CHAPTER 8 Vectors

A vector can be defined as a directed line segment. The word 'direction' distinguishes a vector quantity from other physical quantities like scalars. While a vector is defined by magnitude and direction, a scalar is defined by magnitude alone. The magnitude of any physical quantity is a measure of how big the quantity is, and magnitudes of vectors are represented on paper by how long a line representing a vector is; the bigger the magnitude of a vector, the longer the line representing it. A 7*Newton* force exerted in the direction 070º is said to have a magnitude of 7*Newton*.

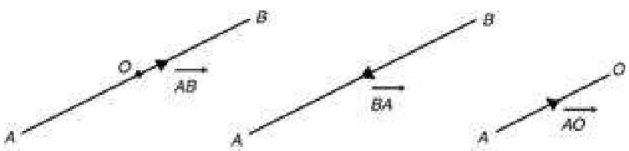

Fig. 8.1

The three diagrams in Figure 8.1 are vectors (as can be seen, the three diagrams are **directed** line segments, just like we defined). Point O is the centre of line AB, and the three vectors are parallel to each other. The sense (i.e. where the arrow is pointing) of the vector in the first diagram shows that it is acting from A to B; thus, the vector is named $\overrightarrow{AB}$. Likewise, the second diagram is $\overrightarrow{BA}$, while the third diagram shows vector $\overrightarrow{AO}$. A careful look at the three diagrams shows that the three vectors are trending in the same direction (since the three are parallel to each other), but the sense of vector $\overrightarrow{BA}$ is opposite that of $\overrightarrow{AB}$; $\overrightarrow{BA}$ and $\overrightarrow{AB}$ are of the same length, therefore, $\overrightarrow{AB} = -\overrightarrow{BA}$. In addition, O is the midpoint

of AB, hence, the vector $\overrightarrow{AO}$ in the third diagram will have a magnitude that is just half the magnitude of vector AB . And since the sense of $\overrightarrow{AB}$ and $\overrightarrow{AO}$ are the same, then, $\overrightarrow{AO} = \frac{\overrightarrow{AB}}{2}$.

Two vectors are said to be equal to each other **if** they are equal in magnitude **and** are pointing in the same direction (i.e. they must be parallel and have their sense in the same direction). Should two vectors be equal in magnitude, but **not** in the same direction, the two verctors are then **NOT** equal.

Position Vector

The position of any point P can be shown by its position vector $\overrightarrow{OP}$ relative to an origin O. This vector is called $\overrightarrow{OP}$.Thus, the position of point $P(x_1, y_1)$, can be shown by the position vector of the point, relative to the origin $O(0, 0)$. The position vector, $\overrightarrow{OP}$, is given by $\overrightarrow{OP} = x_1\hat{\imath} + y_1\hat{\jmath}$ where $\hat{\imath}$ is the unit vector in the O-x direction and $\hat{\jmath}$ is the unit vector in the O-y direction as shown in the diagram below.

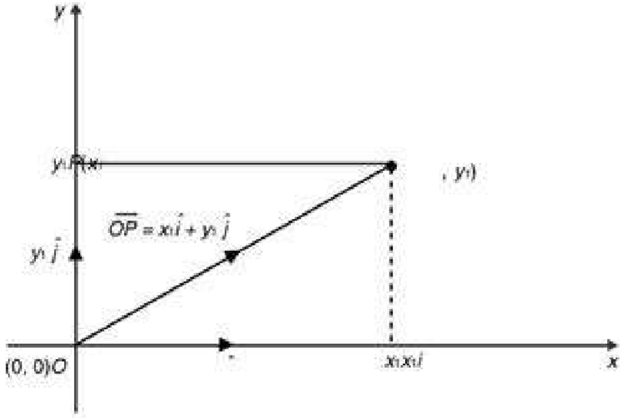

Fig. 8.2

Note that $-\hat{i}$ is the unit vector in the negative x (− x) direction while $-\hat{j}$ is the unit vector in the negative y (− y) direction.

Besides learning about a position vector, a knowledge of what a **free vector** is will also be helpful at the workshop. Unlike the position vectors that are tied to a specified position (for instance point *O*), a free vector is **only** defined by its magnitude and direction. Thus a free vector can be represented at any chosen position, but its magnitude and direction (including its sense) **MUST NOT** change.

Triangular Law of Vectors

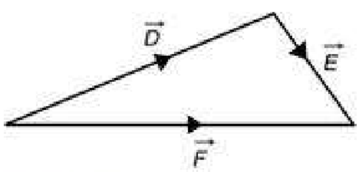

Fig. 8.3

A careful look at Figure 8.3 shows that vectors $\vec{D}$ and $\vec{E}$ are directed clockwise (in the direction in which the hand of your wall clock moves), while $\vec{F}$ seems to be directed anticlockwise, at least compared to the established directions of $\vec{A}$ and $\vec{B}$. For this reason, the relationship between the three vectors is expressed as $\vec{F} = \vec{D} + \vec{E}$, provided $\vec{D}$, $\vec{E}$ and $\vec{F}$ are written in the i, j form (for instance, $xi + yj$). This explains the triangular law of vectors.

Parallelogram Law of Vectors

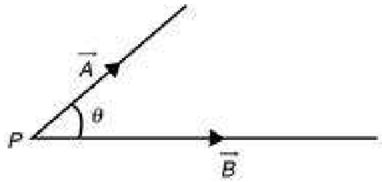

Fig. 8.4(a)

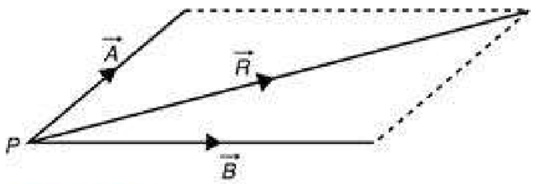

Fig. 8.4(b)

Figure 8.4(a) shows two vectors $\vec{A}$ and $\vec{B}$, acting away from point P, with angle θ between them. Imagine that these two vectors are force vectors. The resultant, $\vec{R}$ of the two forces will be a single force that will have the same effect as the combined effect of the two forces, and the direction of R will be the same as the direction that the combined effect of the two forces will push a body. R is rightly represented in figure 8.4b as the diagonal vector on the parallelogram If A is seen as a free vector, it can be represented on the opposite side of the parallelogram as in Figure 8.5 below.

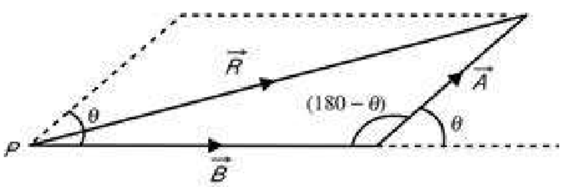

Fig. 8.5

With reference to the triangle law of vectors, $\vec{R} = \vec{A} + \vec{B}$, provided the three vectors are in the i, j form.

The angle θ between vectors $\vec{A}$ and $\vec{B}$, in Figure 8.5 is equal to the external angle shown (corresponding angles are equal); thus, the angle opposite $\vec{R}$ will be $(180 - \theta)$. Therefore, if the magnitude of $\vec{A}$ and $\vec{B}$, and the angle between them are known, the magnitude of the resultant can be calculated using the cosine rule as follows:

$$|\vec{R}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|cos(180 - \theta).$$

Scalar (dot) Product

Given two vectors $\vec{A}$ and $\vec{B}$ such that $\vec{A} = pi + qj$ and $\vec{B} = xi + yj$, the dot product of vectors $\vec{A}$ and $\vec{B}$ is given by $\vec{A} \cdot \vec{B} = (pi + qj) \cdot (xi + yi) = (px + qy)$ units. $\vec{A} \cdot \vec{B}$ is also defined thus: $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|$ $\cos\theta$; where θ is the acute angle between vectors $\vec{A}$ and $\vec{B}$.

Hence, $A \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta = (pi + qj) \cdot (xi + yj)$.

1. Two forces of magnitudes $3N$ and $7N$ have a resultant of magnitude $5N$. Calculate, correct to one decimal place, the angle between the two forces. **(WAEC)**

Workshop

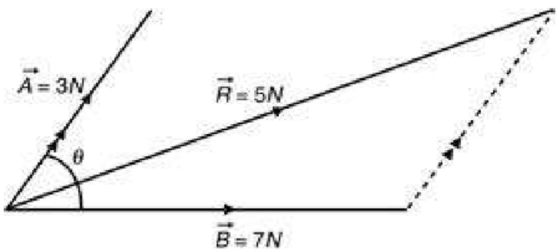

Fig. 8.6(a)

Recall that vectors are specified by their magnitude and direction, therefore, vector $\vec{A}$ in Figure 8.6a can be redrawn having the same magnitude and direction as below.

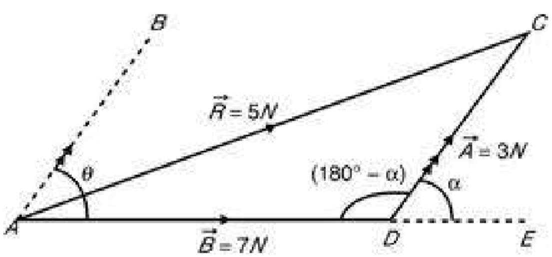

Fig. 8.6(b)

From Figure 8.6(b), *ABCD* is a parallelogram, so line *AB* is parallel to *CD*. Also, line *AE* is a straight line cutting lines *AB* and *DC*, so, angle θ corresponds to angle α. Then,θ = α (*corresponding angles are equal*).

Note that in redrawing $|\vec{A}| = 3N$ as drawn in Figure 8.6(b), the arrow showing the sense of vector $\vec{A}$ must point in the same direction as it was in Figure 8.6(a).

From Figure 8.6(b), the magnitudes of the vectors marking the sides of triangle *ACD* are related by the cosine rule as:

$$|\vec{R}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos(180° - \alpha);$$

$$|\vec{R}|^2 - |\vec{A}|^2 - |\vec{B}|^2 = -2|\vec{A}||\vec{B}| \cos(180° - \alpha);$$

$$\frac{|\vec{R}|^2 - |\vec{A}|^2 - |\vec{B}|^2}{-2|\vec{A}||\vec{B}|} = \cos(180° - \alpha); \frac{5^2 - (3^2) - (7^2)}{-2(3)(7)}$$

$$= \cos(180° - \alpha);$$

Note that $(-3)^2 = -3 \times -3 = +9$ *while;*
$-3^2 = -(3^2) = -(3 \times 3) = -9.$

$$\frac{25 - 9 - 49}{-42} = \cos(180° - \alpha); \cos(180° - \alpha) = \frac{-33}{-42}$$

$$= 0.7857;\ 180° - \alpha$$

$$= \cos^{-1} 0.7857 = 38.22°;\ 180° - \alpha$$

$$= 38.22°;\ -\alpha = 38.22° - 180°$$

$$= -141.78°;$$

$\alpha = 141.78°$. Recall that $\theta = \alpha$ (*corresponding angles are equal*). Therefore, the angle, θ, between the two forces correct to one decimal place is $141.8°$.

You can see that from our result that the diagrams initially drawn seem to be wrong, so we can redraw the two vectors and their resultant as shown in Figure 8.7.

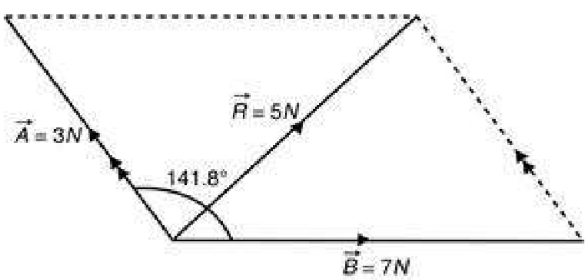

Fig. 8.7

2. $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ are two vectors in the XY plane. If V is the mid-point of $\overrightarrow{AB}$, find CV. **(WAEC)**

Workshop

Vectors $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ can be represented as shown in Figure 8.8.

From Figure 7.11, since $\overrightarrow{CB}$ is known, to find $\overrightarrow{CV}$, we have to first know $\overrightarrow{VB}$ $\overrightarrow{AV} = \overrightarrow{VB} = \frac{1}{2}\left(\overrightarrow{AB}\right)$. This is because V is half way between points A and B, $\overrightarrow{AV}$ and $\overrightarrow{VB}$ are of the same length and both points in the same direction as shown in Figure 8.8.

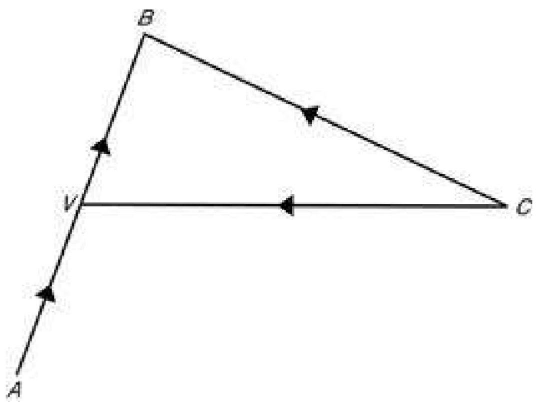

Fig. 8.8

Hence, $\overrightarrow{AV} = \overrightarrow{VB} = \frac{1}{2}\left(\overrightarrow{AB}\right) = \frac{1}{2}\begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{-4}{2} \\ \frac{6}{2} \end{pmatrix}$,

$\overrightarrow{VB} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

A careful look at triangle CVB, in Figure 8.8, shows that the arrows showing the senses of vectors $\overrightarrow{CV}$ and $\overrightarrow{VB}$ seem to be directed in the clockwise direction while the arrow showing the sense of vector $\overrightarrow{CB}$ seems to be directed opposite (anticlockwise) with respect to the sense of $\overrightarrow{CV}$ and $\overrightarrow{VB}$, hence, $\overrightarrow{CV} + \overrightarrow{VB} = \overrightarrow{CB}$ (triangle law of vectors).

