# Chapter 12

Chapter 12

# Calculus 1: Differentiation

**OBJECTIVES**

At the end of the chapter, students should be able to:

1. state the definition of the limit of a function.
2. determine the limit of any function at any given point.
3. explain the meaning of differentiation.
4. differentiate any given function using the first principle.
5. recognise the standard derivatives of some functions.
6. apply the rules of differentiation to find derivatives of functions.
7. apply differentiation in real-life situations.

## I. Limit of a Function

Consider the function $f(x) = \dfrac{1}{1 + x^2}$.

We can see that the function rapidly approaches the value zero (0) as $x$ becomes large (positive or negative). We say that the limit of $f(x)$ is zero as *e*xtends to infinity. Limits are very important because the process of differentiation involves the formation of a limit.

### (i) Definition of a limit

We say that the function $f(x)$ tends to a limit $l$ as $x$ tends to $a$, if we can make $f(x)$ as close as we like to the number $l$ simply by taking $x$ suffi ciently close to $a$ (additionally $f(x)$ remains close to $l$ as $x$ becomes even closer to $a$). We write $f(x) \to l$ as $x = a$ or lim$n \to \infty$ $(f(x)) \to l$

**Note**

1. The definition does not imply that $f(0) = l$.

In fact, this may be true, but often $f(a)$ does not exist. For example, consider the function

$$f(x) = \frac{x - 2}{x^2 - 4}.$$

We note that $f(2)$ does not exist, but for $x \neq 2$, we have:

$f(x) = \frac{1}{x+2}$.

Therefore, we can make $f(x)$ as close as we like to the number 2 simply by taking $x$ sufficiently close to 2, that is:

$$\frac{x-2}{x^2-4} \to \frac{1}{4} \text{ as } x \to 2 \text{ or } \lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right) = \frac{1}{4}$$

2. The second definition of limit extends the idea of a limit to the case when $a$ is not finite. Here, we say that the function $f(x)$ tends to a limit $l$ *as* $x$ tends to infi nity if we can make $f(x)$ as close as we like to the number $l$ simply by taking $x$ sufficiently large.

We write $f(x) \to l$ as $x \to \infty$ or $\lim_{n\to\infty}(f(x)) = l$.

**Theorem**

Given that $\lim_{n\to\infty}(f(x)) = m$, and $\lim_{n\to\infty}(g(x)) = n$, then

1. $\lim_{x\to a}(\lambda_1 f(x) \pm \lambda_2 g(x)) = \lambda_1 m \pm \lambda_2 n$, where $\lambda_1$ and $\lambda_2 \partial$ are constants.
2. $\lim_{x\to a}\{f(x)\cdot g(x)\} = mn.$
3. $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{m}{n}$ provided that $n \neq 0$.

## Worked Example 1

Evaluate the following limits:

(a) $\lim_{x\to 3}(x^2+3)$ (b) $\lim_{x\to 1}\left\{\frac{x^2-1}{x-1}\right\}$

**SOLUTION**

(a) $\lim_{x \to 3} (x^2 + 3) = 3^2 + 3 = 9 + 3 = 12$

(b) $\lim_{x \to 1} \left\{ \frac{x^2 - 1}{x - 1} \right\} = \lim_{x \to 1} \left\{ \frac{(x + 1)(x - 1)}{(x - 1)} \right\}$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

$$\therefore \lim_{x \to 1} \left\{ \frac{x^2 - 1}{x - 1} \right\} = 2$$

## Worked Example 2

Evaluate $\lim_{n \to \infty} \left\{ \frac{x^2}{x^2 + 3x + 2} \right\}$

**SOLUTION**

$$\lim_{x \to \infty} \left\{ \frac{x^2}{x^2 + 3x + 2} \right\}$$

**Method 1**

Divide both numerator and denominator by *x*2 to have

$$\lim_{x \to \infty} \left\{ \frac{\frac{x^2}{x^2}}{\frac{x^2 + 3x + 2}{x^2}} \right\} = \lim_{x \to \infty} \left\{ \frac{1}{1 + \frac{3}{x} + \frac{2}{x^2}} \right\}$$

$$= \frac{1}{1} = 1$$

**Note:** $\lim_{n \to \infty} \left\{ \frac{3}{x} \right\} = 0$ and $\lim_{n \to \infty} \left\{ \frac{2}{x^2} \right\} = 0$

**Method 2**

We put $x = \frac{1}{n}$ and take the limit as $n \to o$ (i.e. $x \to \infty$). Hence, we have

$$\lim_{n \to \infty} \frac{x^2}{x^2 + 3x + 2} = \lim_{x \to 0} \left\{ \frac{\frac{1}{n^2}}{\frac{1}{n^2} + \frac{3}{n} + 2} \right\}$$

$$= \lim_{n \to 0} \left\{ \frac{1}{1 + 3n + 2n^2} \right\}$$

$$= 1$$

$$\therefore \lim_{x\to\infty}\left\{\frac{x^2}{x^2+3x+2}\right\}=1$$

## Exercise 1

Evaluate the following limits:

