Chapter 12: Trigonometry

OBJECTIVES

At the end of the chapter, students should be able to:

1. Derive the Sine Rule.
2. Apply the Sine Rule.
3. Derive the Cosine Rule.
4. Apply the Cosine Rule.

I. Sine Rule

In any triangle, the angles are denoted by the upper case letters such as *A, B* and *C* while the sides opposite (facing) these angles are denoted by the lowercase letters such as *a, b* and *c*, respectively as shown in Figure 12.1.
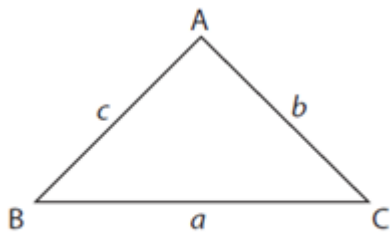

Figure 12.1

The Sine Rule states that: given any triangle ABC, the ratio of the sines of two angles is equal to the ratio of the sides opposite those angles which is represented as shown below:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The Sine Rule

(i)   *Given:* Any acute-angled Î"ABC as shown in Figure 12.2.
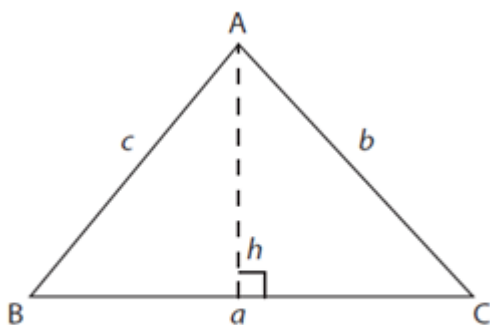

Figure 12.2

Required to prove:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Construction: Draw a perpendicular line from A to BC and call it *h*. *Proof:* Using the triangle in Figure 12.2,

$$\sin B = \frac{h}{c} \text{ and } \sin C = \frac{h}{b}$$

⇒ $h = c \sin B$ and $h = b \sin C$

⇒ $h = c \sin B = b \sin C$
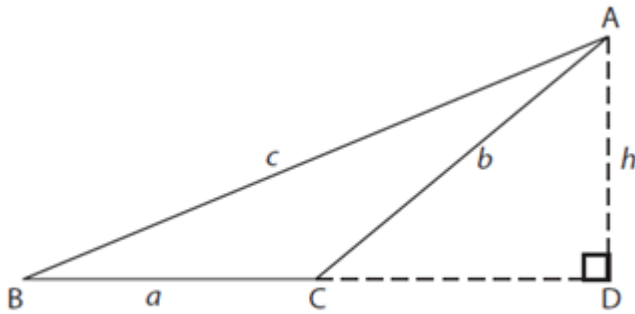
∴ $c \sin B = b \sin C$

Dividing both sides by sin B and sin C

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(ii)   *Given:* Any obtuse-angled Î”ABC as shown in Figure 12.3.

Figure 12.3

Required to prove: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ Note: Î”A CË† B + Î”A CË† D = 180Â°
(supplementary angles)

Hence, $\sin(180° - C) = \frac{h}{b}$

$\sin C = \frac{h}{b}$ [since sin(180 − α)] = sin α

⇒ $b \sin C = h$

i.e $h = c \sin B = b \sin C$

∴ $c \sin B = b \sin C$

Divide both sides by sin A and sin C

⇒ $\frac{c}{\sin C} = \frac{a}{\sin A}$

Similarly,

$\sin A = \frac{h}{c} \Rightarrow h = c \sin A$

and

$\sin C = \frac{h}{b} \Rightarrow h = b \sin C$

∴ $h = c \sin A = b \sin C$ ⇒ $c \sin A = b \sin C$. Divide both sides by sin A and sin C

$$\Rightarrow \frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \boxed{\text{To find the unknown sides}}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \boxed{\text{To find the unknown angles}}$$