The essence of this explanation is to make you understand the triangular law of vectors. Do not be mislead that the forces are moving clockwise or anticlockwise around the triangle; the forces are directed on a straight line as shown in Figure 8.8.

*Note this vector addition holds **only if** vectors $\overrightarrow{CV}$, $\overrightarrow{VB}$ and $\overrightarrow{CB}$ are written in vector form as in this question or, if the vectors are in the i, j form.*

$$\overrightarrow{CV} = \overrightarrow{CB} - \overrightarrow{VB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-(-2) \\ -3-3 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2 \\ -6 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}. \text{ So, } \overrightarrow{CV} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}.$$

3. Two forces of magnitudes $50N$ and $100N$ have a resultant of magnitude $120N$. Calculate, correct to the nearest degree, the angle between the two forces. **(WAEC)**

Workshop

Forces are vector quantities so let us represent the two forces in the question as vectors $\vec{A}$ and $\vec{B}$ and their resultant as $\vec{R}$. From the question, the magnitudes of $\vec{A}$, $\vec{B}$ and $\vec{R}$ are respectively, $|\vec{A}| = 50N$, $|\vec{B}| = 100N$ and $|\vec{R}| = 120N$. Vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ can be represented as shown in Figure 8.9(a).

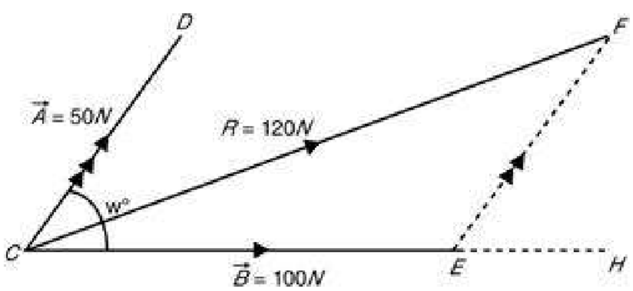

Fig. 8.9(a)

As a vector is defined by its magnitude and direction, vector $\vec{A}$ can be redrawn on the opposite side of the parallelogram, while it still has the same magnitude $50N$ and also makes angle W (same direction) with line CH as shown in Figure 8.9(b).

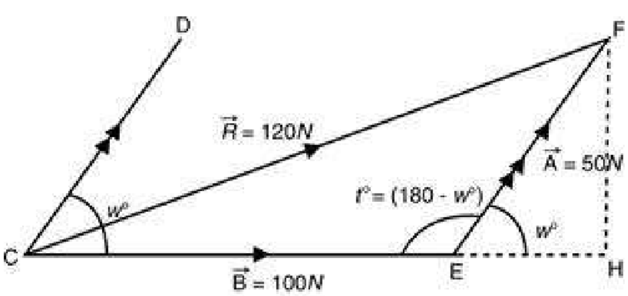

Fig. 8.9(b)

As shown in Figure 8.9(a), the angle between forces $\vec{A}$ and $\vec{B}$ is W. This is because CD is parallel to EF, and CH is a straight line cutting CD and EF, therefore, angle W is equal to angle $H\hat{E}F$(*corresponding angles are equal*). So, $H\hat{E}F = W$. If, $H\hat{E}F = W$, then $C\hat{E}F = 180° - W$ (*sum of angles on a straight line is* 180). Considering triangle CEF, by the cosine rule, $|\vec{R}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos (180° - W)$;

$|\vec{R}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos t$; $|\vec{R}|^2 - |\vec{A}|^2 - |\vec{B}|^2 = -2|\vec{A}||\vec{B}| \cos t$;

$$\frac{|\vec{R}|^2 - |\vec{A}|^2 - |\vec{B}|^2}{-2|\vec{A}||\vec{B}|} \cos t; \cos t$$

$$= \frac{120^2 - 50^2 - 100^2}{-2(50)(100)} = \frac{1900}{-10{,}000} = -0.19.$$

Thus, cos t = −0.19, to know t, we will first find an angle α such that, cos α = 0.19 in the four figure table. cos α = 0.19; α = cos-10.19. From the **four-figure table**, $\cos^{-1}0.19 = 79.15º$ so, cos79.15º = 0.19.

In the trigonometrical quadrants, cosθ is negative in the second and third quadrant. cos t = −0.19, so, the two possible values of angle t will be an angle in the second quadrant or an angle in the third quadrant where cosθ is always negative.

Thus, for the second quadrant, cos t = 0.19 recall that cos 79.15º = 0.19 − cos 79.15º = − 0.19; hence, cos t = − 0.19 = −cos 79.15º, also recall that in the second quadrant, cosθ = −cos(180º − θ), so that cos t = −cos(180º − t) = −cos 79.15º; so, −cos(180º − t) = −cos 79.15º; cos(180º − t) = cos 79.15º; 180 − t = 79.15º; t = 180º − 79.15º = 100.85º

Moreover, in the third quadrant, cos t = −0.19.

Recall that cos 79.15º = 0.19º −cos79.15º = −0.19º hence, cos t = −0.19 = −cos 79.15º, also recall that in the third quadrant, cosθ = −cos(θ − 180º), so that cos t = −cos(t − 180º) = −cos79.15º; so, −cos(t − 180º) = −cos 79.15º; cos(t − 180º) = cos 79.15º; t − 180º = 79.15º; t = 79.15º + 180º = 259.15º.

Therefore, the possible values of cos-1(−0.19) are 100.85º and 259.15º.

From Figure 8.9b, angle t = 180º − W is greater than 90º but less than 180º. Thus, the value of angle t that is appropriate for the diagram is 100.85º.

Recall that t = 180º − W; t = 180º − W = 100.85º. W = 180º − 100.85º = 79.15º. Therefore, the angle W between the two forces A and B is 79º to the nearest degree.

4. The position vectors of points P, Q, R and S relative to the origin are $\begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}1\\4\end{pmatrix}$ respectively.

   (a) Show that PQRS is a rectangle.

(b) Find the perimeter of the figure. **(WAEC)**

Workshop

(a) The position of point $P(x_1, y_1)$ can be shown by the position vector of the point relative to origin $O(0, 0)$ given by, $\vec{OP} = (x_1 - 0)_i + (y_1 - 0)_j = x_{1i} + y_{1j}$. Please, have it in mind that, any vector $xi + yj$ can be written in the form $\begin{pmatrix} x \\ y \end{pmatrix}$. Given that the position vector of a point A, relative to

the origin is $\begin{pmatrix} x_a \\ y_a \end{pmatrix}$, the coordinate of point A will be (x_a, y_a). For this reason, if the position vectors of points P, Q, R *and* S relative to the origin are $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ respectively, the co-ordinates of points P, Q, R *and* S will be $(-2, -1)$, $(0, -1)$, $(3, 2)$, and $(1, 4)$, respectively.

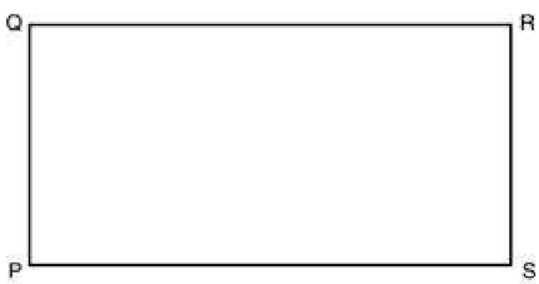

Fig. 8.10

> *Note: To make a rough sketch of the rectangle, the points must be labelled in the order mentioned in the question.*

For a shape to be a rectangle, the length of its opposite sides **must** be equal and adjacent sides must be perpendicular to each other. And so, for shape $PQRS$ to be called a rectangle, length $|QR|$ **must** be equal to length $|PS|$, length $|QP|$ **must** be equal to length $|RS|$ and line QP **must** be perpendicular to line PS. Given two points, $A(x_1, y_1)$ and $B(x_2, y_2)$ the distance $|AB|$ between the two points is expressed as

$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$; therefore,

$$|QR| = \sqrt{(3-0)^2 + (2-(-1))^2}$$
$$= \sqrt{3^2 + (2+1)^2} = \sqrt{3^2 + 3^2} = \sqrt{18};$$

$$|PS| = \sqrt{(1-(-2))^2 + (4-1)^2} = \sqrt{(1+2)^2 + 3^2}$$
$$= \sqrt{3^2 + 3^2} = \sqrt{18};$$

$$|QP| = \sqrt{(-2-0)^2 + (1-(-1))^2}$$
$$= \sqrt{(-2)^2 + (1+1)^2} = \sqrt{4+4} = \sqrt{8};$$

$$|RS| = \sqrt{(1-3)^2 + (4-2)^2} = \sqrt{(-2)^2 + 2^2}$$
$$= \sqrt{4+4} = \sqrt{8}.$$

From Figure 8.10, line *QR* is opposite line *PS* and the lengths of lines *QR* and *PS* are of equal length ($\sqrt{18}$ *units each*), line *QP* is opposite line *RS* and the lengths of lines *QP* and *RS* are equal ($\sqrt{8}$ *units each*), but these findings only show that *PQRS* is a parallelogram in which a rectangle is **just one** of the parallelograms.

For *PQRS* to be a rectangle, $|QR|$ = $|PS|$ and $|QP|$ = $|RS|$ as we have proven, and *QP* must be perpendicular to *PS* and *RS* must also be perpendicular to *PS*. Two lines are perpendicular to each other if the product of the gradients of the two lines is equal to -1. Recall that the gradient of a line passing through points (x_1, y_1) and (x_2, y_2) is given by gradient,

$$m = \frac{y_2 - y_1}{x_2 - y_1}.$$

Hence, for a line *QP* passing through points, $Q(0, -1)$ and $P(-2, 1)$, m_{QP} $= \frac{1-(-1)}{(-2)-0} = \frac{2}{-2} = -1$. Also, for a line *PS* passing through points, $P(-2, 1)$ and $S(1, 4)$, $m_{PS} = \frac{4-1}{1-(-2)} = \frac{3}{3} = 1$.

$m_{QP} \times m_{PS} = -1 \times 1 = -1$. The product of the gradients of lines m_{QP} and m_{PS} is -1; therefore, *QP* is perpendicular to *PS*. *QP*

is perpendicular to *PS* and $|R| = |PS|$, then, *RS* is also perpendicular to PS, therefore, *PQRS* is a rectangle.

(b) Perimeter = $|PQ| + |QR| + |RS| + |SP|$

$= \sqrt{8} + \sqrt{18} + \sqrt{8} + \sqrt{18}$.

Note that because length is a scalar quantity, length |PS| = length |S|P|, length |QP| = length |PQ|, e.t.c, while if PS and QP are vectors, PS = –SP, QP = –PQ, e.t.c.

Let $\sqrt{8} = x$ so that $\sqrt{8} + \sqrt{8} = x + x = 2x = 2\sqrt{18}$.

Hence, $\sqrt{8} + \sqrt{8} = 2\sqrt{8}$; also $\sqrt{18} + \sqrt{18} = 2\sqrt{18}$. $\sqrt{8} + \sqrt{8} + \sqrt{18} + \sqrt{18} = 2\sqrt{8} + 2\sqrt{18}$

$= 2\sqrt{4 \times 2} + 2\sqrt{9 \times 2} = 2\sqrt{4}\,\sqrt{2} + 2\sqrt{9}\,\sqrt{2}$

$= 2 \times 2\sqrt{2} + 2 \times 3\sqrt{2} = 4\sqrt{2} + 6\sqrt{2}$.

Also, if $\sqrt{2} = y$; $4y + 6y = 10y$; thus, $4\sqrt{2} + 6\sqrt{2} = 10\sqrt{2}$.

Therefore, the perimeter of the figure is $10\sqrt{2}$ units.

5. Forces of magnitudes 2, 5 and 4 units lie in the direction 045°, 090° and 135° respectively. Find:

(a) the magnitude,

(b) the direction of their resultant. **(WAEC)**

Workshop

(a) By the polygon law of forces, the resultant of these three forces is a single force *R*, that will have an effect that is equal to the effect of the three forces combined. For example, Tola, Ayo and Ojo can combine their strength to carry a bag of rice in a particular direction. If, moving at the same pace as the children, their father alone can carry the same bag in the same direction as the children did, then, it can be said that the fathers strength is the resultant *R* of the strengths of his three children.

