



# Fuzzy AHP-based multicriteria decision making systems using particle swarm optimization

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## ABSTRACT

This paper presents a fuzzy optimization model to solve multicriteria decision making (MCDM) systems based on a fuzzy analytic hierarchy process (fuzzy AHP). To deal with the imprecise judgments of decision makers, a fuzzy AHP decision making model is proposed as an evaluation tool, where the expert's comparison judgments are translated into fuzzy numbers. Unlike the conventional fuzzy AHP methods, the proposed method drives exact weights from consistent and inconsistent fuzzy comparison matrices, which eliminate the need of additional aggregation and ranking procedures. The proposed method transforms a fuzzy prioritization problem into a constrained nonlinear optimization model. An improved particle swarm optimization (PSO) is applied to solve the optimization model as a nonlinear system of equations. Several illustrative examples using existing fuzzy AHP methods are given to demonstrate the effectiveness of the proposed method.

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## 1. Introduction

Decision making expert systems are often complex and multifaceted. In recent years, tools for modeling decision making have improved significantly, and multicriteria decision making (MCDM) models are widely considered to be very useful in resolving conflicts related to the decision making process. Since Bellman and Zadeh (1970) developed the theory of decision behavior in a fuzzy environment, various methods have been developed for handling multicriteria decision making systems (Beynon, Cosker, & Marshall, 2001; Chen & Chen, 2005; Chen & Lee, 2010; Chen & Wang 2009; Fu, 2008; Hua, Gong, & Xu, 2008; Kahraman & Cebi, 2009; Kulak, 2005; Kwon & Kim, 2004; Kwon, Kim, & Lee, 2007; Lin, Hsu, & Sheen, 2007; Mikhailov, 2003; Tacker & Silvia, 1991; Yager, 1991, 1992).

Multicriteria decision making deals with the problem of choosing the best alternative, that is, the one with the highest degree of satisfaction for all the relevant criteria or goals. In order to obtain the best alternative a ranking process is required. Extensively adopted in MCDM, the analytic hierarchy process (AHP) has successfully been applied to the ranking process of decision making problems (Saaty, 1988). The main advantage of the AHP is its inherent ability to handle intangibles, which are present in any decision making process. Also, the AHP less cumbersome mathematical calculations and, it is more easily comprehended in comparison with other methods. Triantaphyllou and Lin (1996)

and Duran and Aguilo (2007) summarized the following advantages for AHP: (1) it is the only known MCDM model that can measure the consistency in the decision maker's judgments; (2) the AHP can also help decision makers to organize the critical aspects of a problem in a hierarchical structure, making the decision process easy to handle; (3) pairwise comparisons in the AHP are often preferred by the decision makers, allowing them to derive weights of criteria and scores of alternatives from comparison matrices rather than quantify weights/scores directly; (4) AHP can be combined with well-known operation research techniques to handle more difficult problems; (5) AHP is easier to understand and can effectively handle both qualitative and quantitative data. However, in many practical situations, the human preference model is uncertain and decision makers might be reluctant or unable to assign exact numerical values to the comparison judgments. Although the use of the discrete scale of 1–9 for performing pairwise comparative analysis has the advantage of simplicity, a decision maker may find it extremely difficult to express the strength of his preferences and to provide exact pairwise comparison judgments. AHP is thus ineffective when applied to ambiguous problems. Since the real world is highly ambiguous, some researchers apply fuzzy AHP as an extension of conventional AHP and employ fuzzy set theory to handle uncertainty and overcome this limitation.

Recently, Mikhailov (2003) proposed a new fuzzy AHP method namely fuzzy preference programming (FPP) approach for deriving priorities from fuzzy comparison judgments. Cakir and Canbolat (2008) developed a web-based package based on the Mikhailov's prioritization method. However, the steps in the FPP method make it a little complicated from a computational point of view. This

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idea—deriving crisp priorities from fuzzy judgment matrices—is a new way to deal with the prioritization problem from fuzzy reciprocal comparisons in the fuzzy AHP. We describe a simple prioritization method which can derive exact priorities from fuzzy comparison judgments by applying an efficient nonlinear optimization tool for solving the fuzzy optimization model.

There are three objectives of this paper. First, we propose a modified fuzzy optimization model to deal with shortfalls of the AHP method in handling the uncertainties and imprecision of multicriteria decision making systems. Secondly, we construct a fuzzy prioritization method which can derive exact priorities from consistent and inconsistent fuzzy comparison matrices. Thirdly, we apply an improved particle swarm optimization (PSO) method to solve the fuzzy optimization model as a system of nonlinear equations.

## 2. Preliminaries

Multicriteria decision making provides techniques for comparing and ranking different outcomes, even though a variety of criteria with different units are used. This is a very important advantage over traditional decision making methods where all criteria need to be converted to the same unit. Another significant advantage of most MCDM techniques is that they have the capacity to analyze both quantitative and qualitative evaluation criteria together. Several methods exist for MCDM (Vincke, 1992). Among these methods, the most popular ones are ELECTRE (Roy, 1991), PROMETHEE (Brans & Vincke, 1985), TOPSIS (Hwang & Yoon, 1981), and the AHP methods (Saaty, 1980).

