CHAPTER 7 Rectangular Cartesian Coordinates

The house address is about the only means by which a new visitor can locate a particular house. In the same vein, positions are located in mathematics (and sciences generally) by using coordinates. In this chapter, we will be dealing with the rectangular cartesian coordinates.

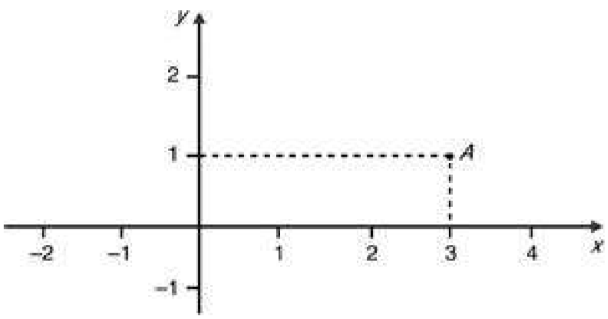

Fig. 7.1

The two dotted lines emanating from point *A*, in Figure 7.1, are perpendicular to the *x* and *y*-axes, and both of these lines form a rectangle with the *x* and *y*-axes. The height and width of this rectangle are the parameters used in giving the location of *A* on the certesian coordinate. A scientist named Cartesian brought about (invented) this coordinate system. Hence, the coordinate of point *A* in Figure 7.1 is (3,1).

> *Note that it is a convention in Mathematics that the x-coordinate of a point is written first before the y-coordinate, just like it is written for point A above.*

It takes joining two points together to get a straight line; a straight line can be vertical, horizontal or oblique, so we shall be looking at the property of a line that tells how oblique (slanted) a line is. This property is the steepness of a line and it is commonly called gradient in mathematics.

The gradient (steepness) of a line can be quantified by finding the ratio of the change in its vertical height to the change in its horizontal width between **any** two points on the line.

From Figure 7.2 below, the change in vertical height between points *A* and *B* will be the difference in vertical height between points *A* and *B*, which is 4 – 1 = 3.

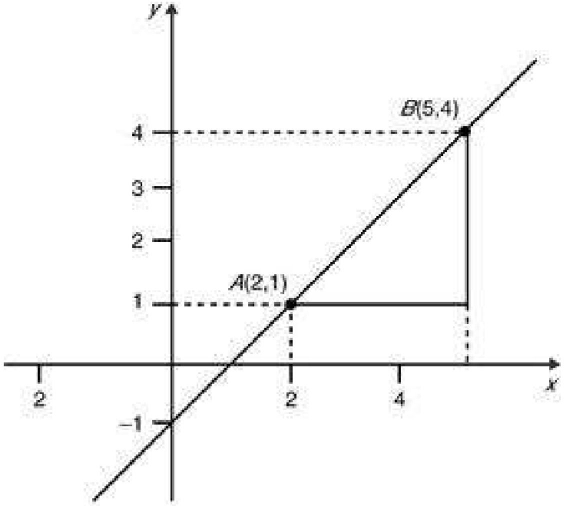

Fig. 7.2

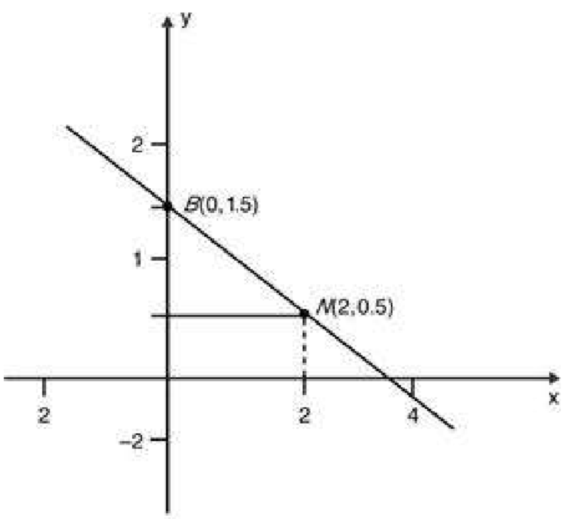

Fig. 7.3

The change in horizontal width will be the difference in width between points A and B, which is 5 – 2 = 3. Then, the gradient of the line will be

$$\frac{\textit{change in vertical height}}{\textit{change in horizontal width}} = \frac{4-1}{5-2} = \frac{y_b - y_a}{x_b - x_a}$$
$$= \frac{3}{3} = 1.$$

In summary, the gradient of a line passing through points with coordinates (x_1, y_1) and (x_2, y_2) is expressed as $\frac{y_2 - y_1}{x_2 - x_1}$.

Thus, by the same explanation, the gradient of the line in Figure 7.3 above will be calculated as follows: (x_1, y_1) = (2, 0.5) and $(x2, y2)$ = (0, 1.5);

$$\text{gradient, } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1.5 - 0.5}{0 - 2} = -\frac{1}{2}$$

The gradient of the line in Figure 7.2 is positive while that of Figure 7.3 is negative. This is due to the orientation of the lines, so keep this rule in mind: the gradient of a line that rises from left to right (for example Figure 7.2) is a positive gradient, while the gradient of a line that rises from right to left (for example Figure 7.3) is a negative gradient.

Relationships between the Gradients of Parallel and Perpendicular Lines

Two lines l_1 and l_2 are said to be parallel to each other if their respective gradients, m_1 and m_2 are equal.

Lines l_3 and l_4 are said to be perpendicular to each other if the product of their gradients is equal to −1, that is l_3 is perpendicular to l_4 if $m_3 \times m_4 = -1$.

Distance between Two Points

The distance *d*, between any two points with coordinates (x_1, y_1) and (x_2, y_2) is expressed as

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Equation of a Line

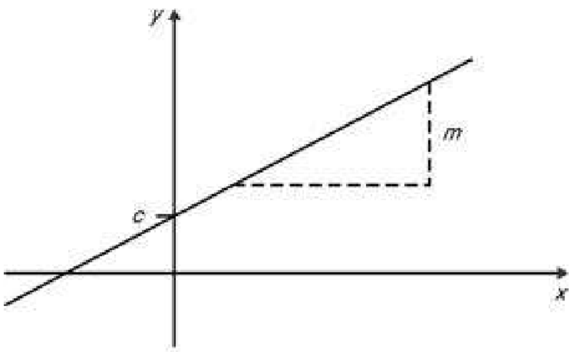

Fig. 7.4

The equation of a line with gradient, *m*, that intersects the vertical axis (*y*-axis) at *c* is expressed as $y = mx + c$ (Figure 7.4).

Circle

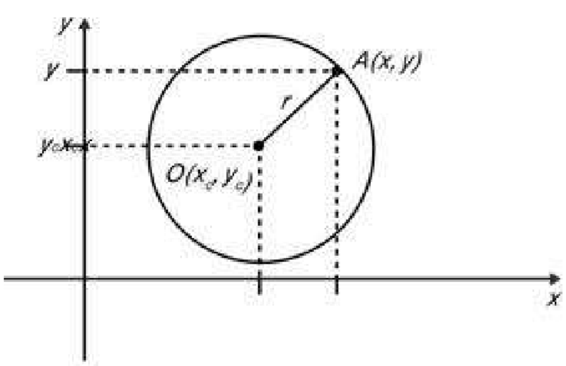

Fig. 7.5

The distance *r* between the centre $O(x_c, y_c)$ and any point with coordinate $A(x, y)$ on its circumference (as shown in Figure 7.5) is given by

$r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$. Squaring both sides gives

$r^2 = \left(\sqrt{(x - x_c)^2 + (y - y_c)^2}\right)^2; r^2 = (x - x_c)^2 + (y - y_c)^2.$

This is the equation of a circle with centre coordinates (*xc*, *yc*) and radius *r*. This equation can be further simplified to arrive at the general equation of a circle.

$r^2 = (x - x_c)^2 + (y - y_c)^2$; $r^2 = x^2 - 2x_c x + x_c^2 + y^2 - 2y_c y + y_c^2$;

$r^2 = y^2 + x^2 - 2y_c y - 2x_c x + y_c^2 + x_c^2$;

$0 = y^2 + x^2 - 2y_c y - 2x_c x + y_c^2 + x_c^2 - r^2$;

$y^2 + x^2 - 2y_c y - 2x_c x + y_c^2 + x_c^2 - r^2 = 0$;

$y^2 + x^2 + 2(-y_c)y + 2(-x_c)x + y_c^2 + x_c^2 - r^2 = 0$.

Let $-y_c = f$, $-x_c = g$ and $y_c^2 + x_c^2 - r^2 = c$, so that $y^2 + x^2 + 2fy + 2gx + c = 0$. This is the general equation of a circle with centre coordinates $(-g, -f)$.

Lines, Gradients and Distances

1. The points $D(2, 1)$, $E(1, 5)$ and $F(-6, -1)$ are the midpoints of the sides AC, AB and BC respectively of a triangle ABC.

(a) Show that ΔDEF is right-angled;

(b) Find the coordinates of B;

(c) Find the equation of AC. **(WAEC)**

Workshop

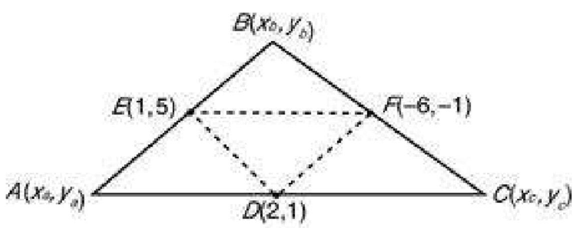

Fig. 7.6

(a) For ΔDEF in Figure 7.6 to be a right angled triangle, any two of lines DE, EF and FD must be perpendicular to each other. For two lines to be perpendicular to each other, the product of their gradients must be equal to −1. Gradient of line

$$DE = m_{DE} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-1}{1-2} = \frac{4}{-1} = -4.$$

$$\text{Also, } m_{EF} = \frac{-1-5}{-6-1} = \frac{-6}{-7} = \frac{6}{7};$$

$$m_{FD} = \frac{1-(-1)}{2-(-6)} = \frac{1+1}{2+6} = \frac{2}{8} = \frac{1}{4}.$$

$$\text{Therefore, } m_{DE} \times m_{FD} = -4 \times \frac{1}{4} = -1.$$

The product of the gradient of lines DE and FD making up two sides of the triangle DEF is −1; therefore, DE is perpendicular to FD, which confirms that ΔDEF is a right angled triangle.

(b) The coordinates (x_m, y_m) of the midpoint of a line joining two points $P(x_1, y_1)$ and

$Q(x_2, y_2)$ is given by: $x_m = \frac{\quad}{2}$ and $y_m = \frac{y_1 + y_2}{2}$. From Figure 7.6, as $E(1, 5)$ is the midpoint of line AB, then

$x_E = \frac{x_A + x_B}{2} = 1$ and $y_E = \frac{y_A + y_B}{2} = 5$;

$x_A + x_B = 2 \times 1$ and $y_A + y_B = 2 \times 5$;

$x_A + x_B = 2$ and $y_A + y_B = 10$..................(1)

$x_B = 2 - x_A$ and $y_B = 10 - y_A$.................(1*b*)

Also, $F(-6, -1)$ is the midpoint of line BC, thus, $x_F = \frac{x_B + x_C}{2} = -6$ and $y_F = \frac{y_B + y_C}{2} = -1$;

$x_B + x_C = -12$ and $y_B + y_C = -2$ ....(2)

Recall that from equations (1b), $x_B = (2 - x_A)$ and $y_B = (10 - y_A)$;

replace x_B and y_B in equations (2) with $(2 - x_A)$ and $(10 - y_A)$

respectively, to get $(2 - x_A) + x_c = -12$ and $(10 - y_A) + y_c = -2$.

$x_C - x_A + 2 = -12$ and $y_C - y_A + 10 = -2$

$x_C - x_A = -14$ and $y_C - y_A = -12$............(3)

Moreover, $D(2, 1)$ is the mid-point of line AC, therefore,

$x_D = \frac{x_A + x_C}{2} = 2$ and $y_D = \frac{y_A + y_C}{2} = 1$;

$x_A + x_C = 4$ and $y_A + y_C = 2$ ....(4)

Recall that, $x_C - x_A = -14$ and $y_C - y_A = -12$..(3)

and $x_A + x_C = 4$ and $y_A + y_C = 2$..............(4)

Subtract equation (4) from (3) to get $x_C - x_A - (x_A + x_C) = -14 - 4$ and $y_C - y_A - (y_A + y_C) = -12 - 2$;

$x_C - x_A - x_A - x_C = -18$ and $y_C - y_A - y_C = -14$;

$-2x_A = -18$ and $-2y_A = -14$; $x_A = \frac{-18}{-2}$ and $y_A = \frac{-14}{-2}$;

$xA = 9$ and $yA = 7$. Therefore, the coordinates of A is $(xA, yA) = (9, 7)$. Recall from equations (1b) above that $xB = 2 - xA$ and $yB = 10 - yA$; thus, $xB = 2 - 9$ and $yB = 10 - 7$; $xB = -7$ and $yB = 3$.