Let the resultant of the three forces be represented by *R*. Then, the horizontal component of *R* will be *Rx*, and can be expressed as:

$$R_x = \Sigma F_x = 2\cos 45^\circ + 5\cos\theta^\circ + 4\cos 45^\circ$$
$$= 2\left(\frac{1}{\sqrt{2}}\right) + 5(1) + 4\left(\frac{1}{\sqrt{2}}\right).$$

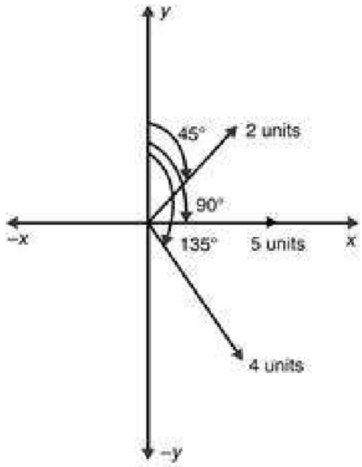

Fig. 8.11(a)

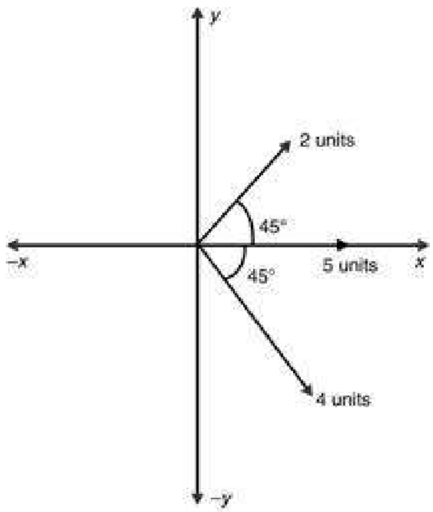

Fig. 8.11(b)

Note that the angles 45°, 0° and 45° are the acute angles the forces 2, 5 and 4 units make with the x-axis (either along the +x or along the –x).

$$= \frac{2}{\sqrt{2}} + 5 + \frac{4}{\sqrt{2}} = \frac{2 + 5\sqrt{2} + 4}{\sqrt{2}} = \frac{6 + 5\sqrt{2}}{\sqrt{2}}$$

$$= \frac{6 + 5\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2} + 5\sqrt{2}\sqrt{2}}{\sqrt{2}\ \sqrt{2}}$$

$$= \frac{6\sqrt{2} + 5\sqrt{2 \times 2}}{\sqrt{2 \times 2}} = \frac{6\sqrt{2} + 5\sqrt{4}}{\sqrt{4}} = \frac{6\sqrt{2} + 5(2)}{2}$$

$$= \frac{6\sqrt{2}}{2} + \frac{10}{2} = 3\sqrt{2} + 5.$$

The vertical component of R will be R_y, expressed as:

$$R_y = \Sigma F_y = 2\sin 45° + 5\sin 0° + (-4\sin 45°).$$

Note that the resolution of the 4 unit force to the vertical, falls on the ***negative*** *y-axis, hence, its resolution to the vertical is –4 sin 45°.*

$$R_y = 2\left(\frac{1}{\sqrt{2}}\right) + 50(0) - 4\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} - \frac{4}{\sqrt{2}} = \frac{2-4}{\sqrt{2}}$$

$$= \frac{-2}{\sqrt{2}} = \frac{-2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{-2\sqrt{2}}{2} = -\sqrt{2}.$$

The magnitude of $R = |R| = \sqrt{R_x^2 + R_y^2}$

$$= \sqrt{(3\sqrt{2}+5)^2 + (-\sqrt{2})^2}$$

$$\sqrt{(9\sqrt{2}\sqrt{2} + 15\sqrt{2} + 15\sqrt{2} + 25) + (-\sqrt{2} \times -\sqrt{2})}$$

$$= \sqrt{18 + 30\sqrt{2} + 25 + \sqrt{4}}.$$

Recall that $15x + 15x = 30x$, so $15\sqrt{2} + 15\sqrt{2}$ will be equal to $30\sqrt{2}$.

$$= \sqrt{43 + 30\sqrt{2} + 2} = \sqrt{45 + 30\sqrt{2}} = \sqrt{87.426}$$
$$= 9.35 \text{ units.}$$

Therefore, the magnitude of the resultant of the three forces is 9.35 units of force.

Note that the unit of the forces is not written in Newton (N) because the question did not mention the unit of the three forces. So, their unit could have been in Newton or any other unit of force.

(b) Let the direction of the resultant R relative to the x-axis, be θ, as shown in Figures 8.12(a) and (b) below.

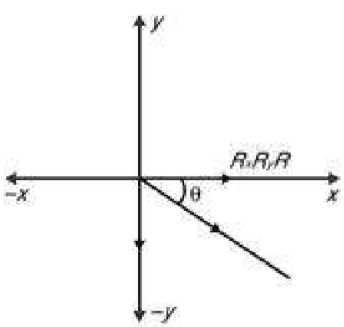

Fig. 8.12(a)

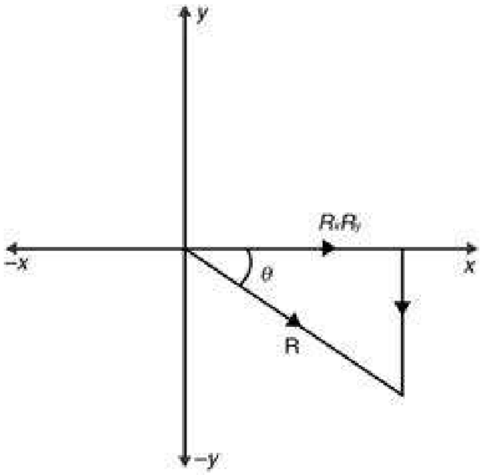

Fig. 8.12(b)

Note: R_y was drawn, in Figure 8.12(a), facing the −y axis because the value of R_y obtained is negative (i.e − $\sqrt{2}$).

From Figures 8.12(a) and (b), $\tan\theta = \dfrac{opposite}{adjecent}$

$$= \frac{R_y}{R_x} = \frac{\Sigma F_y}{\Sigma F_x} = \frac{\sqrt{2}}{3\sqrt{2}+5} = 0.1530,$$

$\theta = \tan^{-1} 0.1530 = 8.7°$.

*Carefully note that in calculating direction θ, since R_x and R_y **had been drawn facing the directions** based on the sign they bear and resultant, **R had been drawn in the right quadrant**, we **do not need** to include the negative sign in front of the value of R_y to calculate θ as the negative sign only shows that R_y is facing the direction of the negative y-axis which is already shown on the diagram.*

Therefore, the resultant force makes angle θ = 8.7º with the positive *x*-axis as shown in Figure 8.12(b).

*Note that the direction of the resultant force **R** can be measured, clockwise from the positive y-axis: by this measurement, the direction of the resultant R, from the diagram, will be 90° + 8.7° = 98.7°.*

6. Given that $p = 2i + 3j$; $q = i + j$; $r = i - 2j$; find:

(a) $|2p - 3q + r|$;

(b) the unit vector in the opposite direction of $2p - 3q + r$;

(c) the angle between *p* and *r*, correct to the nearest degree. **(WAEC)**

Workshop

(a) If $p = 2i + 3j$, $q = i + j$, $r = i - 2j$,
$\therefore 2p = 2(2i + 3j) = 4i + 6j$.

$3q = 3(i + j) = 3i + 3j$. $2p - 3q + r$
$= 2(2i + 3j) - 3(i + j) + (i - 2j)$

$= 4i + 6j - 3i - 3j + i - 2j = 4i - 3i$
$+ i + 6j - 3j - 2j = 2i + j$;

$|2p - 3q + r| = |2i + j| = \sqrt{2^2 + 1^2}$
$= \sqrt{4 + 1} = \sqrt{5}$.

(b) Given any vector D, the unit vector in the direction of D, will be

$\hat{D} = \frac{1}{|D|}(D)$. Let $B = 2p - 3q + r = 2i + j$; then, the unit vector in the direction of B will be

$$\hat{B} = \frac{1}{|B|}(B) = \frac{1}{|2i - j|}(2i + j) = \frac{1}{\sqrt{5}}(2i + j)$$

The unit vector, opposite in direction to B, will be $-B = -\left(\frac{1}{\sqrt{5}}(2i + j)\right) = \frac{1}{\sqrt{5}}(-2i - j)$
$= -\frac{2}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j$.

Rationalizing the coefficients of i and j, we get

$$-\frac{2}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j = -\left(\frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\right)i - \left(\frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\right)j$$
$$= -\frac{2\sqrt{5}}{5}i - \frac{\sqrt{5}}{5}j.$$

(c) $p \cdot r = (2i + 3j) \cdot (i - 2j) = |p||r|\cos\theta$.

Where θ is the acute angle between vectors p and r.

$(2i + 3j) \cdot (i - 2j) = 2(1) + 3(-2) = 2 - 6$
$= -4$. Hence, $p \cdot r = -4$

$|p| = |2i + 3j| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$;

$|r| = |i - 2j| = \sqrt{1^2 + (-2)^2} = \sqrt{1 + 4} = \sqrt{5}$;

$p \cdot r = -4 = |p||r|cos\theta = (\sqrt{13})(\sqrt{5})\cos\theta$;
$(\sqrt{13})(\sqrt{5})\ cos\theta = -4$;

$\cos\theta = \frac{-4}{\sqrt{13}\sqrt{5}} = -0.4961$; $\cos\theta = -0.4961$; now, we want to know θ when $\cos\theta$ is negative. $\cos\theta = -0.4961$, to know θ, we will first find an angle α such that, $\cos\alpha = 0.4961$ in the four figure table, If $\cos\alpha = 0.4961$; then, $\alpha = \cos^{-1} 0.4961$. From the four figure table, $\cos^{-1} 0.4961 = 60.26°$;

Hence, $\cos 60.26 = 0.4961$. Now, let's take a look at the four trigonometric quadrants.

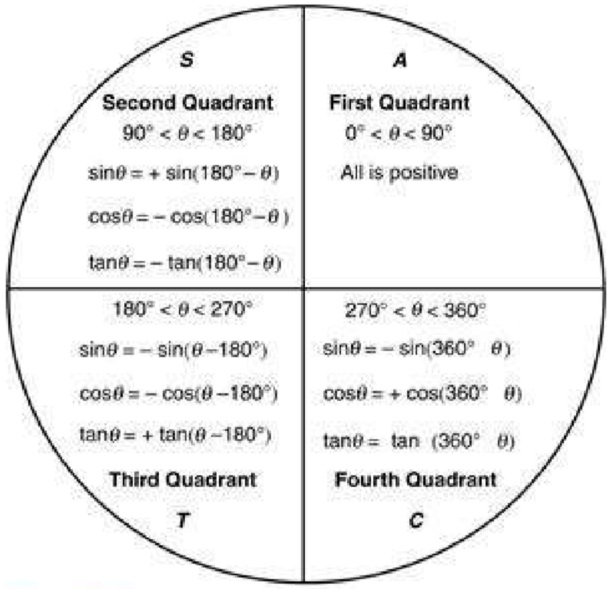

Fig. 8.13a

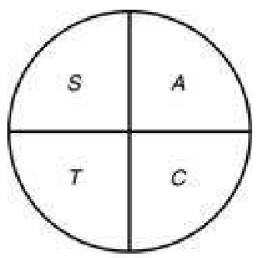

Fig. 8.13b

As shown in Figures 8.13a and b above, for angles in the first quadrant, (angles between zero and 90°), the cosine, sine and tangent of these angles are all positive. For example,

$$\cos 30° = +\frac{\sqrt{3}}{2},$$

$$\sin 45° = +\frac{1}{\sqrt{2}},$$

and $\tan 60° = +\sqrt{3}$.

The symbol ***A*** in the quadrant means **ALL** is positive.

For angles in the second quadrant (angles between 90° and 180°), only the sine of these angles is positive, the cosine and tangent of these angles are negative. For example,

$$\sin 120° = +\sin(180° - 120°) = +\sin 60° = +\frac{\sqrt{3}}{2}$$

$$\cos 120° = -\cos(180° - 120°) = -\cos 60° = -\frac{1}{2}$$

$$\tan 120° = -\tan(180° - 120°) = -\tan 60° = -\sqrt{3}$$

The symbol ***S*** means **only the sine** of angles in the quadrant is positive.

In the third quadrant (angles between 180° and 270°), only the tangent of the angles is positive; the sine and cosine of the angles are negative. For example,

$$\sin 210° = -\sin(210° - 180°) = -\sin 30° = -\frac{1}{2}$$

$$\cos 210° = -\cos(210° - 180°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

$$\tan 210° = +\tan(210° - 180°) = +\tan 30° = -\frac{1}{\sqrt{3}}$$

The symbol ***T*** in the quadrant means **only the tangent** of the angles is positive.

For the fourth quadrant (angles between 270º and 360º), only the cosine of these angles is positive; the sine and tangent of these angles are negative. For example,

$$\sin 315° = -\sin(360° - 315°) = -\sin 45° = -\frac{1}{\sqrt{2}}$$

$$\cos 315° = +\cos(360° - 315°) = +\cos 45° = +\frac{1}{\sqrt{2}}$$

$$\tan 315° = -\tan(360° - 315°) = -\tan 45° = -1.$$

The symbol ***C*** in the quadrant means **only the cosine** of the angles is positive.

From the explanation above, we can see that cosine of angles in the second and third quadrants are negative. We knew earlier that $\cos\theta = -0.496$. So, the two possible values of angle θ will be an angle in the second quadrant or an angle in the third quadrant where $\cos\theta$ is negative.

Thus, for the second quadrant, where $\cos\theta$ is negative, $\cos\theta = -0.496$ recall that $\cos 60.26º = 0.4961$; therefore, $-\cos 60.26º = -0.4961$.

Hence,$\cos\theta = -0.4961 = -\cos 60.26º$. Also, recall that, in the 2nd quadrant,

$\cos\theta = -\cos(180º - \theta)$, so that $\cos\theta = -\cos(180º - \theta) = -\cos 60.26º$;

Then, $-\cos(180º - \theta) = -\cos 60.26º$; $\cos(180º - \theta)$

$= \cos 60.26º$; $180º - \theta = 60.26º$; $\theta = 180º - 60.26º$

$= 119.74º$.

For the third quadrant where, $\cos\theta$ is also negative, $\cos\theta = -0.4961$; recall that $\cos 60.26º = 0.4961$, therefore, $-\cos 60.26º = -0.4961$. Thus,

$\cos\theta = -0.4961 = -\cos 60.26º$ moreover, recall that in the 3rd quadrant $\cos\theta = -\cos(\theta - 180º)$,

so that $\cos\theta = -\cos(\theta - 180º) = -\cos 60.26º$; $-\cos(\theta - 180º) = -\cos 60.26º$; $\cos(\theta - 180º) =$

$\cos 60.26º$; $\theta - 180º = 60.26º$; $\theta = 180º + 60.26º = 240.26º$.

