Linear Programming

What is Linear Programming?

- A method for solving certain types of minimization/maximization problems.
- The problems must be expressible by a system of inequalities
- The inequalities must have very specific constraints.

Our first example:

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & \end{array}$$

$$\begin{array}{rcr} 4x_1 - x_2 & \leq & 8 \\ 2x_1 + x_2 & \leq & 10 \\ 5x_1 - 2x_2 & \geq & -2 \\ x_1, x_2 & \geq & 0 \end{array}$$

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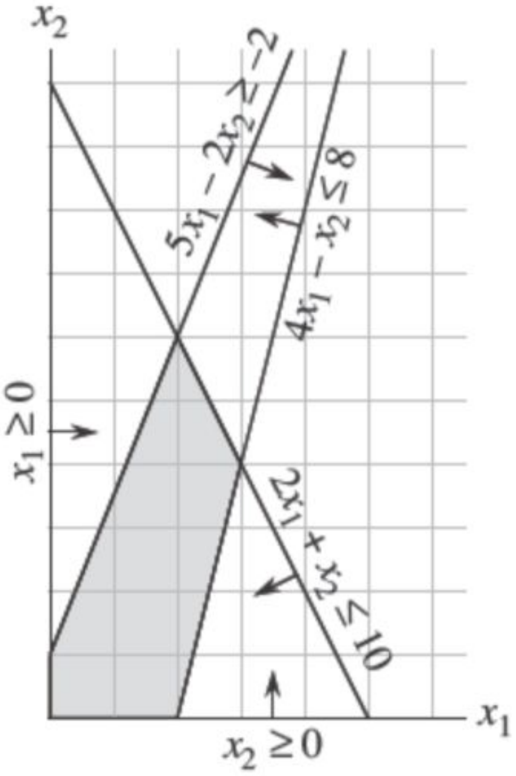
Let's graph it!



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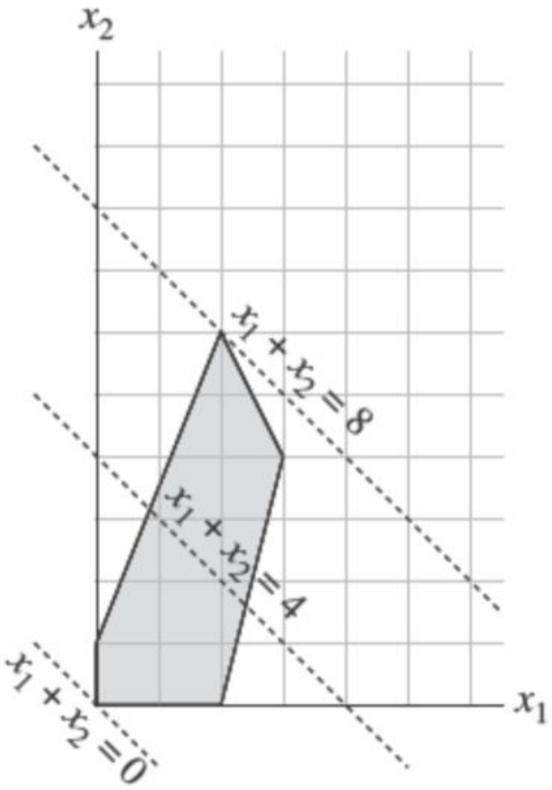
And, we are done...