Therefore, the coordinates of B is $(xB, yB) = (-7, 3)$

(b) For us to know the equation of line AC, we need to know the gradient of line AC, and a point on line AC. We already know two points on line AC, which are $D(2, 1)$ and $A(9, 7)$; we can use any one of the two points to calculate the equation of the line. Also know that, if any one of these two points is used to calculate the equation of line AC, you will arrive at the same answer as these two points are on line AC. We already know two points on line AC; therefore, we can find its gradient as

$$m_{AC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-1}{9-2} = \frac{6}{7}.$$

The equation of a line of gradient m, with the line passing through a point $(x1, y1)$ is expressed as

$y - y1 = m(x - x1)$ Thus, the equation of line AC with gradient $\frac{6}{7}$ and the line passing through point (2, 1) will be

$$(y-1) = \frac{6}{7}(x-2); (y-1)$$
$$= \frac{6(x-2)}{7}; 7(y-1) = 6(x-2);$$
$$7y - 7 = 6x - 12; 7y - 6x - 7 + 12 = 0;$$
$$7y - 6x + 5 = 0.$$

Therefore, the equation of line AC is

$$7y - 6x + 5 = 0.$$

Note that you can use any point lying on a line (along with the gradient of the line) to calculate the equation of that line. Therefore, if you make use of point (9, 7), instead of (2, 1) in the above calculation, you will get the same equation $7y - 6x + 5 = 0$.

2. The equation of the sides PQ, PR and QR of a triangle PQR are:

$2x + 7y + 3 = 0$;

$3x - 8y + 23 = 0$;

$5x - y - 11 = 0$ respectively.

Find the:

(i) coordinates of the points P, Q and R;

(ii) area of ΔPQR, using determinant method. **(WAEC)**

Workshop

(i) Equation of line PQ is $2x + 7y + 3 = 0$;

equation of line PR is $3x - 8y + 23 = 0$

Equation of line QR is $5x - y - 11 = 0$.

These equations can be rewritten as.

$2x + 7y = -3$ ....(i)

$3x - 8y = -23$ ....(ii)

$5x - y = 11$ .... (iii)

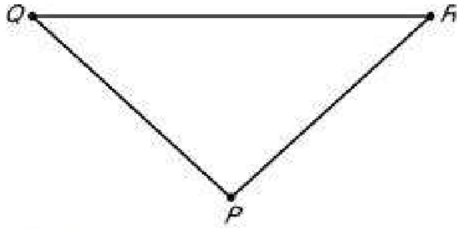

Fig. 7.7

As shown in Figure 7.7, lines *PQ* and *PR* will meet at point *P*, and we can know the co-ordinates of *P* by solving the equations of lines *PQ* and *PR* simultaneously.

This is because point *P* is a point which is common to the two lines as drawn in the diagram. So, the coordinates (x, y) of *P*, will satisfy both equations.

$2x + 7y = -3$ ....(*i*)

$3x - 8y = -23$ ....(*ii*)

Solving equations (*i*) and (*ii*) we get $x = -5$ and $y = 1$; therefore, the coordinates of *P* are $(-5, 1)$.

Note that you are expected to show all the steps (workings) in an exam.

Lines *PR* and *QR* will meet at point *R*, so the co-ordinates of *R* can be known by solving the equation of lines *PR* and *QR* simultaneously.

$3x - 8y = -23$ ....(*ii*)

$5x - y = 11$ .... (*iii*)

Solving equations (*ii*) and (*iii*) simultaneously, get $x = 3$ and $y = 4$; therefore, the coordinates of *R* are $(3, 4)$. Lines

PQ and *QR* will meet at point *Q* so, the co-ordinates of *Q* can be known by solving

the equations of lines *PQ* and *QR* simultaneously.

$2x + 7y = -3$ ....(*i*)

$5x - y = 11$ .... (*iii*)

Solving equations (*i*) and (*iii*) simultaneously, we find, again, that the coordinates of *Q* are $(2, -1)$. Therefore, the co-ordinates of *P*, *Q* and *R* are $(-5, 1)$, $Q(2, -1)$ and $R(3, 4)$ respectively.

(ii) The area of any triangle *PQR*, with verticesmat $P(x_P, y_P)$, $Q(x_Q, y_Q)$, $R(x_R, y_R)$, using determinant method, is expressed as

$$Area = \frac{1}{2}\begin{vmatrix} x_P & y_P & 1 \\ x_Q & y_Q & 1 \\ x_R & y_R & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[x_P[(y_Q \times 1) - (y_R \times 1)] - y_P[(x_Q \times 1) - (x_R \times 1)] + 1[(x_Q \times y_R) - (x_R \times y_Q)]\right].$$

Thus, the area of triangle *PQR*, with vertices at $P(-5, 1)$, $Q(2, -1)$ $R(3, 4)$ will be expressed as follows:

$$Area = \frac{1}{2}\begin{vmatrix} -5 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[(-5)[(-1\times 1)-(4\times 1)]-(1)[(2\times 1) - (3\times 1)]+(1)[(2\times 4)-(3\times -1)]];$$

$$Area = \frac{1}{2}[(-5)(-1-4)-1(2-3) + 1(8-[-3])];$$

$$Area = \frac{1}{2}[-5(-5)-1(-1)+1(11)]$$

$$= \frac{1}{2}[25+1+11] = \frac{1}{2}(37);$$

$$Area = \frac{37}{2} = 18.5 \text{ square units.}$$

Note that, since the unit of area is not specified in the question i.e whether the area is in square meter (m2) or square centimetre (cm2) or some other units of area, the right thing to do is to leave the units as square units in the answer.

3. Find the coordinates of the points on the curve $y = x^2(3 - x)$ where the tangent is parallel to the x-axis. **(WAEC)**

Workshop

First, understand that a tangent is a **line**. Now let us try to draw a line parallel to the x-axis.

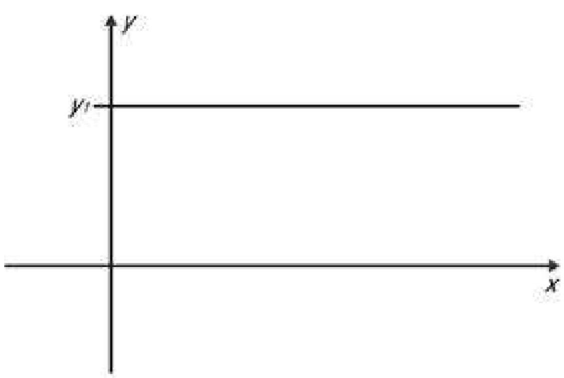

Fig. 7.8

Look at this graph closely and you will find out that, while the value of x is changing along the graph, the value of y is constant (which is equal to $y1$), hence, change in y = 0. Recall that the gradient of a line having equation $y = f(x)$ is expressed as $\frac{dy}{dx} = \frac{change\ in\ y}{change\ in\ x}$. Thus, the gradient of the line above $= \frac{change\ in\ y}{change\ in\ x}$. But, change in y is equal to zero (from the graph), therefore, gradient of the graph i.e $\frac{0}{change\ in\ x} = 0$. Therefore, the gradient of a line, parallel to the x-axis is zero. Thus, where the tangent to the curve is parallel to the x-axis, the gradient of the tangent is zero.

Moreover, recall that the gradient of the tangent to a curve at a point, is equal to the gradient of the curve at that same point. Hence, the gradient of the curve $y = x^2(3 - x)$, at the point where the tangent is parallel to the x-axis, is also zero. The gradient, at any point on the curve, is given by $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d(x2(3-x))}{dx} = \frac{d(3x2 - x3)}{dx} = 6x - 3x2$$

At the point, where the tangent is parallel to the x-axis, the gradient of the curve is zero, so, *gradient* =

$6x - 3x^2 = 0$; $3x(2 - x) = 0$; $3x = 0$ *or* $2 - x = 0$; $x = \frac{0}{3}$ *or* $-x = -2$; $x = 0$ *or* $x = 2$ Recall that, $y = x2\ (3 - x)$, therefore, when $x = 0$, $y = x2\ (3 - x) = 0^2\ (3 - 0) = 0(3) = 0$. Then, the coordinate of one of the points is (0, 0). When $x = 2$, $y = 22(3 - 2) = 4(1) = 4$. The coordinates of the other point is (2, 4) The coordinates of the points on the curve $y = x^2\ (3 - x)$ where the tangent is parallel to the x-axis, are (0, 0) and (2, 4).

4. (a) $F(7, 3)$, $G(-4, 1)$ and $H(-3, -2)$ are vertices of the ΔFGH.

Show that ΔFGH is isosceles. **(WAEC)**

Workshop

(a) $F(7, 3)$, $G(-4, 1)$, $H(-3, -2)$

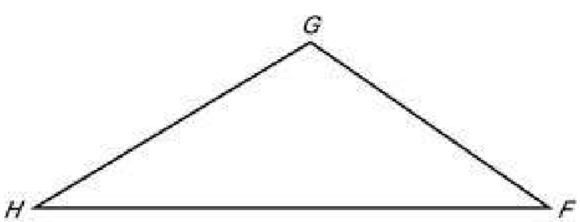

Fig. 7.9

For a triangle to be isosceles, the lengths of **two** of its sides *must* be equal. And so, for ΔFGH to be an isosceles triangles *two* of its side lengths must be equal. Given the co-ordinates, $A(x_1, y_1)$ and $B(x_2, y_2)$, of two points A and B, the distance between points A and B can be calculated as $|AB| = \sqrt{(x2 - x1)2 + (y2 - y1)2}$.

Therefore, $|FG| = \sqrt{(-4-7)^2 + (1-3)^2}$

$$= \sqrt{121 + 4} = \sqrt{125},$$

$$|GH| = \sqrt{(-3-(-4))^2 + (-2-1)^2}$$

$$= \sqrt{(-3+4)^2 + (-3)^2}.$$

$$|GH| = \sqrt{(+1)^2 + (-3)^2}$$

$$= \sqrt{1+9} = \sqrt{10}, \text{ and}$$

$$|FH| = \sqrt{(-3-7)^2 + (-2-3)^2}$$

$$= \sqrt{(-10)^2 + (-5)^2}$$

$$= \sqrt{100+25} = \sqrt{125}.$$

From the lengths of the sides of the triangle calculated, $|FG| = |FH|$ and these two sides are *not* equal to the third side $|GH|$. The lengths of **only two** sides of triangle *FGH* are equal; thus, triangle *FGH* is an isosceles triangle.

5. Given that $x^2 - 3xy + 2y^2 - y = 1$, find the equation of the tangent to the curve at the point (0, 1). **(WAEC)**

Workshop

To get the equation of tangent at point (0, 1), we need to get the slope, $(\frac{dy}{dx})$ of the curve at this point. The equation can be rewritten as $x^2 - 3xy + 2y^2 - y - 1 = 0$.

Then, the gradient of the curve can be calculated as

$$\frac{d(x^2 - 3xy + 2y^2 - y - 1)}{dx} = \frac{d(0)}{dx};$$

$$\frac{d(x^2)}{dx} + \frac{d(-3xy)}{dx} + \frac{d(2y^2)}{dx} + \frac{d(-y)}{dx} + \frac{d(-1)}{dx} = 0.$$

$$since\ \frac{d(0)}{dx} = 0.$$

The equation $x2 - 3xy + 2y2 - y = 1$ is an implicit function since we **cannot** make y the subject of the formula of the equation. If you try to make y the subject of the formula, you will still find y on the side of the equation where all terms should be in x. So, the relationship between x and y, in this case, is **implicit**. As the equation is an **implicit function**, we will find $\frac{dy}{dx}$ differentiating implicitly.

