robotics engineering system

s1522921

October 2019

1 question 1

1.1

there are two matrix [a] and [b]. The relationship between them can be expressed as:

$$[\mathbf{a}] = [{}^{\mathbf{a}}\mathbf{R}_{\mathbf{b}}][\mathbf{b}] \tag{1}$$

where, ${}^{\mathbf{a}}\mathbf{Rb} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix}$

that means matrix [a] and [b] have the same magnitude but different directions.

Let matrix [b] equal to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ The relationship can be re-writted as:

$$[a] = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_4 \\ r_7 \end{bmatrix} \tag{2}$$

the magnitude of [a] is equal to that of [b] which is 1, thus:

$$\sqrt{{r_1}^2 + {r_4}^2 + {r_7}^2} = 1 \tag{3}$$

Similarly, let [b]equal to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ the results can be shown as:

$$\sqrt{{r_2}^2 + {r_5}^2 + {r_8}^2} = 1 \tag{4}$$

$$\sqrt{{r_3}^2 + {r_6}^2 + {r_9}^2} = 1 \tag{5}$$

According to the equations (3-5), $||r_i|| \leq 1$ where i=1,2,...9

1.2

Rotation matrix can be expressed as:

$$\mathbf{R} = \mathbf{I} + sin(\theta)[\mathbf{u}]_{\mathbf{x}} + (1 - cos(\theta)[\mathbf{u}]^{\mathbf{2}}{}_{\mathbf{x}}) \tag{6}$$

where, $[u]_x = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$

$$\mathbf{R} = \begin{bmatrix} cos(\theta) + u^2{}_x(1 - cos(\theta)) & u_x u_y(1 - cos(\theta)) - u_z sin(\theta) & u_x u_z(1 - cos(\theta)) + u_y sin(\theta) \\ u_y u_x(1 - cos(\theta) + u_z sin(\theta) & cos(\theta) + u^2{}_y(1 - cos(\theta)) & u_y u_z(1 - cos(\theta)) - u_x sin(\theta) \\ u_z u_x(1 - cos(\theta)) - u_y sin(\theta) & u_z u_y(1 - cos(\theta)) + u_x sin(\theta) & cos(\theta) + u^2{}_z(1 - cos(\theta)) \end{bmatrix} \tag{7}$$

as it has shown in the equation(8), -k would be changed to positive sign by ${u_k}^2$. $cos(\theta)$ is an even function, so $cos(\theta)$=$cos(-\theta)$. Even though, $sin(\theta)$ is an odd function, which mean $sin(-\theta) = sin(\theta)$, the product of $-u_k$ and $sin(-\theta)$ is the same as that of u_k and $sin(\theta)$.
Therefore,

$$\mathbf{R}_{\mathbf{k},\theta} = \mathbf{R}_{-\mathbf{k},-\theta} \tag{8}$$

1.3

A rotation from b to a around the x-axis can be shown as:

$$[\mathbf{a}] = [{}^{\mathbf{a}}\mathbf{R}_{\mathbf{b}}][\mathbf{b}] \tag{9}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & -sin(\alpha) \\ 0 & sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

As it is shown above, the first row represents the projections/transformations showing x,y,z axis of b in the x axis of a. The second row represents the projections/transformations showing x,y,z axis of b in the y axis of a. the third row represents the projections/transformations showing x,y,z axis of b in the z axis of a.

1.4

The trace of a rotation matrix can be expressed as:

$$r_1 1 + r_2 2 + r_3 3 = \lambda_1 + \lambda_2 + \lambda_3 = 1 + e^{-i\theta} + e^{i\theta} \tag{10}$$

λs are the eigenvalues whose values are 1, $e^{-i\theta}$ and $e^{i\theta}$. θ here represents the angle of rotation. The eigenvector of a rotation matrix represents the axis of rotation.