1. (a) $\lim_{x\to 3}\left\{\frac{2x+4}{x^2-2}\right\}$

   (b) $\lim_{x\to 2}\left\{\frac{x^2-4}{x^2-5x+6}\right\}$

2. (a) $\lim_{x\to 2}\left\{\frac{2x-4}{x^2+2}\right\}$

   (b) $\lim_{x\to 2}\left\{\frac{x-2}{x^2-4}\right\}$

3. (a) $\lim_{y\to\infty}\left\{\frac{7y_7-7}{8y^2+8}\right\}$

   (b) $\lim_{x\to\infty}\left\{\frac{x^3+5}{2x^4-1}\right\}$

4. (a) $\lim_{x\to\infty}\left\{\frac{x^4+5}{2x^4-1}\right\}$

   (b) $\lim_{x\to\infty}\left\{\frac{x^2}{x-1}\right\}$

5. Given that

$$g(z)=\frac{z^2-9}{z^2-4z+3}$$

Evaluate $\lim_{z\to a}\{g(z)\}$ for the following:

(a) $a=2$ (b) $a=3$

(c) $a=-3$ (d) $a=-1$

# II. The Derivate of a Function

## (i) Meaning of differentiation/ derived function

Given the function $f(x)$, we can define a related function denoted by $f'(x)$ as follows:

$$f'(x) = \lim_{\Delta x \to 0} \left\{ \frac{f(x+\Delta x) - f(x)}{\Delta x} \right\}$$ provided that the limit exists.

**Note**

1. $f'(x)$ is called the fi rst derived function of $f(x)$ or simply the derivative of $f(x)$. The process of forming $f'(x)$ from $f(x)$ is called **differentiation** with respect to $(x)$.
2. If $y = f(x)$, we may write the derivative of $y$ with respect to $x$ as $\frac{dy}{dx} = f'(x)$ where $f'(x)$ is pronounced as '$f$ prime of $x$'
3. $\frac{dy}{dx}$ is called the differential coefficient of $y$ with respect to $x$.
4. $\frac{dy}{dx}$ can be interpreted as a change in $y$ with respect to change in $x$.
5. It must be understood at this stage that $\frac{dy}{dx}$ is merely a symbol for the derivative and does not imply a division of $dy$ by $dx$. That is $\frac{dy}{dx} \neq dy \div dx.$

## (ii) Differentiation from the first principle

Given the function $y = f(x)$. To differentiate the function from the fi rst principle, we proceed as follows:

(a) Let $\Delta y$ and $\Delta x$ be little (infi nitesimal) increment in $y$ and $x$, respectively, that is, $y + \Delta y = f(x + \Delta x)$

(b) Subtract $y$ from both sides to have

$\Delta y = f(x + \Delta x) - f(x)$

(c) Divide both sides by $\Delta x$ to have

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(d) Take the limits of both sides

$$\left( \text{As } \Delta x \to 0, \frac{\Delta y}{\Delta x} \to \frac{dy}{dx} \right)$$

$$\therefore \frac{dx}{dy} = \lim_{\Delta x \to 0} \left\{ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right\}$$

**Note:** The procedures from (a) to (d) are the procedures required when differentiating from the first principle.

## Worked Example 3

Differentiate the following functions from the first principle:

(a) $y = x^2$ (b) $y = x^3$

(c) $y = x^{\frac{1}{2}}$

### SOLUTION

(a) $y = x^2$

Let $\Delta y$ and $\Delta x$ be little increment in $y$ and $x$, respectively.

$\therefore y + \Delta y = (x + \Delta x)2 = x2 + 2x(\Delta x) + (\Delta x)2$

Subtract $y$ from both sides

$\therefore \Delta y = x2 + (\Delta x) + \Delta x)2 - x2$

$= 2x(\Delta x) + (\Delta x)2$

Divide throughout by $\Delta x$

$$\therefore \frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

Take the limits of both sides

$$\therefore \lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \{2x + \Delta x\} = 2x$$

$$\therefore \frac{dy}{dx} = 2x$$

(b) $y = x3$

Let $\Delta y$ and $\Delta x$ be infinitesimal increment in $y$ and $x$, respectively.

$\Rightarrow y + \Delta y = (x + \Delta x)3$

$= x3 + 3x2(\Delta x) + 3x(\Delta x)2 + (\Delta x)3$

Subtract $y$ from both sides to have

$\Delta y = 3x2(\Delta x) + 3x(\Delta x)2 + (\Delta x)3$

Divide throughout by $\Delta x$ to have

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x(\Delta x) + (\Delta x)^2$$

Take the limits of both sides to have

$$\lim_{\Delta x \to 0} \left( \frac{\Delta y}{\Delta x} \right) = \lim_{\Delta x \to 0} 3x^2 + 3x(\Delta x) + (\Delta x)^2$$

$\therefore \frac{dy}{dx} = 3x^2$

(c) $y = x^{\frac{1}{2}}$

$y + \Delta y = (x + \Delta x)^{\frac{1}{2}}$ $(\Delta y, \Delta x$

## Worked Example 4

Given that $f(x) = x4$. Use the definition of $f'(x)$ to show that $f'(x) = 4x3$.

