Making Decisions Against an Opponent

- Two people/teams in competition
- Outcome depends on both sides' simultaneous decisions
- Many examples in business, sports, economics, and even everyday life

Simpsons Decision-Making

vs.

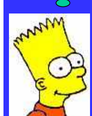
Bart Lisa

Rock-Paper-Scissors

Good old rock… You can always count on rock.

Poor, predictable Bart…always chooses rock.

Lisa's Optimal Choice



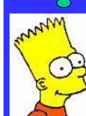
	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

1 = Bart wins -1 = Lisa wins

What if…?

Good old rock… 60% of the time, you can count on rock.

?



	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

1 = Bart wins -1 = Lisa wins

What if…?

Good old rock… 60% of the time, you can count on rock.

Paper!

	Rock	Paper	Scissors
60% Rock	0	-1	1
20% Paper	1	0	-1
20% Scissors	-1	1	0
Average =	0	-0.4	+0.4

1 = Bart wins -1 = Lisa wins

So, what's the outcome?

	Rock	Paper	Scissors
Average =	0	-0.4	+0.4

- Lisa should choose the column with the best results (paper), and wins 40% more
- In general, Bart's overall result equals the value of Lisa's best (lowest) column.
- But… if Bart knows Lisa will choose paper, he should choose scissors… and so on

Bart’s Mathematical Model

x = probability that Bart chooses rock,
y = probability that Bart chooses paper,
z = probability that Bart chooses scissors

	Rock	Paper	Scissors
(x) Rock	0	-1	1
(y) Paper	1	0	-1
(z) Scissors	-1	1	0
Average =	y - z	z - x	x - y

v = total value (smallest of column totals)

- Let’s use a Linear Programming Model

Linear Programming Models

- “Objective Function” - minimize or maximize
- “Constraints” - mathematical restrictions
- All linear functions
  - No powers (x^2)
  - No square roots, trig functions, logs, etc.
  - No variables multiplied by each other
- Find values for the variables that
  - satisfy all constraints
  - give the best (optimal) objective function value

Bart’s Linear Program

	Rock	Paper	Scissors
(x) Rock	0	-1	1
(y) Paper	1	0	-1
(z) Scissors	-1	1	0
Average =	y - z	z - x	x - y

Maximize v

Subject to

$v \le y - z$ (Rock)

$v \le z - x$ (Paper)

$v \le x - y$ (Scissors)

$x + y + z = 1$

$x \ge 0, y \ge 0, z \ge 0$

Lisa’s Mathematical Model

a = probability that Lisa chooses rock,
b = probability that Lisa chooses paper,
c = probability that Lisa chooses scissors

	Rock(a)	Paper(b)	Scissors(c)	Avg.
Rock	0	-1	1	c - b
Paper	1	0	-1	a - c
Scissors	-1	1	0	b - a

d = total value (largest of row totals)

- Another Linear Programming Model

Lisa’s Linear Program

	Rock(a)	Paper(b)	Scissors(c)	Avg.
Rock	0	-1	1	c - b
Paper	1	0	-1	a - c
Scissors	-1	1	0	b - a

Minimize d

Subject to

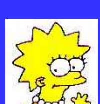
$d \ge b - c$ (Rock)

$d \ge c - a$ (Paper)

$d \ge a - b$ (Scissors)

$a + b + c = 1$

$a \ge 0, b \ge 0, c \ge 0$



Two Optimization Models

- Bart’s Model 
  - Maximize outcome
  - Outcome = lowest column average
  - Solution: x = y = z = 1/3
  - Value = 0
- Lisa’s Model 
  - Minimize outcome
  - Outcome = highest row average
  - Solution: a = b = c = 1/3
  - Value = 0

Note: The two values will always be equal

A More Interesting Example: 2-point conversions in football

- Offense gets the ball at 3 yard line, has one play to score
- Many offensive play choices
- Many defensive strategies

So, let's simplify a little bit…

Simplified Model: Run or Pass

- Offense can choose to run or to pass
- Defense can choose run defense or pass defense

	Run Defense	Pass Defense
Run	30%	50%
Pass	80%	40%

Success Rates

What do you expect?

Two Linear Programs

	Run (a) Defense	Pass (b) Defense
Run (x)	30%	50%
Pass (y)	80%	40%

Offense

Maximize v

Subject to $v \le .3x + .8y$

$v \le .5x + .4y$

$x + y = 1$

$x \ge 0,\ y \ge 0$

Defense

Minimize d

Subject to $d \ge .3a + .5b$

$d \ge .8a + .4b$

$a + b = 1$

$a \ge 0,\ b \ge 0$

Defense's Optimization

Minimize d

Subject to $d \ge .3a + .5b$

$d \ge .8a + .4b$

$a + b = 1$

$a \ge 0,\ b \ge 0$

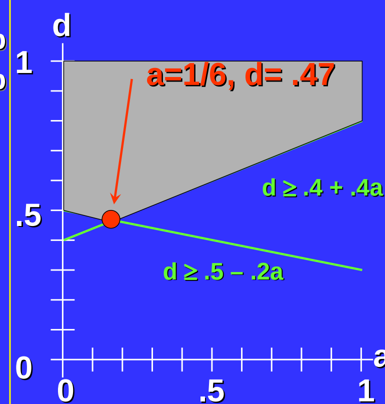
- Substitute $b = 1 - a$

Minimize d

Subject to $d \ge .5 - .2a$

$d \ge .4 + .4a$

$0 \le a \le 1$



Offense's Optimization

Maximize v

Subject to $v \le .3x + .8y$

$v \le .5x + .4y$

$x + y = 1$

$x \ge 0,\ y \ge 0$

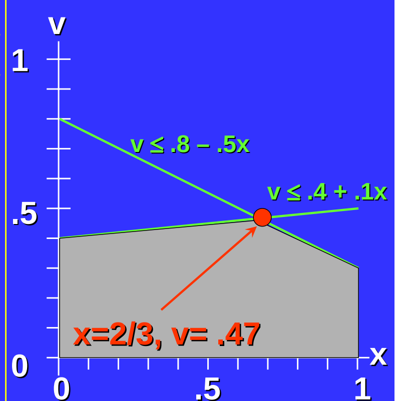
- Substitute $y = 1 - x$

Maximize v

Subject to $v \le .8 - .5x$

$v \le .4 + .1x$

$0 \le x \le 1$



Summary

- Start with a complex problem (making decisions against an opponent)
- Create a simplified mathematical model (Optimization/linear programming)
- Use the model to suggest a strategy (in this case, a surprising strategy!)

Slides at www.isye.gatech.edu/~jsokol/2030.ppt