the following equations:

(i) $\sin x + 2\cos x = 0$

(ii) $\sin x = 2.1 - 2\cos x$

(d) From the graph, find y when $x = 171°$. **(WASCE)**

5. (a) Copy and complete the table of values for $y = 3 \sin 2x$ correct to 1 decimal place.

Table 12.5

x	0°	30°	60°	90°	120°	150°	180°
y	0					−0.3	0

(b) Using a scale of 2 cm to 30° on the x-axis and 1 cm to 0.1 unit on the y-axis, draw the graph of $y = 3 \sin 2x$ for $0° \leq x \leq 180°$.

(c) Using your graph, identify the following:

(i) Solution to the equation $y = 3 \sin 2x$.

(ii) Minimum and maximum values of y.

SUMMARY

In this chapter, we have learnt the following:

◆ That all the values of sine and cosine lies between +1 and –1.

◆ How we can use the concept of unit circle to determine the trigonometrical ratio of any given angle (General Angles).

◆ That all trigonometrical ratios are positive in the first quadrant while only sine is positive in the second, only tangent is positive in the third and only cosine is positive in the fourth quadrant.

◆ That an angle can be measured clockwise or anti-clockwise from its origin. It is done either by adding or subtracting from a complete rotation of 360°.

◆ How to plot the sine, cosine and tangent graphs from 0° to 360°.

GRADUATED EXERCISES

1. (a) Copy and complete the table of values for the function $y = 4 \cos 2x$ for $0° \leq x \geq 180°$.

Table 12.6

x	0°	30°	60°	90°	120°	150°	180°
y			–2			2	

(b) Using a scale of 2 cm to 30° on the x-axis and 1 cm to 1 unit on the y-axis, draw the graph of $y = 4 \cos 2x$ for $0° \leq x \leq 180°$.

(c) Use your graph to solve the equation $4 \cos 2x - 1 = 0$. (WASSCE)

2. (a) Copy and complete the following table of values for $y = 9 \cos x + 5 \sin x$ to 1 decimal place.

Table 12.7

x	0°	30°	60°	90°	120°	150°	180°	210°
y		10.3			–0.2	–5.3		–10.3

(b) Using a scale of 2 cm to 30° on the x-axis and 2 cm to 1 unit on the y-axis, draw the graph of $y = 9 \cos x + 5 \sin x$ for $0° \leq x \geq 210°$.

(c) Use your graph to solve the equation:

(i) $9 \cos x + 5 \sin x = 0$

(ii) $9 \cos x + 5 \sin x = 3.5$, correct to 1 decimal place.

(d) Find the maximum value of y correct to 1 decimal place.

3. (a) Copy and complete the table of values for $y = 3 \sin x + 2 \cos x$ for $0° \leq x \geq 360°$.

Table 12.8

x	0°	60°	120°	180°	240°	300°	360°
y	2.00						2.00

(b) Using a scale of 2 cm to 60 units on x-axis and 2 cm to 1 unit on the y-axis, draw the graph of $y = 3 \sin x + 2 \cos x$ for $0° \leq x \leq 360°$.

(c) Use your graph to solve the equation $3 \sin x + 2 \cos x = 1.5$.

(d) Find the range of values of x for which $3 \sin x + 2 \cos x < -1$.

4. (a) Plot the graph of $y = 5 \cos x - 2 \sin x$ from $x = 0°$ to $x = 360°$ at an interval of 30° on the x-axis and 1 cm to 1 unit on the y-axis.

(b) Use your graph to

(i) Solve the equation $y = 5 \cos x - 2 \sin x$.

(ii) Find the values of x at $Y = 2.5$.

5. (a) Copy and complete Table 12.9 of values for the function:

$Y = 2(\sin x + \cos x)$ for $0° \leq x \leq 360°$.

(b) Using a scale of 2 cm to 30° on the x axis and 2 cm to 2 units on the y axis, plot the graph of $y = 2(\sin x + \cos x$ for $0° \leq x \leq 360°$

(c) Use your graph to solve the equation $2(\sin x + \cos x) - 2 = 0$.

Table 12.9

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	360°
y	2.0		2.7			−0.7						2.0