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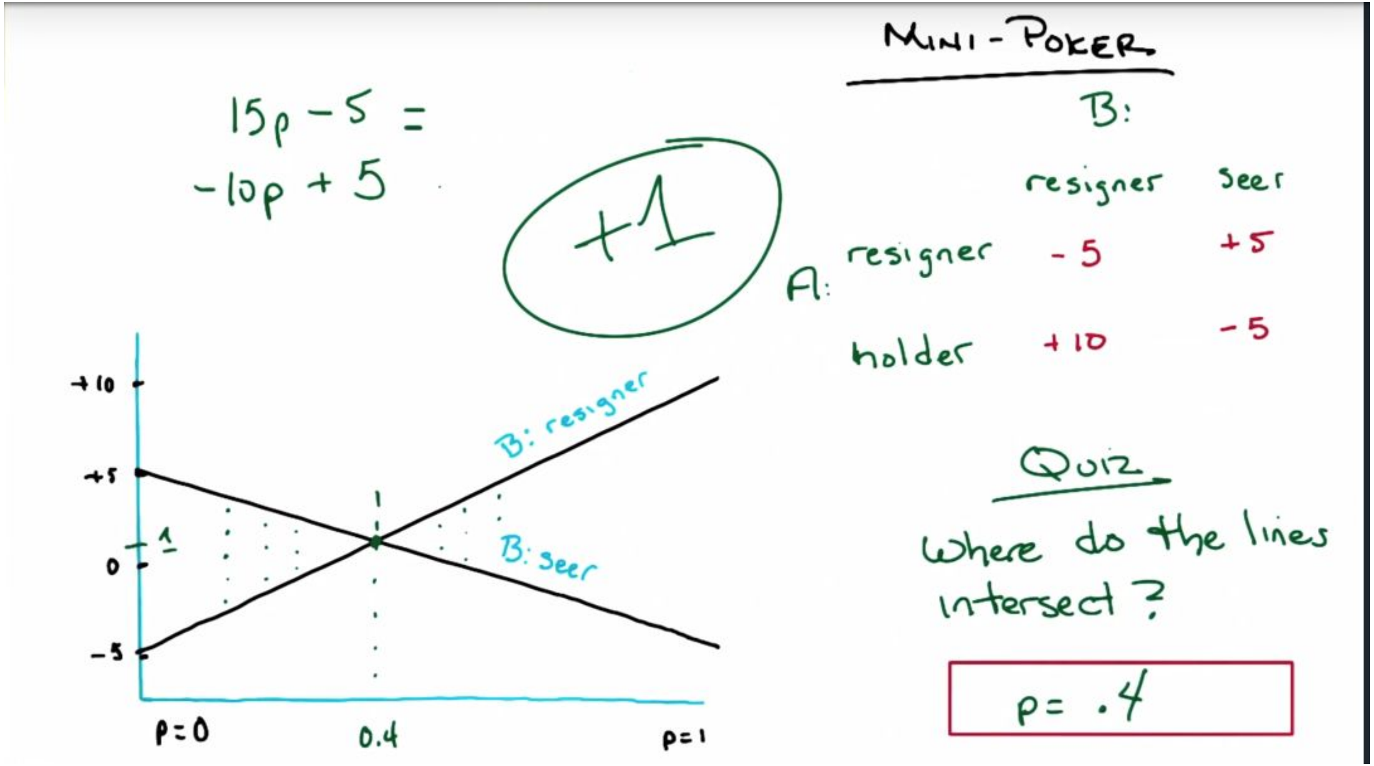
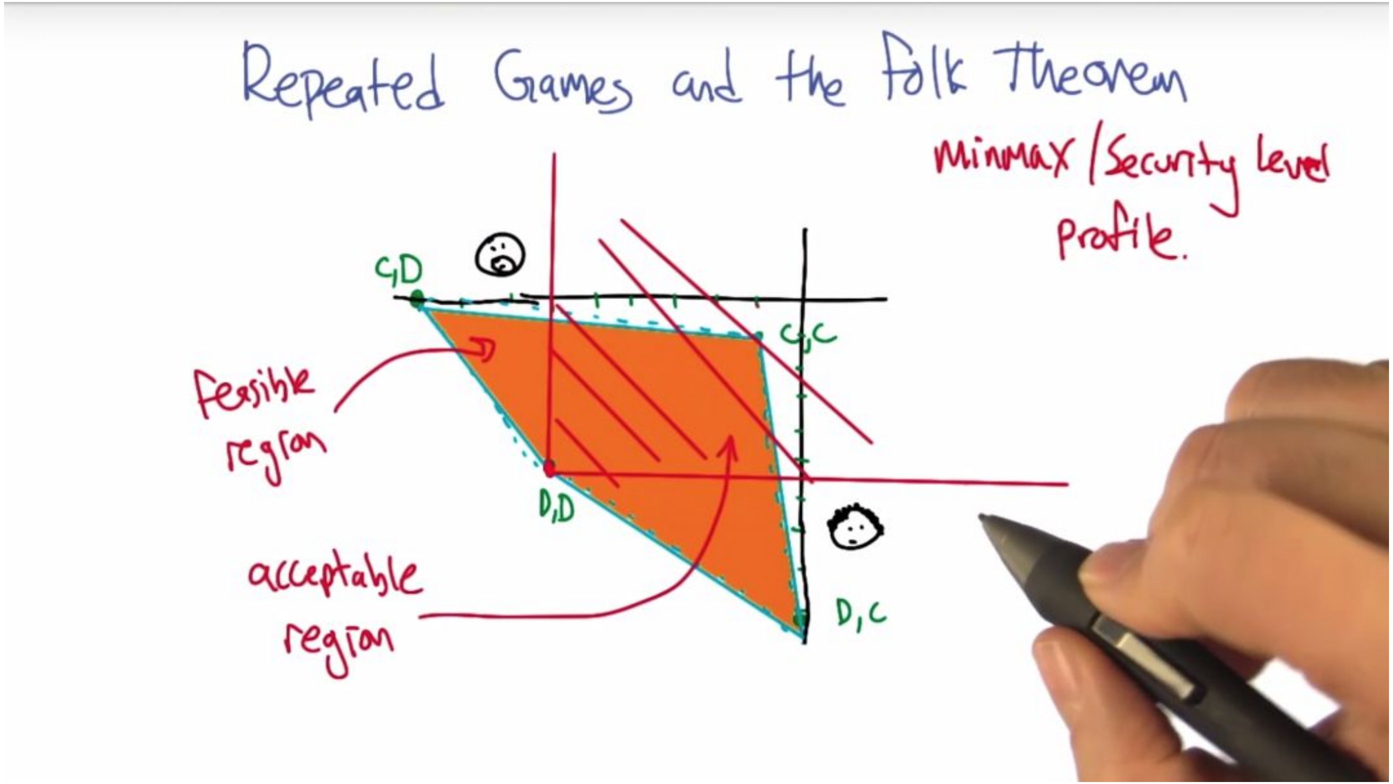
This should look familiar