2 question 2

2.1

It is asked to convert a rotation matrix into Z-Y-Z Euler angle representation.

$$\mathbf{R} = \mathbf{R_z}\mathbf{R_y}\mathbf{R_z} = \tag{11}$$

$$\begin{bmatrix} cos(\alpha) & -sin(\alpha) & 0 \\ sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix} \begin{bmatrix} cos(\gamma) & -sin(\gamma) & 0 \\ sin(\gamma) & cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos(\alpha)cos(\beta)cos(\gamma) - sin(\alpha)sin(\gamma) & -cos(\alpha)cos(\beta)sin(\gamma) - sin(\alpha)cos(\gamma) & cos(\alpha)sin(\beta) \\ cos(\beta)sin(\alpha)sin(\gamma) + cos(\alpha)sin(\gamma) & -cos(\beta)sin(\alpha)sin(\gamma) + cos(\alpha)cos(\gamma) & sin(\alpha)sin(\beta) \\ -sin(\beta)cos(\gamma) & sin(\beta)sin(\gamma) & cos(\beta) \end{bmatrix}$$

the rotation matrix is given below:

$$\mathbf{R} = \begin{bmatrix} -\frac{\sqrt{3}}{4} - \frac{\sqrt{6}}{8} & -\frac{\sqrt{2}}{4} & -\frac{3}{4} + \frac{\sqrt{2}}{8} \\ \frac{1}{4} - \frac{3\sqrt{2}}{8} & \frac{-\sqrt{6}}{4} & \frac{\sqrt{3}}{4} + \frac{\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{8} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \end{bmatrix} \tag{12}$$

Comparing the equation (11) and (12), the angles can be calculated by:

$$cos(\beta) = \frac{\sqrt{2}}{4} ==> \beta = 1.2094rador - 1.2094rad \tag{13}$$

$$sin(\beta)sin(\gamma) = \frac{\sqrt{2}}{2} ==> \gamma = 0.8571rador - 2.2845rad \tag{14}$$

$$cos(\alpha)sin(\beta) = -\frac{3}{4} + \frac{\sqrt{2}}{8} ==> \alpha = 2.2304rador - 0.9112rad \tag{15}$$

so the first case is: $[\alpha, \beta, \gamma]$=$[2.2304, 1.2094, 0.8571]$
The second case is: $[\alpha, \beta, \gamma]$=$[-0.9112, -1.2094, -2.2845]$

2.2

there is always more than one sequence of rotations about the three principles axes that result in the same orientation of an object. As the results shown in the question 2.1, there are two sets of values for α, β, γ in the non-degenerate case of $cos(\theta) \neq 0$. Due to there are more than one way to achieve a desired rotation, a rotation matrix can have more than one Z-Y-X Euler angle representation with different rotational angles$[\alpha, \beta, \gamma]$

2.3

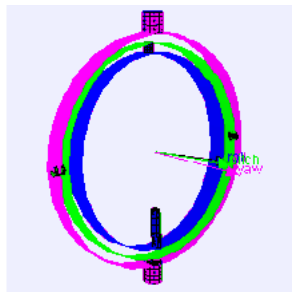


Figure 1: Gimbal lock

When using Euler angle representation, there are more than one sequence of rotation about the three principle axes which result in the same orientation of an object.Therefore, it might fall into gimbal lock in practical. As it shows in the figure 1, in the condition of Gimbal lock, that means it will lose one degree of freedom in the three dimensional mechanism that occurs when the axes of two of the three gimbals are rotated into a aligned configuration.
In order to avoid this condition in practical, it is necessary to use quaternion rather than euler angle representation. It is because in quaternion representation, it has 4 parameters to represent 3D rotation. It has enough parameters to avoid any uncertainties (gimbal lock) in practical.