Therefore, the possible values of $\cos{-1}(-0.4961)$ are 119.74º and 240.26º.

Hence, the vectors p and r make angles 119.74º, 240.26º with each other as shown in the diagram below.

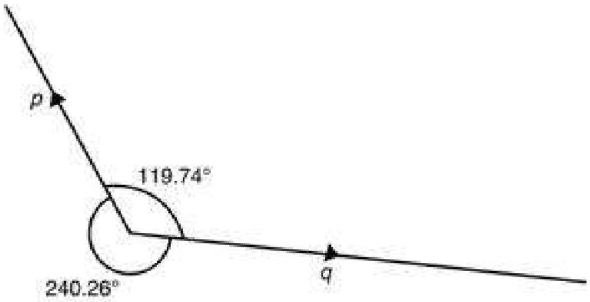

Fig. 8.14

Note that for exam purposes, you are advised to give the smaller of the two angles as your answer, even though the two values are correct.

Therefore, the angle vector *p* makes with vector *r* is 120º to the nearest degree.

7. The position vectors of P and Q with respect to a fixed point are $P = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$. If M divides PQ in the ratio 3:1, find the position vector of M with respect to the fixed point. (WAEC)

Workshop

The position of any point P can be shown by its position vector $\overrightarrow{OP}$, relative to an origin O. This vector is called $\overrightarrow{OP}$. Thus, the position of point $P(x_1, y_1)$, can be shown by the position vector of the point, relative to the origin $O(0, 0)$. The position vector, $\overrightarrow{OP}$, is given by $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j}$ where $\hat{i}$ is the unit vector in the O-x direction and $\hat{j}$ is the unit vector in the O-y direction as shown in the diagram below.

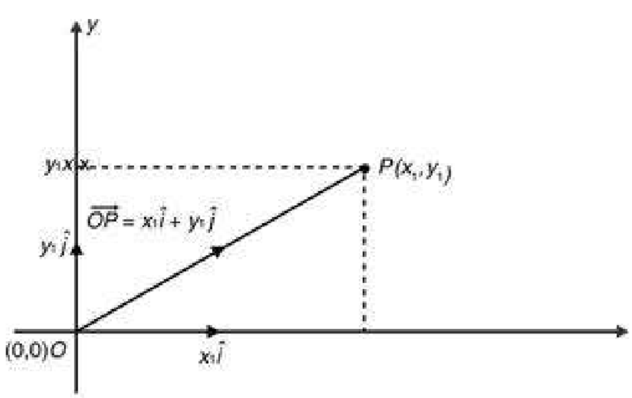

Fig. 8.15

Note that $-\hat{i}$ is the unit vector in the negative x (−x) direction while $-\hat{j}$ is the unit vector in the negative y (−y) direction.

Thus, position vector $p = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ will be the position vector that defines the position of point P, relative to the origin O; therefore, the coordinate of point P will be (2, −1). Also, position vector $q = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, will be the position vector that defines the position of point Q, relative to the origin O; the coordinate of point Q will, therefore, be (3, 5).

*Note that, for any position vector written in the form $a = \begin{pmatrix} v \\ w \end{pmatrix}$ vector, a, has component $v\hat{i}$ on the horizontal axis (x-axis), while it has component $w\hat{j}$ on the vertical axis (y-axis). Hence, the point A, that is being defined by vector 'a', will have coordinates $(x_A, y_A) = (v, w)$, and **not** (w, v); please **note** carefully.*

The coordinates of any point $C(x_c, y_c)$, that divides a line that joins point $A(x_A, y_A)$ to $B(x_B, y_B)$, in the ratio $m{:}n$, is expressed as

$$x_c = \frac{mx_A + nx_B}{m+n} \text{ and } y_c = \frac{my_A + ny_B}{m+n}.$$

Fig. 8. 16(a)

Fig. 8. 16(b)

Carefully note that in Figure 8.16(a), point C divides line AB in the ratio 1:3, hence,

$$x_c = \frac{1(x_A) + 3(x_B)}{1+3} \text{ and } y_c \frac{1(y_A) + 3(y_B)}{1+3}$$

However, in Figure 8.16(b), point D divides line AB in the ratio 3:1, so that,

$$x_c = \frac{3(x_A) + 1(x_B)}{3+1} \text{ and } y_c = \frac{3(y_A) + 1(y_B)}{3+1}$$

If we consider the line in Figure 8.16(a), as line *BA*, then, point *C* divides line *BA* in the ratio 3:1,

hence, $x_C = \frac{3(x_B) + 1(x_A)}{3+1}$ and $y_C = \frac{3(y_B) + 1(y_A)}{3+1}$.

If we consider the line in Figure 8.16(b) as line *BA*, then, point *D* divides line *BA* in the ratio 1:3,

thus, $x_C = \frac{1(x_B) + 3(x_A)}{1+3}$ and $y_C = \frac{1(y_B) + 3(y_A)}{1+3}$

This illustration is simple enough for you to understand line division using ratio. If you seem not to get the drift, please go back to study the diagram with the explanation; you do not need to memorise it.

Summary: To divide line *AB* in the ratio *m* : *n*, so as to determine the coordinates of the point dividing *AB* into ratio *m* : *n*, x_A and y_A must be multiplied by *m*, while x_B and y_B must be multiplied by *n*, as explained above. However, to divide line *BA* in the ratio *m* : *n*, so as to determine the coordinates of the point that divides *BA* in that ratio, x_B and y_B must be multiplied by *m*, while x_A and y_A must be multiplied by *n* as explained above.

Back to the question, we were told that *M* divides *PQ* in the ratio 3:1. This means that, in calculating the coordinates of *M*, x_P and y_P will be multiplied by 3, while x_Q and y_Q will be multiplied by 1 as shown below. Coordinates of point *P* = (x_P, y_P) = (2, −1), while Coordinates of point *Q* = (x_Q, y_Q) = (3, 5)

$x_M = \frac{mx_P + nx_Q}{m+n} = \frac{3(2) + 1(3)}{3+1} = \frac{6+3}{4} = \frac{9}{4}$ and

$y_M \frac{my_P + ny_Q}{m+n} = \frac{3(-1) + 1(5)}{3+1}$

$= \frac{-3+5}{4} = \frac{2}{4} = \frac{1}{2}$. Therefore, the coordinate of

point *M* are $\left(\frac{9}{4}, \frac{1}{2}\right)$.

As the coordinate of point *M* are $\left(\frac{9}{4}, \frac{1}{2}\right)$, thus, the position vector, which shows point *M* with respect to the fixed point *O*, will be $\begin{pmatrix} \frac{9}{4} \\ \frac{1}{2} \end{pmatrix}$. Since point *M* divides *PQ* in the ratio 3:1, the position vector *M*, with respect to the fixed point, is $\begin{pmatrix} \frac{9}{4} \\ \frac{1}{2} \end{pmatrix}$.

8. *ABCD* is a square. Forces of 5, 10 √2 and 10 Newtons act along directions, $\overrightarrow{AB}$, $\overrightarrow{AC}$and $\overrightarrow{AD}$ respectively. Find:

(a) the magnitude,

(b) the direction of the force that will keep the system in equilibrium. **(WAEC)**

(a)

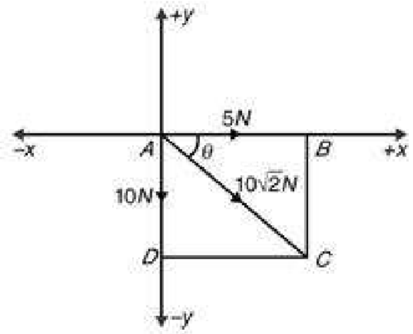

Fig. 8.17

As shown in Figure 8.17, ABCD is a square; thus, angle DAB will be 90°. Also, since ABCD is a square, the diagonal AC will bisect (*divide*) angle DAB into two equal angles θ. Hence, $\theta = \frac{90^\circ}{2} = 45^\circ$. Let the resultant of the three forces in question, be R.

The resultant of these three forces is a single force, that has the same net effect as the combined effort of these three forces.

Now, let us try to resolve each of these three forces into their respective vertical and horizontal components.

$R_x = 5\cos\theta^\circ + 10\sqrt{2}\cos 45^\circ + 10\cos 90^\circ$;

$R_x = 5(1) + 10\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + 10(0)$; $R_x = 5 + \frac{10\sqrt{2}}{\sqrt{2}} + 0$
$= 5 + 10 = 15.$

Note that the resolution of the 5N force, to the horizontal, is facing the positive x-axis (+x-axis) direction. Also, the resolution of the $10\sqrt{2}N$ force, to the horizontal, is facing the positive x-axis (+x-axis) direction, as seen in the diagram, and that is why 5 cos0° and 10 $\sqrt{2}$cos 45° are bearing positive signs. Also, note that the 10N force is acting parallel to the vertical axis, so it ***does not*** *have a component on the horizontal axis; therefore, any sign could be allotted to its component on the horizontal axis, since its component, on the horizontal axis, is zero.*

$R_y = 5\sin 0^\circ - 10\sqrt{2}\sin 45^\circ - 10\sin 90^\circ = 5(0) - \left(10\sqrt{2} \times \frac{1}{\sqrt{2}}\right) - 10(1)$

$= 0 - \frac{10\sqrt{2}}{\sqrt{2}} - 10 = -10 - 10 = -20.$

$10\sqrt{2}$ sin 45° and 10 sin 90° are bearing negative signs, because the resolution of these two forces, to the vertical, is facing the negative y-axis (−y) direction, as shown in the diagram. Also, note that the 5N force is parallel to the horizontal axis, so it ***does not*** *have a*

component on the vertical axis; thus, any sign could be allotted to its component, on the vertical axis, since this component is zero.

$|R| = \sqrt{R_x^2 + R_y^2} = \sqrt{15^2 + (-20)^2} = \sqrt{225 + 400} =$

$\sqrt{625} = 25N.$

The force that will keep the system in equilibrium, must be equal, in magnitude (but opposite in direction) to the resultant of the three forces. So, the magnitude of the force that will keep the system in equilibrium is 25N

(b)

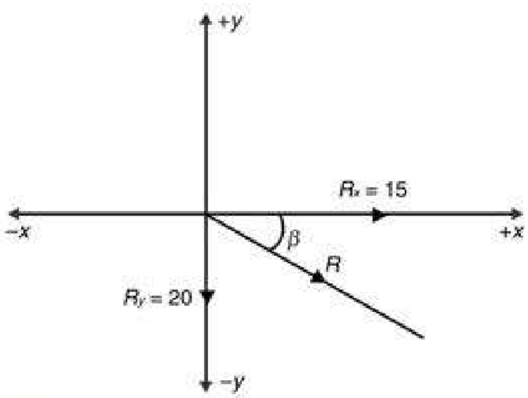

Fig. 8.18(a)

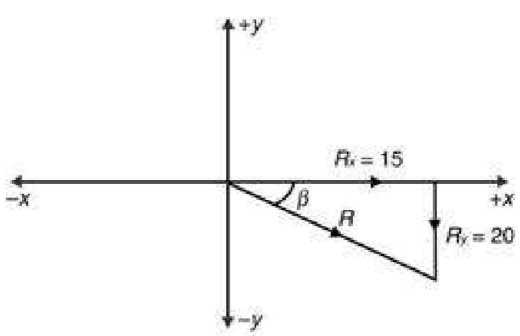

Fig. 8.18(b)

Note that you do not need to put the negative sign in front of R_y = 20 in the diagram above because R_y is already facing the negative y(−y) direction.

Figures 8.18(a) and (b) are one and the same since forces (and vectors generally) are defined by their magnitude and direction, and the 20N force is drawn in the two diagrams, having the same magnitude 20N, and facing the same direction, which is the direction of the negative y-axis. Note hat, Ry is shown facing the negative y direction, because it bears a negative sign. Looking at Figure 8.18(b), the angle, β, that R makes with the x-*axis* can be calculated as follows:

$$\tan\beta = \frac{opposite}{adjacent} = \frac{R_y}{R_x} = \frac{20}{15} = 1.333;$$

$\beta = \tan^{-1}1.333 = 53.12°.$

*Also, note that once you have represented a vector in the right quadrant, and in the right direction, you do **not** need to include any negative sign while working out its direction.*

The direction of the resultant, *R*, measured clockwise from the positive *y*-axis, as drawn in Figure 8.18(b), can be calculated as follows:

90º + β = 90º + 53.12º = 143.12º. The force needed to keep the system in equilibrium, must be equal in magnitude, but act opposite to the direction of the resultant of the 3 forces. Two forces acting directly opposite to each other, will make angle 180º (angle on a straight line) with each other. So, the direction of the force needed to keep the system in equilibrium is

143.12º + 180º = 323.12º, measured clockwise from the positive *y*-axis.

9. Given that ***p*** = (10*N*, 060º) and; ***q*** = (8*N*, 120º) are two forces acting at a point, find the:

(a) magnitude;

(b) direction, of their resultant. **(WAEC)**

Workshop

Figure 8.19(a) shows the 10*N* and 8*N* forces acting at a point. Recall that a vector is defined by

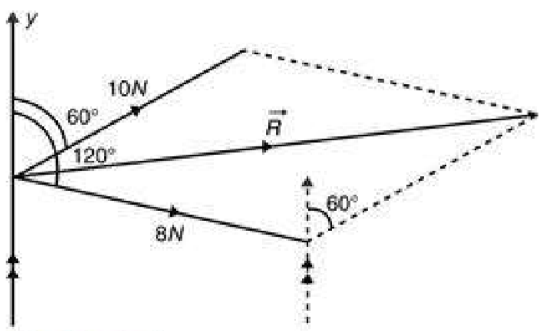

Fig. 8.19(a)

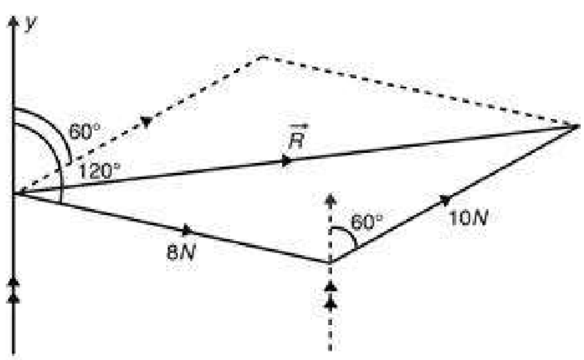

Fig. 8.19(b)

its magnitude and direction, thus, the 10*N* force can, as well, be represented on the opposite side of the parallelogram, as shown in Figure 8.19(b) above. The force has the same magnitude (10*N*) and is still in the same direction 060º, measured clockwise from the positive *y*-axis).