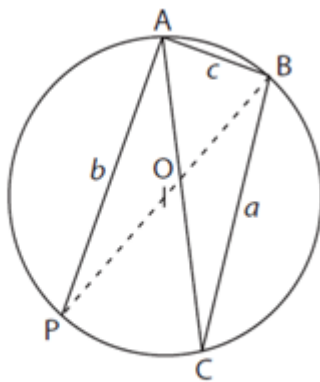(iii)   *Given:* Circumscribed circle about an acute- and an obtuse-angled Î"ABC.
Required to prove:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Construction: Draw the circumscribed circle with centre O and radius R of an acute- and an obtuse-angled Î"ABC, then draw the diameter BP and join AP.
As shown in Figure 12.4

(a)

Figure 12.4

Proof: In Figure 12.4(a) A PË† B = A CË† B (angles on the same segment)

$$\text{Hence, } \sin P = \frac{c}{BP} = \frac{c}{2r}$$

$$\Rightarrow \sin C = \frac{c}{2r} \text{ (since } \hat{p} = \hat{c})$$

$$\Rightarrow 2r = \frac{c}{\sin C}$$

While in Figure 12.4(b), A PË† B + A CË† B = 180Â° (opposite angles of a cyclic quadrilateral are supplementary)

Then, $\sin P = \dfrac{c}{BP} = \dfrac{c}{2r}$

$\Rightarrow \sin(180° - C) = \dfrac{c}{2r}\{\text{since } \hat{P} = (180° - C)\}$

$\Rightarrow 2r = \dfrac{c}{\sin(180° - C)}$
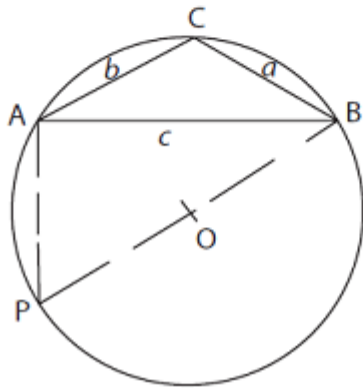
Thus, in all figures, $2r = \dfrac{c}{\sin C}$

Similarly, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = 2r$

$\therefore$ In all cases, solving $\triangle ABC$

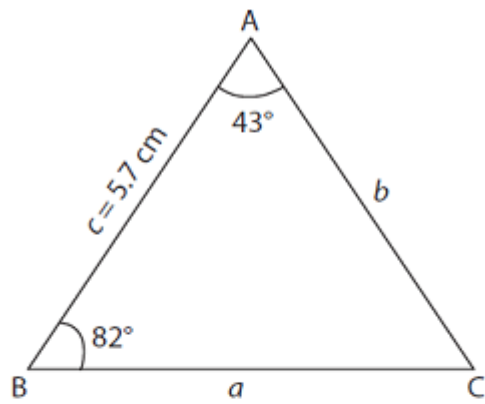$$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2r$$

(b)

The Sine Rule formula is used to solve a triangle that is not right-angled and when (1) two sides and an angle facing one of them are given and (2) two angles and any of the sides are given.
Worked Example 1
In Î"ABC, $A = 43Â°$, $B = 82Â°$ and c = 5.7 cm. Find (i) C (ii) a (iii) b
SOLUTION
First, sketch the information

Figure 12.5

(i) $C = 180° - (82° + 43°)$ (sum of angles in a triangle)

$= 180° - 125°$

$= 55°$

(ii) $\dfrac{a}{\sin 43°} = \dfrac{5.7}{\sin 55°}$

$\Rightarrow a = \dfrac{5.7 \times \sin 43°}{\sin 55°}$

At this point, we need to use logarithms of sine as shown below:

No.	Log
5.7	0.7559
sin 43°	+$\overline{1}$.8338
5.7 × sin 43°	0.5897
sin 55°	$\overline{1}$.9134
	0.6763
	4742
	+ 3
4.745	4745

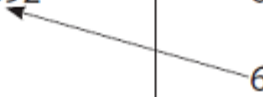$$\therefore a = 4.7 \text{ cm (1 d.p.)}$$

(iii) $\dfrac{b}{\sin 82°} = \dfrac{5.7}{\sin 55°}$

$$\Rightarrow b = \frac{5.7 \times \sin 82°}{\sin 55°}$$

No.	Log
5.7	0.7559
sin 82°	+$\overline{1}$.9958
	0.7517
sin 55	$\overline{1}$.9134
	0.8383
6.892	6887
	+ 5
	6892

$\therefore b = 6.9$ cm (1 d.p.).