Given the advantages of integral structure, simple theory, and ease-of-operation, AHP is a popular tool for MCDM wherein complicated decision problems and non-structural situations are divided into hierarchical elements. The decision problem is constructed as a hierarchical structure in which the overall goal of decision is located at the highest level, and  $m$  alternatives,  $A_1, A_2, \dots, A_m$ , are at the lowest level (Fig. 1). The  $n$  criteria,  $C_1, C_2, \dots, C_n$ , (sub-criteria) form the internal layers of the hierarchy. The preferences of decision makers are elicited in the form of ratios using pairwise comparison matrices. The judgment matrices of criteria or alternatives are defined by rating the relative importance of elements based on a standard scale (where 1 = equally important; 3 = weak importance; 5 = strong importance; 7 = demonstrated importance; 9 = absolute importance). Comparing any two elements at the same level of hierarchy, a decision maker can provide a numerical value  $a_{ij}$  for the ratio of the elements' importance. Each set of comparisons for a level with  $n$  elements requires  $n(n-1)/2$  judgments, which are further used to construct a positive reciprocal comparison matrix  $A = \{a_{ij}\} \in \mathcal{R}^{n \times n}$ , where  $a_{ji} = 1/a_{ij}$ , and  $a_{ij} > 0$ , for  $j = 1, 2, \dots, n$ ,  $i = 1, 2, \dots, n$ . A priority vector  $w = (w_1, w_2, \dots, w_n)^T$  may be obtained from the comparison matrix by applying a priori-

tization method. Several methods for deriving the local weights of criteria and the local scores of alternatives from judgment matrices have been developed (Mikhailov, 2000). At this stage the consistency of each pairwise comparison is checked. A final aggregation of local priorities is performed to rank the alternatives.

In summary, AHP is consistent, structured and intuitive. However, the AHP is criticized for its inability to accommodate uncertainty in the decision making process. Critics argue that it would be cognitively demanding to ask a decision maker to express his/her preference as a discrete numerical value in the pairwise comparison matrices. The main problem lies in the fact that, since all comparisons are done by importance comparisons, it loses the possible nuances of describing criteria as fuzzy sets. Only the relative importance of each criterion to all others is measured. In other word, decision makers often face uncertain and fuzzy cases when considering the relative importance of one element to another. Hence, it is difficult to derive the crisp ratios from pairwise comparison matrices stated above. Fuzzy set based (van Laarhoven & Pedrycz, 1983) approach has been suggested to overcome the inability of AHP to handle uncertainties.

### 2.1. Fuzzy judgments

Fuzzy sets have been applied as an important tool to represent and treat the uncertainty in various situations (Zadeh, 1965). A major contribution of fuzzy set theory is its capability of representing vague or uncertain data in a natural form. This capability is the reason for its success in many applications. A fuzzy set is a class of objects with a continuum of grades of membership. Linguistic terms are represented by membership functions, valued in the real unit interval, which translate the vagueness and imprecision of human thought related to the proposed problem. In the literature, triangular and trapezoidal fuzzy numbers are usually used to capture the vagueness of the parameters related to the topic. The arithmetic operations of these types of fuzzy numbers can be found in Zimmermann (1994). In this study, the triangular fuzzy numbers (TFNs) were used to represent the fuzzy relative importance. A TFN is graphically shown in Fig. 2 and can be described as:

$$\mu_{\tilde{N}}(x) = \begin{cases} \frac{x-l}{m-l}, & l \leq x \leq m \\ \frac{u-x}{u-m}, & m \leq x \leq u \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

in which the parameters,  $l$ ,  $m$ , and  $u$  respectively denote the smallest possible value, the most promising value, and the largest possible value that describe a fuzzy event. The TFN  $\tilde{N}$  is often represented as  $(l, m, u)$ .

Assume that the fuzzy comparison judgments are decided with respect to the linguistic (non-numerical) judgments of decision

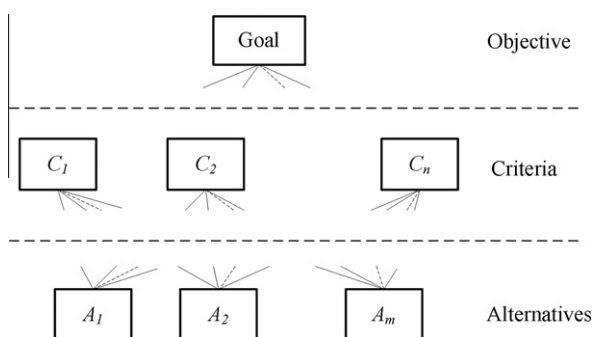


Fig. 1. Hierarchical structure for AHP-based MCDM.