3 question3

3.1

The Quaternion to rotation matrix can be expressed:

$$
{}^{0}R_{1} = \begin{bmatrix} 1-2q^2{}_y-2q^2{}_z & 2q_xq_y-2q_zq_w & 2q_xq_z+2q_yq_w \\ 2q_xq_y+2q_zq_w & 1-2q^2{}_x-2q^2{}_z & 2q_yq_z-2q_xq_w \\ 2q_xq_z-2q_yq_w & 2q_yq_z+2q_xq_w & 1-2q^2{}_x-2q^2{}_y \end{bmatrix} \tag{16}
$$

The trace of the matrix can be obtained by summing up the diagonal elements,

$$
\begin{aligned}
Trace &= 1-2q^2{}_y-2q^2{}_z+1-2q^2{}_x-2q^2{}_z+1-2q^2{}_x-2q^2{}_y \\
&= 4-4q^2{}_x-4q^2{}_y-4q^2{}_z \\
&= 4(1-q^2{}_x-q^2{}_y-q^2{}_z) \\
&= m_00+m_11+m_22+1
\end{aligned} \tag{17}
$$

A unit quaternion can be expressed as:

$$
||q|| = {q_w}^2+{q_x}^2+{q_y}^2+{q_z}^2 = 1 \tag{18}
$$

because the trace is greater than zero and , then the result is:

$$q_w = \frac{\sqrt{Trace}}{2} \tag{19}$$
$$q_x = (m_21 - m_12)/(4q_w) \tag{20}$$
$$q_y = (m_02 - m_20)/(4q_w) \tag{21}$$
$$q_z = (m_10 - m_01)/(4q_w) \tag{22}$$

In the question, the rotation matrix is:

$$R = \begin{bmatrix} -\frac{\sqrt{3}}{4} - \frac{\sqrt{6}}{8} & -\frac{\sqrt{2}}{4} & -\frac{3}{4} + \frac{\sqrt{2}}{8} \\ \frac{1}{4} - \frac{3\sqrt{2}}{8} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{4} + \frac{\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \end{bmatrix} \tag{23}$$

Therefore, its quaternion terms are:

$$q_w = \frac{\sqrt{Trace+1}}{2} = 0.0223 \tag{24}$$
$$q_x = (m_21 - m_12)/(4q_w) = -0.3604 \tag{25}$$
$$q_y = (m_02 - m_20)/(4q_w) = 0.4397 \tag{26}$$
$$q_z = (m_10 - m_01)/(4q_w) = 0.8224 \tag{27}$$

3.2

the rotation quaternion is:

$$q(\theta, \vec{v}) = q_w + q_x + q_y + q_z \tag{28}$$
$$= [cos(\theta/2), \vec{v}sin(\theta/2)]$$

Now, replacing θ with θ+2π (they are equivalent to each other) and

$$q(-\theta, -\vec{v}) = [cos(-\theta/2), -\vec{v}sin(-\theta/2)] = [cos(\theta/2), \vec{v}sin(\theta/2)] \tag{29}$$

so, q are the same as $q(-\theta, -\vec{v})$

$$R = \begin{bmatrix} 1 - 2{q^2}_y - 2{q^2}_z & 2q_xq_y - 2q_zq_w & 2q_xq_z + 2q_yq_w \\ 2q_xq_y + 2q_zq_w & 1 - 2{q^2}_x - 2{q^2}_z & 2q_yq_z - 2q_xq_w \\ 2q_xq_z - 2q_yq_w & 2q_yq_z + 2q_xq_w & 1 - 2{q^2}_x - 2{q^2}_y \end{bmatrix} \tag{30}$$

now, inserting -q into the equation (30)

$$\begin{bmatrix} 1 - 2(-q_y)^2 - 2(-q_z)^2 & 2(-q_x)(-q_y) - 2(-q_z)(-q_w) & 2(-q_x)(-q_z) + (-2q_y)(-q_w) \\ 2(-q_x)(-q_y) + 2(-q_z)(-q_w) & 1 - 2(-q_x)^2 - 2(-q_z)^2 & 2(-q_y)(-q_z) - 2(-q_x)(-q_w) \\ 2(-q_x)(-q_z) - 2(-q_y)(-q_w) & 2(-q_y)(-q_z) + 2(-q_x)(-q_w) & 1 - 2(-{q_x}^2) - 2(-{q_y}^2) \end{bmatrix} = R \tag{31}$$

The result of equation (31) is the same as that of equation (30), so rotation quaternion q and -q are equivalent.