Worked Example 2 Worked Example 2
In $\triangle$PQR, $R = 37°$, $p = 5.5$ cm, $r = 3.9$ cm. Find (i) P and (ii) Q.
SOLUTION
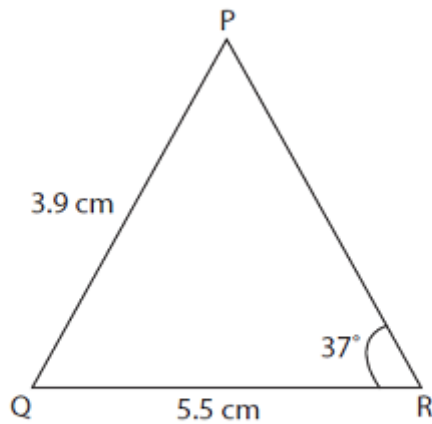
Figure 12.6

(i) Arrange the formula such that the unknown should be stated first.

That is, $\dfrac{\sin P}{p} = \dfrac{\sin R}{r}$

$\Rightarrow \dfrac{\sin P}{5.5} = \dfrac{\sin 37°}{3.9}$

$\Rightarrow \sin P = \dfrac{5.5 \times \sin 37°}{3.9}$

$P = \sin^{-1}\left(\dfrac{5.5 \times \sin 37°}{3.9}\right) = 62.72°$

i.e $P = 62.72°$

Exercise 1

Find the values of the unknown in the following:

1. In ΔABC, $A = 39°$, $B = 25°$, $b = 13$ cm.
2. In ΔABC, $B = 80°$, $C = 41°$, $c = 9.5$ cm.
3. In ΔQPR, $P = 60°$, $R = 15°$, $q = 10.5$ cm.
4. In ΔQPR, $R = 65°$, $q = 13$ cm, $r = 19$ cm.
5. In ΔABC, $B = 100°$, $C = 29°$, $a = 26.5$ cm.
6. In ΔXYZ, $X = 31.5°$, $Z = 27.5°$, $x = 3.7$ cm.
7. In Δ XYZ, Z = 55°, $x= 3.5$ cm, $z= 5.4$ cm.
8. In ΔQPR, P Q̂ R = 84°, Q P̂ R = 43° and /PQ/ = 5 cm. Find /QR/ in cm, correct to 1 decimal place. (WAEC)
9. Solve the following triangles:
(a) $A = 35.5°$, $a = 10.6$ cm, $b = 15.5$ cm.
(b) $B = 110°$, $b = 8.2$ cm, $a = 4.5$ cm.
10. In ΔABC, Calculate the values of c, B and C given that; $A = 39°$, $a = 8.2$ m and $b = 3.6$ m.

II. Cosine Rule

In any given triangles with the usual notation A, B and C as shown in Figure 12.7,
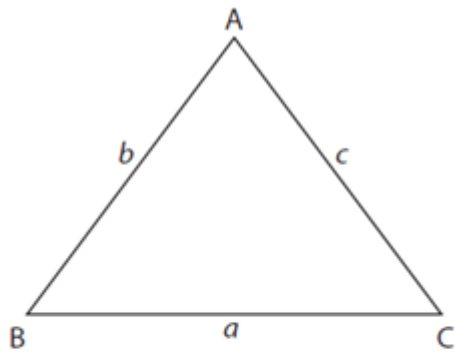
Figure 12.7