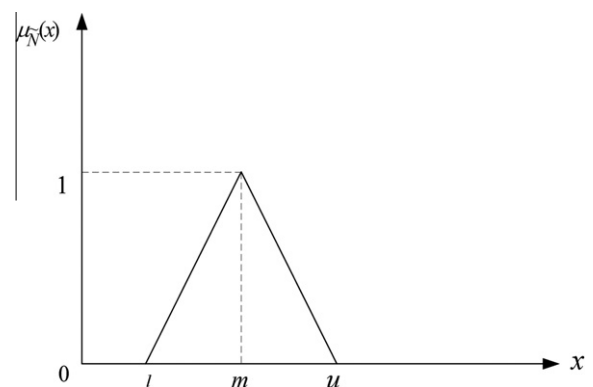


Fig. 2. Triangular fuzzy number.

makers. Their imprecise and uncertain assessments can be translated into corresponding triangular fuzzy numbers. Introduction of fuzzy linguistic variables instead of exact, crisp values will help a decision maker to use non-numerical terms for his/her highly subjective judgments and it can incorporate the imprecision related to the decision maker's preference. Hence, it will eliminate the drawback of the static structure of the fundamental scale in capturing uncertainty in the comparisons. In this paper, we use the fuzzy judgments employed in Wang, Chu, and Wu (2007) as depicted in Table 1.

## 2.2. Fuzzy AHP

Fuzzy AHP uses fuzzy set theory to express the uncertain comparison judgments as a fuzzy numbers. The main steps of fuzzy AHP are as follows:

- (1) *Structuring decision hierarchy.* Similar to conventional AHP, the first step is to break down the complex decision making problem into a hierarchical structure.
- (2) *Developing pairwise fuzzy comparison matrices.* Consider a prioritization problem at a level with  $n$  elements, where pairwise comparison judgments are represented by fuzzy triangular numbers  $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$ . As in the conventional AHP, each set of comparisons for a level requires  $n(n-1)/2$  judgments, which are further used to construct a positive fuzzy reciprocal comparison matrix  $\tilde{A} = \{\tilde{a}_{ij}\}$ , such that:

$$\tilde{A} = \{\tilde{a}_{ij}\} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \dots & \dots & \tilde{a}_{nn} \end{pmatrix} \quad (2)$$

- (3) *Consistency check and deriving priorities.* This step checks for consistency and extracts the priorities from the pairwise comparison matrices. In existing fuzzy AHP methods, only a few past studies have addressed the issue of checking for inconsistencies in pairwise comparison matrices. According to Buckley (1985), a fuzzy comparison matrix  $\tilde{A} = \{\tilde{a}_{ij}\}$  is consistent if  $\tilde{a}_{ik} \otimes \tilde{a}_{kj} \approx \tilde{a}_{ij}$ , where  $i, j, k = 1, 2, \dots, n$ , and  $\otimes$  is fuzzy multiplication, and  $\approx$  denotes fuzzy equal to. Once the pairwise comparison matrix,  $\tilde{A}$ , passes the consistency check, fuzzy priorities  $\tilde{w}_i$  can be calculated with conventional fuzzy AHP methods. Then, the priority vector  $(w_1, w_2, \dots, w_n)^T$  can be obtained from the comparison matrix by applying a prioritization method.
- (4) *Aggregation of priorities and ranking the alternatives.* The final step aggregates local priorities obtained at different levels of the decision hierarchy into composite global priorities for the alternatives based on the weighted sum method. If there are  $i$  alternatives and  $j$  criteria, then the final global priority of alternative  $i$  is given as:

$$A_i = \sum_{j=1}^n w_j a_{ij} \quad (3)$$

where  $w_j$  is the weight of criterion  $j$  and  $a_{ij}$  is the evaluation of alternative  $A_i$  against criterion  $j$ . The higher the value  $A_i$ , the more preferred the alternative. However, if the priorities are fuzzy as in the conventional fuzzy AHP, then an appropriate ranking procedure should be applied to defuzzify the rank of alternatives.

The existing fuzzy AHP methods mainly differ on the employed fuzzy judgments in above-stated step 2 or the developed fuzzy prioritization method in step 3, or both. van Laarhoven and Pedrycz (1983) used a triangular membership function, and developed a fuzzy version of the logarithmic least squares method. Buckley (1985) proposed fuzzy priorities of comparison ratios similar to the proposed approach of Wagenknecht and Hartmann (1983), and Buckley employed trapezoidal membership functions, claiming that such numbers are more easily understood by experts. Boender, de Grann, and Lootsma (1989) proposed an approach for local priority normalization. Chang (1996) introduced an extent analysis method for the synthetic extent values of the pairwise comparisons and applied a simple arithmetic mean algorithm to find fuzzy priorities from comparison matrices, whose elements are represented by triangular fuzzy numbers. Wang, Luo, and Hua (2008), using several numerical examples, showed that the priority vectors determined by the extend analysis method do not represent the relative importance of decision criteria or alternatives. Rather it is a method for showing to what degree the priority of one decision criterion or alternative is bigger than those of all others in a fuzzy comparison matrix. These methods have some common characteristics.