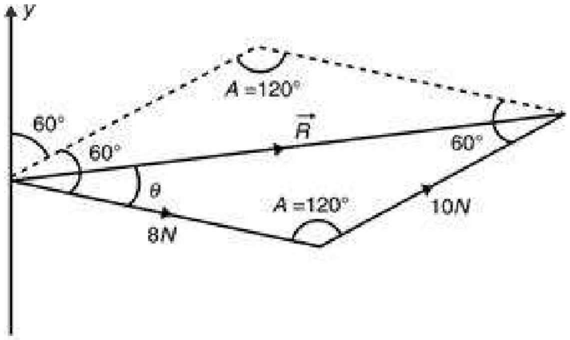

Fig. 8.20

From the original diagram, Figure 8.19a, the angle between the 8 and 10*N* forces is 060º, this angle is equal to the angle opposite to it, as opposite angles of a parallelogram are equal. Let the other two opposite angles be *A*°, as shown in Figure 8.20. Recall that the sum of the **internal** angles of a parallelogram (which is 4 sided) add up to 360º, thus,

$A + A + 60º + 60º = 360º$; $2A + 120º = 360º$;

$A = 120º$;

(i) By the cosine rule: $a^2 = b^2 + c^2 - 2bc(cosA)$ where *a*, *b* and *c* are the lengths of the sides of the triangle and *A* is the angle opposite the side with length *a*. Therefore, from figure 8.20,

$|R|^2 = 8^2 + 10^2 - 2(8)(10)(\cos 120º)$.

Recall that $|\vec{R}|$ is the magnitude of vector $|\vec{R}|$, and $\vec{R}$ can be seen, as a measure of the length of vector, $\vec{R}$

$A = 120º$ is in the second quadrant, and in this quadrant, cosθ is negative, that is, $\cos A = (180º - A)$.

Hence, $\cos 120º = -\cos(180º - 120º) = -\cos 60º = -\frac{1}{2}$

Therefore, $\cos 120° = -\frac{1}{2}$, so that $|\vec{R}|^2 = 8^2 + 10^2 -$
$2(8)(10)(\cos 120°)$

$= 64 + 100 - \left(160 - \left(\frac{1}{2}\right)\right) = 64 + 100 + 80 = 244$;

$|\vec{R}| = \sqrt{244} = 15.6N$.

Therefore, the magnitude of the resultant of the 8 and 10*N* forces is 15.6*N*.

(ii) From the Figures 8.19a, b and 8.20, to know the direction of $\vec{R}$, we need to know angle θ. By the sine rule, which states that $\frac{a}{\sin A} = \frac{b}{\sin B}$, where *a* and *b* are the lengths of two sides of a triangle, and *A* and *B* are the angles opposite *a* and *b* respectively.

Then, $\frac{10}{\sin\theta} = \frac{|\vec{R}|}{\sin 120°}$; $10\sin 120° = |\vec{R}| \sin\theta$;

$\sin\theta = \frac{10 \sin 120°}{|\vec{R}|} = \frac{10\sin 120°}{15.6}$.

θ = 120° is in the second quadrant, and in this quadrant, sin θ is positive, i.e, sin θ = +sin(180 – θ). Hence, sin 120° = +sin(180° –120°) = sin60 = $\frac{\sqrt{3}}{2}$.

Therefore, $\sin\theta = \frac{10\sin 120^\circ}{15.6} = \frac{10\left(\frac{\sqrt{3}}{2}\right)}{15.6} = \frac{10\sqrt{3}}{2\times 15.6}$ $= 0.5551$;

$\theta = \sin{-1}\ 0.5551$; $\theta = 33.7^\circ$.

From Figure 8.20, the direction of the resultant, $\vec{R}$, measured clockwise from the positive y-axis $= 120 - \theta = 120 - 33.7 = 86.3^\circ$. Therefore, the direction of the resultant, $\vec{R}$, is 86.3°, measured clockwise from the positive y-axis.

10. In a triangle PQR, $\overrightarrow{RQ} = p$, $\overrightarrow{RP} = q$ and $\overrightarrow{QP} = r$, use the vector method to prove that $r^2 = p^2 + q^2 - 2pq\cos R$. **(WAEC)**

Workshop

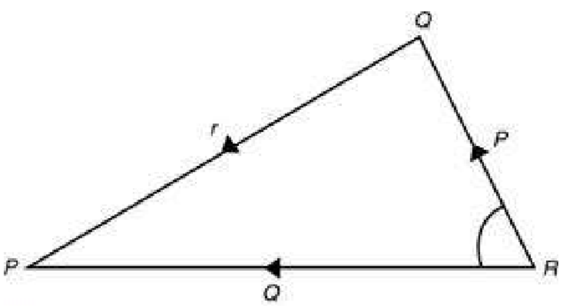

Fig. 8.21(a)

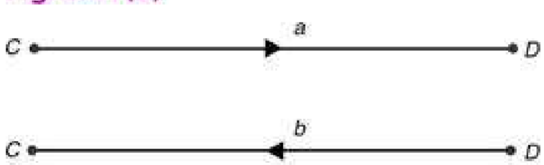

Fig. 8.21(b)

In Figure 8.21b, vector $a = \overrightarrow{CD}$, while vector $b = \overrightarrow{DC}$. Vector $\overrightarrow{CD}$ is not equal to $\overrightarrow{DC}$, although the two vectors have equal magnitudes (*as shown by their equal lengths*); however, the two are not in the same direction. So, take good note of this, while representing vectors. From figure 8.21a, vectors p and r are directed in the anticlockwise direction, while q is in the clock-wise direction. Thus, considering the vectors on triangle PQR, by the triangular law of vectors, $p + r = q$; $r = q - p$; $r.r = (q - p).(q-p) = q.q - q.p - p.q + p.p$;

$r.r = q.q + p.p - q.p - q.p$ (recall that, $a.b = b.a$), so that

$r.r = q.q + p.p - 2q.p$ Again, recall that, $a.a = |a|^2$, where $|a|$ is the magnitude of vector a; then, $r.r = |r|^2 = q.q + p.p - 2q.p = |q|^2 + |p|^2 - 2q.p$.

Also, recall that given two vectors a and b, $a,b = ab \cos\theta$, *where θ is the angle between* a *and* b. For this reason, $r.r = |r|^2 = |q|^2 + |p|^2 - 2|q|\,|p| \cos R$;

where R is the acute angle between vectors p and q.

Therefore, $|r|^2 = |q|^2 + |p|^2 - 2|q|\,|p| \cos R$; (proved)

11. If $s = 3i + 5j$, $t = xi + yj$, and $s + 2t = (4i + 5j)$, find the values of x and y. **(WAEC)**

Workshop

$s = 3i + 5j$, $t = xi + yj$, $s + 2t = (4i + 5j)$, $s + 2t = 3i + 5j + 2(xi + yj) = 4i + 5j$; $s + 2t = 3i + 5j + 2xi + yj = 4i + 5j$; $(3 + 2x)i + (5 + 2y)j = 4i + 5j$ by comparing coefficients, we get $3 + 2x = 4$; $2x = 4 - 3$; $2x = 1$; $x = \frac{1}{2}$; $5 + 2y = 5$

$2y = 5 - 5 = 0$; $y = 0$. Therefore, the values of x and y are $\frac{1}{2}$ and 0 respectively.

12. The position vectors of points P and Q are:

    p = (i + 3j)

    q = (9i + 8j) respectively. Find:

    (a) $\overrightarrow{PQ}$

    (b) correct to the nearest degree, the angle between **p** and **q**,

    (c) the length of the projection of **q** on **p**. **(WAEC)**

Workshop

(a) The position vectors, p and q, of points P and Q, are as shown in Figure 8.22 below.

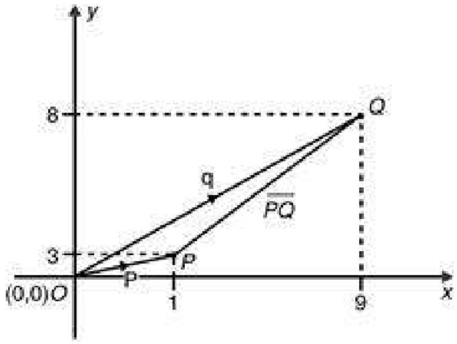

Fig. 8.22

View vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{PQ}$, as if they are moving around the triangle OPQ in figure 8.22. You will notice that vectors $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ seem to be moving anticlockwise while $\overrightarrow{OQ}$ is moving clockwise. Therefore, $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ (triangular law of vectors (forces)).

Note that vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{PQ}$ are really **not** *moving around the triangle, this explanation was made for you understand the trianglular law of vectors (forces). Vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{PQ}$ are acting straight, as indicated in Figure 8.22.*

Hence, $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$. Recall that $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$, so that

$\mathbf{p} + \overrightarrow{PQ} = \mathbf{q}$; $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = 9i + 8j - (i + 3j) =$
$9i - i + 8j - 3j$.

Therefore, $\overrightarrow{PQ} = 8i + 5j$.

Note that if you had been asked to find $\overrightarrow{QP}$, then $\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$, and **not** $\mathbf{q} - \mathbf{p}$.

(b) $\mathbf{p}.\mathbf{q} = |\mathbf{p}|\,|\mathbf{q}| \cos\theta = (i + 3j).(9i + 8j) = (1 \times 9) + (3 \times 8)$; $\mathbf{p}.\mathbf{q} = |i + 3j|\,|9i + 8j| \cos\theta = (1 \times 9) + (3 \times 8)$, where θ is the angle between vectors $\mathbf{p}$ and $\mathbf{q}$. $(\sqrt{1^2 + 3^2})(\sqrt{9^2 + 8^2}) \cos\theta = 9 + 24$; $\sqrt{10}\,\sqrt{145}$

$\cos\theta = 33$;

$$\cos\theta = \frac{33}{\sqrt{10}\,\sqrt{145}} = \frac{33}{38.08} = 0.8666;$$

$$\theta = \cos^{-1} 0.8666 = 30°.$$

Therefore, the angle between **p** and **q** is 30º.

(c) To determine the length (magnitude) of the projection of vectors **A** on vector **B**, simply find the dot product of vector **A**, and the unit vector, in the direction of vector B, on which **A** is to be projected. Therefore, the length of the projection of **A** on B = **A** . $\hat{\mathbf{B}}$, where $\hat{\mathbf{B}}$ is the unit vector, in the direction of vector B. Thus, the projection of **q** on **p** will be $\mathbf{q} \cdot \hat{\mathbf{p}}$.

$$\hat{\mathbf{p}} = \frac{1}{|\mathbf{P}|}(\mathbf{p}) = \frac{1}{\sqrt{1^2+3^2}}(i+3j) = \frac{1}{\sqrt{10}}(i+3j) =$$

$$\frac{i}{\sqrt{10}} + \frac{3j}{\sqrt{10}};\text{ thus,}$$

$$\mathbf{q} \cdot \hat{\mathbf{p}} = (9i+8j).\left(\frac{i}{\sqrt{10}} + \frac{3j}{\sqrt{10}}\right) = 9\left(\frac{1}{\sqrt{10}}\right) + 8\left(\frac{3}{\sqrt{10}}\right) =$$

$$\frac{9}{\sqrt{10}} + \frac{24}{\sqrt{10}} = \frac{33}{\sqrt{10}}$$

Rationalising $\frac{33}{\sqrt{10}}$, we get $\frac{33}{\sqrt{10}}$, $= \frac{33}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$ $= \frac{33\sqrt{10}}{10}$.

Therefore, the length of the projection of **q** on **p** is $\frac{33\sqrt{10}}{10}$ *units*.

13. Four coplanar forces of magnitudes $5N$, $5\sqrt{3}\ N$, $10N$, and $10\sqrt{3}\ N$ act on a body in the directions 300º, 330º, 000º and 120º respectively.

    (a) Find the magnitude of the resultant force.

    (b) If the mass of the body is 6 kg, find its acceleration. **(WAEC)**

Workshop

(a)

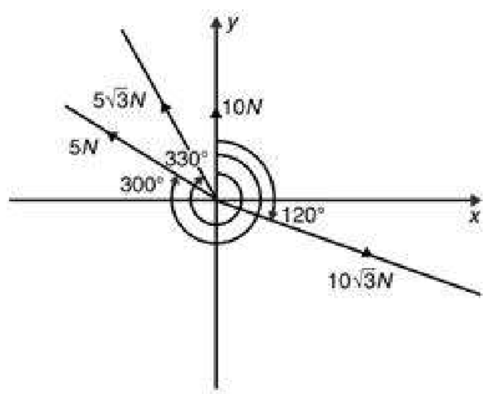

Fig. 8.23(a)

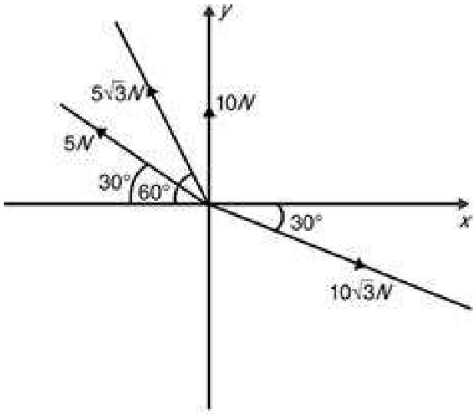

Fig. 8.23(b)

Let R be the magnitude of the resultant of the four forces. The **horizontal** component of resultant, R, can be calculated by summing the horizontal components of the four forces in question.