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

These are used to find the sides of the triangle. To calculate the angles from the formula $a2 = b2 +c2 â^` 2bc \cos A$, Cos A will be isolated to form the formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

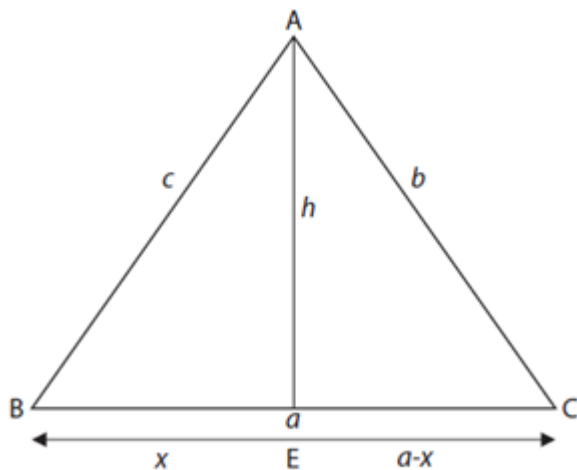Similarly, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

and $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Let us now prove the cosine rule.

Given: An acute-angled Î"ABC as shown in Figure 12.8.

Figure 12.8(a) Acute-angled triangle

Required to prove: $a^2 = b^2 + c^2 - 2bc\cos A$

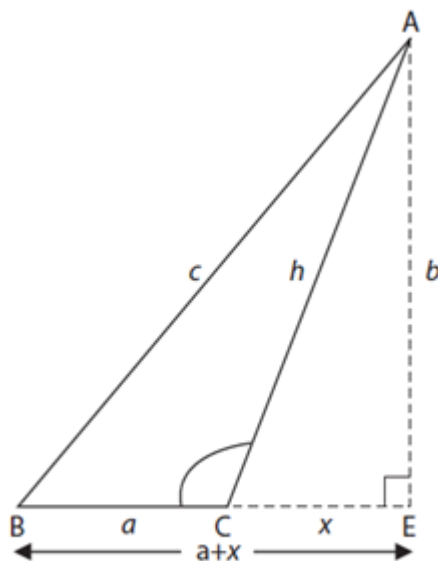*Construction:* Draw $\overline{AE} \perp \overline{BC}$. Denote /AE/ by h, /BE/ by x and /CE/ by $a - x$.

Proof: $b^2 = (a - x)^2 + h^2$ (Pythagoras' theorem)

$= a^2 - 2ax + x^2 + h^2$

$\Rightarrow a^2 - 2ax + c^2$ (since in $\triangle ACE$, $c^2 = x^2 + h^2$)

$= a^2 + c^2 - 2ac \cos B$ (in $\triangle ABE$, $\cos B = \frac{x}{c}$)

Given: An obtuse-angled $\triangle ABC$ with angle $\Rightarrow x = c \cos B$

Figure 12.8(b) Obtuse-angled triangle

Construction: Produce

$\overline{BC}$ to E and join A to E. Denote /AE/ by h, /CE/ by x and /BE/ by $(a + x)$.
Proof: $c^2 = (a + x)2 + h2$ (Pythagoras'™ theorem) $= a2 + 2ax + x2 + h2 = a2 + 2ax + b2$ (since in Î"ACE, $b2 = x2 + h2$)

> In $\triangle ACE$
>
> $\cos C = \dfrac{x}{h}$
>
> $= \cos(180° - c)$
>
> $= -\cos C$
>
> $\Rightarrow x = b \cos C$

