Mathematics Cheat Sheet

Friend

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1 Numbers

1.1 Natural Numbers $\mathbb{N}$

The set of natural numbers represents the process of counting. Whether or not 0 is part of $\mathbb{N}$ depends on the definition and may vary. If 0 is not included, the set is defined as:

$$\mathbb{N} = \{1, 2, 3, 4, \dots, n, n+1, \dots\} \tag{1.1}$$

How can we perform arithmetic with natural numbers? Addition and multiplication are unrestricted. We say that $\mathbb{N}$ is closed under addition and multiplication. Other operations, such as subtraction and division, are not universally applicable because negative numbers are not part of the natural numbers. A subset of $\mathbb{N}$ is the set of **prime numbers**, defined as:

$$\mathbb{P} = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots\} \tag{1.2}$$

Prime numbers are only divisible by 1 and themselves!

1.2 Integers $\mathbb{Z}$

The set of integers is obtained by extending the natural numbers to include negative numbers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \tag{1.3}$$

Now subtraction is also possible without restriction.

1.3 Rational and Irrational Numbers $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$

To perform unrestricted division, we need fractions:

- $$\mathbb{Q}_+ = \left\{\frac{a}{b} \mid a, b \in \mathbb{N}, b \neq 0\right\} \tag{1.4}$$

Including negative fractions, we get the set of rational numbers:

- $$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\} \tag{1.5}$$

- In $\mathbb{Q}$, all basic arithmetic operations are allowed.
- $\mathbb{Q}$ includes all positive and negative fractions, as well as all terminating decimal fractions (e.g., -3.75) and repeating decimal fractions (e.g., 0.6666...).

One operation is not fully allowed within the rational numbers: taking square roots, since it can lead to infinite numbers that cannot be expressed as fractions. These numbers are called **irrational numbers**, e.g.,

$$\sqrt{2} = 1.41421356\ldots \tag{1.6}$$

1.4 Real Numbers $\mathbb{R}$

By combining the rational and irrational numbers, we get the real numbers $\mathbb{R}$. However, taking square roots of negative numbers is not defined. For example,

$$\sqrt{-4} \tag{1.7}$$

is not defined, and such numbers are not included in $\mathbb{R}$.

1.5 Complex Numbers $\mathbb{C}$

A complex number z is represented as a pair of real numbers:

$$x + iy \mid x, y \in \mathbb{R}, \quad i = \sqrt{-1} \tag{1.8}$$

The important feature of the imaginary unit i is that it allows us to take the square root of negative numbers. A complex number $z \in \mathbb{C}$ can also be written as the pair (x, y), where x is the real part and y is the imaginary part. Thus, the set of complex numbers $\mathbb{C}$ can be geometrically represented as pairs of real numbers (x, y) on the complex plane (also called the Gaussian plane), as shown in the figure below.
The addition of two complex numbers is defined as:

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2) \cdot i \tag{1.9}$$

and the subtraction as:

$$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2) \cdot i \tag{1.10}$$

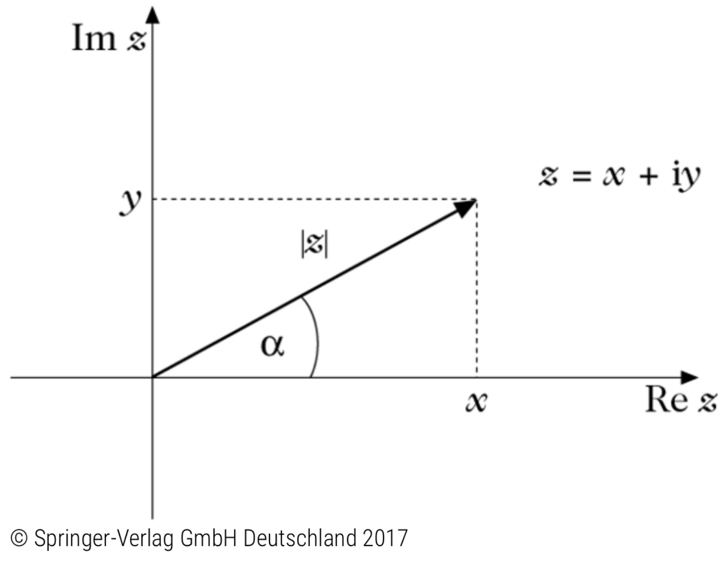

Figure 1.1: Gaussian Plane; $c \in \mathbb{C}$ as a real number pair (x, y)