First, they derive priorities from fuzzy comparison matrices. However, the approach of constructing fuzzy reciprocal matrices, taken by analogy from the crisp prioritization methods leads to some problems, as demonstrated in the next section. In addition, in some cases the decision-maker might be unwilling or unable to provide all fuzzy comparisons necessary to construct full comparison matrices.

Secondly, except for Chang (1996), all these methods derive fuzzy priorities and, after aggregating, the final scores of the alternatives are also represented as fuzzy numbers or fuzzy sets. Due to the large number of multiplication and additional operations, the resulting fuzzy scores have wide supports and overlap over a large range. As shown by Boender et al. (1989) and Gogus and Boucher (1997), the normalization procedure used in some of these methods may even result in irrational final fuzzy scores, where the normalized upper value is less than the normalized mean value, which is less than the normalized lower value.

Finally, the fuzzy prioritization methods mentioned above require an additional fuzzy ranking procedure in order to compare the final fuzzy scores. The different ranking procedures, however, often give different ranking results (Bortolan & Degani, 1985).

To overcome the shortcomings of the fuzzy prioritization methods above, Mikhailov (2003) proposed an FPP for deriving priorities from fuzzy comparison judgments that eliminates some of the drawbacks of the existing fuzzy prioritization methods. This approach does not require the construction of complete fuzzy comparison matrices and it can derive priorities from an incomplete set of fuzzy judgments. The proposed approach is also invariant to the specific form of the fuzzy sets used to represent the judgments. By using  $\alpha$ -cuts, the initial fuzzy judgments are transformed into a series of interval judgments. The FPP method is employed to transform the prioritization problem into a fuzzy linear program which can derive crisp priorities from the interval judgments, corresponding to each  $\alpha$ -cuts level, thus eliminating the need for an additional fuzzy ranking procedure. An aggregation process of

**Table 1**  
Fuzzy judgment scores in fuzzy AHP.

Fuzzy judgments	Fuzzy score
About equal	(1/2, 1, 2)
About $x$ times more important	$(x-1, x, x+1)$
About $x$ times less important	$(1/(x+1), 1/x, 1/(x-1))$
Between $y$ and $z$ times more important	$(y, (y+z)/2, z)$
Between $y$ and $z$ times less important	$(1/z, 2/(y+z), 1/y)$

$x = 2, 3, 9$  &  $y, z = 1, 2, \dots, 9$  &  $y < z$ .

the optimal priorities derived at the different  $\alpha$ -level is also needed for obtaining overall crisp scores of the prioritization elements. However, the steps in the FPP method are complicated from a computational point of view. This idea—deriving crisp priorities from fuzzy judgment matrices—shows a new way to deal with the prioritization problem from fuzzy reciprocal comparisons in the fuzzy AHP. In the subsequent sections, we describe a simple fuzzy optimization model which can derive exact priorities from fuzzy comparison judgments.

### 3. Fuzzy optimization model

Suppose that a fuzzy judgment matrix  $\tilde{A}$  is constructed as in Eq. (2). The elements of the judgment matrix are pairwise comparison ratios represented by fuzzy triangular numbers  $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$ , where  $i$  and  $j = 1, 2, \dots, n$ . Moreover, it is assumed that  $l_{ij} < m_{ij} < u_{ij}$  when  $i \neq j$ . If  $i = j$ , then  $\tilde{a}_{ij} = \tilde{a}_{ji} = (1, 1, 1)$ . Hence, an exact priority vector  $(w_1, w_2, \dots, w_n)^T$  derived from  $\tilde{A}$  must satisfy the fuzzy inequalities:

$$l_{ij} \lesseqgtr \frac{w_i}{w_j} \lesseqgtr u_{ij} \quad (4)$$

where  $w_i > 0$ ,  $w_j > 0$ ,  $i \neq j$  and the symbol  $\lesseqgtr$  means “fuzzy less than or equal to”. To measure the degree of satisfaction for different crisp ratios  $w_i/w_j$  with respect to the double side inequality of Eq. (4), a new membership function can be defined as:

$$\mu_{ij}(w_i/w_j) = \begin{cases} \frac{m_{ij} - (w_i/w_j)}{m_{ij} - l_{ij}}, & 0 < \frac{w_i}{w_j} \leq m_{ij} \\ \frac{(w_i/w_j) - m_{ij}}{u_{ij} - m_{ij}}, & \frac{w_i}{w_j} \geq m_{ij} \end{cases} \quad (5)$$

where  $i \neq j$ . Unlike the triangular membership function in Eq. (1), the value of  $\mu_{ij}(w_i/w_j)$  may be larger than one, and it is linearly decreasing over the interval  $(0, m_{ij})$  and linearly increasing over the interval  $[m_{ij}, \infty)$ . A smaller value of  $\mu_{ij}(w_i/w_j)$  indicates that the exact ratio  $w_i/w_j$  is more acceptable.