**SOLUTION**

$f(x) = x^4$ *and* $f(x + \Delta x) = (x + \Delta x)^4$

$$f(x + \Delta x) - f(x) = (x + \Delta x)^4 - x^4$$

$$= x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)4 - x^4$$

$$= 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4$$

$$\therefore \frac{f(x+\Delta x) - f(x)}{\Delta x} = 4x^3 + 6x(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3$$

From definition, $\lim\limits_{n \to \infty} \left\{ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right\}$

$$= \lim_{n \to \infty} = \{4x^3 + 6x(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\}$$

$$= 4x^3$$

$\therefore f'(x) = 4x^3$ (from definition).

## Exercise 2

Differentiate the following functions from the first principle:

1. $y = x$
2. $y = x^5$
3. $y = x^{\frac{1}{3}}$
4. $y = \frac{1}{x}$
5. $y = \frac{1}{x^2}$

6. Use the definition of a derivative to obtain $f'(x)$ for the following functions:

   (a) $f(x) = \frac{1}{x^3}$ (b) $f(x) \frac{1}{2x}$

7. Show that the derivative of $kf(x)$ is $k\,f'(x)$, where $k$ is a constant. Hence, write down the derivative of the functions $4x^3$ and $\frac{1}{x^3}$.

8. Confirm that $z^n - x^n = (z - x) \sum_{k=1} z^{n-k}$, $x^{k-1}$ where $n$ is a positive integer, and evaluate $\lim_{z \to x} \frac{z^n - x^n}{z - x}$.

   Hence, show that for $f(x)$, $= x^n$, a positive integer, $f'(x) = nx^{n-1}$

## III. Differentiation using Formula

From Worked Examples, we observe that

(a) If $y = x^2$, $\frac{dy}{dx} = 2x = 2x^{2-1}$

(b) If $y = x^3$, $\frac{dy}{dx} = 3x^2 = 3x^{3-1}$

\

Similarly, in Worked Example 4, we see that if

$y = x^4$, $\frac{dy}{dx} = 4x^3 = 4x^{4-1}$.

Hence, in general, if

if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$,

where $n$ is an integer.

This rule can also be written in compact form by eliminating $y$ as follows:

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

We should note that the formula above can be extended to fractional values of $n$.

$$\frac{dy}{dx} x^{-\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} \text{ and } \frac{dy}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

We should also note that differentiation of a constant $C$ (say) is zero. That is, if $y = C$,

then $\frac{dy}{dx} = \frac{d}{dx}(C) = 0.$

## (i) The derivatives of polynomials

**Theorem**

Let $f(x)$ *and* $g(x)$ be two differentiable functions. Then

1. $\frac{d}{dx}(Cf(x)) = C\frac{d}{dx}(fx)$, where $C$ is a constant.

2. $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x) \pm \frac{d}{dx}(g(x)).$

## Worked Examples 5

Find $\frac{dy}{dx}$ if

(a) $y = 3x^4 + 2x^2 - 4.$

(b) $y = 2\sqrt{x} - \frac{1}{x}.$

### SOLUTION

(a) $y = 3x^4 + 2x^2 - 4$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^4 + 2x^2 - 4)$$

$$= \frac{d}{dx}(3x^4) + \frac{d}{dx}(2x^2) - \frac{d}{dx}(4)$$

$$= 3\frac{d}{dy}(x^4) + 2\frac{d}{dx}(x^2) - \frac{d}{dx}(4)$$

$$= 12x^3 + 4x$$

(b) $y = 2\sqrt{x} - \frac{1}{x} = 2x^{\frac{1}{2}} - x^{-1}$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^{\frac{1}{2}} - x^{-1})$$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^{\frac{1}{2}}) - \frac{d}{dx}(x^{-1})$$

$$= 2\frac{d}{dx}(x^{\frac{1}{2}}) - \frac{d}{dx}(x^{-1})$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^2}$$

## Worked Example 6

Differentiate the following functions with respect to *x:*

(a) $y = 7x^3 + 4x^2 - 3x + 18.$

(b) $y = \sqrt{x} + 3\sqrt{x}.$

### SOLUTION

(a) $y = 7x^3 + 4x^2 - 3x + 18$

$$\frac{dy}{dx} = \frac{d}{dx}(7x^3 + 4x^2 - 3x + 18)$$

$$= 21x^2 + 8x - 3$$

(b) $y = \sqrt{x} + \sqrt[3]{x} = x^{\frac{1}{2}} + x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}} + x^{\frac{1}{3}}\right) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{-2}{3}}$$

$$= \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{3x^{\frac{2}{3}}}$$

## Exercise 3

Find the derivatives of the following:

1. $y = x^2 + 3$
2. $y = \frac{3}{x} - \frac{2}{x^2}$
3. $y = 3 + 6x + \frac{1}{x}$
4. $y = \frac{1}{x^2} + \frac{1}{2x^3} + \frac{1}{3x^4}$
5. $y = \frac{5}{x^3} - \frac{2}{x^2} + \frac{2}{x}$
6. $y = 2x^2 - 3x - 7$
7. $y = \frac{4}{x}$
8. $y = x^3 - 5x + 3$
9. $y = 5 - 3x^2$
10. $y = 3 + \frac{1}{2x}$
11. $y = 2x^2 + 7x + 3$
12. $y = x + \frac{1}{x^2}$
13. Find $\frac{dp}{dv}$ when $P = \frac{2}{\sqrt{v}} - \frac{3}{3\sqrt{v}} + 14$.
14. Find $\frac{ds}{dt}$ when $S = t^6 - t^2 + 1 - t^{-2} + t^{-4}$.
15. Find $\frac{du}{dr}$ when $U = 12(7 + r)^2$.