Now, we can implicitly differentiate the term $-3xy$ in the equation by using the product rule that states that: $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ where u and v are functions of x. For us to differentiate $-3xy$ with respect to x, let $u = -3x$, so that

$$\frac{du}{dx} = -3x^{1-1} = -3x^0 = -3 \text{ and } v = y^1, \therefore\ \frac{dv}{dx} = y^{1-1}\frac{dy}{dx} = y^0\frac{dy}{dx}$$

$$= 1\frac{dy}{dx}. \text{ Therefore, } \frac{d(uv)}{dx} = \frac{d(-3xy)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= -3x\left(1\frac{dy}{dx}\right) + y(-3) = -3x\frac{dy}{dx} - 3y; \text{ thus, } \frac{d(-3xy)}{dx}$$

$$= -3x\frac{dy}{dx} - 3y.$$

Carefully note that to differentiate any term in y, with respect to x, $\frac{dy}{dx}$ must be included in the result. For example, if you want to differentiate y^3 with respect to x, the answer will be $3y^2\frac{dy}{dx}$. This is what is called implicit differentiation. Also, if $y^4 + x^2 + 1 = 0$ is differentiated with respect to x, the answer is $4y^3\frac{dy}{dx} + 2x^1$ $+ 0 = 0$ $4y^3\frac{dy}{dx} + 2x = 0$; $4y^3\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{4y^3}$.

Recall that, $\frac{d(-3xy)}{dx} = -3x\frac{dy}{dx} - 3y$ so that

$$\frac{d(x^2)}{dx} + \frac{d(-3xy)}{dx} + \frac{d(dy^2)}{dx} + \frac{d(-y)}{dx} + \frac{d(-1)}{dx}$$

$$= 2x + \left(-3x\frac{dy}{dx} - 3y\right) + 4y\frac{dy}{dx} + \left(-1\frac{dy}{dx}\right) + (-0) = 0;$$

$$2x - 3x\frac{dy}{dx} - 3y + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0;$$

$$2x - 3y - 3x\frac{dy}{dx} + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0;$$

$$-3x\frac{dy}{dx} + 4y\frac{dy}{dx} - \frac{dy}{dx} = 3y - 2x;$$

$$\frac{dy}{dx}(4y - 3x - 1) = 3y - 2x;$$

$\therefore \frac{dy}{dx} = \frac{3y - 2x}{4y - 3x - 1}$. At point (0, 1),

$$\frac{dy}{dx} = \frac{3y - 2x}{4y - 3x - 1} = \frac{3(1) - 2(0)}{4(1) - 3(0) - 1} = \frac{3}{3} = 1.$$

The slope of the curve at point (0, 1) is equal to the slope of the **tangent** to the curve at the same point (0, 1). For this reason, the slope of the tangent to the curve at point (0, 1) is 1. Point (0, 1) will also be a point on the tangent to the curve, as the tangent to the curve at this point **must** touch the point (0, 1) on the curve.

The equation of a line with slope m and a known point, $(x1, y1)$, lying on the line is given by $y - y1 = m(x - x1)$ We now know the slope of the tangent, and point (0, 1) is a point on the tangent (*which is a line*), then, the equation of the tangent will be calculated as $y - y1 = m(x - x1)$; $y - 1 = m\ (x - 0)$; $y - 1 = 1(x - 0)$; $y - 1 = x$; $y - x - 1 = 0$.

Therefore, the equation of the tangent to the curve at point (0, 1) is $y - x - 1 = 0$.

6. A fixed point $A(m, n)$ lies on the curve $y = x^2 + 1$ and the line $y = x + 7$.

(a) Find:

(i) the two possible coordinates of A,

(ii) the distance between the two locations of A.

(b) Sketch the curve and draw the line.

(c) Find the area of the finite region enclosed between the line and the curve. **(WAEC)**

Workshop

(a)(i)For the coordinate (m, n), m is the x-component and n is the y-component of the point. If point $A(m, n)$ lies on the curve $y = x2 + 1$, then we can put the values (m, n) in the respective positions of x and y in the equation of the curve as $n = m2 + 1$ ....(i)

Moreover, point $A(m, n)$ lies on the line $y = x + 7$; therefore, we can, again, put the values (m, n) in their respective positions in the equation of the line as $n = m + 7$ ....(ii)

Now, the problem has been broken down into two simultaneous equations with two unknown; thus, the values of m and n can be calculated as we will do shortly.

$n - m^2 = 1$ ....(i)

$n - m = 7$ ....(ii)

Subtract (ii) from (i), to get: $n - m_2 - (n - m) = 1 - 7$; $n - m_2 - n + m = -6$

$-m_2 + m + 6 = 0$; $-m_2 + 3m + 6 = 0$; $-m(m - 3) - 2(m - 3) = 0$;

$(m - 3)(-m - 2) = 0$; $(m - 3) = 0$ or $(-m - 2) = 0$; $m = 3$ or $-m = 2$;

$m = 3$ or $m = -2$. In equation (ii), $n - m = 7$; $n = m + 7$;

when $m = -2$, $n = m + 7 = -2 + 7 = 5$; when $m = 3$, $n = m + 7 = 3 + 7 = 10$.

Therefore, the two possible coordinatess of A are $(-2, 5)$ and $(3, 10)$.

Note: if the quadratic curve $y = x^2 + 1$ is plotted on the same graph with line $y = x + 7$, the line will cross the curve at two points, which are points $(-2, 5)$ and $(3, 10)$.

(ii) Distance between any two points, (x_1, y_1) and (x_2, y_2), is expressed as $\sqrt{(x_2 - x_1)_2 + (y_2 - y_1)_2}$. Hence, the distance between points $(-2, 5)$ and $(3, 10)$ will be $\sqrt{[3 - (-2)]_2 + (10 - 5)_2}$

$= \sqrt{(3 + 2)_2 + 5_2}$;

$\sqrt{5_2 + 5_2} = \sqrt{50} = \sqrt{5 \times 5 \times 2} = 5\sqrt{2}$.

Therefore, the distance between the two locations of A is $5\sqrt{2}$ *units*.

(b) To be able to sketch the curve $y = x^2 + 1$, we need to know the intercepts of the curve on the x and y-axes, and the turning point of the curve. Recall that at the point where a curve (or a line) intercepts the x-axis, the y-value of the point is zero. Thus, at the points where the curve, $y = x2 + 1$, intercepts the x-axis, the value of y is zero ($y = 0$) so that $y = x^2 + 1 = 0$; $x^2 = -1$; $x = \pm\sqrt{1}$.

Note that the values $x = +\sqrt{-1}$ and $-\sqrt{-1}$ are ***not*** *real values (because one* ***cannot*** *find the value of $\sqrt{-1}$); they are complex values which* *have n o real parts but imaginary parts, i.e they are not real values like 4, –2, 3.5, e.t.c.*

Since the values of x (i.e. $x = + \;\; -1$ and $-\sqrt{-1}$) are **not** real, the curve, $y = x^2 + 1$, will not cross the x-axis. This is because the equation of the curve has complex roots.

*Note that if the root of any quadratic equation is in the form $\sqrt{-a}$ (where a could be any number), then the curve drawn from the equation will **not** touch the x-axis; it will float over the x-axis.*

Also, at the point where the curve intercepts the y-axis, the value of x is zero ($x = 0$) (*confirm this from figure* 7.10). So, $y = x^2 + 1 = 02 + 1 = 1$.

Then, the intercept of the curve on the y-axis is 1. At the turning point of a curve, $\frac{dy}{dx} = 0$, therefore, $\frac{dy}{dx} = \frac{d(x_2 + 1)}{dx}$ $= 2x = 0$; $x = \frac{0}{2} = 0$. Thus the value of x at the turning point is zero. At the turning point, $x = 0$, then $y = x^2 + 1 = 0^2 + 1 = 1$. Therefore, the coordinates of the turning point are (0, 1). Recall that $\frac{dy}{dx} = 2x$; $\frac{d^2y}{dx^2} = 2$. $\frac{d^2y}{dx^2}$ is greater than zero; therefore the turning point is a minimum turning point. Having known these parameters, the graph of $y = x_2 + 1$ can be drawn, and is shown in Figure 7.10 below.

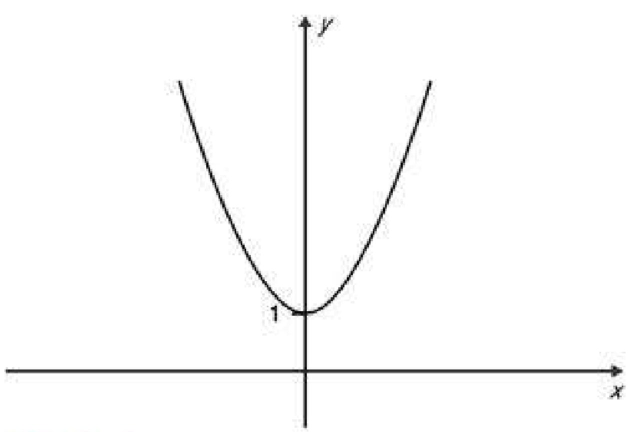

Fig. 7.10

To sketch the graph of line $y = x + 7$, we need to know the intercepts of the line on the x and y axes. At the point where line $y = x + 7$ intercepts the y-axis, the value of $x = 0$; (*see Figure* 7.11). Therefore, $y = x + 7 = 0 + 7 = 7$. At the point where line $y = x + 7$ intercepts the x-axis, the value of $y = 0$; i.e $y = 0 = x + 7$; $x = 0 - 7$; $x = -7$.

Now that we know all these necessary parameters to sketch the graph of the line, the graph can be sketched on the curve of $y = x^2 + 1$ as below.

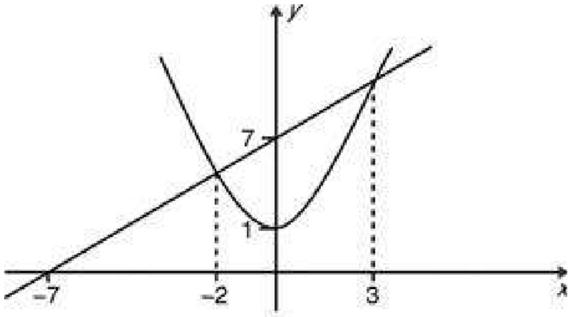

Fig. 7.11

Please note that we had already calculated the possible coordinates of point A that lie on both the line and the curve. These points are points $(-2, 5)$ *and* $(3, 10)$, *and they are the points where the line and the curve meet. The x-values of these points are as shown in Figure 7.11.*

(c)

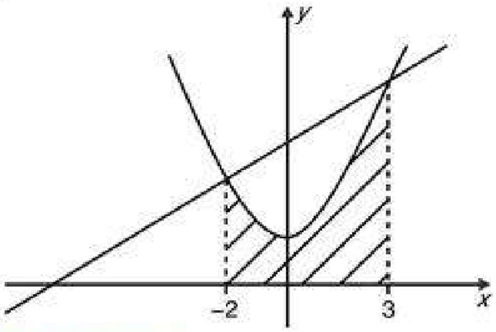

Fig. 7.12(a)

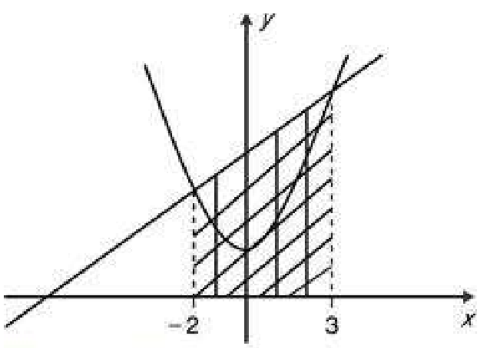

Fig. 7.12(b)

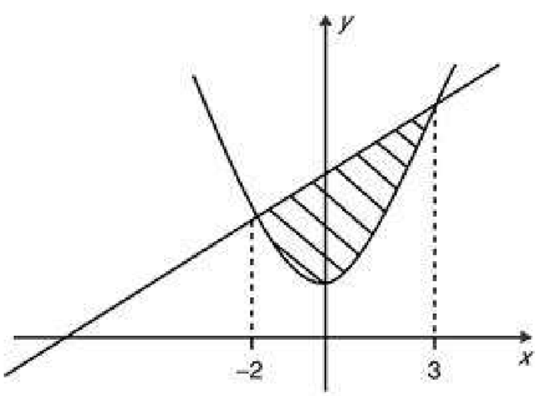

Fig. 7.12(c)