Udacity Reinforcement Learning
Lesson 12: 11A Game Theory
16. Center Game

Seriously, really familiar…



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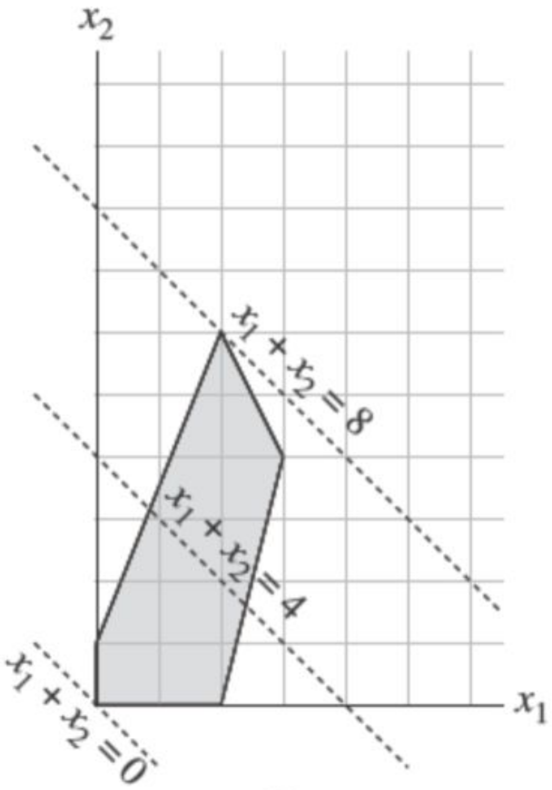
Lesson 13: 11B Game Theory Reloaded
13. Security Level Profile

So, how do we use a computer to solve this?



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Linear Programming!

- Get yourself a LP solver: glpk, CVXOPT, SCPSolver, etc...

- Express your problem as a system of inequalities.

- Run your solver on the resulting matrix.

Back to our example

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & \end{array}$$

$$\begin{aligned} 4x_1 - x_2 &\le 8 \\ 2x_1 + x_2 &\le 10 \\ 5x_1 - 2x_2 &\ge -2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

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Almost there

$$
\begin{array}{ll}
\text{minimize} & -x_1 - x_2 \\
\text{subject to} & \\
& \begin{aligned}
4x_1 - x_2 &\leq 8 \\
2x_1 + x_2 &\leq 10 \\
5x_1 - 2x_2 &\geq -2 \\
x_1, x_2 &\geq 0
\end{aligned}
\end{array}
$$

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"Standard Form"

$$\begin{aligned} \text{minimize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & \end{aligned}$$

$$\begin{aligned} 4x_1 - \quad x_2 &\leq \quad 8 \\ 2x_1 + \quad x_2 &\leq \quad 10 \\ -5x_1 + 2x_2 &\leq \quad 2 \\ x_1, x_2 \quad &\geq \quad 0 \end{aligned}$$