Multiplication is defined as:

$$z_1 \cdot z_2 = (x_1 + x_2) \cdot (y_1 - y_2) \cdot i = x_1 y_1 + x_1 y_2 \cdot i + y_1 x_2 \cdot i + x_2 y_2 \cdot i^2 \tag{1.11}$$

which simplifies to:

$$z_1 \cdot z_2 = (x_1 y_1 - x_2 y_2) + (x_1 y_2 + y_1 x_2) \cdot i \tag{1.12}$$

If z is a complex number, then z^* is its complex conjugate. In the representation below, the real part of z is reflected. In particular, we have

$$i^* = -i, \quad z^* = x - y \cdot i \tag{1.13}$$

Multiplying by the complex conjugate gives the magnitude of z:

$$|z| = \sqrt{x^2 + y^2} \tag{1.14}$$

The division of complex numbers is defined as:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} \tag{1.15}$$

This operation is a bit cumbersome and can be avoided when possible. If it cannot be avoided, multiply the numerator and denominator separately and simplify using the definition of i.
A different representation is possible using polar coordinates:

$$z = a + i \cdot b \quad a, b, r \in \mathbb{R}, \theta \in [0, 2\pi]| \Leftrightarrow z = r \cdot (\cos(\theta) + i \cdot \sin(\theta)) \Leftrightarrow r \cdot e^{i\theta} \tag{1.16}$$

1.5.1 Special Numbers

π - The Circle Constant

3.1415926535... is the irrational number π[1]. Pi describes the ratio of the circumference to the diameter of a circle. Many formulas involve π:

$$\text{Circumference} \quad U = \pi \cdot d = 2 \cdot \pi \cdot r \tag{1.17}$$

$$\text{Area} \quad A = \pi \cdot r^2 \tag{1.18}$$

$$\text{Volume} \quad V = \frac{4}{3} \cdot \pi \cdot r^3 \tag{1.19}$$

e - Euler's Number

Euler's number is the base of the natural logarithm $\ln(e) = 1$. Euler's number can be approximated as

$$e = 2.71828 \tag{1.20}$$

but like π, it does not have an exact solution. Named after the Swiss mathematician and physicist Leonhard Euler (1707-1783), e is crucial for exponential functions.

[1] Pi's digits omitted for brevity.

2 Fundamentals & Arithmetic Laws

A binary operation can be defined as a way in which two objects determine a third. The operation is abstractly expressed with '$\circ$'. The **law of closure states** that the result of an operation on two elements of a set is also an element of that set. That allows to define operations such as *addition* and *multiplication*. Depending on the **algebraic structure**, operations differ in outcome (e.g. $1+1 \neq 2 \in \mathbb{F}_2 = 0, 1$). Although operations may yield different outcomes, axioms are fundamental and true for most structures. *Note:* Having to prove a certain law holds for a specific case is often required during exercises in analysis.

2.1 Axioms

Axioms establish that algebraic structures have operations (addition and multiplication), and that these operations behave in specific, predictable ways. The most fundamental laws that apply to **most** structures are: See this article for more information.

2.1.1 Commutative Law

The order of the operation does not matter, e.g.,

$$2 \circ 3 = 3 \circ 2 \tag{2.1}$$

.

2.1.2 Associative Law

The grouping of three numbers does not affect the result of the operation, e.g.,

$$(2 \circ 3) \circ 4 = 2 \circ (3 \circ 4) \tag{2.2}$$

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2.1.3 Distributive Law

The handling of parentheses depends on the number set and the type of operation. For addition and multiplication in the set of real numbers $\mathbb{R}$, both operations are distributive. Thus,

$$2 \odot (3 \oplus 4) = 2 \odot 3 \oplus 2 \odot 4 \tag{2.3}$$

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2.1.4 Inequality Laws

Inequalities change depending on the operation:

- Adding/subtracting a constant:

$$a > b \Rightarrow a + c > b + c \tag{2.4}$$

- Multiplying by a positive constant:

$$a > b \Rightarrow a \cdot c > b \cdot c \tag{2.5}$$

- Multiplying by a negative constant reverses the inequality:

$$a > b \Rightarrow a \cdot (-c) < b \cdot (-c) \tag{2.6}$$

2.1.5 Identity Laws

These laws describe the neutral element in an operation:

- For addition:

$$a + 0 = a \tag{2.7}$$

- For multiplication:

$$a \cdot 1 = a \tag{2.8}$$

2.1.6 Inverse Laws

These laws describe how to reverse an operation:

- For addition:

$$a + (-a) = 0 \tag{2.9}$$

- For multiplication:

$$a \cdot a^{-1} = 1, \tag{2.10}$$

  where $a{-}1 = \frac{1}{a}$ and $a \neq 0$

2.1.7 Zero Laws

The number zero has special properties:

$$a \cdot 0 = 0 \tag{2.11}$$

Division by zero is undefined.