Therefore, $R_x = 10\sqrt{3}\cos 30° + 10\cos 90° - 5\cos 30° - 5\sqrt{3}\cos 60°$.

> *Please note that the resolution of the 5 and $5\sqrt{3}$ forces to the horizontal (x-axis) are negative, because the resolution of these two forces, to the horizontal, is facing the negative (–x) x-direction. Also, understand that the angles used in the calculation are the angles the forces make with the horizontal (x-axis), as seen in Figure 8.23(b).*

$$R_x = \left(10\sqrt{3} \times \frac{\sqrt{3}}{2}\right) + 10(0) - \left(5 \times \frac{\sqrt{3}}{2}\right) - \left(5\sqrt{3} \times \frac{1}{2}\right)$$

$$= 15 - \frac{5\sqrt{3}}{2} - \frac{5\sqrt{3}}{2}$$

$$= 15 - \left(\frac{5\sqrt{3}}{2} + \frac{5\sqrt{3}}{2}\right) = 15 - \left(\frac{5\sqrt{3} + 5\sqrt{3}}{2}\right)$$

$= 15 - \frac{10\sqrt{3}}{2} = 15 - 5\sqrt{3}$. Recall that $5x + 5x = 10x$

$\therefore 5\sqrt{3} + 5\sqrt{3} = 10\sqrt{3}$. Therefore, $R_x = 15 - 5\sqrt{3}$.

$R_y = -10\sqrt{3}\sin 30° + 10\sin 90° + 5\sin 30° + 5\sqrt{3}\sin 60°$;

$$R_y = -\left(10\sqrt{3} \times \frac{1}{2}\right) + 10(1) + \left(5 \times \frac{1}{2}\right) + \left(5\sqrt{3} \times \frac{\sqrt{3}}{2}\right)$$

$$= -5\sqrt{3} + 10 + \frac{5}{2} + \frac{15}{2} = 20 - 5\sqrt{3}.$$

> *Note that the resolution of the $10\sqrt{3}$ force to the vertical (y-axis) is negative, because the resolution of this force, on the vertical, is facing the negative y (–y) direction.*

$$R = \sqrt{R_y^2 + R_x^2} = \sqrt{(20 - 5\sqrt{3})^2 + (15 - 5\sqrt{3})^2}$$

$$= \sqrt{400 - 200\sqrt{3} + 75 + 225 - 150\sqrt{3} + 75};$$

$$R = \sqrt{775 - 350\sqrt{3}} = \sqrt{775 - 606.22};$$

$$R = \sqrt{168.78} = 12.99N.$$

Therefore, the magnitude of the resultant of the four forces is $12.99N$.

(b) The combined effect of the four forces on the body, is the resultant force, R, of the four forces. Hence, the single force, causing the body to move, is $R = F = ma$;

$$R = ma;\ a = \frac{R}{m} = \frac{12.99}{6} = 2.2ms^{-2}.$$

Therefore, the acceleration of the body is $2.2ms^{-2}$, in the direction of the resultant force, R.

14. The vectors $U = (i + 8j)$ and $V = (7i + 4j)$ represent two adjacent sides of a parallelogram with one vertex at the origin O. Find:

(a) the coordinates of the fourth vertex.

(b) the angles of the parallelogram.

(c) the length of the longer diagonal. **(WAEC)**

Workshop

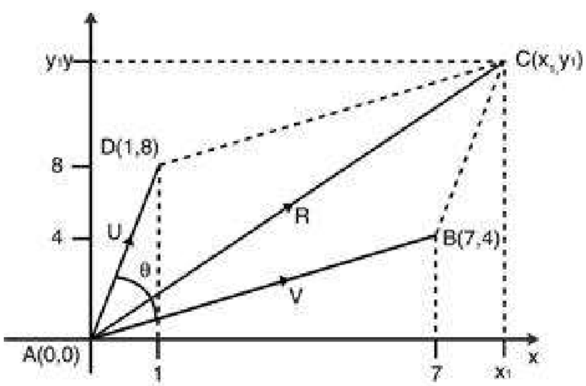

Fig.8.24(a)

Recall that a vector is defined by its magnitude and direction, therefore, vector U can be redrawn (having the same magnitude and direction) on the opposite side of the parallelogram, as shown in Figure 8.24(b).

Looking critically at Figure 8.24(b), it can be seen that vector U is defined by the same magnitude (shown, in this case, by its length) and in the same direction, θ, with respect to vector V.

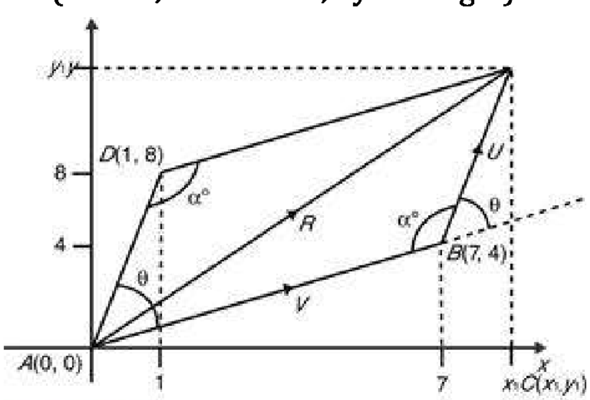

Fig.8.24(b)

(a) From Figure 8.24(b), R is the resultant of vectors U and V;

Look carefully at triangle ABC in Figure 8.24(b), you will notice that $R = U + V$ (trianglular law of vectors (or forces)).

To be sure that R is truly equal to U + V, look at Figure 8.24(b) to see that the sense (direction of the arrow) of resultant R is different from the sense of vectors U and V. While R seems to be directed clockwise, U and V seem to be directed in the anticlockwise direction.

In addition, know that the 'clockwise' and 'anticlockwise' was used to better explain the solution, in the real sense the vectors are not moving around the triangle, but they are acting in straight lines as shown in Figure 8.24(a) and (b). However, the explanation was made to enable you understand the triangular law of vectors (or forces).

Also, note that you can add U to V, to get R, ***if, and only if****, U and V are in the i, j form.*

Hence, $R = U + V = (i + 8j) + (7i + 4j) = 8i + 12j$.

So, the vector $R = 8i + 12j$*. Since R starts from the origin, (0, 0), the coordinates of R will be (8, 12). Therefore, the coordinates of the fourth vertex of theparallelogram is* $(x_1, y_1) = (8, 12)$.

(b) Recall that $U \,.\, V = |U|\,|V| \cos\theta$, where θ is the acute angle between vectors U and V.
Also, $U.V = (i + 8j).(7i + 4j)$, hence,

$U.\ V = (i + 8j).\ (7i + 4j) = |U||V| \cos;\theta$

$1(7) + 8\ (4) = \sqrt{1^2 + 8^2}\ \sqrt{7^2 + 4^2}\ \cos\theta$

$= 7 + 32\ \sqrt{65}\ \sqrt{65}\ \cos\theta = 65\ \cos\theta$

$39 = 65\ \cos\theta;\ \cos\theta\ \frac{39}{65} = 0.6;$
$\theta = \cos^{-1} 0.6 = 53°.$

The opposite angles of a parallelogram are equal, then the angle opposite θ in the parallelogram is also 53°. Let the other two opposite angles of the parallelogram be a°, as these opposite sides are also equal; thus, 53° + 53° + a° + a° = 360° (sum of internal angles of a parallelogram)

$106° + 2a° = 360°;\ 2a° = 254°;\ a° = \frac{254°}{2} = 127°.$

Therefore, the angles of the parallelogram are 53°, 127°, 53° and 127°.

(c) The length of the longer diagonal willbe the length of vector R, which is the magnitude of R. Recall that R = 8i + 12 j; therefore, the magnitude of

$R = |R| = \sqrt{8^2 + 12^2} = \sqrt{64 + 144} = \sqrt{208} =$ 14.4 units

So, the length of the longer diagonal is 14.4 *units*.

Note that the final answer was written as 14.4units because the unit of the length of the diagonal – whether it is in Newton, Pascal ms^{-1}*, or some other units of vectors – is* ***not*** *known.*

15. Given that $p = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$ and $q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, find a vector such that $|U| = 35$ and U is in the direction of $p - 3q$. **(WAEC)**

Workshop

$p = \begin{pmatrix} -12 \\ 5 \end{pmatrix}, q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, |U| = 35.$

Recall that a vector, U, can be known by multiplying the unit vector in its direction (i.e, $\hat{U}$) by the magnitude of U (i.e $|U|$). Thus, $U = |U|\ \hat{U}$.

Because the vector U is in the direction of vector $p - 3q$, the unit vector in the direction of vector $p - 3q$, will be the unit vector in the direction of vector U. Then, we can calculate U, by multiplying the unit vector in the direction of $p - 3q$, by the magnitude of vector U.

$$p - 3q = \begin{pmatrix} -12 \\ 5 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -12 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -12 - 3 \\ 5 - -3 \end{pmatrix}$$

$= \begin{pmatrix} -15 \\ 8 \end{pmatrix}$. Recall that the unit vector in the direction of any vector A is given by $\frac{1}{|A|}(A)$, so,

the unit vector in the direction of $p - 3q$, will be $\frac{1}{|p - 3q|}(p - 3q)$;

$|p - 3q| = \sqrt{(-15)^2 + 8^2} = \sqrt{289} = 17$. Unit vector in $p - 3q$ direction will be

$$\frac{1}{17}(p - 3q) = \frac{1}{17}\begin{pmatrix} -15 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{-15}{17} \\ \frac{8}{17} \end{pmatrix}.$$ Given a vector A, having magnitude $|A|$, the unit vector $\hat{A}$, in the direction of, A is given by $\hat{A} = \frac{1}{|A|}(A) = \frac{A}{|A|}$, so that $A = |A|\hat{A}$.

Therefore, $U = |U|\,\hat{U} = |U|\begin{pmatrix} \frac{-15}{17} \\ \frac{8}{17} \end{pmatrix} = (35) - \begin{pmatrix} \frac{-15}{17} \\ \frac{8}{17} \end{pmatrix}$

$= \begin{pmatrix} 35\left(-\frac{15}{17}\right) \\ 35\left(\frac{8}{17}\right) \end{pmatrix} = \begin{pmatrix} \frac{-525}{17} \\ \frac{280}{17} \end{pmatrix}$.

16. The position vectors of points P and Q relative to the origin are: **p** = –i – 4j, **q** = 2i – 4j.

(a) Find a unit vector parallel to the resultant of **p** and **q**.

(b) Calculate | 2**p** – 3**q** | , leaving your answer in surd form. **(WAEC)**

Workshop

(a) As **p** and **q** are position vectors, vectors **p** and **q** can each be represented as shown in Figure 8.25.

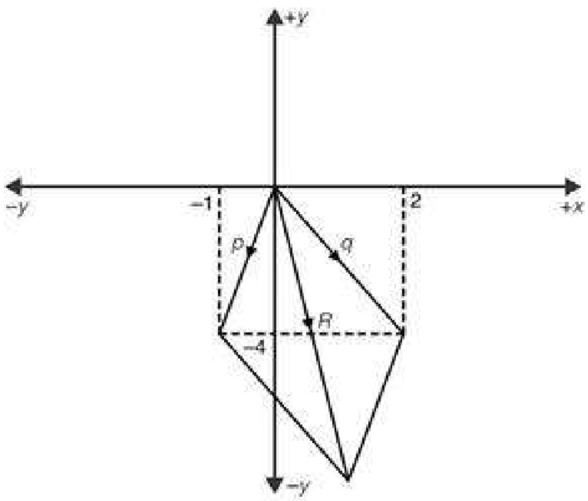

Fig.8.25

From figure 8.25, because p and q are in vector form, the resultant, R, of p and q will be expressed as R = p + q = $-i - 4j + (2i - 4j)$; R = $-i + 2i - 4j - 4j = i - 8j$.

Note that if the magnitudes of **p** *and* **q** *alone were given, the magnitude of* **R** *would have been calculated as* $\mathbf{R}^2 = \mathbf{p}^2 + \mathbf{q}^2 - 2\mathbf{pq}\cos\theta$*, and* θ *would have been the angle opposite vector* **R**.

The unit vector parallel to the resultant, **R**, is the unit vector, $\hat{\mathbf{R}}$ in the direction of vector **R**, which can be calculated as

$\hat{\mathbf{R}} = \frac{1}{|\mathbf{R}|}(\mathbf{R})$. $|\mathbf{R}| = \sqrt{1^2 + (-8)^2} = \sqrt{1 + 64}$

$= \sqrt{65}$; $\hat{\mathbf{R}} = \frac{1}{\sqrt{65}}(i - 8j) = \frac{i}{\sqrt{65}} - \frac{8j}{\sqrt{65}}$.

Therefore, the unit vector parallel to the resultant of **p** and **q** is $\frac{i}{\sqrt{65}} - \frac{8j}{\sqrt{65}}$.