Figure 7.12(a) shows the area bounded by the curve $y = x^2 + 1$, line $x = -2$, line $x = 3$ and the x-axis. Figure 7.12(b) shows the area bounded by the line $y = x + 7$, line $x = -2$, line $x = 3$ and the x-axis. Figure 7.12(c) shows the area of the finite region enclosed between the line and the curve.

$$\begin{pmatrix}\textit{Area of the finite}\\ \textit{region enclosed}\\ \textit{between the line}\\ \textit{and the curve}\end{pmatrix} = \begin{pmatrix}\textit{Area bounded}\\ \textit{by line } y = x + 7,\\ \textit{line } x = -2,\\ \textit{line } x = 3 \textit{ and}\\ \textit{the } x\textit{-axis}\end{pmatrix} - \begin{pmatrix}\textit{Area bounded}\\ \textit{by curve}\\ y = x^2 + 1,\\ \textit{line } x = -2,\\ \textit{line } x = 3 \textit{ and}\\ \textit{the } x\textit{-axis}\end{pmatrix}$$

Recall that the area bounded by a curve $y = f(x)$, line $x = a$, line $x = b$ and the x-axis is given by $A = \int_a^b f(x)dx$.

$$\begin{pmatrix}\textit{The Area of the finite}\\ \textit{region enclosed between}\\ \textit{the line and the curve}\end{pmatrix} = \int_{-2}^{3}(x + 7)\cdot dx - \int_{-2}^{3}(x^2 + 1)\cdot dx =$$

$$\left(\frac{x^2}{2} + 7x\right)_{-2}^{3} - \left(\frac{x^3}{3} + x\right)_{-2}^{3}$$

$$= \left[\frac{3^2}{2} + 7(3) - \left(\frac{(-2)^2}{2} + 7(-2)\right)\right]$$

$$- \left[\frac{3^3}{3} + 3 - \left(\frac{(-2)^3}{3} + (-2)\right)\right]$$

$$= \frac{9}{2} + 21 - \left[\frac{4}{2} - 14\right] - \left[\frac{27}{3} + 3 - \left(-\frac{8}{3} - 2\right)\right]$$

$$= \frac{9}{2} + 21 - \frac{4}{2} + 14 - \frac{27}{3} - 3 + \left(-\frac{8}{3} - 2\right)$$

$$= \frac{9}{2} + 21 - 2 + 14 - 9 - 3 - \frac{8}{3} - 2$$

$$= 4\frac{1}{2} + 21 - 2 + 14 - 9 - 3 - \frac{8}{3} - 2 = 20\frac{5}{6}.$$

Therefore, the area of the finite region enclosed between the line $y = x + 7$ and curve $y = x^2 + 1$ is $20\frac{5}{6}$ *square units* [(*units*)^2].

Remember that area can be measured in cm^2, m^2, e.t.c, however, the unit that is to be used is not specified in the question, hence, it is safe to give our answer as square units.

7. (a) Find the equation of the normal to the curve $y = x^2 - 4x - 2$, at the point where the curve cuts the y-axis.

(b) At what point does the normal meet the x-axis? **(WAEC)**

Workshop

(a) $y = x^2 - 4x - 12$; the slope of this curve at any point (x, y) is given by $\frac{dy}{dx} = \frac{d(x^2 - 4x - 12)}{dx} = 2x - 4$. The slope of a curve at point (x, y) is equal to the slope of the tangent to the curve, at that same point. Hence, the slope of the tangent to the curve $y = x^2 - 4x - 12$, at any point (x, y), is $\frac{dy}{dx}$ $2x - 4$. At the point where this curve cuts the y-axis, the value of x is zero so that $y = x^2 - 4x - 12 = 02 - 4(0) - 12$; $y = -12$.

*Note that at the point where **any curve** cuts the x-axis, the y-coordinate of that point is zero, while at the point where **any curve** cuts the y-axis, the x-coordinate of that point is zero.*

Therefore, the coordinates of the point where the curve crosses the y-axis are (0, −12). The slope of the tangent to the curve at point (0, −12) will be $\frac{dy}{dx} = 2x - 4 = 2(0) - 4 = -4$ (as $x = 0$ at point (0, −12)).

$$\begin{pmatrix}\text{Slope of tangent to a}\\ \text{curve at point } (x_1, y_1)\end{pmatrix} \times \begin{pmatrix}\text{Slope of normal to the}\\ \text{curve at point } (x_1, y_1)\end{pmatrix} = -1$$

Thus,

$$\begin{pmatrix}\text{Slope of tangent to}\\ \text{the curve in question,}\\ \text{at point } (0, -12)\end{pmatrix} \times \begin{pmatrix}\text{Slope of normal to}\\ \text{the curve in question,}\\ \text{at point } (0, -12)\end{pmatrix} = -1$$

Hence, $(-4) \times \begin{pmatrix}\text{Slope of normal to}\\ \text{the curve in question,}\\ \text{at point } (0, -12)\end{pmatrix} = -1.$

$$\begin{pmatrix}\text{Slope of normal to the curve}\\ \text{in question, at point } (0, -12)\end{pmatrix} = \frac{-1}{-4} = \frac{1}{4}.$$

Since this normal is the normal to the curve at point (0, −12), it means that the point(0, −12) also lies on the normal to the curve. So, having known the slope and a point on the normal, we can determine the equation of the normal (which is a line) by the formula $y - y1 = m(x - x1)$; in the

case of this question, $m = \frac{1}{4}$, $y_1 = -12$ and $x_1 = 0$,

$\therefore y - (-12) = \frac{1}{4}(x - 0);\ y + 12 = \frac{x}{4};$

$4y + 48 = x;\ 4y - x + 48 = 0$

Therefore, the equation of the **normal** to the curve, at the point where the curve cuts the y-axis, is $4y - x + 48 = 0$.

(b) At the point where the **normal** meets the x-axis, the value of y at the point is zero($y = 0$). Recall that the equation of the normal is $4y - x + 48 = 0$; thus, when $y = 0$, $4y - x + 48 = 4(0) - x + 48 = 0$; $x = 48$. Therefore, the point where the normal meets the x-axis is (48, 0).

Note that points are defined in x and y values as (x, y), and not in x value alone or y value alone.

8. Given that $3x^2 + 2xy - 5y2 = -5$ find the:

(a) tangent

(b) normal, at the, point (−3, 2). **(WAEC)**

Workshop

(a) $3x^2 + 2xy - 5y^2 = -5$; $3x^2 + 2xy - 5y^2 + 5 = 0$

Carefully note that, to differentiate any term in y with respect to x, $\frac{dy}{dx}$ must be included in the *result. For example, if you want to differentiate y^3 with respect to x, the answer will be $3y^2\frac{dy}{dx}$. This is what is called implicit differentiation. Also, if $y^4 + x^2 + 1 = 0$ is differentiated, with respect to x, the answer will be $4y^3\frac{dy}{dx} + 2x^1 + 0 = 0$; $4y^3\frac{dy}{dx} + 2x = 0$; $4y^3\frac{dy}{dx} = -2x$; $\frac{dy}{dx} = \frac{-2x}{4y^3}$.*

The slope of the curve, $3x^2 + 2xy - 5y^2 + 5 = 0$, at any point is the derivative of the equation of the curve given by

$$\frac{d(3x^2 + 2xy - 5y^2 + 5)}{dx} = \frac{d(0)}{dx};$$

$$\frac{d(3x^2 + 2xy - 5y^2 + 5)}{dx}$$

$$= 6x + \left[2y + 2x\frac{dy}{dx}\right] - 10y\frac{dy}{dx} = \frac{d(0)}{dx};$$

$$= 6x + 2y + 2x\frac{dy}{dx} - 10y\frac{dy}{dx} = 0.$$

Note that 2xy was differentiated using the product rule of differentiation.

The slope, $\frac{dy}{dx} = \frac{-6x - 2y}{2x - 10y}$, of this curve at any point (x, y) is equal to the slope of the tangent to this curve, at that same point (x, y); therefore, the slope of the tangent, at any point $(x, y) = \frac{-6x - 2y}{}$.

$$\left(\begin{array}{l}\textit{the slope of the tangent}\\ \textit{to the curve in ques-}\\ \textit{tion at point } (-3, 2)\end{array}\right) = \frac{-6(-3) - 2(2)}{2(-3) - 10(2)}$$

$$= \frac{18 - 4}{-6 - 20} = \frac{14}{-26} = -\frac{7}{13}.$$

The point (−3, 2) is a point on the curve and the tangent to the curve at this same point will also touch the point, hence, the tangent at point (−3, 2) will also have point (−3, 2) as a point on it. Then, the equation of the tangent to the curve, in question at point (−3, 2), can be calculated using the formula $y - y1 = m(x - x_1)$, (*recall that a tangent to a curve is a line*); where $(x_1, y_1) = (-3, 2)$ and slope, $m = -\frac{7}{13}$

Then, $y - 2 = -\frac{7}{13}(x - -3)$;

$$y - 2 = \frac{-7(x + 3)}{13};$$

$$13(y - 2) = 7(x + 3);$$

$$13y - 26 = -7x - 21;\ 13y + 7x - 5 = 0$$

Therefore, the equation of the tangent to the curve in question at point (−3, 2) is $13y + 7x - 5 = 0$.

(b) The product of the tangent to a curve at a point and the normal to the curve at that same point is −1. Hence, $m_{tangent}$ (*at point* (−3, 2)) × m_{normal} (*at point* (−3, 2)) = −1;

$$\frac{-7}{13} \times m_{normal} = -1 \therefore m_{normal} = -1 \times \frac{13}{-7} = \frac{13}{7}$$

Hence, the slope of the normal, at point (−3, 2) is $\frac{13}{7}$.

Also, the normal to the curve, at point (−3, 2) will also touch the point (−3, 2); thus, (−3, 2) is a point on the normal to the curve, at point (−3, 2). $m = \frac{13}{7}$; (x_1, y_1) = (−3, 2); recall that the equation of a line, having gradient m, and passing through point (x_1, y_1) is given by $y - y1 = m\ (x - x_1)$.

$$\therefore y - 2 = \frac{13}{7}(x - -3); y - 2 = \frac{13(x+3)}{7};$$

$$7(y-2) = 13(x+3); 7y - 14 = 13x + 39;$$

$$7y - 13x - 53 = 0.$$

Therefore, the equation of the normal to the curve, at point (−3, 2) is $7y - 13x - 53 = 0$.

9. Two straight lines are $3x + 2y - 5 = 0$ and $x - 2y - 7 = 0$. Find the equation of the line through their point of intersection and parallel to the line $x + 2y - 5 = 0$. **(WAEC)**

Workshop

Let l be the line we want to find its equation. From the question, line l passes through the point of intersection of the lines $3x + 2y - 5 = 0$ and $x - 2y - 7 = 0$, so, the **point** where these two lines intersect will be a **point** on line l. We can know the coordinates of this point of intersection of the two lines by solving the equations of the two intersecting lines simultaneously, since the coordinate, (x, y) of the point will be common to both lines.

$3x + 2y = 5$ ....*(i)*

$x - 2y = 7$ ....*(ii)*

Add the two equations to get: $3x + 2y + (x - 2y) = 5 + 7$; $3x + 2y + x - 2y = 12$;

$4x + 2y - 2y = 12$; $4x = 12$; $x = \frac{12}{4} = 3$. Put $x = 3$ into equation *(i)* to get $3(3) + 2y = 5$; $2y = 5 - 9$; $2y = -4$;

$y = -\frac{4}{2} = -2$.

So, the coordinates of the point of intersection of the two lines is (3, −2) and this is a point on line l, since line l passes through the intersection of the two lines in question.

Also, from the question, the line l is parallel to line $x + 2y - 5 = 0$. We can rewrite this equation in the gradient-intercept form, $y = mx + c$; where m is the gradient of the line; so, $x + 2y - 5 = 0$ can be rewritten as $2y = -x + 5$; $y = -\frac{1}{2}x + \frac{5}{2}$.

Comparing this equation with $y = mx + c$ shows that the gradient of this line, parallel to l, is $-\frac{1}{2}$. If two lines are parallel to each other, they will have the same gradient. So, the gradient of line l will also be $-\frac{1}{2}$. Hence, the gradient of line l is $-\frac{1}{2}$ and line l passes through point (3, −2).