To find the values of the elements of the priority vector  $(w_1, w_2, \dots, w_n)^T$ , all exact ratios  $w_i/w_j$  should satisfy  $n(n-1)/2$  fuzzy comparison judgments, i.e.  $l_{ij} \lesseqgtr w_i/w_j \lesseqgtr u_{ij}$ , as possible as they can, where  $i$  and  $j = 1, 2, \dots, n$ ,  $i \neq j$ , and  $\sum_{i=1}^n w_i = 1$ . This last requirement is the main constraint of the conventional AHP method. Hence, the problem of crisp priorities assessment in Eq. (5) is transformed into an optimization problem. Assuming that the system of nonlinear equations in Eq. (5) is soluble and its solution is  $(w_1, w_2, \dots, w_n)^T$ , the solution is equivalent to minimizing a master function described as follows:

$$\begin{aligned} \min J(w_1, w_2, \dots, w_n) &= \min \sum_{i=1}^n \sum_{j=1}^n \left[ \mu_{ij}^2 \left( \frac{w_i}{w_j} \right) \right] \\ &= \min \sum_{i=1}^n \sum_{j=1}^n \left[ \delta \left( m_{ij} - \frac{w_i}{w_j} \right) \left( \frac{m_{ij} - (w_i/w_j)}{m_{ij} - l_{ij}} \right)^2 \right. \\ &\quad \left. + \delta \left( \frac{w_i}{w_j} - m_{ij} \right) \left( \frac{(w_i/w_j) - m_{ij}}{u_{ij} - m_{ij}} \right)^2 \right] \end{aligned} \quad (6)$$

subject to:

$$\sum_{k=1}^n w_k = 1, \quad w_k > 0, \quad k = 1, 2, \dots, n \quad (7)$$

where  $i \neq j$  and  $\delta$  is Heaviside function defined as:

$$\delta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad (8)$$

The proposed prioritization model is a constrained nonlinear optimization model. General optimization algorithms limited to convex regular functions cannot be applied to this optimization problem.

Therefore, an efficient optimization algorithm which can solve difficult optimization problems may be employed. We apply PSO to solve the system of nonlinear equations (Eqs. (6)–(8)). The PSO method will be described in the next section.

In some cases, decision makers are unable or unwilling to give all pairwise comparison judgments of  $n$  elements. Provided that the known fuzzy set of pairwise comparisons involve  $n$  elements, such as  $\{\tilde{a}_{12}, \tilde{a}_{13}, \dots, \tilde{a}_{1n}\}$  or  $\{\tilde{a}_{21}, \tilde{a}_{31}, \dots, \tilde{a}_{n1}\}$ , the priority vector  $w = (w_1, w_2, \dots, w_n)^T$  can still be derived using the optimization problem above (Wang et al., 2007). Thus, the proposed method can prioritize an incomplete comparison judgment set, which is an interesting advantage over the conventional fuzzy AHP methods.

In order to measure the degree of consistency of the fuzzy comparison matrix, we follow the method proposed by Mikhailov (2003), in which the consistency can be evaluated using the optimal priority vector  $w^* = (w_1^*, w_2^*, \dots, w_n^*)^T$ . Mikhailov (2003) developed a nonlinear fuzzy prioritization algorithm for his proposed FPP method. The prioritization problem of the FPP method uses two main assumptions. First, if  $\tilde{P}$  is the fuzzy feasible area on the  $(n-1)$ -dimensional simplex  $Q^{n-1}$ , with respect to the optimization problem, the membership function of the fuzzy feasible area is defined as:

$$\mu_{\tilde{P}}(w) = \min \{ \mu_{ij}(w) | i = 1, 2, \dots, n-1; j = 2, 3, \dots, n; j > i \} \quad (9)$$

Second, a selection rule to determine a priority vector with the highest degree of the membership function given in Eq. (9) may be specified. Mikhailov (2003) used a maximization problem as the prioritization method and describes that  $\mu_{\tilde{P}}(w)$  is a convex set and always a priority vector  $w^* \in Q^{n-1}$  could be found that has a maximum degree of membership in  $\tilde{P}$ , which corresponds to the maximum fuzzy feasible area:

$$\mu_{\tilde{P}}(w^*) = \max \min_{ij} \{ \mu_{ij}(w) | w \in Q^{n-1} \} \quad (10)$$