3.3

The Conditions when two arbitrary rotation matrices R_a and R_b become commutative
1. Two of arbitrary rotation matrices are exactly the same
2. one of the arbitrary matrice is an identity matrix

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\mathbf{R} = \mathbf{R}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \tag{32}$$

3. R_a and R_b are rotating along the same axis
4. if two of arbitary roation matrices are 2×2 matrices, because in 2D rotation, they are only rotating along one axis which is the axis coming out of the page.

4 question 4

Please refers to the coding

5 question5

5.1

There are four transformations:

$$\mathbf{Rot_{x_i}}(\alpha_\mathbf{i}) = \begin{bmatrix}1 & 0 & 0 & 0\\0 & cos(\alpha_i) & -sin(\alpha_i) & 0\\0 & sin(\alpha_i) & cos(\alpha_i) & 0\\0 & 0 & 0 & 1\end{bmatrix} \tag{33}$$

$$\mathbf{Trans_{x_i}}(\alpha_\mathbf{i}) = \begin{bmatrix}1 & 0 & 0 & \alpha_i\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \tag{34}$$

$$\mathbf{Rot_{z_{i-1}}}(\theta_i) = \begin{bmatrix}cos(\theta_i) & -sin(\theta_i) & 0 & 0\\sin(\theta_i) & cos(\theta_i) & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \tag{35}$$

$$\mathbf{Trans_{z_i-1}}(\mathbf{d_i}) = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & d_i\\0 & 0 & 0 & 1\end{bmatrix} \tag{36}$$

Now combing the four transformations:

$$^{\mathbf{i-1}}T_i = \mathbf{Trans_{z_i-1}}(\mathbf{d_i})\mathbf{Rot_{z_{i-1}}}(\theta_i)\mathbf{Trans_{x_i}}(\alpha_\mathbf{i})\mathbf{Rot_{x_i}}(\alpha_\mathbf{i}) \tag{37}$$

$$
{}^{\mathbf{i-1}}T_i = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i)cos(\alpha_i) & sin(\theta_i)sin(\alpha_i) & a_i cos(\theta_i) \\ sin(\theta_i) & cos(\theta_i)cos(\alpha_i) & -cos(\theta_i)sin(\alpha_i) & a_i sin(\theta_i) \\ 0 & sin(\alpha_i) & cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The transformation from the end effector to the base reference frame can be obatined by combining the n transformation where 5 is here in this case.
four D-H parameters have been obtained in the following table:

i	$\alpha(rad)$	a (m)	$\theta(rad)$	d (m)
1	$\pi/2$	0	$0{+}\theta_1$	0.147
2	π	0.155	$\pi/2+\theta_2$	0
3	π	0.135	$0{+}\theta_3$	0
4	$\pi/2$	0	$\pi/2+\theta_4$	0
5	0	0	$0{+}\theta_5$	0.218

After inseting the Standard D-H parameters. the new transformation can be expressed as :

$$
\mathbf{{}^0T_5 = {}^0T_1{}^1T_2{}^2T_3{}^3T_4{}^4T_5} \tag{38}
$$

$$
= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0.655 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

5.2

The modified Denavite-hartenberg convention
The modified D-H parameters the transformation from the the end effector to the base frame can be expressed:

$$
\mathbf{{}^{i-1}T_i = Rot_{x_{i-1}}}(\alpha_{\mathbf{i-1}})\mathbf{Trans_{x_{i-1}}(a_{i-1})Rot_{z_i}}(\theta_{\mathbf{i}})\mathbf{Trans_{z_i}(d_i)} \tag{39}
$$