$c2 = a2 + b2 + 2a$ (â^'b cos C)
$c2 = a2 + b2$ â^'2ab cos C
â^´ In all cases

$a2 = b2 + c2 − 2bc \cos A$
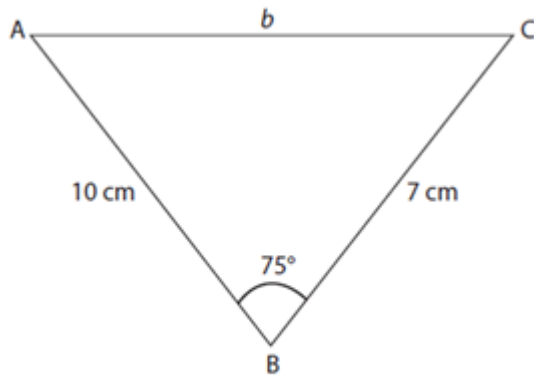$b2 = a2 + c2 − 2ac \cos B$
$c2 = a2 + b2 − 2ab \cos C$

Cosine Rule is used to solve triangles that are not right-angled but when (1) two sides and an included angle are given and (2) all the three sides are given.

Worked Example 3

In ΔABC, $a = 7$ cm, $c = 10$ cm, $B = 75°$. Find

    (i) b (ii) A and (iii) C.

SOLUTION

Figure 12.9

(i) $b2 = a2 + c2 − 2ac \cos B$ (Cosine Rule)

$b2 = 72 + 102 − 2 × 7 × 10 × \cos 75°$

$= 49 + 100 − 140 × \cos 75°$

$= 149 − 140 × 0.2588$

$= 149 − 36.232$

$= 112.768$

$$b = \sqrt{112.786} = 10.6192$$

$$b = 10.62 \text{ cm (2 d.p.)}$$

(ii) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ (Sine Rule)

$$\dfrac{\sin A}{7 \text{ cm}} = \dfrac{\sin 75°}{10.6192}$$

$$\sin A = \frac{7 \times \sin 75°}{10.6192}$$

$$\sin A = \frac{7 \times 0.9659}{10.6192}$$

$$= \frac{6.7613}{10.6192}$$

$$= 0.6367$$

$$A = \sin^{-1} 0.6367$$

$$\therefore A = 39.5462°$$

$$\therefore A = 39.55° \text{ (2 d.p.)}$$

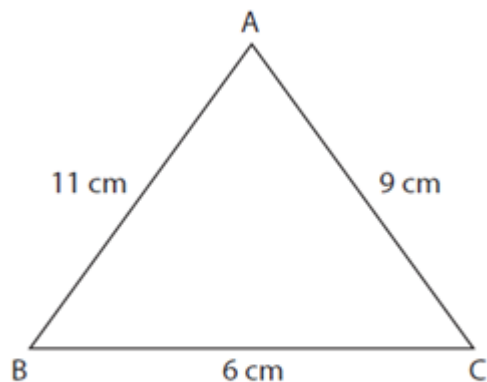(iii) $C = 180° - (75 + 39.5462)$ (sum of angles in a triangle)

$$= 180° - 114.5462$$

$$= 65.4538$$

$$\therefore C = 65.45° \text{ (2 d.p.)}$$

Worked Example 4
Find the angles of the Î"ABC given that
$a = 6$ cm, $b = 9$ cm and $c = 11$ cm.
SOLUTION

Figure 12.10

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ (Cosine Rule)}$$

$$\cos A = \frac{9^2 + 11^2 - 6^2}{2 \times 9 \times 11}$$

$$= \frac{81 + 121 - 36}{198} = \frac{166}{198}$$

$$= 0.8384$$

$$A = \cos^{-1} 0.8384$$

$$A = 33.03°$$

$$\cos B = \frac{6^2 + 11^2 - 9^2}{2 \times 6 \times 11}$$

$$= \frac{36 + 121 - 81}{132}$$

$$= \frac{76}{132}$$

$$= 0.5758$$

$$B = \cos^{-1} 0.5758$$

$$B = 54.84° \text{ (2 d.p.)}$$

$$A + B + C = 180° \text{ (sum of angles in a triangle)}$$

$$33.03 + 54.84 + C = 180°$$

$$C = 180 - (33.03 + 54.84)$$

$$C = 92.13° \text{ (2 d.p.)}$$

Exercise 2

Solve the triangles below:

1.
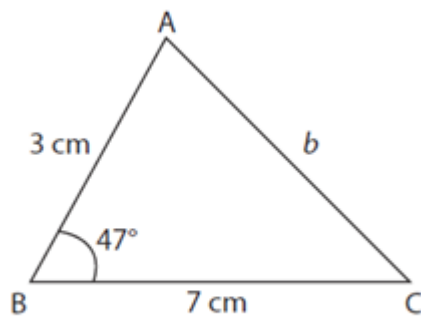
Figure 12.11a

2.
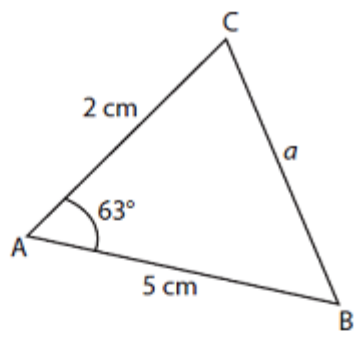
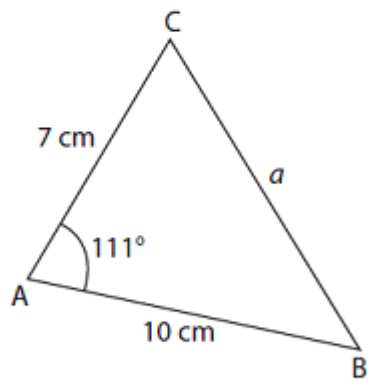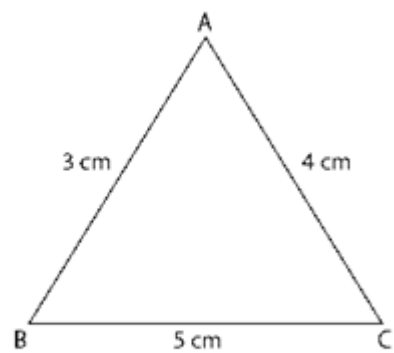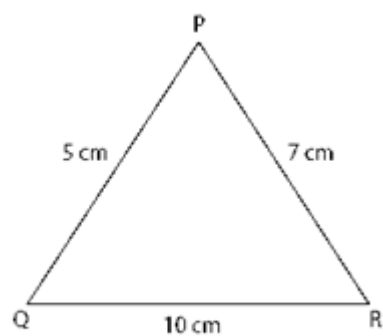
Figure 12.11b

3.
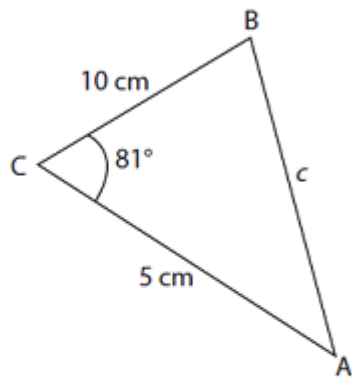
Figure 12.11c

4.

Figure 12.11d

5.

Figure 12.11e

6.

3 cm   4 cm

B   5 cm   C

Figure 12.12a

8.

5 cm   7 cm

Q   10 cm   R

Figure 12.12c

7.

X   5 cm   Y

7 cm   8 cm

Z

Figure 12.12b

9.