# IV. Rules of Differentiation

## (i) The function of a function rule

Given that $y$ is a function of a variable $U$, and $V$ is a function of $x$ (i.e. $y$ depends on $x$), then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

The above rule is known as **chain rule**

## Worked Example 7

Find the first-order derivative of $y$ = (2$x$2 + 7$x$ + 3)2.

**SOLUTION**

$y = (2x^2 + 7x + 3)^2$

Let $u = 2x^2 + 7x + 3, y = u^2$

$$\frac{du}{dx} = 4x + 7, \frac{dy}{du} = 2u$$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 2u(4x + 7)$$

$$= 2(4x + 7)(2x^2 + 7x + 3)$$

## Worked Example 8

Find $\frac{dp}{dx}$ if $P = (1 - x^2)^{\frac{3}{2}}$.

**SOLUTION**

$$P = (1 - x^2)^{\frac{3}{2}}$$

Let $z = 1 - x^2$

$\therefore P = z^{\frac{3}{2}}$

$$\frac{dx}{dy} - 2x \frac{dp}{dx} = \frac{3}{2} z^{\frac{1}{2}}$$

$$\frac{dp}{dx} = \frac{dp}{dz} \cdot \frac{dz}{dx} = \frac{3}{2} z^{\frac{1}{2}} (-2x)$$

$$= 3x(1 - x^2)^{\frac{1}{2}}$$

## Worked Example 9

Given that $y = \frac{3}{1+3x^2}$, determine the rate of change in $y$ with respect to $x$ when $x = 1$.

**SOLUTION**

$$Y = \frac{3}{1+3x^2}$$

Let $u = 1 + 3x^2 \; \frac{du}{dx} = 6x$

$$\therefore\ y = \frac{3}{u} = 3u^{-1}, \frac{dy}{dx} = -\frac{3}{u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -\frac{3}{u^2} \cdot 6x = -\frac{18x}{u^2}$$

$$= \frac{-18x}{(1+3x^2)^2}$$

When $x = 1, \frac{dy}{dx} = -\frac{18}{16} = -\frac{9}{8}$

### (ii) Product rule

If $y = u(x)v(x)$ then $\frac{dy}{dx} = \frac{vdu}{dx} + \frac{udv}{dx}$

## Worked Example 10

Differentiate $y$ = (4 + 2$x$) (1 − x − 4$x$2)2 with respect to $x$.

**SOLUTION**

$$y = (4 + 2x)(1 - x - 4x^2)^2$$

Let $u = 4 + 2x, \frac{du}{dx} = 2$

$$v = (1 - x - 4x^2)^2,$$

$$\frac{dv}{dx} = 2(1 - x - 4x^2)(-1 - 8x)$$
$$= - 2(1 - x - 4x^2)(1 + 8x)$$

$$\frac{dy}{dx} = \frac{vdu}{dx} + \frac{udv}{dx}$$

$$= 2(1 - x - 4x^2)^2 + (4 + 2x)$$
$$-2(1 - x - 4x^2)\ (1 + 8x)$$
$$= 2(1 - x - 4x^2)^2 - 2(4 + 2x)$$
$$(1 - x - 4x^2)(1 + 8x)$$

$$= 2(1 - x - 4x^2)[(1 - x - 4x^2)$$
$$- (4 + 2x)(1 + 8x)]$$
$$= 2(4x^2 - x - 1)(2x^2 + 35x + 3)$$

### (iii) Quotient rule

If $y = \frac{u(x)}{v(x)}$, then

$$\frac{dy}{dx} = \frac{\frac{vdu}{dx} - \frac{udv}{dx}}{v^2}$$

## Worked Example 11

Find $\frac{dy}{dx}$ if $y = \frac{6x^3 - 5x^2 + 1}{3x^2}$.

### SOLUTION

$$y = \frac{6x^3 - 5x^2 + 1}{3x^2}$$

Let $u = 6x^3 - 5x^2 + 1,$

$$\frac{du}{dx} = 18x^2 - 10x$$

$$v = 3x^2, \frac{dv}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{\frac{vdu}{dx} - \frac{udv}{dx}}{v^2}$$

$$\therefore \frac{d}{dx} = \left\{\frac{6x^3 - 5x^2 + 1}{3x^2}\right\}$$

$$= \frac{3x^2(18x^2 - 10x) - (6x^3 - 5x^2 + 1) \cdot 6x}{(3x^2)}$$

$$= \frac{54x^4 - 30x^3 - 36x^4 + 30x^3 - 6x}{9x^4}$$

$$= \frac{18x^4 - 6x}{9x^4} = \frac{2(9x^3 - 3)}{9x^3}$$

$$= \frac{2(3x^2 - 1)}{3x^2}$$

## Worked Example 12

What is the differential coefficient of $y = \frac{3x}{x-1}$ with respect to *x*?