With these two parameters known, we can find the equation of line l as follows:

$y - y_1 = m(x - x_1)$; where $y_1 = -2$, $x_1 = 3$

and $m = \frac{-1}{2}$

$$y - (-2) = \frac{-1}{2}(x - 3);$$

$$y + 2 = \frac{-1(x-3)}{2};$$

$$2(y + 2) = -1(x - 3);$$

$$2y + 4 = -x + 3;$$

$$2y + x + 4 - 3 = 0;$$

$$2y + x + 1 = 0.$$

Therefore, the equation of the line, *l*, passing through the point of intersection of the lines $3x + 2y - 5 = 0$ and $x - 2y - 7 = 0$ and also parallel to the line $x + 2y - 5 = 0$, is $2y + x + 1 = 0$.

10. Find the equation of the tangent to the curve *y* = _ 2*x*_*x* _ −+_ _1 1_ , *x* ≠ − 12_ , at the point (1, 0). **(WAEC)**

Workshop

The slope of the curve $y = \frac{x-1}{2x+1}$, $x \neq -\frac{1}{2}$, at point (1, 0) is equal to the slope of the tangent to this curve at this same point.

> *Note that $x \neq -\frac{1}{2}$ means x is **not** equal to $-\frac{1}{2}$, and this means that all other countable values of x except $-\frac{1}{2}$ satisfies the equation. When $x = -\frac{1}{2}$, $y = \frac{x-1}{2x+1} = \frac{-\frac{1}{2} - 1}{2\left(-\frac{1}{2}\right) + 1}$ $= \frac{-\frac{3}{2}}{-\frac{2}{2} + 1} = \frac{-\frac{3}{2}}{-1 + 1} = \frac{-\frac{3}{2}}{0}$; you can see that this is indeterminate (because the result of any number divided by zero cannot be determined, so, $x = -\frac{1}{2}$ does not satisfy the equation $y = \frac{x-1}{2x+1}$.*

Therefore, the slope of the curve $y = \frac{x-1}{2x+1}$ at any point (x, y) on the curve is given by $\frac{dy}{dx}$. By the quotient rule, if $y = \frac{u}{v}$, where u and v are functions of x,

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v_2}$. In this case, $u = x - 1$; $\frac{du}{dx} = 1$ and $v = 2x + 1$; $\frac{dv}{dx} = 2$.

So, by using the quotient rule,

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v2} = \frac{(2x+1)(1)-(x-1)(2)}{(2x+1)2}$

$= \frac{2x+1-(2x-2)}{(2x+1)2} = \frac{2x+1-2x+2}{(2x+1)2}$

$= \frac{3}{(2x+1)2}$. At point $(1, 0)$, $x = 1$ *and* $y = 0$; $\frac{dy}{dx}$ at point $(1, 0) = \frac{3}{} = \frac{3}{32} = \frac{3}{9} = \frac{1}{3}$.

Therefore, the slope of the curve at point (1, 0) is $\frac{1}{3}$. Hence, the slope of the tangent to the curve $y = \frac{x-1}{2x+1}$ at point (1, 0) is also $\frac{1}{3}$.

Besides, the tangent to a curve $y = f(x)$ at a point (x_1, y_1) will touch the point (x_1, y_1). So, point (x_1, y_1) will be a point on the tangent to the curve. Then, point (1, 0) is a point on the tangent to the curve $y = \frac{x-1}{2x+1}$ at point (1, 0). The slope of the tangent to the curve at this point is $\frac{1}{3}$. Therefore, having known the gradient and one point on the tangent is known, we can find the equation of the tangent as $y - y1 = m(x - x1)$, where $(x1, y1) =$

$(1, 0)$ *and slope* $m = \frac{1}{3}$; $y - 0 = \frac{1}{3}(x-1)$; $y = \frac{x-1}{3}$;

$3y = x - 1$; $3y - x + 1 = 0$.

Therefore, the equation of the tangent to the curve at point (1, 0) is $3y - x + 1 = 0$.

11. The gradient of the tangent to the curve $y = 4x^3$ at points P and Q is 108. Find the coordinates of P and Q. **(WAEC)**

Workshop

The gradient m, of a curve at any point (x, y) is equal to the gradient of the tangent to the curve at that same point (x, y). The gradient to the curve $y = 4x^3$ at any point (x, y) is $\frac{dy}{dx} = 4(3x^{3-1}) = 4(3x^2)$ = $12x2$. Hence, the gradient of the tangent to this curve at any point (x, y) is also $\frac{dy}{dx} = 12x2$. From the question, at point P and Q, the gradient of the tangent is 108, so at points P and Q $\frac{dy}{dx} = 108 = 12x2$;

$x2 = \frac{108}{12} = 9$; $x2 = 9$; $x = \pm\sqrt{9} = \pm 3$; $x = 3$,

i.e $x = +3$ or $x = -3$. Therefore, the x-coordinates of points P and Q are $+3$ and -3.

Recall that $y = 4x^3$; thus, when $x = -3$; $y = 4x^3 = 4(3)^3 = 4(27) = 108$;

when $x = -3$; $y = 4x^3 = 4(-3)^3 = 4(-27) = -108$. So, the y-coordinates of points P and Q are 108 and –108.

Therefore, the coordinates of points P and Q are (+3, 108) and (−3, −108).

12. The normal to the curve $y = 2x^2 + x - 3$ at the point (2, 7) meets the x-axis at point P. Find the coordinates of P. **(WAEC)**

Workshop

To know the co-ordinates of *P*, we must first find the equation of the normal to the curve $y = 2x^2 + x - 3$ at point (2, 7), after which we will find the point *P* where the normal intersects the *x*-axis. The normal to a curve at any point (x, y) is a line perpendicular to the tangent to the curve at that point (x, y). Recall that, the slope of the tangent to a curve at any point (x, y) is equal to the slope of the curve at that same point (x, y); thus, the slope of the curve at any point (x, y)

$= \frac{dy}{dx} = \frac{d(2x_2 + x - 3)}{dx} = 2(2x) + 1 = 4x + 1$. At point (2, 7), $x = 2$; so, $\frac{dy}{dx}$ *at point* $(2, 7) = 4x + 1 =$

$4(2) + 1 = 9$. Recall that the slope of the curve at point (2, 7) is equal to the slope of the tangent to the curve at that same point (2, 7); therefore, the slope of the tangent to the curve at point (2, 7) is 9. As earlier mentioned, the normal to the curve at point (2, 7) is perpendicular to the tangent to the curve at this same point (2, 7). And for two lines perpendicular to each other with slopes $m1$ and m_2, $m_1 \times m_2 = -1$, so, $9 \times m_{\text{normal}} = -1$;

$m_{normal} = -\frac{1}{9}$. Therefore, the slope of the normal to the curve at point (2, 7) is $-\frac{1}{9}$. Since the normal touches point (2, 7) on the curve, then point (2, 7) is a point on the normal (remember a normal is also a line). We know gradient mnormal $= -\frac{1}{9}$, and we also know that point (2, 7) lies on the normal. We can use the gradient (slope) and one point formula to find the equation of a line (the normal) as follows:

$y - y_1 = m(x - x_1)$; $m = -\frac{}{9}$, $y_1 = 7$, $x_1 = 2$; therefore,

$y - 7 = -\frac{1}{9}(x - 2)$; $y - 7 = \frac{-(x-2)}{9}$; $9(y - 7) = -x + 2$;

$9y - 63 = -x + 2$; $9y = -x + 2 + 63$; $9y = -x + 65$;

$y = -\frac{1}{9}x + \frac{65}{9}$. Recall the general equation of a line, with gradient m, that intercepts the y-axis at c is $y = mx + c$, then compare the equation of the normal with this general equation to see that the gradient of the normal is $-\frac{1}{9}$ while its intercept on y–axis is $+\frac{65}{9}$ so we can make a rough sketch of the normal as below.

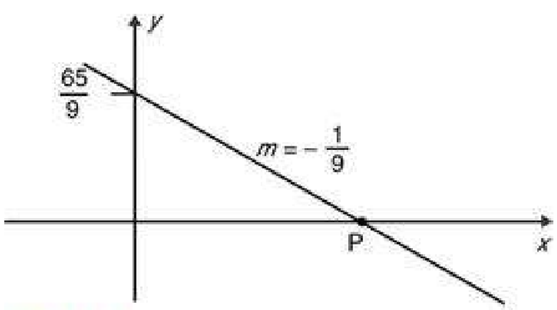

Fig. 7.13

Figure 7.13 shows that, at the point where the line meets the x–axis, $y = 0$. Thus, at the point where the line crosses the x-axis, $y = -\frac{1}{9}x + \frac{65}{9} = 0$; $-\frac{1}{9}x = -\frac{65}{9}$; $\frac{-x}{9} = -\frac{65}{9}$; $-9x = -65 \times 9$; $x = \frac{-65 \times 9}{-9} = 65$. Hence, at the point where the normal meets the x–axis, $y = 0$ while $x = 65$. Therefore, the co-ordinate of point P is (65, 0).

Note that at any point where a curve or straight line crosses the x-axis, the corresponding value of y at the point is zero. Also, at any point where a curve or straight line crosses the y-axis, the corresponding value of x at the point is zero. You can check the rough sketch again (Figure 7.13) to confirm this.

13. The equation of a curve is $x(y^2 + 1) - y(x^2 + 1) + 4 = 0$. Find the:

(a) gradient of the curve at any point (x, y);

(b) equation of the tangent to the curve at the point $(-1, -3)$. **(WAEC)**

Workshop

(a) The gradient of any curve $y = f(x)$ at point (x, y) is given by $\frac{dy}{dx}$. This equation is an implicit function because we **cannot** make y the subject of the formula of the equation. If we try to make y the subject of the formula, y will still be found on that side of the equation where all terms should be in x. So, the relationship between x and y in this case is implicit.

Then, since the equation is an implicit function, we will find $\frac{dy}{dx}$ using implicit differentiation method. We can implicitly differentiate the first and second term of the equation by using the product rule that states thus: to differentiate a function, uv (where u and v *are*

functions of x), $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

The equation in question can be differentiated by differentiating the individual terms as follows: considering the term

$x(y_2 + 1)$, $u = x$, $\frac{du}{dx} = 1$ and $v = y_2 + 1$,

$\frac{dv}{dx} = 2y\frac{dy}{dx}$.

Note that to differentiate any term in y with respect to x, $\frac{dy}{dx}$ *must be included in the result. For example, if you want to differentiate* y^3 *with respect to x, the answer will be* $3y_2\frac{dy}{dx}$. *This is what is called implicit differentiation. Also, if* $y^4 + x_2 + 1$ *is differentiated with respect to x the answer will be* $4y_3\frac{dy}{dx} + 2x_1 + 0$.

Hence, $\dfrac{d[x(y^2+1)]}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$= x\left(2y\frac{dy}{dx}\right) + (y^2+1)(1)$$

$$= 2xy\frac{dy}{dx} + y^2 + 1.$$

Also, to differentiate $y(x^2+1)$; $u = y$; $\dfrac{du}{dx} =$

$y^{1-1}\dfrac{dy}{dx} = y^0\dfrac{dy}{dx} = 1\dfrac{dy}{dx}$ and $v = x^2+1$, $\dfrac{dv}{dx} = 2x$.

Therefore, $\dfrac{d(y(x^2+1))}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$= y(2x) + (x^2+1)\left(1\dfrac{dy}{dx}\right) = 2xy + (x^2+1)\dfrac{dy}{dx}.$

Also, $\dfrac{d(4)}{dx} = 0$, as 4 is a constant. So,

$x(y^2+1) - y(x^2+1) + 4 = 0;$

$\dfrac{d(x(y^2+1) - y(x^2+1) + 4)}{dx} = \dfrac{d(0)}{dx}.$

Recall that; $\dfrac{d(0)}{dx}$, so that,

$\dfrac{d(x(y^2+1) - y(x^2+1) + 4)}{dx} =$

$$\frac{d(x(y^2+1))}{dx} - \frac{d(y(x^2+1))}{dx} + \frac{d(4)}{dx} = 0;$$

$\left(2xy\dfrac{dy}{dx} + y^2 + 1\right) - \left(2xy + (x^2+1)\dfrac{dy}{dx}\right) + 0 = 0;$

$2xy\dfrac{dy}{dx} + y^2 + 1 - 2xy - (x^2+1)\dfrac{dy}{dx} = 0;$

$2xy\dfrac{dy}{dx} - (x^2+1)\dfrac{dy}{dx} = 2xy - y^2 - 1;$

$\dfrac{dy}{dx}(2xy - (x^2+1)) = 2xy - y^2 - 1;$

$\dfrac{dy}{dx}(2xy - x^2 - 1) = 2xy - y^2 - 1;$

$\dfrac{dy}{dx} = \dfrac{2xy - y^2 - 1}{2xy - x^2 - 1}$. Therefore, the gradient of the curve with equation $x(y^2+1)$ $- y(x^2+1) + 4 = 0$ at any point (x, y)

$= \dfrac{dy}{dx} = \dfrac{2xy - y^2 - 1}{2xy - x^2 - 1}.$

(b) The gradient (slope) of the tangent to a curve at a point is also the gradient of the curve at that same point. For this reason, the gradient of the tangent to the curve in question at any point (x, y) is also equal to $\frac{2xy - y^2 - 1}{2xy - x^2 - 1}$.