Mikhailov (2003) refers to  $\lambda^* = \mu_{\tilde{P}}(w^*)$  as a “consistency index” and explains that this value is intended to measure the level of satisfaction from the optimized priority vector  $w^*$ . When  $\lambda^*$  is positive, it indicates that all solution ratios completely satisfy the fuzzy judgments, i.e.  $l_{ij} \lesseqgtr w_i^*/w_j^* \lesseqgtr u_{ij}$ , which means that the initial set of fuzzy judgments is rather consistent. The  $\lambda^*$  takes values between 0 and 1 which shows the degree of inconsistency in the decision maker's judgments. The principles we use for developing our prioritization approach are similar to Mikhailov's method. However, with respect to the definition of our new fuzzy number presented in Eq. (5), we try to find a priority vector  $w^* \in Q^{n-1}$  that has a minimum degree of membership, that is:

$$\mu_{\tilde{P}}(w^*) = \min \min_{ij} \{ \mu_{ij}(w) | w \in Q^{n-1} \} \quad (11)$$

Accordingly, we refer to the  $\gamma^* = \mu_{\tilde{P}}(w^*)$  as the consistency index of pairwise comparisons. Later in this paper, we will examine the proposed consistency index  $\gamma^*$ , comparing with  $\lambda^*$ , by performing several illustrative examples.

### 4. Particle swarm optimization

PSO, first proposed by Kennedy and Eberhart (1995), is a simple model of social learning whose emergent behavior has found popularity in solving difficult optimization problems (e.g. Wang, Che, & Wu, 2010). The initial analogy had two cognitive aspects, individual learning and learning from a social group. The original idea was to simulate the social behavior of a flock of birds trying to reach an unknown destination (fitness function), e.g., the location of food resources with flying through the field (search space).



In an  $n$ -dimensional search space,  $S \subseteq R^n$  assume that the swarm consists of  $N$  particles. The position of the  $i$ th particle is an  $n$ -dimensional vector  $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in S$ . The velocity of this particle is also an  $n$ -dimensional vector  $v_i = (v_{i1}, v_{i2}, \dots, v_{in}) \in S$ . The best previous position visited by the  $i$ th particle is a point in  $S$ , denoted as  $p_i = (p_{i1}, p_{i2}, \dots, p_{in}) \in S$ . Let  $g$  be the index of the particle that attained the best previous position among the entire swarm, and  $t$  be the iteration counter. In standard PSO, the swarm is manipulated according to the following update equations

$$v_{id}(t+1) = v_{id}(t) + c_1 r_1 (p_{id}(t) - x_{id}(t)) + c_2 r_2 (p_{gd}(t) - x_{id}(t)) \quad (12)$$

$$x_{id}(t+1) = x_{id}(t) + v_{id}(t+1) \quad (13)$$

where  $i = 1, 2, \dots, N$  is the particle's index,  $d = 1, 2, \dots, n$  indicates the particle's  $d$ th components,  $c_1$  and  $c_2$  are the positive constants referred to as cognitive and social parameters, respectively, and  $r_1$  and  $r_2$  are random numbers uniformly distributed in  $[0, 1]$ , denoted as  $r_1, r_2 \in [0, 1]$ . Consider the  $d$ th dimension of the search space,  $d = 1, 2, \dots, n$ . The right-hand-side of Eq. (12) consists of three parts (Shi, 2004). The first part  $v_{id}(t)$  is the momentum part. The second part is the "cognitive" part which represents personal thinking of itself-learning from its own flying experience. The third part is the "social" part which represents the collaboration among particles-learning from group flying experience. In fact, the sum of the last two parts, i.e.,  $c_1 r_1 (p_{id}(t) - x_{id}(t)) + c_2 r_2 (p_{gd}(t) - x_{id}(t))$ , can be considered as the newly gained velocity term towards a potential position in the promising region around  $p_{gd}(t)$  and  $p_{id}(t)$ . Consequently, summing the momentum part and gained velocity part results in the current velocity  $v_{id}(t+1)$ . However, these two terms do not consider the influence of the feasible region. For example each or both of these two components might be so large that the corresponding particle would leave far away from the feasible region.

One problematic characteristic of PSO is its propensity to converge, prematurely, on early best solutions. Many strategies have been developed in attempts to overcome this but by far the most popular are inertia and constriction (Shi & Eberhart, 1998)

$$v_{id}(t+1) = \omega v_{id}(t) + c_1 r_1 (p_{id}(t) - x_{id}(t)) + c_2 r_2 (p_{gd}(t) - x_{id}(t)) \quad (14)$$

where  $\omega$  is a parameter called inertia weight. Later work (Eberhart & Shi, 2000) indicates that the optimal strategy is to initially set  $\omega$  to 0.9 and reduce it linearly to 0.4, allowing initial exploration followed by acceleration toward an improved global optimum. Constriction (Clerc & Kennedy, 2002),  $\chi$ , alleviates the requirement to clamp the velocity and is applied as follows:

$$v_{id}(t+1) = \chi \{v_{id}(t) + c_1 r_1 (p_{id}(t) - x_{id}(t)) + c_2 r_2 (p_{gd}(t) - x_{id}(t))\} \quad (15)$$

where

$$\chi = \frac{2}{|2 - \varphi - \sqrt{\varphi^2 - 4\varphi}|}, \quad \varphi = c_1 + c_2 > 4.$$

Eberhart and Shi (2000) used constant  $\varphi = 4.1$  to ensure convergence. Hence, the values  $\chi = 0.72984$  and  $c_1 = c_2 = 2.05$  are obtained.