### SOLUTION

$$y = \frac{3x}{x-1}$$

Let $u = 3x, \frac{du}{dx} = 3$

$v = x - 1, \frac{dv}{dx} = 1$

$$\frac{d}{dx}\left(\frac{3x}{x-1}\right) = \frac{(x-1)\cdot 3 - 3x \cdot 1}{(x-1)^2}$$

$$= \frac{3x - 3 - 3x}{(x-1)^2} = \frac{-3}{(x-1)^2}$$

**Exercise 4**

1. Find $\frac{df}{dx}$ if $f(x) = \sqrt{x + \frac{1}{x}}$.

Differentiate the following functions with respect to $x$:

2. $(x^2 + 2)^{12}$
3. $(x^3 + 4)^2$
4. $(x^3 + 2)^{12}$
5. $(4x^3 + 7x + 3)^6$
6. $\frac{1}{(1-x)^2}$
7. $\sqrt{3x^2 + 4x - 3}$

Evaluate the following:

8. $\frac{d}{dv}\left(\sqrt{1 + \sqrt{v}}\right)$
9. $\frac{d}{dx}\left(\frac{9}{1+9}\right)$
10. $\frac{d}{dt}(3 - t - 4t^2)^8$
11. $\frac{d}{dx}(x^2 + 2x + 1)^{\frac{5}{2}}$
12. $\frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$

13. $\frac{d}{dx}\left\{\left(\frac{1-3}{1+x}\right)^2\right\}$

14. $\frac{d}{dx}\left(x^2\sqrt{1-x^2}\right)$

15. $\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right)$

16. $\frac{d}{dx}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$

17. $\frac{d}{dx}\left((4-x)\left(\sqrt{4+x}\right)\right)$

18. $\frac{d}{dx}\left(\frac{x^2}{x-1}\right)$

19. $\frac{d}{dx}\{(t^2+2t-1)(3t^2-5t+2)\}$

20. $\frac{d}{dx}\left(\frac{z^2+z-1}{z^2+z+1}\right)$

# V. Differentiation of Trigonometric, Exponential and Logarithmic Functions

## (i) Differentiation of trigonometric functions

Note that $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\tan x) = \sec^2 x$

### Worked Example 13

Differentiate $y = \cos 6x$ with respect to $x$.

**SOLUTION**

$y = \cos 6x, \qquad Let\ u = 6x$

$\therefore\ y = \cos u$

$\frac{dy}{du} = -\sin u$ and $\frac{du}{dx} = 6$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (using chain rule)

$\frac{d}{dx}(\cos 6x) = -6 \sin u = -6 \sin 6x.$

## Worked Example 14

Find $\frac{dy}{dx}$ if $y = (x^2 + 1)^3 \sin^2 x.$

**SOLUTION**

$y = (x^2 + 1)^3 \sin^2 x$

Let $u = (x^2 + 1)^3$, $v = \sin^2 x$

$\frac{du}{dx} = 3(x^2 + 1) \cdot 2x = 6x(x^2 + 1)^2$ and

$\frac{dv}{dx} = 2\sin x \cos x$

$\frac{dy}{dx} = \frac{vdu}{dx} + \frac{udv}{dx}$ (product rule)

$\frac{dy}{dx} = \sin^2 x \cdot 6x(x^2 + 1)^2 + (x^2 + 1)^3 \cdot 2\sin x \cos x$

$= 6x \sin^2 x(x^2 + 1)^2 + 2(x^2 + 1)^3 \sin x \cos x$

$= 2\sin x\,(x^2 + 1)^2 + [3x \sin x + (x^2 + 1)\cos x]$

**Note**

1. $\frac{d}{dx}(\text{cosec } x) = -(\text{cosec } x \cot x)$
2. $\frac{d}{dx}(\sec x) = \sec x \tan x$
3. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

## (ii) Differentiation of exponential functions

### Exponential functions

If $y = e^x, \frac{dy}{dx} = e^x$

However,

If $y = e^{-x}, \frac{dy}{dx} = -e^x$

If $y = e^{nx}, \frac{dy}{dx} = ne^{nx}$, where $n$ is an integer

### Logarithmic functions

If $y = \log_e x = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

## Worked Example 15

Differentiate the following with respect to *x:*

(a) $y = e^{2x}$

(b) $y = e^{-x^2}$

(c) $y = \ln(x^2 \cos x)$

**SOLUTION**

(a) $y = e^{2x}$

Let $u = 2x$, $y = e^{u}$

$$\frac{du}{dx} = 2, \qquad \frac{dy}{du} = e^{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ (chain rule)}$$