Then the gradient of the tangent to the curve at point (−1, −3)

$$= \frac{2(-1)(-3) - (-3)^2 - 1}{2(-1)(-3) - (-1)^2 - 1} = \frac{2(3) - 9 - 1}{2(3) - 1 - 1}$$

$$= \frac{6 - 9 - 1}{6 - 1 - 1} = \frac{-4}{4} = -1.$$

The tangent to the curve at point (−1, −3) will touch point (−1, −3); therefore, point (−1, −3) is a point on the tangent to the curve at point (−1, −3). Now, because we know the gradient and a point on this tangent, we can find the equation of the tangent to the curve at point (−1, −3) by the formula; $y - y_1 = m(x - x_1)$ where $y_1 = -3$, $x_1 = -1$ and $m = -1$, where m is the gradient of the tangent. $y - (-3) = -1(x - (-1))$; $y + 3 = -1(x + 1)$; $y + 3 = -x - 1$; $y + x + 3 + 1 = 0$; $y + x + 4 = 0$.

Therefore, the equation of the tangent to the curve having equation $x(y^2 + 1) - y(x^2 + 1) + 4 = 0$ at point (−1, −3) is $y + x + 4 = 0$.

Conic Section

1. A circle intersects the x-axis at the points $P(5, 0)$ and $Q(11, 0)$. If the centre of the circle is on the line $3x + 4y = 8$, find:

   (a) the coordinates of the centre of the circle.

   (b) the equation of the circle.

   (c) the equation of each of the tangents to the circle at P and Q;

   (d) the coordinates of the points of intersection of the tangents in (c). **(WAEC)**

Workshop

(a) Let the coordinates of the centre C of the circle be (x_c, y_c) as shown below.

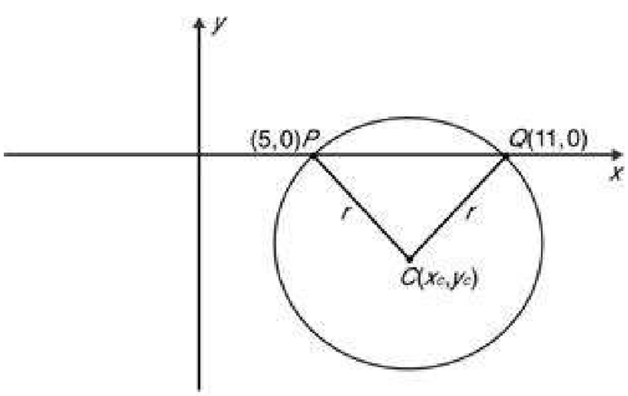

Fig 7.14

The two points P and Q are on the circumference of the circle, hence, distance $|CP| = |CQ|$, as $|CP|$ and $|CQ|$ are radii. Recall that the distance between two points (x_1, y_1), (x_2, y_2) is given by $\sqrt{(y_2 - y_1)_2 + (x_2 - x_1)_2}$; therefore, the distance between $C(x_c, y_c)$ and $P(5, 0)$ will be $|CP| = \sqrt{(0 - y_c)_2 + (5 - x_c)_2}$, while the distance $|CQ|$ between $C(x_c, y_c)$ and $Q(11, 0)$ will be $|CQ| = \sqrt{(0 - y_c)_2 + (11 - x_c)_2}$.

Since $|CP| = |CQ| = radius$, then

$$\sqrt{(0 - y_c)_2 + (5 - x_c)_2} = \sqrt{(0 - y_c)_2 + (11 - x_c)_2}$$

Square both sides of the equation to get

$$\left(\sqrt{(0 - y_c)_2 + (5 - x_c)_2}\right)^2 = \left(\sqrt{(0 - y_c)_2 + (11 - x_c)_2}\right)^2;$$

$$(0 - y_c)_2 + (5 - x_c)_2 = (0 - y_c)_2 + (11 - x_c)_2;$$

$$(-y_c)_2 + (5 - x_c)_2 = (-y_c)_2 + (11 - x_c)_2;$$

$$y_c^2 + 25 - 10x_c + x_{c2} = y_c^2 + 121 - 22x_c + x_{c2};$$

$$25 - 10x_c = y_c^2 - y_{c2} + x_{c2} - x_{c2} + 121 - 22x_c;$$

$$25 - 10x_c = 121 - 22x_c;$$

$$12x_c = 121 - 25 = 96;$$

$$x_c = \frac{96}{12} = 8.$$

The centre of the circle, $C(x_c, y_c)$, is a point on the line, therefore, the equation of the line can be written in terms of x_c and y_c as follows: $3x + 4y = 8$, then (x_c, y_c) will satisfy the equation so that $3x_c + 4y_c = 8$; $3(8) + 4y_c = 8$; $4y_c = 8 - 24$; $4y_c = 16$; $y_c = -4$.

Therefore, the coordinates of the circle's centre are $(8, -4)$.

(b) From question (*a*), *radius*,

$$r = \sqrt{(x_c - 5)^2 + (y_c - 0)^2}.$$

Square both sides of the equation to get

$$r_2 = (x_c - 5)^2 + (y_c - 0)^2 = (8 - 5)_2 + (-4 - 0)_2 = 3_2 + (-4)_2 = 9 + 16 = 25;$$

$r^2 = 25$. The equation of a circle with centre (x_c, y_c) and radius r is given by

$r^2 = (y - y_c)^2 + (x - x_c)^2$;

$r^2 = (y - -4)^2 + (x - 8)^2 = 25$;

$(y + 4)^2 + (x - 8)^2 = 25$;

$y^2 + 8y + 16 + x^2 - 16x + 64 = 25$;

$y^2 + x^2 + 8y - 16x + 80 = 25$;

$y^2 + x^2 + 8y - 16x + 55 = 0$;

Therefore, the equation of the circle is

$y^2 + x^2 + 8y - 16x + 55 = 0$.

(c) The general equation of the circle with centre (x_c, y_c) is $y^2 + x^2 + 2fy + 2gx + c = 0$; where $g = -x_c$ and $f = -x_c$. So, we can re-write the equation of the circle in question in this form as $y^2 + x^2 + 2(4)y + 2(-8)x + 55 = 0$. Compare this, with the general equation, to see that, $g = (-8)$, $f = 4$ and $c = 55$. The general equation of the tangent to a **circle** at point (x^1, y^1) on the circle's circumference is expressed as follows:

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$; thus, the tangent to the circle at point (5, 0) will be $x(5) + y(0) + (-8)(x + 5) + 4(y + 0) + 55 = 0$

$5x - 8x - 40 + 4y + 55 = 0$; $4y - 3x + 15 = 0$.

Hence, the equation of the tangent to thecircle at point, $P(5, 0)$ is $4y - 3x + 15 = 0$. The tangent to the circle at point (11, 0) will be $x(11) + y(0) + (-8)(x + 11) + 4(y + 0) + 55 = 0$; $11x - 8x - 88 + 4y + 55 = 0$; $4y + 3x - 33 = 0$. Therefore, the equation of the tangent to the circle at point $Q(11, 0)$ is $4y + 3x - 33 = 0$.

Alternative method

Recall that the slope of a curve at a point is **equal** to the slope of the tangent to the curve at **that** same point. Therefore, the slope of the circle at point (5, 0) will be equal to the slope of the **tangent** to the circle at the same point, (5, 0). Having gotten the equation of the circle, which is $y^2 + x^2 + 8y - 16x + 55 = 0$, the slope, $\frac{dy}{dx}$, of a circle at point (x, y) can be calculated as follows: as y cannot be made the subject of the equation of the circle, we will have to differentiate the equation of the circle using **implicit** differentiation method. Differentiating the equation of the circle implicitly, we get

$$\frac{d}{dx}(y^2 + x^2 + 8y - 16x + 55) = \frac{d}{dx}(0);$$

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^2) + \frac{d}{dx}(8y) - \frac{d}{dx}(16x) + \frac{d}{dx}(55) = \frac{d}{dx}(0);\ 2y\frac{dy}{dx} + 2x + 8\frac{dy}{dx} - 16 + 0 = 0.$$

Note: Given that y is a function of x (i.e $y = f(x)$), to differentiate a term in y by implicit differentiation method, one ***must*** *include* $\frac{dy}{dx}$ *in the differential as we did above. For example, given that* $y^3 + 2x + y = 0$; $\frac{d}{dx}(y^3 + 2x + y) = \frac{d}{dx}(0)$; $\frac{d}{dx}(y^3) + \frac{d}{dx}(2x) + \frac{d}{dx}(y) = \frac{d}{dx}(0)$; $3y^2\frac{dy}{dx} + 2 + 1\frac{dy}{dx} = 0$; $\frac{dy}{dx}(3y^2 + 1) = -2$; $\frac{dy}{dx} = \frac{-2}{3y+1}$.

Back to the problem, $2y\frac{dy}{dx} + 2x + 8\frac{dy}{dx} - 16 + 0 = 0$; therefore, $\frac{dy}{dx}(2y + 8) + 2x - 16 = 0$; $\frac{dy}{dx}(2y + 8) = 16 - 2x$; $\frac{dy}{dx} = \frac{16-2x}{2y+8}$. Hence, at any point (x, y) on the circle in question, the slope, $\frac{dy}{dx}$, of the circle is $\frac{16-2x}{2y+8}$. The slope of the circle at point (5, 0) will be $\frac{16-2(5)}{2(0)+8} = \frac{16-10}{8} = \frac{6}{8} = \frac{3}{4}$.

So, the slope of the tangent to the circle in question at point (5, 0) is $\frac{3}{4}$. Since the tangent to the circle at point (5, 0) will touch the circle at this point, then (5, 0) is also a point on the tangent to the circle. Now we know the slope and a point on the tangent, thus, the equation of the tangent can be calculated as follows: $y - y_1 = m(x - x_1)$, where $(x_1, y_1) = (5, 0)$ and $m = \frac{3}{4}$; $y - 0 = \frac{3}{4}(x - 5)$; $y = \frac{3(x-5)}{4}$; $4y = 3(x - 5)$; $4y = 3x - 15$;

$4y - 3x + 15 = 0$. So the equation of the tangent to the circle at point (5, 0) is $4y - 3x + 15 = 0$. At point (11, 0), the slope of the circle will be

$$\frac{dy}{dx} = \frac{16-2x}{2y-8} = \frac{16-2(11)}{2(0)+8} = -\frac{6}{8} = -\frac{3}{4}.$$

By the same argument, the slope of the tangent to the circle at point (11, 0) will be $-\frac{3}{4}$. Hence, the equation of the tangent at this point will be expressed as $y - y_1 = m(x - x_1)$;

$$y - 0 = -\frac{3}{4}(x - 11); y = \frac{3(x-11)}{4}; 4y = -3(x - 11);$$

Therefore, the equation of the tangent to the circle at point (11, 0) is $4y + 3x - 33 = 0$.

(d) The point where the two lines (*tangents*) intersect is a common point to both lines; therefore, the values of x and y for this point will satisfy both equations. Furthermore, the x and y values of this point of intersection can be known by solving the equations of the two lines simultaneously to get the values of x and y common to these two points.

$4y - 3x = -15$ .... (i)

$4y + 3x = 33$ .... (ii)

Subtracting equation (ii) from (i) gives

$4y - 3x - (4y + 3x) = -15 - 33;$

$4y - 3x - 4y - 3x = -48;$

$4y - 4y - 3x - 3x = -48;$

$-6x = -48;$

$$x = \frac{-48}{-6} = 8$$

Put $x = 8$ into equation (i) to get $4y - 3(8) = -15;$

$4y = 9; y = \frac{9}{4}$

Therefore, the co-ordinates of the point of intersection of the tangents are $\left(8, \frac{9}{4}\right)$.