To apply PSO for solving the system of nonlinear equations, Eq. (5), let the form of nonlinear equations be:

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \dots \\ f_n(x_1, x_2, \dots, x_n) = 0 \end{cases} \quad (16)$$

The system of nonlinear equations (16) is equal to the optimization problem (17):

$$\min f(x) = \sum_{i=0}^n f_i^2(x) \quad (17)$$

Thus, the master function presented in Eq. (6) is similar to Eq. (17), and PSO can be applied to solve the optimization problem (6). The PSO algorithm is presented as follows:

- (1) Setup up the control parameters, and iteration  $t = 1$ .
- (2) Initialize position  $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in S$ , and velocity  $v_i = (v_{i1}, v_{i2}, \dots, v_{in}) \in S$  of each particle  $i$ .
- (3) Update position of each particle  $p_i = (p_{i1}, p_{i2}, \dots, p_{in}) \in S$ .
- (4) Evaluate objective (fitness) function of each particle  $f(\bar{x}_i)$ .
- (5) Update personal best position  $p_{id}(t)$  for each particle and swarm best position  $p_{gd}(t)$ .
- (6) If  $f(\bar{x}_i) < p_{gd}(t)$ , output the best position (global solution).
- (7) Otherwise, update iteration,  $t = t + 1$  and repeat the steps 3–6.

## 5. Illustrative examples

For an illustration and justification of our approach, we have performed several case studies to compare our results with the existing fuzzy AHP methods. MATLAB is used for implementing the PSO algorithm for solving the nonlinear optimization to derive the priorities from pairwise comparison matrices. Hereafter, we refer to our proposed method as PSO-fuzzy AHP.

**Example 1.** Consider the example given by Boender et al. (1989), where one decision maker provides the matrix of pairwise comparisons for three different criteria;  $\bar{a}_{21} = (2.5, 3, 3.5)$ ,  $\bar{a}_{31} = (4, 5, 6)$ ,  $\bar{a}_{32} = (1.5, 2, 2.5)$ , in which, particularly some of the interval judgments are inconsistent. The fuzzy solutions applied by three other existing fuzzy AHP methods are represented in Table 2. Applying the proposed PSO-fuzzy AHP method the exact weights of criteria are obtained as depicted in Table 3. It is indicated that the solution ratios lie in the corresponding interval judgments of pairwise comparisons. Moreover, the exact weights and consistency index obtained from both linear FPP and nonlinear FPP developed by Mikhailov (2003) are represented in Table 3 for comparison. It is seen that the results of proposed PSO-fuzzy AHP method are reasonable and the comparison matrix is slightly consistent since both  $\lambda^*$  or  $\gamma^*$  are between 0 and 1.

**Example 2.** In order to compare the performance of our approach with another existing method for deriving crisp priorities, we will consider the example given by extent fuzzy AHP method (Chang, 1996), which is a modification of the problem originally presented by van Laarhoven (Dubois & Prade, 1979). The problem describes

**Table 2**

Fuzzy solutions for Example 1 presented by existing fuzzy AHP methods.

Existing fuzzy AHP methods	$\bar{w}_1$	$\bar{w}_2$	$\bar{w}_3$
Buckley (1985)	(0.09, 0.11, 0.15)	(0.24, 0.31, 0.42)	(0.42, 0.58, 0.77)
van Laarhoven and Pedrycz (1983)	(0.09, 0.11, 0.13)	(0.25, 0.31, 0.40)	(0.44, 0.58, 0.75)
Boender et al. (1989)	(0.10, 0.11, 0.12)	(0.28, 0.31, 0.35)	(0.50, 0.58, 0.66)

**Table 3**

Exact solutions for Example 1 using FPP methods (Mikhailov, 2003) and PSO-fuzzy AHP.

Methods for exact solutions	$w_1$	$w_2$	$w_3$	Consistency index
Linear FPP (Mikhailov, 2003)	0.1095	0.3126	0.5779	$\lambda^* = 1.001$
Nonlinear FPP (Mikhailov, 2003)	0.1093	0.3121	0.5786	$\lambda^* = 0.708$
PSO-fuzzy AHP	0.1098	0.3131	0.5771	$\gamma^* = 0.705$