$$= 2e^{u} = 2e^{2x}$$

(b) $y = e^{-x^3}$,

Let $t = -x^3$, $\qquad y = e^{t}$

$$\frac{dt}{dx} = -3x^2, \frac{dy}{dt} = e^{t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -3x^2\, e^{-x^3}$$

$$\frac{dy}{dx} = -3x^2\, e^{-x^3}$$

(c) $y = \ln(x^2 \cos x)$

Now let $w = x^2 \cos x$, $\therefore\ y = \ln w$

$$\frac{dw}{dx} = 2x \cos x - x^2 \sin x \text{ (product rule)}$$

$$\frac{dy}{dw} = \frac{1}{w}$$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dx} \text{ (chain rule)}$$

$$\therefore \frac{d}{dx}(\ln (x^2 \cos x)) = \frac{1}{w}(2x\cos x - x^2 \sin x)$$

$$= \frac{2x \cos x - x^2 \sin x}{x^2 \cos x}$$

$$= \frac{2\cos x - x \sin x}{x \cos x}$$

## Exercise 5

Differentiate the following with respect to $x$:

1. $y = \sin 3x$
2. $y = \cos\left(\frac{x}{4}\right)$
3. $y = \sin(1 - 2x)$
4. $y = \sin x^2$
5. $y = \sin^3 2x$

Differentiate the following:

6. $\frac{d}{dt}(\sin^3 t)$
7. $\frac{d}{d\theta}\left(\sqrt{\sin 2\theta}\right)$
8. $\frac{d}{dx}\left(\frac{1}{\cos x}\right)$
9. $\frac{d}{dt}\{4t - \sin^2(2(1 - t))\}$
10. $\frac{d}{dt}(R\sin(wt + \varnothing))$
11. $\frac{d}{dx}\left(\frac{x^2 + 2x}{\sin x}\right)$

Differentiate the following with respect to $x$:

12. $y = e^{\sqrt{x}}$
13. $y = e^{\sin 3x}$
14. $y = e^{2x} \ln x$
15. $y = \ln 2x + 2 \ln x$
16. $y = \ln\left(\sqrt{x^2 + 2y + 1}\right)$
17. $y = \frac{e^{2x}}{1 + x^3}$

Obtain the following derivatives:

18. $\frac{d}{dt}\{\ln(e^t\sqrt{1+t^2})\}$

19. $\frac{d}{dx}\{\ln(1+e^{2x})\}$

20. $\frac{d}{du}\left\{\ln\left(\frac{\sec u}{u}\right)\right\}$

## VI. Applications of Differentiation in Real-Life Situations

### (i) Derivative of a function as the shape or gradient of the function

The derivative of a given function at any point is otherwise known as the gradient or shape of the function at the point.

### Worked Example 16

Find the gradient of the curve $y = x2$ at the point (1, 10).

**SOLUTION**

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

At the point

$$(1, 10), \frac{dy}{dx} = 2 \times 1 = 2.$$

Hence, the gradient of the curve at the point (1, 10) is 2.

### Worked Example 17

Find the equation of the tangent and normal to the curve

$y = \frac{9}{1+2x}$ at point (1, 3).

**SOLUTION**

$$y = \frac{9}{1+2x}$$

$$\frac{dy}{dx} = \text{slope of the curve}$$

$$- -18(1+2x)^{-2}$$

$$= \frac{-18}{(1+2x)^2}$$

At the point (1, 3), the slope of the curve

$= -\frac{18}{9} = -2.$

Hence, the equation of the tangent at point (1, 3) is

$\frac{y-y_1}{x-x_1}$, that is $\frac{y-3}{x-1} = \frac{1}{2}$

$$2(y-3) = x-1$$

$$\therefore x - 2y + 5 = 0$$

## (ii) Maximum and minimum points

Let $y = f(x)$ be a differentiable function (i.e. $f'(x)$ exist). We now define the following:

(a) If $f'(a) > 0$, $f(x)$ is said to be increasing at the point $x = a$.

(b) If $f'(a) < 0$, $f(x)$ is said to be decreasing at the point $x = a$.

(c) If $f'(a) = 0$, $f(x)$ is said to be stationary at the point $x = a$ and the point with coordinates $(a, f(a))$ is called a **stationary** or **turning point** of the curve $y = f(x)$.

There are three types of stationary (turning) points, namely, a maximum point, a minimum point and point of inflexion (or horizontal point).

(i) If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at the point $(x, y)$ , then the point is called a maximum point of the function and the value of $y$ is called a maximum value of the function.

(ii) If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at the point $(x, y)$, then the point is called a minimum point and the value of $y$ is called a minimum value of the function.

(iii) If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at the point $(x, y)$, then the point is called a point of inflexion or horizontal point.

## Worked Examples 18

Find the coordinates of the stationary points of the curve if $y = x4(6 - x2)$. Determine the nature of each.

### SOLUTION

$$y = x^4(6 - x^2)$$
$$= 6x^4 - x^6$$
$$\frac{dy}{dx} = 24x^3 - 6x^5$$
$$= 6x^3(4 - x^2)$$
$$= 6x^3(2 + x)(2 - x)$$

At the turning point, $\frac{dy}{dx} = 0$ i.e. $6x^3(2 + x)(2 - x) = 0$.