(b) $|2\mathbf{p} - 3\mathbf{q}| = |2(-i - 4j) - 3(2i - 4j)|$

$$= |-2i - 8j - 6i + 12j|$$

$$= |-8i + 4j| = \sqrt{(-8)^2 + 4^2}$$

$$= \sqrt{64 + 16} = \sqrt{80}$$

$$= \sqrt{16 \times 5} = 4\sqrt{5}.$$

Therefore, $|2\mathbf{p} - 3\mathbf{p}| = 4\sqrt{5}$ *units*.

17. The position vectors of points A and B relative to the origin O are $\overrightarrow{OA} = i + 7j$ $\overrightarrow{OB} = 5i + 5j$.

(a) Show that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ have equal magnitudes.

(b) If points C and D have position vectors given by:

$\overrightarrow{OC} = 2\overrightarrow{OA}$,

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB}$,

express, in terms of i and j

(i) $\overrightarrow{OC}$,

(ii) $\overrightarrow{OD}$.

(c) Calculate the angle between $\overrightarrow{OD}$ and $\overrightarrow{BC}$.

(d) Find a unit vector in the direction of $\overrightarrow{DO}$.

(WAEC)

Workshop

(a) $\overrightarrow{OA} = i + 7j$; $|\overrightarrow{OA}| = \sqrt{1^2 + 7^2} = \sqrt{1 + 49}$
$= \sqrt{50}$;

$\overrightarrow{OB} = 5i + 5j$;

$|\overrightarrow{OB}| = \sqrt{5^2 + 5^2} = \sqrt{25 + 25} = \sqrt{50}$.

Thus, $|\overrightarrow{OA}| = |\overrightarrow{OB}| = \sqrt{50}$ *units*.

Note that it is safe to write $\sqrt{50}$ units since it is not known whether the vector is a force, velocity, momentum, or any other vector.

(b) (i) Point C has position vector given by $\overrightarrow{OC} = 2\overrightarrow{OA}$, $\overrightarrow{OC}$ can be expressed in terms of i and j as $\overrightarrow{OC} = 2\overrightarrow{OA} = 2(i + 7j) = 2i + 14j$.

(ii) Point D has position vector given by $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB}$, $\overrightarrow{OD}$ can be expressed in terms of i and j as $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB} = i + 7j + (5i + 5j) = i + 5i + 7j + 5j = 6i + 12j$

(c)

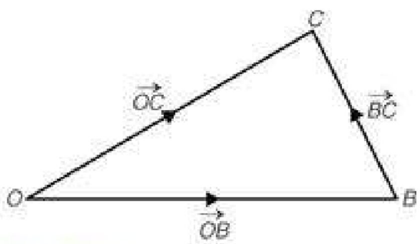

Fig. 8.26

A look at Figure 8.26 shows that $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$, this is because while $\overrightarrow{OC}$ seems to be directed in the clockwise direction, $\overrightarrow{OB}$ and $\overrightarrow{BC}$ seem to be directed in the anticlockwise direction. Hence, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$ (triangular law of vectors).

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{BC} = 2i + 14j - (5i + 5j)$;

$\overrightarrow{BC} = 2i + 14j - 5i - 5j = 2i - 5i + 14j - 5j$
$= -3i + 9j$; therefore,

$\overrightarrow{BC} = -3i + 9j$ and $\overrightarrow{OD} = 6i + 12j$ from b(*ii*) above.

Recall that $\overrightarrow{A \cdot B} = |\vec{A}| + |\vec{B}| \cos\theta$, where θ is the acute angle between vectors $\vec{A}$ and $\vec{B}$.

Hence, $\overrightarrow{OD} \cdot \overrightarrow{BC} = \overrightarrow{OD}\,\overrightarrow{BC} \cos\theta$, where θ is the angle between $\overrightarrow{OD}$ and $\overrightarrow{BC}$.

$$\cos\theta = \frac{\overrightarrow{OD} \cdot \overrightarrow{BC}}{|\overrightarrow{OD}||\overrightarrow{BC}|} = \frac{(6i + 12j) \cdot (-3i + 9j)}{\sqrt{6^2 + 12^2}\,\sqrt{(-3)^2 + 9^2}};$$

$$= \frac{6(-3) + 12(9)}{\sqrt{36 + 144}\,\sqrt{9 + 81}}$$

$$= \frac{-18 + 108}{\sqrt{180} \times \sqrt{90}}$$

$$= \frac{-18 + 108}{\sqrt{180 \times 90}}$$

$$= \frac{90}{\sqrt{2 \times 90 \times 90}}$$

$$= \frac{90}{90\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

Therefore, the angle between $\overrightarrow{OD}$ and $\overrightarrow{OB}$ is 45°.

(d) Vector $\overrightarrow{OD} = 6i + 12j$

Know that vector $\overrightarrow{OD} = -(\overrightarrow{OD})$

$= -(6i + 12j) = -6i - 12j$.

The unit vector in the direction of $\overrightarrow{OD}$ is expressed as

$$\hat{DO} = \frac{1}{|\overrightarrow{OD}|}(\overrightarrow{DO})$$

$$= \frac{1}{\sqrt{(-6)^2 + (-12)^2}}(-6i - 12j)$$

$$= \frac{1}{\sqrt{36 + 144}}(-6i - 12j);$$

$$\hat{DO} = \frac{1}{\sqrt{180}}(-6i - 12j)$$

$$= \frac{1}{6\sqrt{5}}(-6i - 12j)$$

$$= \frac{-6}{6\sqrt{5}}i - \frac{12}{6\sqrt{5}}j$$

$$= \frac{-1}{\sqrt{5}}i - \frac{2}{\sqrt{5}}j$$

Therefore, the unit vector in the direction of $\overrightarrow{DO}$ is $\hat{DO} = \frac{-1}{\sqrt{5}}i - \frac{2}{\sqrt{5}}j$.

18. The position vectors of the points A, B, C and H are $3i + 2j$, $2i + 3j$, $3i - 2j$, and $8i + 3j$ respectively. If D is the midpoint of BC,

(a) find the position vector of D;

(b) show that $|\overrightarrow{AH}| = 2\,|\overrightarrow{OD}|$ where O is the origin. **(WAEC)**

Workshop

The position (*location*) of any point P can be shown by its position vector relative to an origin O. This vector is called $\overrightarrow{OP}$. Threfore, the position of point $P(x_1, y_1)$ can be shown by the position vector of the point relative to origin $O(0, 0)$ given by $\overrightarrow{OP} = (x_1 - 0)\hat{i} + (y_1 - 1)\hat{j} = x_1\hat{i} + y_1\hat{j}$ where $\hat{i}$ is the unit vector in the O-x direction and $\hat{j}$ is the unit vector in the O-y direction as drawn in Figure 8.27.

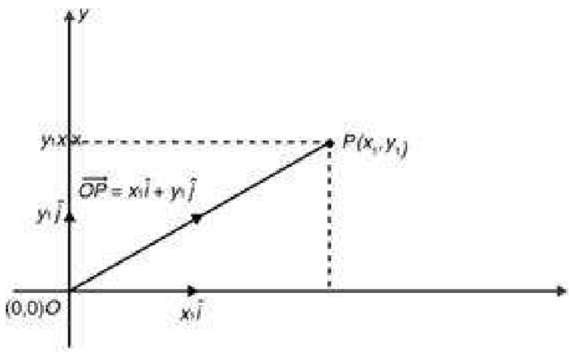

Fig 8.27

> *Note that the coefficient of unit vector $\hat{i}$ must come first, when writing the co-ordinates of P. This is because $x_1\hat{i}$ is the component of the position vector of point, P, on the x-axis*

$A = 3i + 2j$; $B = 2i + 3j$; $C = 3i - 2j$; $H = 8j + 3j$. Given that the position vector of a point P is $x_1\hat{i} + y_1\hat{j}$, then, the coordinate of point P will be (x_1, y_1).

Hence, the co-ordinates of points A, B, C and H are $A(3, 2)$, $B(2, 3)$, $C(3, 2)$ and $H(8, 3)$ respectively.

(a) We know that point B has co-ordinates $(2, 3)$ and point C has coordinates, $(3, -2)$, and the question informs that D is the mid point of BC, then, the co-ordinate of point D can be calculated as follows: the coordinate (x_m, y_m) of the midpoint of a line joining two points (x_p, y_p) and (x_q, y_q) is given by $x_m = \dfrac{x_p + x_q}{2}$ and $y_m = \dfrac{y_p + y_q}{2}$.

$$x_d = \frac{x_b + x_c}{2} = \frac{2 + 3}{2} = \frac{5}{2};$$

$$y_d = \frac{y_b + y_c}{2} = \frac{3 + (-2)}{2} = \frac{3 - 2}{2} = \frac{1}{2}.$$

The coordinates of point D are $\left(\frac{5}{2}, \frac{1}{2}\right)$. The position of D relative to the origin, (0, 0), will be $\left(\frac{5}{2}-0\right)i+\left(\frac{1}{2}-0\right)j=\frac{5}{2}i+\frac{1}{2}j$.

Therefore, the position vector of D will be $\frac{5}{2}i+\frac{1}{2}j$.

(b) Given two points, $p(x_p, y_p)$ and $q(x_q, y_q)$, the vector $\overrightarrow{pq}$ along line pq is given by $\overrightarrow{pq}=(x_q-x_p)i+(y_q-y_p)j$ while $\overrightarrow{qp}=(x_p-x_q)i+(y_p-y_q)j$.

Given points A (3, 2) and H (8, 3), then vector $\overrightarrow{AH}=(8-3)i+(3-2)j=5i+j$.

$|\overrightarrow{AH}|=\sqrt{5^2+1^2}=\sqrt{26}$.

Note that $\overrightarrow{pq}=(x_q-x_p)i+(y_q-y_p)j$ *and* **not** $(x_p-x_q)i+(y_p-y_q)j$.

Given points, $O(0, 0)$ (*origin*) and $D\left(\frac{5}{2}, \frac{1}{2}\right)$, position vector $\overrightarrow{OD}$ will be

$\overrightarrow{OD}=\left(\frac{5}{2}-0\right)i+\left(\frac{1}{2}-0\right)j=\left(\frac{5}{2}\right)i+\left(\frac{1}{2}\right)j$.

Magnitude, $|\overrightarrow{OD}|=\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{25}{16}+\frac{1}{4}}$

$+\sqrt{\frac{26}{4}}=\frac{\sqrt{26}}{\sqrt{4}}=\sqrt{\frac{26}{2}}$. $2|\overrightarrow{OD}|=2\left(\frac{\sqrt{26}}{2}\right)$

$=\sqrt{26}=|\overrightarrow{AH}|$. Therefore, $\overrightarrow{AH}=2|\overrightarrow{OD}|$.

19. A body of mass 5 kg resting on a smooth horizontal plane, is acted upon by forces $6i + 2j$, $5i + 4j$ and $4i - j$. Calculate the;

(a) velocity of the body;

(b) magnitude of its velocity, after 4 seconds. **(WAEC)**

Workshop

(a) $m=5kg$, $\vec{F}_1=6i+2j$, $\vec{F}_2=5i+4j$, $\vec{F}_3=4i-j$; to know the velocity of the body, we need to know first, a single force, the effect of which will be equal to that of the three forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$ acting on the body. This single force is the resultant force, $\vec{F}_R$. Since $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$

are in vector form (that is the i, j form), resultant, $\vec{F}_R$ can be calculate as follows:

$\vec{F}_R = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = (6i + 2j) + (5i + 4j) + (4i - j) = 6i + 2j + 5i + 4j + 4i - j;$

$= 6i + 5i + 4i + 2j + 4j - j = 15i + 5j.$

Note that the acceleration and velocity of a body are vector quantities because they have magnitude and direction.

Recall that $F = ma$, this formula shows the relationship between the magnitude of force, the magnitude of acceleration and a scalar quantity which is the mass m. The vectors $\vec{F}$ and $\vec{a}$ with the scalar quantity m are also related as $\vec{F} = \overrightarrow{ma}$.

Hence, $\vec{a} = \frac{\vec{F}_R}{m} = \frac{15i + 5j}{5} = \frac{15}{5}i + \frac{5}{5}i$

$= 3i + j.$

The equation of motion $v = u + at$ can be written in vector form as $\vec{v} = \vec{u} + \vec{a}t$, recall that time, t is a scalar quantity. From the question, the body was resting on the plane so that initial velocity, $\vec{u} = 0$; then, $\vec{v} = 0 + \vec{a}t = \vec{a}t$.

So, velocity $\vec{v}$ of the body after 4 seconds is given by $\vec{v} = \vec{a}t = (3i + j)4$
$= 12i + 4j.$

Therefore, the velocity of the body after traveling for 4 *seconds* is $12i + 4j$.

(b) The magnitude of the velocity will be

$\overrightarrow{|v|} = \sqrt{12^2 + 4^2} = \sqrt{144 + 16} = \sqrt{160}$

$= \sqrt{16 \times 10} = \sqrt{16}\sqrt{10} = 4\sqrt{10}\ ms^{-1}$

Therefore, the magnitude of the velocity of the body after 4 seconds is $4\sqrt{10}\ ms^{-1}$.

Note that we were able to write the unit of the magnitude of the velocity in ms^{-1} because mass and time given in the question are in S. I units (kg and seconds respectively).

20. Find, correct to one decimal place, the angle between $P = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $q = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

(WAEC)

Workshop

Please note that p and q are vectors, and vectors are also represented in the way p and q are presented in the question. So, do not make a mistake of calling p and q column matrices.

Recall that, the scalar (dot) product of the vectors p and q is expressed as follows:

$p \cdot q = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = |p|\,|q| \cos\theta$. where θ is the acute angle between vectors p and q. Recall that acute angles are angles less that 90°;

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = (3 \times 3) + (-1 \times 4) = 9 + (-4) = 9 - 4 = 5.$$

Note that if vector $A = \begin{pmatrix} m \\ n \end{pmatrix}$ and $B = \begin{pmatrix} r \\ s \end{pmatrix}$,
$A \cdot B = \begin{pmatrix} m \\ n \end{pmatrix} \cdot \begin{pmatrix} r \\ s \end{pmatrix} = [(m \times r) + (n \times s)]$.