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CVXOPT LP Solver

http://cvxopt.org/userguide/coneprog.html#cvxopt.solvers.lp

cvxopt.solvers.lp(*c*, *G*, *h*[, *A*, *b*[, *solver*[, *primalstart*[, *dualstart*]]]])

Explicit Multipliers

$$
\begin{array}{ll}
\text{minimize} & -1x_1 - 1x_2 \\
\text{subject to} &
\end{array}
$$

$$
\begin{aligned}
4x_1 - \ 1x_2 &\leq \ \ 8 \\
2x_1 + 1x_2 &\leq 10 \\
-5x_1 + 2x_2 &\leq \ \ 2 \\
x_1, x_2 \quad &\geq \ \ 0
\end{aligned}
$$

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CVXOPT LP Parameters

$$\text{minimize} \quad \underbrace{-1x_1 - 1x_2}_{c}$$

$$\text{subject to}$$

$$\begin{array}{rrrcr}
\text{G} & & \text{G} & & \text{h} \\
4x_1 & - & 1x_2 & \leq & 8 \\
2x_1 & + & 1x_2 & \leq & 10 \\
-5x_1 & + & 2x_2 & \leq & 2 \\
 & x_1, & x_2 & \geq & 0
\end{array}$$

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CVXOPT LP Parameters

c $\boxed{[-1, -1]}$

G $\boxed{[4, 2, -5], [-1, 1, 2]}$

h $\boxed{[8, 10, 2]}$

cvxopt.solvers.lp(*c*, *G*, *h*[, *A*, *b*[, *solver*[, *primalstart*[, *dualstart*]]]])

Code

```
from cvxopt import matrix, solvers


c = matrix([-1., -1.])
G = matrix([[4., 2., -5.], [-1., 1., 2.]])
h = matrix([8., 10., 2.])

solution = solvers.lp(c, G, h)
```

https://github.com/axonal/cvxopt-tutorial/blob/master/clrs.py

Just a small bit harder… The game of "chicken"

	L	R
T	6,6	2,7
B	7,2	0,0

The correlated equilibria in this game are described by the probability constraints $\pi_{TL} + \pi_{TR} + \pi_{BL} + \pi_{BR} = 1$ and $\pi_{TL}, \pi_{TR}, \pi_{BL}, \pi_{BR} \geq 0$ together with the following rationality constraints:

$$-1\pi_{TL} + 2\pi_{TR} \geq 0 \qquad -1\pi_{TL} + 2\pi_{BL} \geq 0$$
$$1\pi_{BL} - 2\pi_{BR} \geq 0 \qquad 1\pi_{TR} - 2\pi_{BR} \geq 0$$

Correlated-Q Learning
Greenwald/Hall 2003

Four Inequalities

	3	4
	L	R
1 T	6,6	2,7
2 B	7,2	0,0

The correlated equilibria in this game are described by the probability constraints $\pi_{TL}+\pi_{TR}+\pi_{BL}+\pi_{BR}=1$ and $\pi_{TL},\pi_{TR},\pi_{BL},\pi_{BR}\geq 0$ together with the following rationality constraints:

$$-1\pi_{TL}+2\pi_{TR}\geq 0 \qquad -1\pi_{TL}+2\pi_{BL}\geq 0$$
$$1\pi_{BL}-2\pi_{BR}\geq 0 \qquad 1\pi_{TR}-2\pi_{BR}\geq 0$$

Correlated-Q Learning
Greenwald/Hall 2003

Four Inequalities

	L	R
1 T	6,6	2,7
B	7,2	0,0

Intuitively, the expected reward to the row player of action T is at least that of action B whenever he in fact plays action T.

$$6\pi_{L|T} + 2\pi_{R|T} \geq 7\pi_{L|T} + 0\pi_{R|T}$$

Correlated-Q Learning
Greenwald/Hall 2003

Four Inequalities

$$6\pi_{L|T} + 2\pi_{R|T} \geq 7\pi_{L|T} + 0\pi_{R|T}$$
$$-1\pi_{L|T} + 2\pi_{R|T} \geq 0$$
$$-1\pi_{TL} + 2\pi_{TR} \geq 0$$