Hence, $x = 0$, $x = 2$, $x = -2$ which gives the stationary points (0, 0), (2, 32) and (−2, 32).

$$\frac{d^2y}{dx^2} = 72x^2 - 30x^4$$

When $x = 0, \frac{d^2y}{dx^2} = 0,$
thus the point (0, 0) is the point of inflexion.

When $x = 2, \frac{d^2y}{dx^2} = -192 < 0,$ thus the point (2, 32) is a maximum point.

When $x = -2, \frac{d^2y}{dx^2} = -192 < 0,$ thus the point of (−2, 32) is also a maximum point.

## (iii) Velocity, acceleration and rate of change

**Velocity and acceleration**

The most common application of the concept of the rate of change is velocitywhich is the rate of change in distance with respect to time. On the other hand, acceleration is the rate of change in velocity with respect to time. If a particle moves on a straight line with distance $s$ metre, the velocity of the particle isgiven as $v = \frac{ds}{dt}$ and its acceleration is given as

$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ where $t$ is measured in seconds.

## Worked Example 19

The distance travelled by a moving vehicle is given by $S = 20t + 2t$ where $S$ is measured in kilometres and $t$ in hours.

(a) Obtain an expression for the $\frac{as}{dt}$ km/hr of the vehicle after $t$ hours.

(b) Calculate the speed after the following time: (i) 30 minutes (ii) 2 hours.

### SOLUTION

(a) $S = 20t + 2t^2$

Speed $= \frac{ds}{dt} = \frac{d}{dt}(20t + 2t^2)$

$= (20 + 4t)$ km/hr

(b) (i) At 30 minutes $= \left(\frac{60}{2}\right)$ hour $= \frac{1}{2}$ hour.

Speed $= \left(20 + 4 \times \frac{1}{2}\right)$ km/hr $=$ 22 km/hr.

(ii) At 2 hours: speed $= (20 + 4 \times 2)$ km/hr $=$ 28 km/hr.

**Rate of change**

In many applications, two variables are related by the same law (e.g. $y = f(x)$) and both of the variables are dependent on a third variable, usually time $t$. The rate of change in $y$ with respect to $t$ may be related to the rate of change in $x$ with respect to $t$ by applying the function of a function rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

In a typical example, one of $\frac{dy}{dt}$ will be given and the other will have to be determined at some instant $t$ (when $x$ and $y$ will be known). It will be necessary to establish a relationship between $x$ and $y$ and then differentiate the equation with respect to $t$. This will provide a relationship between the rates of change in $x$ and $y$, and the final step will be to insert the particular numbers corresponding to the given instant.

## Worked Example 20

A large spherical balloon is being inflated such that its volume is increasing at a constant rate of 2 m3/min. Calculate the rate of increase in the radius of the balloon at the instant when the radius is 2 m.

**SOLUTION**

Let the radius and volume of the balloon be denoted by $r$ and $V$, respectively. We are given that

$\frac{dV}{dt} = \frac{2\text{ m}^3}{\text{min}}$ and we need to obtain $\frac{dr}{dt}$. $V$ and $r$ are related by the function $V = \frac{4}{3}\pi r^3$. Applying function of a function rule, we have

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \qquad 4\pi r^2 \left(\frac{dr}{dt}\right)$$

$$\text{Hence, } 2 = 4\pi r^2 \left(\frac{dr}{dt}\right)$$

$$\therefore \quad \frac{dr}{dt} = \frac{2}{4\pi r^2} = \frac{1}{2\pi r^2}$$

At the instant when $r = 2$ m

$$\frac{dr}{dt} = \frac{1}{4\pi} \approx 0.0795 \text{ m/min}$$

## Worked Example 21

A ladder 1.3 m long is resting against a wall. If the foot of the ladder slips awayfrom the base of the wall at a rate of 0.5 m/s, how fast is the top sliding down the wall when the foot is 5 m from the base of the wall?