$$
= \begin{bmatrix} cos(\theta_i) & -sin(\theta_i) & 0 & a_{i-1} \\ sin(\theta_i)cos(\alpha_{i-1}) & cos(\theta_i)cos(\alpha_{i-1}) & -sin(\alpha_{i-1}) & -d_i sin(\alpha_{i-1}) \\ sin(\theta_i)sin(\alpha_{i-1}) & cos(\theta_i)sin(\alpha_{i-1}) & cos(\alpha_{i-1}) & d_i cos(\alpha_{i-1}) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The modified DH parameters for the question is shown in the following table:

i	a (m)	$\alpha(rad)$	$\theta(rad)$	d (m)
1	0	0	$0{+}\theta_1$	0.147
2	0	$\pi/2$	$\pi/2+\theta_2$	0
3	0.155	π	$0{+}\theta_3$	0
4	0.135	π	$\pi/2+\theta_4$	0
5	0	$\pi/2$	$0{+}\theta_5$	0.218
6	0	0	0	0

After inserting the modified D-H parameters, the new transformation can be expressed as:

$$
\begin{aligned}
{}^{0}\mathbf{T}_{5} &= {}^{0}\mathbf{T}_{1}{}^{1}\mathbf{T}_{2}{}^{2}\mathbf{T}_{3}{}^{3}\mathbf{T}_{4}{}^{4}\mathbf{T}_{5} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0.655 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{40}
$$

5.3

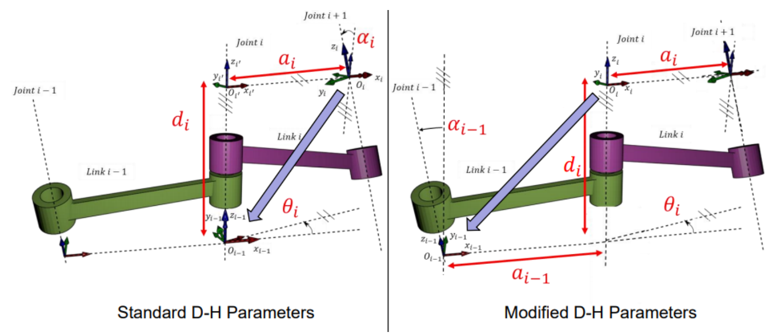


Figure 2: Comparison between Standard and Modified D-H Parameters

As it is shown in the figure 4, for the Standard D-H parameters, Both of the i^{th} frame and the $(i-1)^{th}$is on the i^{th} joint axis. However, for the Modified D-H parameters, the i^{th} frame is on the axis of joint i^{th} and the $(i-1)^{th}$ frame is on the axis of joint $(i-1)^{th}$.

Standard D-H Parameters

$$^{i-1}T_i = Trans_{z_{i-1}}(d_i)\ Rot_{z_{i-1}}(\theta_i)\ Trans_{x_i}(a_i)\ Rot_x(\alpha_i)$$

$$^{i-1}T_i = \begin{bmatrix} cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Modified D-H Parameters

$$^{i-1}T_i = Rot_{x_{i-1}}(\alpha_{i-1})\ Trans_{x_{i-1}}(a_{i-1})\ Rot_{z_i}(\theta_i)\ Trans_{z_i}(d_i)$$

$$^{i-1}T_i = \begin{bmatrix} cos(\theta_i) & -\sin(\theta_i) & 0 & a_{i-1} \\ \sin(\theta_i)\cos(\alpha_{i-1}) & \cos(\theta_i)\cos(\alpha_{i-1}) & -sin(\alpha_{i-1}) & -d_i\sin(\alpha_{i-1}) \\ \sin(\theta_i)\sin(\alpha_{i-1}) & \cos(\theta_i)\sin(\alpha_{i-1}) & \cos(\alpha_{i-1}) & d_i\cos(\alpha_{i-1}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 3: Formula Comparison between Standard and Modified D-H Parameters

As it is shown in the figure 3, for the Standard D-H parameters, it does the translation part first and then rotation part. However in the Modified D-H Parameters, it does the other way around. Two of transformation matrices has been shown in the figure 3 too.