By Bayes Rule:

$$\pi_{xy} = \pi(x|y)\pi(y) = \pi(x \cap y) = \pi(y|x)\pi(x) = \pi_{yx}$$

Four Inequalities

Formula 1

$$6\pi_{L|T} + 2\pi_{R|T} \geq 7\pi_{L|T} + 0\pi_{R|T}$$
$$-1\pi_{L|T} + 2\pi_{R|T} \geq 0$$
$$-1\pi_{TL} + 2\pi_{TR} \geq 0$$

Formula 2

$$7\pi_{L|B} + 0\pi_{R|B} \geq 6\pi_{L|B} + 2\pi_{R|B}$$
$$1\pi_{L|B} - 2\pi_{R|B} \geq 0$$
$$1\pi_{BL} - 2\pi_{BR} \geq 0$$

Formula 3

$$6\pi_{T|L} + 2\pi_{B|L} \geq 7\pi_{T|L} + 0\pi_{B|L}$$
$$-1\pi_{T|L} + 2\pi_{B|L} \geq 0$$
$$-1\pi_{TL} + 2\pi_{BL} \geq 0$$

Formula 4

$$7\pi_{T|R} + 0\pi_{B|R} \geq 6\pi_{T|R} + 2\pi_{B|R}$$
$$1\pi_{T|R} - 2\pi_{B|R} \geq 0$$
$$1\pi_{TR} - 2\pi_{BR} \geq 0$$

note: $\pi_{xy} = \pi(x|y)\pi(y) = \pi(x \cap y) = \pi(y|x)\pi(x) = \pi_{yx}$

Four Inequalities, four variables: it is party time.

$$\pi_{TL} + \pi_{TR} + \pi_{BL} + \pi_{BR} = 1$$

s.t.

$$-1\pi_{TL} + 2\pi_{TR} \geq 0$$
$$1\pi_{BL} - 2\pi_{BR} \geq 0$$
$$-1\pi_{TL} + 2\pi_{BL} \geq 0$$
$$1\pi_{TR} - 2\pi_{BR} \geq 0$$

$$x_1 = \pi_{TL},\ x_2 = \pi_{TR},\ x_3 = \pi_{BL},\ x_4 = \pi_{BR}$$

"Standard Form"?

$$1x_1 + 1x_2 + 1x_3 + 1x_4 = 1$$

s.t.

$$-1x_1 + 2x_2 \geq 0$$

$$1x_3 - 2x_4 \geq 0$$

$$-1x_1 + 2x_3 \geq 0$$

$$1x_2 - 2x_4 \geq 0$$

"Standard Form"?

$$1x_1 + 1x_2 + 1x_3 + 1x_4 = 1$$

s.t.

$$\begin{aligned} 1x_1 - 2x_2 &\leq 0 \\ -1x_3 + 2x_4 &\leq 0 \\ 1x_1 - 2x_3 &\leq 0 \\ -1x_2 + 2x_4 &\leq 0 \end{aligned}$$

New parameters

$$\underbrace{1x_1 + 1x_2 + 1x_3 + 1x_4}_{A} = \underbrace{1}_{b}$$

s.t.

$$\begin{aligned} 1x_1 - 2x_2 &\leq 0 \\ -1x_3 + 2x_4 &\leq 0 \\ 1x_1 - 2x_3 &\leq 0 \\ -1x_2 + 2x_4 &\leq 0 \end{aligned}$$

Old parameters

$$1x_1 + 1x_2 + 1x_3 + 1x_4 = 1$$

s.t.

$$
\begin{array}{rcrcrcrcl}
G & & G & & G & & G & & h \\
1x_1 & - & 2x_2 & + & 0x_3 & + & 0x_4 & \leq & 0 \\
0x_1 & + & 0x_2 & - & 1x_3 & + & 2x_4 & \leq & 0 \\
1x_1 & + & 0x_2 & - & 2x_3 & + & 0x_4 & \leq & 0 \\
0x_1 & - & 1x_2 & + & 0x_3 & + & 2x_4 & \leq & 0
\end{array}
$$

Code

```
from cvxopt import matrix, solvers


c = matrix([1., 1., 1., 1.])
G = matrix([[1., 0., 1., 0.], [-2., 0., 0., -1.],
            [0., -1., -2., 0.], [0., 2., 0., 2.]])
h = matrix([0., 0., 0., 0.])

A = matrix([[1.], [1.], [1.], [1.]])
b = matrix(1.)

solution = solvers.lp(c, G, h, A, b)
```

https://github.com/axonal/cvxopt-tutorial/blob/master/chicken.py

Presentation and Code samples can be found at:
https://github.com/axonal/cvxopt-tutorial