9 cm

10 cm

N

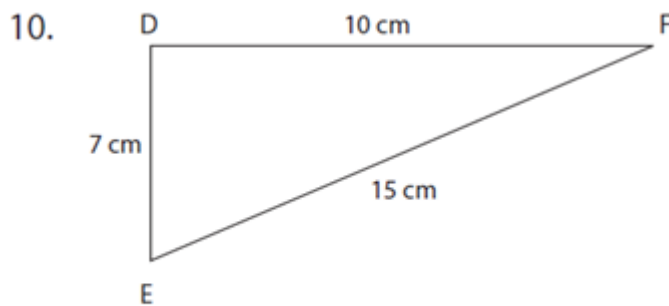6 cm

O

Figure 12.12d

10.
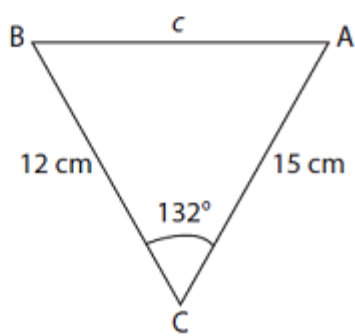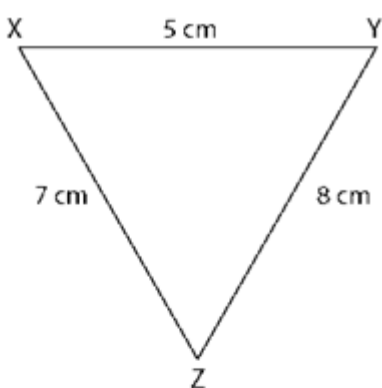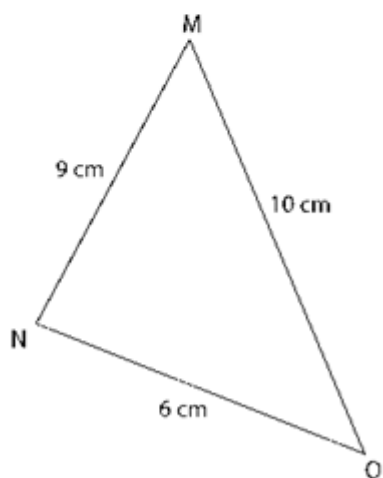
Figure 12.12e

SUMMARY

In this chapter, we have learnt the following:

â— In any given triangle, the angles are conventionally denoted by capital letters A, B and C, while sides opposite these angles are denoted by small letters a, b and c.

â— Sine and Cosine Rules which are used to solve triangles were proved.

â— Sine Rule states that: given any triangle A, B and C, the ratio of the sines of two angles is equal to the ratio of the sides opposite those angles.

Hence,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Which can also be inversely written as:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

â— Given any triangle with the usual notation A, B and C, the Cosine Rule states that:

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$ or

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

GRADUATED EXERCISES

1. Given a pentagon ABCDE, /AB/ = 3 cm, /AE/ = 5 cm, /DE/ = 7 cm, /CD/ = 9 cm, B AÊ† E = 45° and C DÊ† E = 65°. Find
(a) Diagonal BE
(b) Diagonal CE
(c) C EÊ† D
(d) A BÊ† E

2. Given an isosceles Î"ABC such that /AB/ = /AC/ = 5 cm and AB Ê† C = 50°, calculate /BC/.

3. Calculate the values of the angles A and C of Î"ABC where b = 14.35 cm, a = 7.8 cm and B = 115.6°. (WAEC)

4. A student from a spot X observed that two footballers A and B on a fi eld were 25 m apart, if X AÊ† B = 45° and X BÊ† A = 65°,
(a) Calculate angle A XÊ† B.
(b) How far is the student from footballers A and B? Express all answers to the nearest whole number.