5.4

Please, refers to the coding

6 question6

6.1

The first error calculating method in rigid transformation is represented as:

$$\mathbf{T_{error1}} = \mathbf{T^{-1}}_{\mathbf{gt}}\mathbf{T_{est}} = \begin{bmatrix} R_{gt} & t_{gt} \\ 0_{1\times3} & 1 \end{bmatrix}^{-1} \begin{bmatrix} R_{es} & t_{es} \\ 0_{1\times3} & 1 \end{bmatrix} \tag{41}$$

$$= \begin{bmatrix} R_{gt}^T & -R_{gt}^T t_{gt} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} R_{es} & t_{es} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} R_{gt}^T R_{es} & R_{gt}^T t_{es} - R_{gt}^T t_{gt} \\ 0 & 1 \end{bmatrix}$$

where, $\mathbf{T_{gt}}$ is the ground truth of the transformation and $\mathbf{T_{est}}$ is the estimation of the transformation.
The rotation error is$R_{gt}^T R_{es}$ and the translation error is $R_{gt}^T t_{es} - R_{gt}^T t_{gt}$
The second error calculating methods:

$$\mathbf{T_{error1}} = \mathbf{T_{gt}}\mathbf{T_{est}^{-1}} = \begin{bmatrix} R_{gt} & t_{gt} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} R_{es} & t_{es} \\ 0_{1\times3} & 1 \end{bmatrix}^{-1} \tag{42}$$

$$= \begin{bmatrix} R_{gt}^T & t_{gt} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} R_{es}^T & -R_{es}^T t_{es} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} R_{gt} R_{es}^T & -R_{gt} R_{es}^T t_{es} + t_{gt} \\ 0 & 1 \end{bmatrix}$$

the rotation error is:$R_{gt}R_{es}^T$ and the translation error is:$-R_{gt}R_{es}^T t_{es} + t_{gt}$

Now the rotation part is in the form of rotation matrix, let's transform it to the rodrigues representation:
The general rotation matrix is:

$$\mathbf{R} = \begin{bmatrix} cos(\theta) + u^2{}_x(1 - cos(\theta)) & u_x u_y(1 - cos(\theta)) - u_z sin(\theta) & u_x u_z(1 - cos(\theta)) + u_y sin(\theta) \\ u_y u_x(1 - cos(\theta) + u_z sin(\theta) & cos(\theta) + u^2{}_y(1 - cos(\theta)) & u_y u_z(1 - cos(\theta)) - u_x sin(\theta) \\ u_z u_x(1 - cos(\theta)) - u_y sin(\theta) & u_z u_y(1 - cos(\theta)) + u_x sin(\theta) & cos(\theta) + u^2{}_z(1 - cos(\theta)) \end{bmatrix} \tag{43}$$

The angle θ is shown as:

$$\theta = arcos(\frac{R_{1,1} + R_{2,2} + R_{3,3} - 1}{2}) \tag{44}$$

the Rodrigues representation can be expressed as:

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \frac{1}{2sin(\theta)} \begin{bmatrix} R_{3,2} - R_{2,3} \\ R_{1,3} - R_{3,1} \\ R_{2,1} - R_{1,2} \end{bmatrix} \tag{45}$$

By inserting the rotation matrix 6.1 into the angle and rodrigues representation equation6.16.1, it turns out that the angle calculated from the first and second methods are the same, but the rodrigues representation (u) are different. The translation parts are totally different as it shows before.