$p = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$; $|p| = \sqrt{3^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$;

$q = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$; $|q| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$.

Hence, $p \cdot q = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5 = |p|\,|q| \cos\theta = (\sqrt{10})(5)\cos\theta$;

$5 = (\sqrt{10})(5)\cos\theta$; $\cos\theta = \dfrac{5}{5\sqrt{10}}$; $\cos\theta = \dfrac{1}{\sqrt{10}} = 0.3162$;

$\theta = \cos^{-1} 0.3162 = 71.6°$. Therefore, the acute angle between p and q correct to one decimal place is $71.6°$.

Note that to find the value of θ as explained above it is good to calculate the value of cosθ to at least 4 decimal places as we did so as to make the final answer (value of θ) more accurate.

21. $ABCD$ is a square with vertices at $A(0, 0)$, $B(2, 0)$, $C(2, 2)$ and $D(0, 2)$. Forces of magnitudes $10N$, $15N$, $20N$, and $5N$ act along $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$ and $\overrightarrow{AD}$ respectively.

Find the: (a) magnitude; (b) direction; of their resultant. (WAEC)

Alternative method

(a) Let us show these forces by first drawing the x and y-axis showing the points A, B, C, and D and then represent the forces on lines joining these points.

lines joining these points.

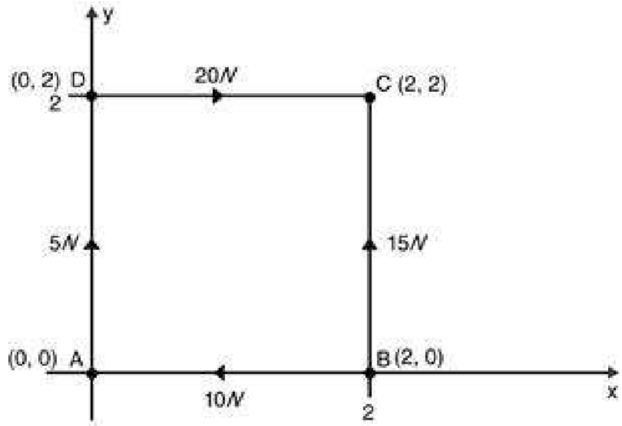

Fig 8.28

*Note that while vector $\overrightarrow{MN}$ is represented as $M \longrightarrow N$, vector $\overrightarrow{NM}$ will be represented as $M \longleftarrow N$. You can now see why vector $\overrightarrow{BA}$ has an arrow facing A and **not** B. This also applies to other vectors drawn in Figure 8.28.*

Now, if these four forces act on a body as represented in Figure 8.28, (in magnitude and direction), the body will move with a force $\vec{R}$ in a direction θ. This force $\vec{R}$ is the resultant force of the four forces while θ is the direction of the resultant force. To know vector $\vec{R}$, we need to know its horizontal component R_x and its vertical component R_y. From Figure 8.28, $ABCD$ is a square with two of its sides parallel to the x-axis and the other two sides parallel to the y-axis. Thus, vectors $\overrightarrow{DC}$ and $\overrightarrow{BA}$ are both parallel to the x-axis so that, $R_x = \Sigma F_x = 20 + (-10) = 20 - 10 = 10N$.

Note that a negative sign assigned to the 10N force because it is pointing to the negative x-axis.

Also, since vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are both parallel to the y-axis,

$R_y = \Sigma F_y = 5 + 15 = 20N.$

Positive signs are allotted to both forces because they are both pointing in the direction of the positive y-axis.

Magnitude of the resultant, $\vec{R}$, is expressed as $|\overrightarrow{R}|$

$= \sqrt{R_x^2 + R_y^2}$

$= \sqrt{(\Sigma F_x^2) + (\Sigma F_y^2)} = \sqrt{10^2 + 20^2} = \sqrt{100 + 400}$

$= \sqrt{500} = 22.36N.$

Therefore, the magnitude of the resultant of the four forces is $22.36N$.

(ii) $R_x = +10N$ while $R_y = +20N$. These two forces can be plotted as shown in figures 8.29(a) and (b) below.

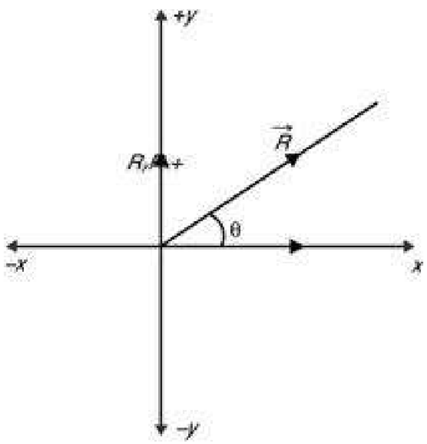

Fig. 8.29(a)

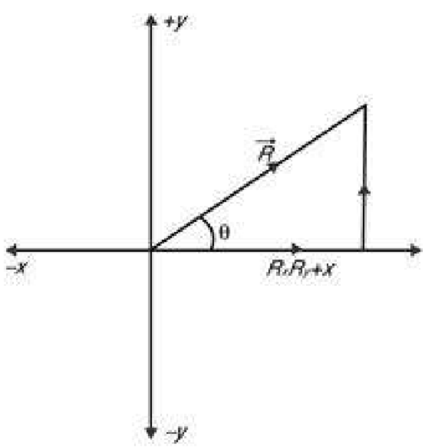

Fig. 8.29(b)

R_y is pointing in the positive y-axis direction because R_y is positive (+20N), also, R_x is pointing in the direction of positive x-axis because R_x is positive (+10N).

Therefore, from figures 8.29(a) and (b) above, the angle that resultant $\vec{R}$ makes with the x-axis is given by $\tan\theta = \frac{Opposite}{Adjacent} = \frac{R_y}{R_x}$; $\tan\theta = \frac{R_y}{R_x}$;

$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{20}{10}\right)$

$= \tan^{-1} 2 = 63.43°.$

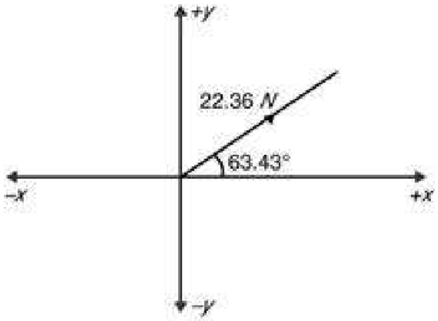

Fig. 8.30

Therefore, the direction of the resultant is the angle 63.43° that it makes with the positive *x*-axis as shown in Figure 8.30. Then again, you can state the direction of *R* as 90° − 63.43° = 63.43° measured clockwise from the positive *y*-axis.

> *Before you write the final answer of the direction of the resultant force in problems like this, try to make a final sketch of the curve to show the quadrant where the resultant force is acting as shown in Figure 8.30 above. For example, if $R_y = -12$ N and $R_x = 5$ N, the direction of $\overrightarrow{R}$ can be known by drawing R_y and R_x on the x and y-axis as below.*

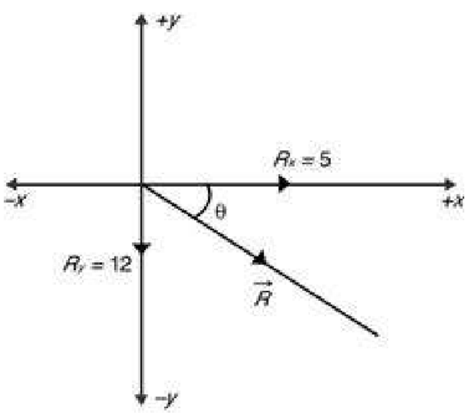

Fig. 8.31 a

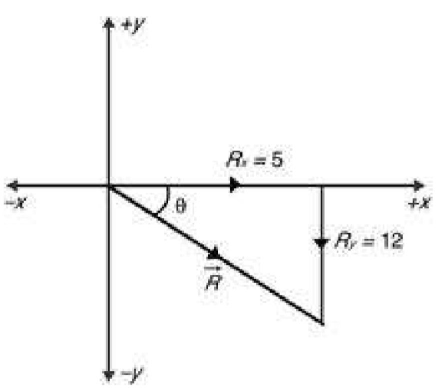

Fig. 8.31 (b)

Since *Ry* is negative, it will point in the direction of the negative *y*-axis as in the diagram. Hence, in this case, $\tan\theta = \frac{R_y}{R_x} = \frac{12}{5}$, this is because R_x and R_y had been drawn facing the right direction, therefore, you do **not** need to include the negative sign in front of 12 to calculate θ **if you are using this method (graphical method)**.

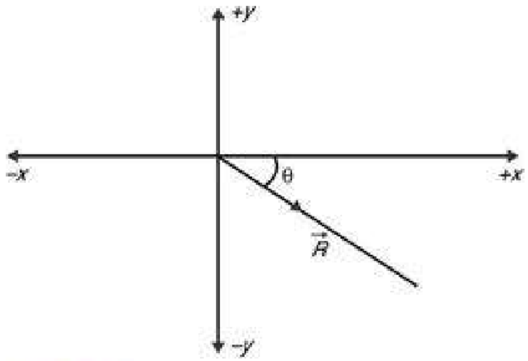

Fig. 8.32

Furthermore, note that, since R_x and R_y have been well represented in the diagram, you do not need to add the negative sign to R_y while calculating direction θ. So, generally, in using this method (graphical method), to calculate θ, even if R_x and R_y are negative, you should not include the negative sign while calculating θ as we did since R_x and R_y have been drawn facing the right direction.

It can be seen that the resultant in this example also makes an angle with the positive *x*-axis (Figure 8.32), but this resultant is acting in another quadrant different from the quadrant the 22.36*N* resultant is acting. Therefore, to make the examiner see that you know what you are doing,

you can quickly make a final sketch of $\vec{R}$ and θ as we did to tell more about the direction, θ. In case you do not want to sketch, the direction of $\vec{R}$ in Figure 8.32 can be reported as $90° + \theta$ measured clockwise from the positive *y*-axis.

22. The position vectors of points *P*, *Q* and *R* are $2i - j$, $i + kj$ and $3i + 2j$ respectively, where *k* is a constant. If the area of triangle *PQR* is 4 square units, find the value of *k*. **(WAEC)**

Workshop

The position of any point *P* can be shown by its position vector relative to an origin *O*. This vector is called $\overrightarrow{OP}$. Thus, the position of point $P(x_1, y_1)$ can be shown by the position vector of the point relative to origin $O(0, 0)$ given by $\overrightarrow{OP} = x_1\hat{i}, + y_1\hat{j}$ where $\hat{i}$ is the unit vector in the O-x direction and $\hat{j}$ is the unit vector in the O-y direction as shown in Figure 8.33.

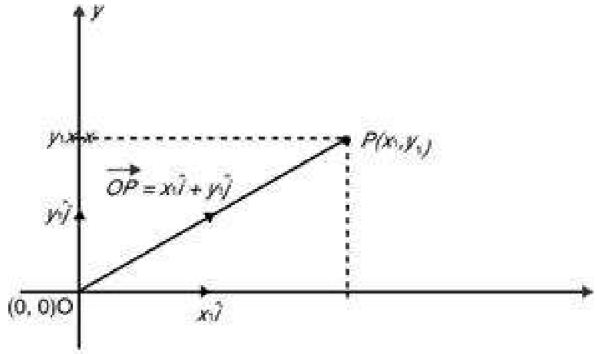

Fig. 8.33

> *Note that $-\hat{i}$ is the unit vector in the negative x $(-x)$ direction while $-\hat{j}$ is the unit vector in the negative y $(-y)$ direction.*

By this explanation, the position vector of point P in question is $2\hat{i} - \hat{j}$ so, the co-ordinates of point P will be $(2, -1)$.

> *Note that the coefficient of unit vector $\hat{i}$ must come first when writing the co-ordinates of point P. This is because $2\hat{i}$ is the component of the position vector of point p on the x-axis.*

By the same explanation, the co-ordinates of point Q will be $(1, k)$ and the co-ordinates of point R will be $(3, 2)$. The area of the triangle formed by joining points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is expressed as follows:

$$Area = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\left(= \frac{1}{2}\begin{bmatrix} x_1[(y_2 \times 1) - (y_3 \times 1)] - y_1[(x_2 \times 1) - \\ (x_3 \times 1)] + 1[x_2 \times y_3) - (x_3 \times y_2)] \end{bmatrix}\right).$$

> *Eh! Do not cram this long formula; it is the same normal method of finding the determinant of a matrix that was applied.*

Hence, the area of triangle PQR will be expressed as :

$$Area = 4 \text{ square units} = \frac{1}{2}\begin{vmatrix} 2 & -1 & 1 \\ 1 & k & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$4 = \frac{1}{2}\begin{bmatrix} 2[(k \times 1) - (2 \times 1)] - (-1)\,[(1 \times 1) - (3 \times 1)] \\ + 1[(1 \times 2) - (3 \times k)] \end{bmatrix}$$

$$4 = \frac{1}{2}\,[2(k-2) + 1(1-3) + 1(2-3k)];$$

$$4 = \frac{1}{2}\,(2k - 4 + 1 - 3 + 2 - 3k);$$

$$4 = \frac{1}{2}\,(-k-4);$$

$$4 = \frac{-k-4}{2}; \quad 4 \times 2 = -k - 4;$$

$$-k - 4 = 8; \quad -k = 8 + 4 = 12;$$

$$k = \frac{12}{-1} = -12.$$

Therefore, the value of k *is* -12.