**SOLUTION**

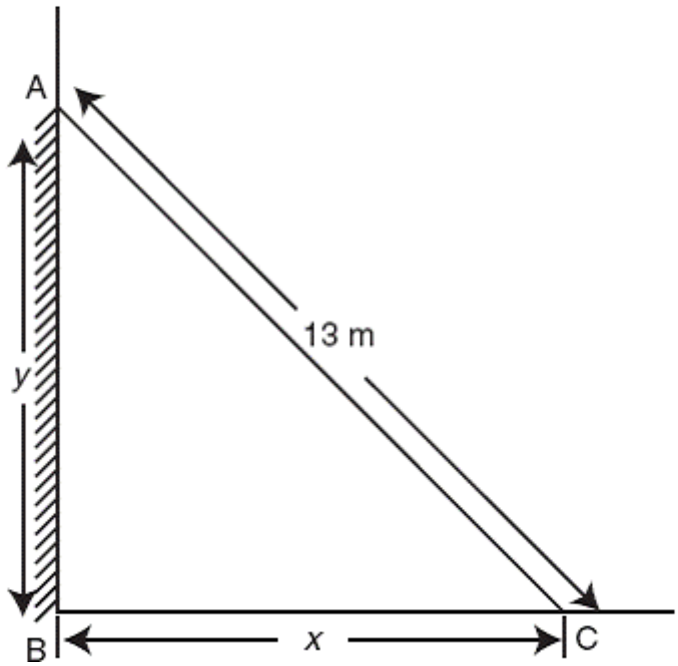


Figure. 12.1

The rate of change in $x$ is given for all

$t\left(\text{i. e}\ \frac{dx}{dt}=0.5\right).$ The rate $\frac{dy}{dt}$ of change in $y$ is required. Therefore, we try to find a relationship between $x$ and $y$ which can be differentiated with respect to $t$ using the function of function rule.

From the right-angled triangle ABC above, we have $x$2 + $y$2 = 132 = 169

$$\therefore \quad y=\sqrt{169-x^2}$$

Using function of a function rule

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dy}$$

$$=-\frac{x}{\sqrt{169-x^2}}\cdot\frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt}=-\frac{5}{\sqrt{144}}\times 0.5\ (\text{when } x=5)$$

$=-\frac{5}{24}$ which means that the top of the ladder is sliding down the wall at the rate $\frac{5}{24}$ m/sec.

## Exercise 6

1. Find the gradient of the curve $y = \frac{1}{x} - 2\sqrt{x}$ at the point (1, 0).

2. Determine the gradient of the curve $y = x(x + 2)(x^2 + 1)$ at the point (2, 3).

3. The position of a moving component is given by $S = A \sin mt + B \cos mt$, where $A$, $B$ and $M$ are constants and $t$ is time. Show that the acceleration $\frac{d^2s}{dt^2}$ is given by $-m^2S$.

4. A particle is projected vertically upwards with a velocity of 50 m/sec. The height $S$ m reached after $t$ seconds is given by $S = 50t - 5t^2$. Show that the acceleration is constant. Find the maximum height reached by the particle.

5. The distance $S$ m travelled in $t$ seconds by a body moving in a straight line is given by $S = t^3 - t^2$. Find
   (a) Its velocity after 3 seconds.
   (b) Its acceleration after 4 seconds.

6. Find the gradient of the curve $y^2x - 4x^2 = 4$ at the point (2, 1).

Find the stationary points for the following functions and determine their nature.

7. $y = x^2 - x + 1$

8. $y = 2x^3 - 3x^2 + 3$

9. $y = x^4 - 8x^2 + 10$

10. $y = (x - 1)(x + 2)$

11. Sheet metal of 120 cm$^3$ is to be used to construct an open-top tank with a square base. Find the dimensions of the tank such that its volume is maximised. What is the maximum volume?

12. Each side of an equilateral triangle is increasing at a rate of 2 cm/sec. Find the area at which its area is increasing when the scale is of length $2\sqrt{3}$ cm.

## SUMMARY

**In this chapter, we have learnt the following:**

❖ The derivative of a function $f(x)$ denoted as $f'(x)$ is given as $f'(x)$ $\lim_{0x \to 0} \left\{ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right\}$.

❖ The derivative of a function at any point is the gradient or slope of the function at the point.

❖ $\frac{d}{dx}(u(x) \pm v(x)) = \frac{d}{dx}(u(x)) \pm \frac{d}{dx}(v(x))$

- $\frac{d}{dx}(f(u(x)) = \frac{df}{du} \cdot \frac{du}{dx}$
- $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $\frac{d}{dx}(e^{nx}) = ne^{nx}$

## GRADUATED EXERCISES

Find the derivatives of the following functions from the first principle:

1. $y = x(x+1)$
2. $y = 3x^3 + 4x + 1$
3. $(2x+1)(2x-1)$
4. $y = x + \frac{1}{2}x^2$
5. $y = x(5x-1)$

Differentiate the following:

6. $y = \frac{1}{x^n}$
7. $y = \frac{1}{x} + 3x^2$
8. $y = (x^2+1)(x^3+1)$
9. $y = \frac{x-2}{\sqrt{x}}$
10. $\frac{(x-1)(\sqrt{x}+2)}{\sqrt[3]{x}}$

11. Give that $pv = 50$, find the value of $\frac{dp}{dv}$ when $v = 7$.

12. Find the gradient of the tangent to the curve $y = x^8 - 4x^2 + 3x + 1$ at the point where $x = 2$.

Find the maximum and minimum values of the following:

13. $y = x + \frac{2}{\sqrt{x}}$

14. $y = 48x - x^2$

15. $y = 8x + \frac{1}{x}$

16. $y = x^8 - 8x + 2$

17. $2x^3 - x^2$

18. The sum of two whole numbers is 36. Find their maximum product.
19. The volume of a right circular cylinder open at one end is 27π cm3. Find its maximum surface area.
20. Sand falls into a conical pile at a rate of 0.4 m3/min. The diameter of the base of the pile is always equal to the height. How fast is the height of the pile $\left(\frac{2}{5\pi}\text{ m/min}\right)$ rising at the instant when it is 2 m high?