6.2

The error using the first method is shown as:

$$\mathbf{T_{est}^{-1}T_{gt}} = \begin{bmatrix} R_{est}^T & -R_{est}^T t_{est} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} R_{gt} & t_{gt} \\ 0_{1\times3} & 1 \end{bmatrix} \tag{46}$$

$$= \begin{bmatrix} R_{est}^T R_{gt} & R_{est}^T t_{gt} - R_{est}^T t_{est} \\ 0_{1\times3} & 1 \end{bmatrix}$$

The error using the second methods is shown as:

$$\mathbf{T_{gt}^{-1}T_{est}} = \begin{bmatrix} R_{gt}^T & -R_{gt}^T t_{gt} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} R_{est} & t_{est} \\ 0_{1\times3} & 1 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} R_{gt}^T R_{est} & R_{gt}^T t_{est} - R_{gt}^T t_{gt} \\ 0_{1\times3} & 1 \end{bmatrix}$$

In the translational part, the rotational matrix will only change the direction but not the magnitude(norm), thus we just need to compare between $t_{gt} - t_{est}$ and $t_{est} - t_{gt}$.
The norms of translation part for the first method can be calculated by:

$$Norm_1 = \sqrt{(x_{gt} - x_{est})^2 + (y_{gt} - y_{est})^2 + (z_{gt} - z_{est})^2} \tag{48}$$

The norms of translation part for the second method can be calculated by:

$$Norm_2 = \sqrt{(x_{est} - x_{gt})^2 + (y_{est} - y_{gt})^2 + (z_{est} - z_{gt})^2} \tag{49}$$

Therefore, $Norm_1$ is equal to $Norm_2$, thus the error in the translation will be the same.

6.3

Recollecting, the estimated transformation matrix is:

$$\mathbf{T_{est}} = \mathbf{{}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5} \tag{50}$$

The error happening at $\mathbf{{}^0T_1}$, so we need to update $\theta_1 = 0 + 0.5°$ and keep the rest of D-H parameters the same.

$${}^\mathbf{0}T_1 = \begin{bmatrix} cos(\theta_1) & -sin(\theta_1)cos(\alpha_1) & sin(\theta_1)sin(\alpha_1) & a_1 cos(\theta_1) \\ sin(\theta_1) & cos(\theta_1)cos(\alpha_1) & -cos(\theta_1)sin(\alpha_1) & a_1 sin(\theta_1) \\ 0 & sin(\alpha_1) & cos(\alpha_1) & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{51}$$

The ground truth transformation matrix will remain the same in the equation

Therefore, the positioning error in the translation component when $\theta_1 = 0{+}0.5°$:

$$T_{error} = T_{gt}^{T} T_{est} = \begin{bmatrix} 1 & -0.0087 & 0 & 0 \\ 0.0087 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{52}$$

The norm of the translation component of T_{error} can be expressed:

$$Norm_{translation} = \sqrt{0^2 + 0^2 + 0^2} = 0 \tag{53}$$

6.4

Recollecting, the estimated transformation matrix is:

$$\mathbf{T_{est}} = \mathbf{{}^{0}T_{1}{}^{1}T_{2}{}^{2}T_{3}{}^{3}T_{4}{}^{4}T_{5}} \tag{54}$$

The error happening at $\mathbf{{}^{3}T_{4}}$, so we need to update $\theta_4 = \frac{\pi}{2} + 0.5°$ and keep the rest of D-H parameters the same.

$$\mathbf{{}^{3}}T_4 = \begin{bmatrix} cos(\theta_4) & -sin(\theta_4)cos(\alpha_4) & sin(\theta_4)sin(\alpha_4) & a_4 cos(\theta_4) \\ sin(\theta_4) & cos(\theta_4)cos(\alpha_4) & -cos(\theta_4)sin(\alpha_4) & a_4 sin(\theta_4) \\ 0 & sin(\alpha_4) & cos(\alpha_4) & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{55}$$

The ground truth transformation matrix will remain the same in the equation

Therefore, the positioning error in the translation component when $\theta_4 = \frac{\pi}{2} + 0.5°$:

$$T_{error} = T_{gt}^{T} T_{est} = \begin{bmatrix} 1 & 0 & 0.0087 & 0.0019 \\ 0 & 1 & 0 & 0 \\ -0.0087 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{56}$$