2. The equation of a circle is $x2 + y2 - 8x + 4y + 15 = 0$. Find:

(a) the coordinates of its centre,

(b) its radius,

(c) the equation of the tangent to the circle at the point $(2, -3)$. **(WAEC)**

Workshop

(a) $x^2 + y^2 - 8x + 4y + 15 = 0$. Rearrange this equation to get $x^2 - 8x + y^2 + 4y = -15$.

By applying completing the square method, which is a method of solving quadratic equations, we get

$$x^2 - 8x + \left(\tfrac{1}{2}(-8)\right)^2 - \left(\tfrac{1}{2}(-8)\right)^2 + y^2 + 4y + \left(\tfrac{1}{2}(4)\right)^2 - \left(\tfrac{1}{2}(4)\right)^2 = -15;$$

$$x^2 - 8x + (-4)^2 - (-4)^2 + y^2 + 4y + (2)^2 - (2)^2 = -15;$$

$$x^2 - 8x + (-4)^2 + y^2 + 4y + (2)^2 = -15 + (-4)^2 + (2)^2;$$

$$(x - 4) + (y + 2) = -15 + 16 + 4;$$

$$(x - 4)^2 + (y + 2)^2 = 5$$

The equation of a circle with radius r and centre coordinates (x_c, y_c) is given as $(x - x_c)^2 + (y - y_c)^2 = r^2$; comparing this with $(x - 4)^2 + (y + 2)^2 = 5$, we see that $-x_c = -4; x_c = 4$ and $-y_c = +2;$ $y_c = -2$. Also, $r^2 = 5; r = \sqrt{5}$ Therefore, the coordinates of the centre of the circle in question is $(x_c, y_c) = (4, -2)$.

(b) The radius of the circle is $\sqrt{5}$ *units of length*, since we are not sure of the unit of the radius (i.e *cm*, *m*, or some other unit of length).

(c) The equation of the tangent to a circle, $x^2 + y^2 + 2gx + 2fy + c = 0$, at point (x_1, y_1), is given by the equation: $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

The equation of the circle in questioncan be rewritten in the general form as $x^2 + y^2 + 2(-4)x + 2(2)y + 15 = 0$. By comparing $x^2 + y^2 + 2(-4)x + 2(2)y + 15 = 0$ with the general equation of a circle, $(x^2 + y^2 + 2gx + 2fy + c = 0)$, we see that: and $g = -4, f = 2$ and $c = 15$.

Hence, the equation of the tangent to the circle in question, at point $(2, -3)$, which is $(x1, y1)$ in this case, will be $x(2) + y(-3) + -4(x + 2) + 2(y + (-3)) + 15 = 0$;

$2x - 3y - 4x - 8 + 2y - 6 + 15 = 0; -2x - y + 1 = 0; 1 - 2x - y = 0$.

Therefore, the equation of the tangent to the circle, at point $(2, -3)$, is $1 - 2x - y = 0$. Please check the solution to question 1c for an alternative method of calculating the tangent to a circle.

3. A circle passes through the points (0, 3) and (4, 1). If the centre of the circle is on the x-axis, find the equation of the circle. **(WAEC)**

Workshop

Recall that, for any point that lies on the x-axis, the y-value of the point will be zero. Hence, as the centre of the circle lies on the x-axis, the y-component of the coordinate of the centre is zero. Thus, if x_c is the x-component of the centre of the circle, the co-ordinate of the centre will be $(x_c, 0)$.

The distances between point (3, 0) and the centre $(x_c, 0)$, will be equal to the distance between points (4, 1) and the centre, $(x_c, 0)$, as these two distances are radii of the circle in question. The distance between any two points (x_1, y_1) and

(x_2, y_2) is expresses as $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$. Thus, the distance between point (0, 3) and the centre $(x_c, 0)$ will be $\sqrt{(3-0)^2 + (0-x_c)^2}$ *radius, r of the circle.* $r = \sqrt{3^2 + (-x_c)^2}$; recall that, $(-x_c)^2 = -x_c \times -x_c = +x_c^2$.

$r = \sqrt{3^2 + x_c^2}; r^2 = \left(\sqrt{3^2 + x_c^2}\right)^2; r^2 = 3^2 + x_c^2$ ..... (*i*)

The distance between point (4, 1) and the centre $(x_c, 0)$ will be $\sqrt{(1-0)^2 + (4-x_c)^2}$ = *radius, r of the circle.*

$r = \sqrt{1^2 + (4-x_c)^2}; r^2 = \left(\sqrt{1^2 + (4-x_c)^2}\right)^2;$

$r^2 = 1^2 + (4 - x_c)^2$ .... (*ii*)

From equations (*i*) and (*ii*), $r^2 = 3^2 + x_c^2 = 1^2 + (4 - x_c)^2$; $9 + x_c^2 = 1 + 16 - 8x_c + x_c^2$; $x_c^2 - x_c^2 +$

$8x_c = 1 + 16 - 9$; $8x_c = 8$; $x_c = 1$; put $x_c = 1$ into equation 1, to get $r^2 = 3^2 + 1^2 = 10$; $r = \sqrt{10}$. Therefore, the coordinates of the circle's centre is (1, 0) and its radius is $\sqrt{10}$ *units*. The equation of a circle with centre (x_c, y_c), having radius, r, is $(y - y_c)^2 + (x - x_c)^2 = r^2$. Then, the equation of the circle in question will be $(y-0)^2 + (x-1)^2 = (\sqrt{10})^2$;

$$y^2 + x^2 - 2x + 1 = 10;$$

$$y^2 + x^2 - 2x + 1 - 10 = 0;$$

$$y^2 + x^2 - 2x - 9 = 0.$$

Therefore, the equation of the circle is $y^2 + x^2 - 2x - 9 = 0$.

4. The equation of a circle is $x^2 + y^2 + 10x + 8y = 0$. Find:

(a) its area in terms of π;

(b) the length of the chord along the x-axis. **(WAEC)**

Workshop

(a) To know the area of a circle, we need to know the radius, then, we will find the circle's area by the formula *Area* = πr^2. The equation of a circle with centre (x_c, y_c) and radius r is given by $(x - x_c)2 + (y - y_c)^2 = r^2$.

The distance S between any two points (x_1, y_1) and (x_2, y_2) is given by $S = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$ Thus, the distance between any point (x, y) at the circumference and the centre (x_c, y_c) of a circle is the radius of the circle, and it is expressed as $r = \sqrt{(y - y_c)^2 + (x - x_c)^2}$. Square both sides and the equation becomes, $(y - y_c)^2 + (x - x_c)^2 = r^2$.

This is how this equation came about.

So, we can rewrite the equation of the circle $x^2, y^2, 10x + 8y = 0$ in the form $(x - x_c)^2 + (y - y_c)^2 = r^2$ by using the method of completing the square. $x^2 + y^2 + 10x + 8y = 0$; $x^2 + 10x + y^2 + 8y = 0$.

By the method of completing the square, we will find the square of half of the coefficient of x and y in the equation of the circle, and add them to the two sides of the equation to keep the equation balanced.

$$x^2 + 10x + \left(\frac{10}{2}\right)^2 + y^2 + 8y + \left(\frac{8}{2}\right)^2 = 0 + \left(\frac{10}{2}\right)^2 + \left(\frac{8}{2}\right)^2;$$

$x^2 + 10x + 5^2 + y^2 + 8y + 4^2 = 5^2 + 4^2$; factorize this equation to get $(x + 5)^2 + (y + 4)^2 = 5^2 + 4^2$; $(x + 5)^2 + (y + 4)^2 = 25 + 16$; $(x + 5)^2 + (y + 4)^2 = 41$.

Recall that $- \times - = +$, for example, $--5 = -(-5) = - \times -5 = +5$; therefore, $(x+5)^2 + (y+4)^2 = 41$ can be rewritten as $(x-(-5))^2 + (y-(-4))^2 = 41$. Comparing $(x-(-5))^2 + (y-(-4))^2 = 41$ with $(x+x_c)^2 + (y+y_c)^2 = r^2$

shows that $r^2 = 41$; $r = \sqrt{41}$. Recall that the area A of a circle is given by $A = \pi r^2 = \pi \times (\sqrt{41})^2 = 41\pi$ *square units*.

Therefore, the area of the circle with equation $x^2 + y^2 + 10x + 8y = 0$ in terms of π is 41π *square (units)*^{2}.

Note that 41π square units was written because it is not known whether the unit of this area is square centimeter (cm^2), or square metre (m^2), or any other unit of area; thus, it is safe to simply put square unit to get maximum mark.

(b) Lets make a rough sketch of the circle to get a better picture of it. Comparing $(x-(-5))^2 + (y-(-4))^2 = 41$ with $(x-x_c)^2 + (y-y_c)^2 = r^2$ shows that $x_c = -5$ and $y_c = -4$. Thus, the coordinate centre of the circle is $(-5, -4)$ and its radius is $\sqrt{41}$. Using these parameters, we can make a rough sketch of the circle as drawn in figure 7.15.

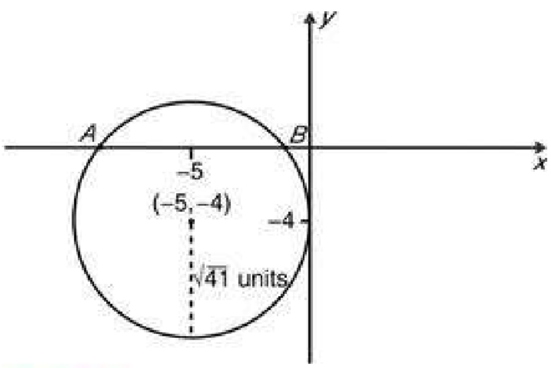

Fig. 7.15

A clear picture of the chord AB, whose length is to be found is shown.

From Figure 7.15, you can see that at the two points where the circle **cross** the x-axis, the corresponding value of y is zero. As points A and B are on the circle, the values of x and y, for these points, will satisfy the equation of the circle. And as the y-coordinates of the two points is zero, the x-coordinates of the two points can be known by putting $y = 0$ into the equation of the circle to get $x^2 + y^2 + 10x + 8y = x^2 + 0^2 + 10x + 8(0) = 0$; $x^2 + 10x = 0$; $x(x + 10) = 0$; $x = 10$ *or* $x + 10 = 0$; $x = 0$ *or* $x = -10$. Recall that the value of y at these two points is zero, therefore, the coordinates of points A and B are $(0, 0)$ and $(-10, 0)$.

If this known value is plotted on the graph, it will become clear that the previous sketch is slightly wrong. This presents no problem however, what is needed is just these two coordinates of points B $(0, 0)$ and A $(-10, 0)$ as drawn in the diagram.

The distance S between any two points and (x_1, y_1) (x_2, y_2) is given by

$S = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$. Therefore, the distance AB will be $AB = \sqrt{(0-0)^2 + (0-(-10))^2} = \sqrt{(+10)^2}$

$= \sqrt{10 \times 10} = 10$ *units*.

Therefore, the length of the chord of the circle along the x-axis is 10 *units* of *length*.

5. The points (7, 3), (2, 8) and (−3, 3) lie on a circle. Find the:

(a) equation;

(b) radius, of the circle. **(WAEC)**

Workshop

(a) The general equation of a circle I expressed as follows $x^2 + y^2 + 2gx + 2fy + c = 0$, where the coordinate of the centre (x_c, y_c) of the circle I given by $(x_c, y_c) = (-g, -f)$ that is, $x_c = -g$ while $y_c = -f$ and $c = x_c^2 + y_c^2 - r^2$, where, is the radius of the circle. Since the circle passes through points (7, 3), (2, 8) and (−3, 3), all these points will satisfy the equation of the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ as

$7^2 + 3^2 + 2g(7) + 2f(3) + c = 0$;

$49 + 9 + 14g + 6f + c = 0$

$2^2 + 8^2 + 2g(2) + 2f(8) + c = 0$;

$4 + 64 + 4g + 16f + c = 0$

$(-3)^2 + 3^2 + 2g(-3) + 2f(3) + c = 0$;

$9 + 9 - 6g + 6f + c = 0$

$14g + 16f + c = -58$ ....(1)

$4g + 6f + c = -68$ ....(2)

$-6g + 6f + c = -18$ ....(3)

Look at equations (1) and (2) to see that we can eliminate c by subtracting equation (2) from equation (1). Subtract equation (2) from equation (1) to get $14g + 6f + c - (4g + 16f + c) = -58 - (-68)$;

$14g + 6f + c - 4g - 16f - c = -58 + 68$;

$10g + 10f = 10$ ....(a).