2.1.8 Absolute Value Properties

The absolute value represents the distance from zero:

- $$|a| \geq 0 \tag{2.12}$$

- $$|a| = a \tag{2.13}$$

- $$|a \cdot b| = |a| \cdot |b|. \tag{2.14}$$

2.2 The Binomial Formulas

(a)
$$(a+b)^2 = a^2 + 2ab + b^2 \tag{2.15}$$

(b)
$$(a-b)^2 = a^2 - 2ab + b^2 \tag{2.16}$$

(c)
$$(a+b) \cdot (a-b) = a^2 - b^2 \tag{2.17}$$

2.3 Set Operations

In set operations, the following laws hold:

- Commutative:

$$A \cup B = B \cup A \tag{2.18}$$

$$A \cap B = B \cap A \tag{2.19}$$

- Associative:

$$(A \cup B) \cup C = A \cup (B \cup C) \tag{2.20}$$

$$(A \cap B) \cap C = A \cap (B \cap C) \tag{2.21}$$

- Distributive:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{2.22}$$

2.4 Powers

A power is a shorthand notation for repeated multiplication by itself.

- $$a^0 = 1 \tag{2.23}$$

- $$a^1 = a \tag{2.24}$$

- $$a^{-1} = \frac{1}{a} \tag{2.25}$$

- $$a^{-n} = \frac{1}{a^n} \tag{2.26}$$

- $$a^n = \frac{1}{a^{-n}} \tag{2.27}$$

- $$a^p \cdot a^q = a^{p+q} \tag{2.28}$$

- $$a^p : a^q = a^{p-q} \tag{2.29}$$

- $$a^q \cdot b^q = (a \cdot b)^q \tag{2.30}$$

- $$a^q : b^q = (a : b)^q \tag{2.31}$$

- $$(a^p)^q = a^{p \cdot q} \tag{2.32}$$

- $$\frac{a^m}{a^n} = a^{m-n} \tag{2.33}$$

- $$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \tag{2.34}$$

- $$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n} \tag{2.35}$$

Building on the basic exponent rules, we have:

- $$(a \cdot b)^n = a^n \cdot b^n \tag{2.36}$$

- $$\left(\frac{a}{b}\right)^{n} = \frac{a^n}{b^n} \tag{2.37}$$

2.5 Roots

The root of a number, when multiplied by itself, gives the number. By default, this refers to square roots, but higher roots (e.g., cubic roots $\sqrt[3]{x}$) are also possible. In terms of powers, the square root is expressed as

$$\sqrt{x} = x^{\frac{1}{2}} \tag{2.38}$$

and

$$\sqrt[n]{x} = x^{\frac{1}{n}} \tag{2.39}$$

e.g.

$$\sqrt[3]{125} = 125^{\frac{1}{3}}. \tag{2.40}$$

If a power has a solution, then

$$x^n = a \Leftrightarrow x = \sqrt[n]{a} \tag{2.41}$$

as in

$$3^4 = 81 \equiv \sqrt[4]{81} = 3. \tag{2.42}$$

Roots also follow these additional rules:

- Nested roots:

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a} \tag{2.43}$$

- For products:

$$\sqrt[m]{a \cdot b} = \sqrt[m]{a} \cdot \sqrt[m]{b} \tag{2.44}$$

Note: The nth root is the inverse function of the power function x^n.

2.6 Logarithms

Question: What number must I raise a to, to get y? Written as

$$\log_a(x) = y. \tag{2.45}$$

Note: The logarithms of zero and negative numbers are not defined!

The logarithm is the inverse function of the exponential function:

$$f(x) = a^x = y, \quad f^{-1}(y) = \log_a(y) = x. \tag{2.46}$$

Thus, the logarithm provides the exponent of the exponential function to the base a. For the exponential function $f(x) = e^x$ with e as the base, the natural logarithm (ln) is defined as

$$f^{-1} \tag{2.47}$$

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Logarithms have specific rules:

- $$\log_a(1) = 0 \tag{2.48}$$

- $$\log_a(a) = 1 \tag{2.49}$$

- $$\log_a(p \cdot q) = \log_a(p) + \log_a(q) \tag{2.50}$$

- $$\log_a\left(\frac{p}{q}\right) = \log_a(p) - \log_a(q) \tag{2.51}$$

- $$\log_a(p^q) = q \cdot \log_a(p) \tag{2.52}$$

- $$\log_a\left(\sqrt[n]{p}\right) = \frac{\log_a(p)}{n} \tag{2.53}$$

- $$\log_a(p) = \frac{\log_b(p)}{\log_b(a)} \tag{2.54}$$

These additional rules expand on logarithmic behavior:

- Change of base:

$$\log_b(a) = \frac{\ln(a)}{\ln(b)} \tag{2.55}$$

- Reciprocal property:

$$\log_a\left(\frac{1}{b}\right) = -\log_a(b) \tag{2.56}$$

Logarithmic scaling is helpful when data varies significantly or when relative differences between values are important. Logarithmic scaling makes patterns easier to discern.