**Table 4**  
Exact solutions for Example 2.

Criteria	$C_1$	$C_2$	$C_3$	$C_4$	Weight priority vector		
					Chang (1996)	Saaty (1980)	PSO-fuzzy AHP
$C_1$	(1, 1, 1)	(0.86, 1.17, 1.56)	(0.67, 1, 1.5)	(0.33, 0.39, 0.49)	0.13	0.2025	0.1962
$C_2$	(0.64, 0.85, 1.16)	(1, 1, 1)	(2.5, 3, 3.5)	(0.95, 1.33, 1.83)	0.41	0.3242	0.2988
$C_3$	(0.87, 1, 1.49)	(0.29, 0.33, 0.40)	(1, 1, 1)	(0.4, 0.5, 0.67)	0.03	0.1465	0.1417
$C_4$	(2.04, 2.56, 3.03)	(0.55, 0.75, 1.05)	(1.49, 2, 2.5)	(1, 1, 1)	0.43	0.3268	0.3633

**Table 5**  
Exact solutions for Example 3.

Criteria	MTY	FLX	ENC	RSK	QTY	Weight priority vector	
						Nonlinear FPP (Jaganathan et al., 2007)	PSO-fuzzy AHP
MTY	(1, 1, 1)	(1/6, 1/5, 1/4)	(1, 2, 3)	(2/5, 1/2, 2/3)	(2/7, 1/3, 2/5)	0.0871	0.0909
FLX	(4, 5, 6)	(1, 1, 1)	(7, 8, 9)	(5/2, 3, 7/2)	(1, 2, 3)	0.4141	0.4104
ENC	(1/3, 1/2, 1)	(1/9, 1/8, 1/7)	(1, 1, 1)	(1/4, 1/3, 1/2)	(1/8, 1/7, 1/6)	0.0472	0.0472
RSK	(3/2, 2, 5/2)	(2/7, 1/3, 2/5)	(2, 3, 4)	(1, 1, 1)	(1/2, 2/3, 1)	0.1582	0.1568
QTY	(5/2, 3, 7/2)	(1/3, 1/2, 1)	(6, 7, 8)	(1, 3/2, 2)	(1, 1, 1)	0.2934	0.2947

MTY: Monetary, FLX: Flexibility, ENC: Environment consciousness, RSK: Risk, QTY: Quality.

choosing a professor among three serious candidates ( $A_1$ ,  $A_2$ , and  $A_3$ ) for vacant post at a university. A committee has convened to decide which applicant is best qualified for the job. The committee has three members and they have identified the following decision criteria: mathematical creativity ( $C_1$ ); creativity implementations ( $C_2$ ); administrative capabilities ( $C_3$ ); and human maturity ( $C_4$ ).

The priority values obtained by the proposed PSO-fuzzy AHP method are compared with the results of Chang's extent fuzzy AHP (Chang, 1996) and represented in Table 4. We have also solved this prioritization problem using the Saaty's conventional AHP method (Saaty, 1980). The mean of fuzzy comparison judgment is used to construct the crisp pairwise comparison matrix in AHP method and eigenvector method is applied for driving priority weights of the criteria. The results from Saaty's conventional AHP method are also depicted in Table 4. As seen, comparing to the conventional AHP, the proposed PSO-fuzzy AHP method allows better modeling of the uncertainty than the Chang's extent fuzzy AHP and is cognitively less demanding for the decision maker.

**Example 3.** As the last example, consider the example given by Jaganathan, Jinson, and Ker (2007). They applied nonlinear FPP method developed by Mikhailov (2003), to investment prioritization in new manufacturing technologies. They considered five criteria: Monetary; Flexibility; Environment consciousness; Risk; and Quality. The pairwise comparison matrix for criteria in the form of triangular fuzzy numbers is represented in Table 5. The local priority of criteria derived by both nonlinear FPP and proposed PSO-fuzzy AHP methods are also shown in Table 5 for comparison. The consistency indices calculated by the nonlinear FPP and the PSO-fuzzy AHP methods are  $\lambda^* = 0.2355$  and  $\gamma^* = 0.2356$ , respectively. As it can be observed from the aforementioned illustrative examples, the proposed PSO-fuzzy AHP method could be used as an efficient method for deriving priorities from fuzzy pairwise comparison judgments.

## 6. Conclusions

This study presents a simple fuzzy optimization model for fuzzy AHP-based multicriteria decision making systems. The fuzzy modification of the AHP method can account for the multiple forms of uncertain pairwise comparison judgments. Unlike several known fuzzy AHP methods, the proposed fuzzy prioritization method can derive crisp priorities based on a nonlinear optimization model

from both consistent and inconsistent pairwise judgments, avoiding fuzzy priority calculation and fuzzy ranking procedures. It allows for performing a prioritization problem based on an incomplete set of judgments. It is less demanding for the decision makers and may appropriately be applied to group decision making.

In the proposed nonlinear optimization method the judgments are represented as triangular fuzzy numbers, and it does not need an additional aggregation procedure. Restructuring the nonlinear optimization problem as a system of nonlinear equations, the PSO is easily applied to derive the optimal solution. The proposed method has been demonstrated by various case studies given in literature and illustrated the power of the method for solving multicriteria decision making systems in many related fields.

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