A look at equations (1) and (3) shows that we can eliminate c by subtracting equation (3) from equation (1). Subtract equation (3) from equation (1) to get $14g + 16f + c - (-6g + 6f + c) = -58 - (-18)$;

$14g + 6f + c + 6g - 6f - c = -58 + 18$;

$20g + 6f + c - c = -40$; $20g = -40$ ......(b).

From equation (b), $g = \frac{-40}{20} = -2$.

Put $g = -2$ into equation (a) to get $10(-2)$

$- 10f = 10$; $-20 - 10f = 10$;

$- 10f = 10 + 20 = 30$; $f =$ _−3100_. $= -3$. Put

$g = -2$ and $f = -3$ into equation (1) to get $14(-2) + 6(-3) + c = -58$; $-28 - 18 + c = -58$; $-46 + c = -58$; $c = -58 + 46 = -12$.

If we put the values of f, g and c into the general equation of a circle, we obtain $x^2 +$

$y^2 + 2(-2)x + 2(-3)y + (-12) = 0$;

$x^2 + y^2 - 4x - 6y - 12 = 0$. Therefore, the equation of the circle passing through the points (7, 3), (2, 8) and (−3, 3) is $x^2 + y^2 - 4x - 6y - 12 = 0$.

(b) As earlier explained, $c = x_c^2 + y_c^2 - r^2$ also recall that, $x_c = -g = -(-2) = 2$ *and* $y_c = -f = -(-3) = 3$ *and* $c = -12$;

Then, $c = x\,c2 + y\,c2 - r2$; $-12 = 22 + 32 - r2$; $-12 = 4 + 9 - r2$; $-12 = 13 - r2$; $-12 - 13 = r2$; $-25 = r2$; $25 = -r2$; $r = \sqrt{25} = 5$.

Therefore, the radius of the circle passing through points (7, 3), (2, 8) and (−3, 3) is 5 *units*.

Alternative method

Let the coordinate of the centre of the circle in question be (x_c, y_c). Points (7, 3), (2, 8) and (−3, 3) lie on the circle; therefore, the distance between **any one** of these three points and the centre, (x_c, y_c) will be the radius, r. In other words, distance between point (7, 3) and centre (x_c, y_c) of the circle will be the radius of the circle, the distance between (2, 8) and the centre, (x_c, y_c) will also be the radius, r, of circle. The same applies to point (−3, 3), as it also lie on the circumference of the circle.

Recall that the distance d, between any two points (x_1, y_1) and (x_2, y_2) is given as

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, so that distance, r between (7, 3) and centre (x_c, y_c) will be

$r = \sqrt{(7 - x_c)^2 + (3 - y_c)^2}$;

$r^2 = \left(\sqrt{(7 - x_c)^2 + (3 - y_c)^2}\right)^2$;

$r^2 = (7 - x_c)^2 + (3 - y_c)^2$;

$r^2 = 49 - 14x_c + x_c^2 + 9 - 6y_c + y_c^2$ ....(*i*)

Point (2, 8) also lie on the circle; therefore, the distance between point (2, 8) and the centre (x_c, y_c) will also be equal to r, thus,

$r = \sqrt{(2 - x_c)^2 + (8 - y_c)^2}$;

$r^2 = (2 - x_c)^2 + (8 - y_c)^2$;

$r^2 = 4 - 4x_c + x_c^2 + 64 - 16y_c + y_c^2$ ....(*ii*)

Again, (−3, 3) also lie on the circle so that,

$r = \sqrt{(-3 - x_c)^2 + (3 - y_c)^2}$;

$r^2 = (-3 - x_c)^2 + (3 - y_c)^2$;

$r^2 = 9 + 6x_c + x_c^2 + 9 - 6y_c + y_c^2$ .... (*iii*)

You will notice that all the expressions to the right side of equation (*i*), (*ii*) and (*iii*) are all equal to r^2, then these three expressions are equal to each other, so that

$r^2 = 49 - 14x_c + x_c^2 + 9 - 6y_c + y_c^2 = 4 - 4x_c + x_c^2$
$\quad + 64 - 16y_c + y_c^2;$

$49 - 14x_c + x_c^2 + 9 - 6y_c + y_c^2 = 4 - 4x_c + x_c^2 + 64$
$- 16y_c + y_c^2;$

$49 - 14x_c + 9 - 6y_c = 4 - 4x_c + 64 - 16y_c + x_c^2$
$- x_c^2 + y_c^2 - y_c^2;$

$49 - 14x_c + 9 - 6y_c = 4 - 4x_c + 64 - 16y_c;$
$-14x_c + 4x_c - 6y_c + 16y_c = 4 + 64 - 49 - 9;$

$10x_c + 10y_c = 4 + 64 - 49 - 9;$
$10x_c + 10y_c = 10$ .... (*a*)

Also, recall from equations (*ii*) and (*iii*) above that

$r^2 = 4 - 4x_c + x_c^2 + 64 - 16y_c + y_c^2$
$\quad = 9 + 6x_c + x_c^2 + 9 - 6y_c + y_c^2;$

$4 - 4x_c + x_c^2 + 64 - 16y_c + y_c^2 = 9 + 6x_c + x_c^2 +$
$\quad 9 - 6y_c + y_c^2;$

$4 - 4x_c + 64 - 16y_c = 9 + 6x_c + 9 - 6y_c + x_c^2 - x_c^2$
$\quad + y_c^2 - y_c^2;$

$4 - 4x_c + 64 - 16y_c = 9 + 6x_c + 9 - 6y_c;$
$-4x_c - 6x_c - 16y_c + 6y_c = 9 + 9 - 4 - 64;$

$-10x_c - 10y_c = 9 + 9 - 4 - 64;$
$-10x_c - 10y_c = -50$ .... (*b*)

$-10x_c + 10y_c = 10$ .... (*a*)

$-10x_c - 10y_c = -50$ .... (*b*)

Solving equation (*a*) and (*b*) simultaneously gives $x_c = 2$ and $y_c = 3$. $(x_c, y_c) = (2, 3)$.

We already know that point (7, 3) lie on the circumference of the circle, therefore, the distance between point (7, 3) and the circle's centre, $(x_c, y_c) = (2, 3)$ will be

$r = \sqrt{(7-2)^2 + (3-3)^2} = \sqrt{5^2 - 0^2} = \sqrt{5^2} = (5^2)^{\frac{1}{2}}$
$\quad = 5^{2\times\frac{1}{2}} = 5^1 = 5.$

The distance between any point (x, y) on the circumference of the circle and the centre of the circle (x_c, y_c) is given by; $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$.

For this problem, we have calculated the coordinates of the centre as (2, 3), and the radius of the circle as 5. Putting these two into $r =$

$\sqrt{(x - x_c)^2 + (y - y_c)^2}$, we get; $r = 5$
$= \sqrt{(x-2)^2 + (y-3)^2}$; $5 = \sqrt{(x-2)^2 + (y-3)^2}$.
Squaring both sides of the equation, we get

$$5^2 = \left(\sqrt{(x-2)^2 + (y-3)^2}\right)^2;$$

$$25 = \left(((x-2)^2 + (y-3)^2)^{\frac{1}{2}}\right)^2$$

$$= ((x-2)^2 + (y-3)^2)^{\frac{1}{2}\times 2};$$

$$25 = ((x-2)^2 + (y-3)^2)^1; \; 25 = (x-2)^2 + (y-3)^2;$$

$$25 = x^2 - 4x + 4 + y^2 - 6y + 9;$$

$$25 = x^2 + y^2 - 4x - 6y + 13; \; 0 = x^2 + y^2 - 4x - 6y$$

$$+ 13 - 25; \; x^2 + y^2 - 4x - 6y - 12 = 0.$$

Therefore the equation of the circle is $x^2 + y^2 - 4x - 6y - 12 = 0$, while the radius of the circle is 5 *units*.

6. The line $2x + 3y = 1$ intersects the circle $2x^2 + 2y^2 + 4x + 9y - 9 = 0$ at points P and Q, where Q lies in the fourth quadrant. Find the coordinates of P and Q. **(WAEC)**

Workshop

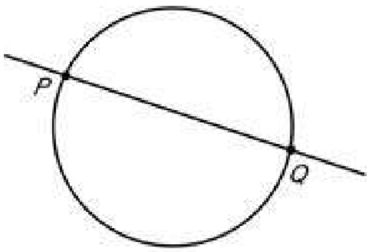

Fig. 7.16

As shown in the Figure 7.16, a line that intersects a circle does this at two points. So we want to find the co-ordinates of these points (P and Q).

To know the coordinates of P and Q, we simply solve the equations of the line and circle simultaneously. This is so because in Figure 7.16, the points P and Q are on the circle and on the line: therefore, they are points common to the circle and the line. These points can be known by finding the coordinates (x, y) common to both equations, and this can be done by solving the two equations simultaneously.

$2x + 3y = 1$ ....(*i*)

$2x^2 + 2y^2 + 4x + 9y - 9 = 0$ ....(*ii*)

From equation (*i*), substitute $x = \frac{1-3y}{2}$ into equation (*ii*) to get

$$2\left(\frac{1-3y}{2}\right)^2 + 2y^2 + 4\left(\frac{1-3y}{2}\right) + 9y - 9 = 0;$$

$$2\left(\frac{(1-3y)^2}{2^2}\right) + 2y^2 + \frac{4}{2}(1-3y) + 9y - 9 = 0;$$

$$\frac{2}{4}(1 - 6y + 9y^2) + 2y^2 + 2(1-3y) + 9y - 9 = 0;$$

$$\frac{1 - 6y + 9y^2}{2} + 2y^2 + 2 - 6y + 9y - 9 = 0;$$

$$\frac{1 - 6y + 9y^2}{2} + 2y^2 + 3y - 7 = 0.$$

Multiply through this equation by 2 to get

$$2\left(\frac{1-6y+9y^2}{2} + 2y^2 + 3y - 7\right) = 2(0);$$

$$2\left(\frac{1-6y+9y^2}{2}\right) 2(2y^2) + 2(3y) + 2(-7) = 2(0);$$

$1 - 6y + 9y^2 + 4y^2 + 6y - 14 = 0$; $13y^2 - 6y + 6y +$
$1 - 14 = 0$; $13y^2 - 13 = 0$; $13y^2 - 13$; $y^2 = \frac{13}{13} = 1$;
$y \pm \sqrt{1}$. Therefore, $y = +1$ *or* -1

when $y = +1$, $x = \frac{1-3y}{2} = \frac{1-3(1)}{2} = \frac{1-3}{2} = \frac{-2}{2} -1.$

Thus, one of the points of intersection of the circle and line is $(-1, 1)$

when $y = -1$, $x = \frac{1-3(-1)}{2} = \frac{1-(-3)}{2} = \frac{1+3}{2}$
$= \frac{-2}{2} = 2.$

Also, the other point is (2, −1). But, we need to know which of the points is *P* and which is *Q* because we were told in the question that *Q* lies in the fourth quadrant. So, we may plot these two points on the *x-y* plane to know which is *Q*, and which is *P*.

Kindly note that you are not expected to show the drawing (Figure 7.17) in the exam. It has been shown to facilitate better understanding.

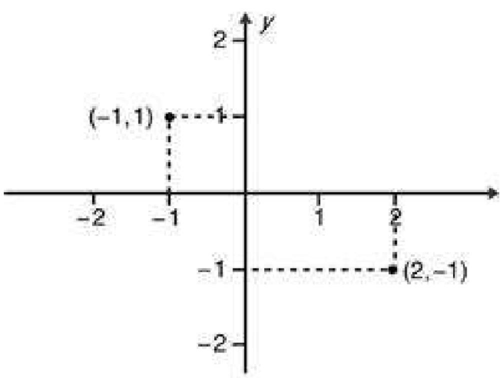

Fig. 7.17

Now, from this plot (Figure 7.17), which of the point is in the fourth quadrant? Definitely point (2, −1) is in the fourth quadrant. So, the point with co-ordinates (2, −1) is point *Q*, while (−1, 1) is the co-ordinate of point *P*.