The norm of the translation component of T_{error} can be expressed:

$$Norm_{translation} = \sqrt{0.0019^2 + 0^2 + 0^2} = 0.0019 \tag{57}$$

6.5

Recollecting, the estimated transformation matrix is:

$$\mathbf{T_{est}} = \mathbf{{}^{0}T_{1}{}^{1}T_{2}{}^{2}T_{3}{}^{3}T_{4}{}^{4}T_{5}} \tag{58}$$

The error happening at $\mathbf{{}^{1}T_{2}}$, so we need to update $\theta_2 = \frac{\pi}{2} + 0.5°$ and keep the rest of D-H parameters the same.

$$\mathbf{{}^{1}}T_2 = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2)cos(\alpha_2) & sin(\theta_2)sin(\alpha_2) & a_2 cos(\theta_2) \\ sin(\theta_2) & cos(\theta_2)cos(\alpha_2) & -cos(\theta_2)sin(\alpha_2) & a_2 sin(\theta_2) \\ 0 & sin(\alpha_2) & cos(\alpha_2) & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{59}$$

The ground truth transformation matrix will remain the same in the equation
Therefore, the positioning error in the translation component when $\theta_2 = \frac{\pi}{2} + 0.5°$:

$$T_error = T_{gt}^T T_{est} = \begin{bmatrix} 1 & 0 & 0.0087 & 0.0044 \\ 0 & 1 & 0 & 0 \\ -0.0087 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{60}$$

The norm of the translation component of T_{error} can be expressed:

$$Norm_{translation} = \sqrt{0.0044^2 + 0^2 + 0^2} = 0.0044 \tag{61}$$

6.6

$$^{\mathbf{i-1}}T_i = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i)cos(\alpha_i) & sin(\theta_i)sin(\alpha_i) & a_i cos(\theta_i) \\ sin(\theta_i) & cos(\theta_i)cos(\alpha_i) & -cos(\theta_i)sin(\alpha_i) & a_i sin(\theta_i) \\ 0 & sin(\alpha_i) & cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{62}$$

According to the equation 62, it shows that different D-H parameters at different joint position which is i, determine different transformations.

The transformation from the end effector to the base reference frame is:

$$^0T_n = {}^0T_1 {}^1T_2 ... {}^{n-2}T_{n-1} {}^{n-1}T_n \tag{63}$$

As you can see from the equation 63, The complete transformation is calculated by different transformations from one joint to the next joint.

Conclusively, the error in each joint mean the errors in four D-H parameters would result in different transformations $^{n-1}T_n$. Therefore the error in position reading in each joint produce different errors.

In a bigger robot, the error in the position reading in each joint is more obvious. The error happening in the joint which is nearer to the base link, will result in larger errors showing in the end-effector.

7 question7

7.1

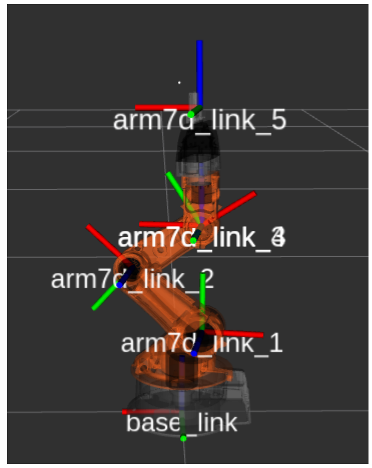
from 2019-11-05 21-28-41.png 

Figure 4: KUKA robot

The results of D-h parameter has been calculated and shown below:

i	α (rad)	a (m)	θ (rad)	d (m)
1	$\pi/2$	0.033	$\frac{170^\circ\times\pi}{180^\circ}+\theta_1$	0.145
2	0	0.155	$\frac{155^\circ\times\pi}{180^\circ}+\theta_2$	0
3	0	0.135	$\frac{-146^\circ\times\pi}{1806\circ}+\theta_3$	0
4	$\pi/2$	0	$\frac{-167.5^\circ}{1806\circ}+\theta_4$	0
5	0	0.002	$0+\theta_5$	0.185

7.2

please, refers to the coding