5. The sides of a triangle are 8.4 cm, 5.9 cm and 6.3 cm. Find the smallest and the largest angles.

6. In the diagram in Figure 12.13, O is the centre of the circle with radius 3.5 cm through points L, M and N . If â^ MLN = 74Â° and â^ MNL = 39Â°, calculate the length LN.
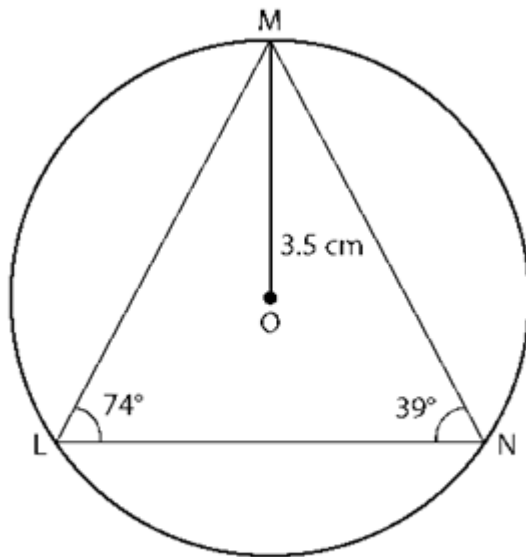
Figure 12.13
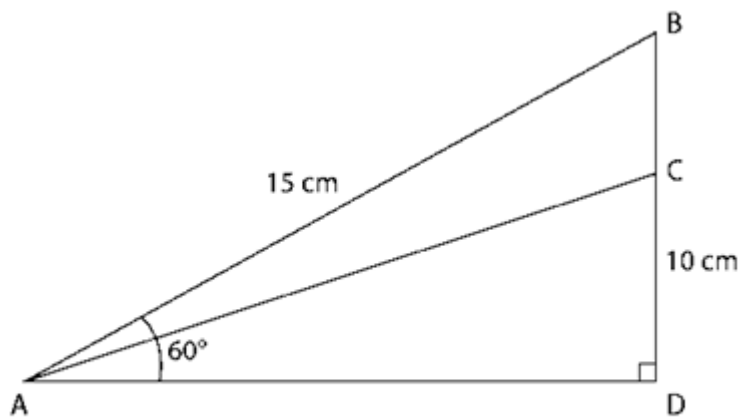
7. In the diagram in Figure 12.14, /PR/ = /RQ/ and /RS/ = 10 cm.
â^ RPS = 70Â° and â^ PQR = 30Â°. Calculate /PS/. (WAEC)
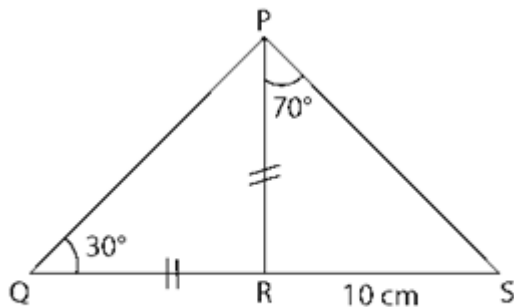
Figure 12.14

8. In the diagram in Figure 12.15, â^ ADB = 90Â°, /AB/ = 15 cm, /DC/ = 10 cm and
â^ BAD = 60Â°. Find
(a) /AD/.
(b) /BC/ correct to 2 d.p.
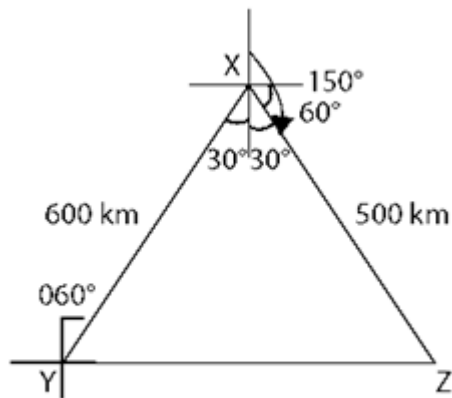(c) angle DAC correct to the nearest degree. (WAEC)

Figure 12.15

9. Find the value of $\hat{I}$ in the diagram in Figure 12.16. (Jamb)
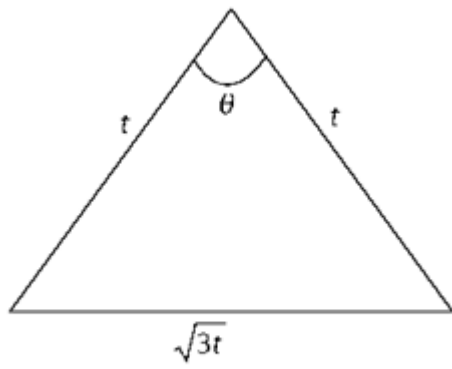
Figure 12.16

10. In the diagram in Figure12.17, the bearing of X from Y is 60°. The bearing of Z from X is 150°. Find /YZ/.

Figure 12.17